This is the published version:


Available from Deakin Research Online:

http://hdl.handle.net/10536/DRO/DU:30000930

Reproduced with the kind permission of the copyright owner.

Copyright : 2006, Sense Publishers
CHAPTER FIFTEEN

Autonomous Looking-In to Support Creative Mathematical Thinking: Capitalising on Activity in Australian LPS Classrooms

INTRODUCTION

The lesson was almost finished when Leon suddenly realised something which changed the direction and intensity of his focus:

Towards the end of the lesson, I started understanding it and I don’t know but for some reason my goal sort of changed a little bit. ... it changed from finishing the work to actually understanding the work [L12-SI-Leon].

During the next lesson, Leon softly exclaimed ‘Oh!’ as what he had puzzled over suddenly became clear. Leon explained in his interview after the lesson: “It just sorta clicks into your head and- and you think oh! I know this now”. When Leon recalled what had occurred, he described features that fitted ‘flow’ conditions:

[I] really didn’t understand ... it was a bit of a challenge ... when I finally did understand it- it really made me feel good about myself [L13-SI-Leon].

Flow is a state of high positive affect during creative activity (Csikszentmihalyi, 1992). It occurs when people work to overcome a ‘spontaneously’ set challenge that is almost out of reach and develop new skills during the process. Leon spontaneously set himself an intellectual challenge: to develop new conceptual understanding thus setting up the conditions for flow specific to mathematical exploration (‘discovering complexity’) that I had previously identified (Williams, 2002a).

Discovering complexity involves a student or group of students spontaneously setting up an intellectual challenge and developing novel conceptual understanding as they overcome that challenge. My use of the term ‘spontaneous’ is consistent with its use by Steffe and Thompson:

We do not use spontaneous in the context of learning to indicate the absence of elements with which the student interacts. Rather we use the term to refer to the non-causality of teaching actions, to the self regulation of the students when interacting. (Steffe & Thompson, 2000, p. 291)
This chapter explores the process of discovering complexity where complex mathematical thinking and high positive affect co-exist during the creative development of novel (to the student) mathematical concepts. In particular, this chapter examines ‘Looking-In activity’ (that Leon undertook just prior to his goal change) and its role in this process.

Leon was a Year 8 student in Australian School 1 in the Learner’s Perspective Study (LPS) who used the term ‘Looking-In’ to describe his own activity. I found this same activity common to two of the other three students who were identified discovering complexity in other Australian LPS classrooms. Some students engaged in Looking-In activity on more than one occasion. Looking-In (Williams, 2004) is an idiosyncratic student activity that supports creative mathematical thinking. It can occur when a student does not possess appropriate ‘cognitive artifacts’ to progress their exploration. Cognitive artefacts include mathematical ideas, and concepts assembled by the student during the process of abstracting (Hershkowitz, Schwarz, & Dreyfus, 2001). When Looking-In, the student focuses on a dynamic visual display generated by another class member and extracts mathematical ideas and/or concepts implicit within that display.

By studying Leon’s Looking-In activity, and the conditions under which it occurred, the nature of Looking-In is elaborated and a theoretical language is developed to discuss the process of discovering complexity (creative mathematical thinking) in which this activity can be embedded. This study of creative mathematical thinking contributes to the body of knowledge about how students can develop relational understanding (Skemp, 1976): learning resulting from the interconnecting of concepts to develop new conceptual understanding rather than the learning of a fragmented set of rules that are applied without awareness of their meanings (instrumental understanding, Skemp, 1976).

This study informs teachers and teacher educators about pedagogy that supports the development of relational understanding, thus increases students’ ‘mathematical literacy’ (Kilpatrick, 2002). Mathematical literacy is the ability to use mathematics flexibly in unfamiliar situations and to recognize mathematics in unfamiliar contexts during work and life in general. To be able to do this, students need to ‘recognize’ the usefulness of previously developed mathematical ideas and concepts in unfamiliar contexts, and/or recognize mathematics embedded within contexts. ‘Recognizing’ is one of the three ‘observable cognitive elements’ found to occur during the process of abstracting (Hershkowitz, Schwarz, & Dreyfus, 2001). Recognizing, and the other two observable cognitive elements ‘building-with’ and ‘constructing’ are illustrated through Leon’s activity in this chapter. Recent focus on the importance of mathematical literacy has led to a valuing of creative student thinking in curriculum documents internationally (e.g., South Africa, Department of Education, 1997; Australia, http://vels.vcaa.vic.edu.au; USA, Kilpatrick, 2002).
STUDY DESIGN

Through the post-lesson interview process, students who explored mathematical complexities to generate new knowledge were identified and social influences upon their thinking were made explicit through their discussion of the lesson video. The probes used to clarify, elaborate, and/or extend student responses in the Australian interviews focused on student reconstruction of their thinking during the lesson and student identification of what they thought had influenced their thinking. The interviews in conjunction with the lesson video were used to identify the periods of time over which student cognitive activity was spontaneous (Williams, 2004) and those intervals of time were reanalysed to identify intervals of creative mathematical activity. The following analysis elaborates terms in this paragraph.

Ericsson and Simon (1980) have shown that verbal reports can provide valid data when attention is given to research design. There were features of the LPS interview probes in Australia that fitted with Ericsson and Simon's (1980) findings about how to generate high quality verbal data associated with cognitive activity. The interviews focused on: (a) using salient stimuli (mixed image lesson video of the student [centre screen] and the teacher [in the corner]) to stimulate student reconstruction); (b) allowing the student to focus the content of the interview; and (c) encouraging students to focus on lesson activity and their own thinking rather than the interviewer asking general questions. Ericsson and Simon state that where the researcher asks specific questions that include constructs the subject has not previously reported, the subject is more likely to “generate answers without consulting memory traces” (Ericsson & Simon, 1980, p. 217). On the other hand, if a subject spontaneously “described one or more specific sub-goals, and these were both relevant to the problem and consistent with other evidence of the solution process, ...” (Ericsson & Simon, 1980, p. 217) there was stronger evidence that the reported activity occurred. The multi-source data collection techniques in the LPS supported such analysis.

To study social and personal influences upon creative student thinking, simultaneous analysis of cognitive, social, and affective elements of the process of abstracting was undertaken in conjunction with analysis of student responses to successes and failures (resilience or optimism, Seligman, 1995). Seligman identified the engineering of flow situations as a way to increase student perception of their ability to overcome perceived failures. Connections between flow and optimism thus have pedagogical implications. There are three dimensions to optimism (Permanent-Temporary, Pervasive-Specific, Personal-External). Optimistic children see success as permanent, pervasive, and personal and failures as temporary, specific, and external. It was expected that optimism would be associated with student inclination to explore unfamiliar mathematics because problem solving in mathematics can be perceived as the encountering of failures along the pathway towards success. Discourse analysis (Säljö, 1999) of post-lesson interviews was undertaken to find indicators of optimism, and analysis of student cognitive activity to see how optimism was enacted. This enabled study of the
theorised link between optimism and flow situations. The dimensions of optimism are elaborated through Leon’s activity in this chapter.

SITE AND SUBJECTS

Australian School 1, the government school Leon attended in Melbourne, contained students from many cultural backgrounds (the 22 came from at least 12 cultural backgrounds). The families had generally been in Australia for more than one generation or had arrived in Australia more than two years prior to this research study. Leon’s teacher (Mrs Milano) generally commenced her lessons with a whole class activity followed by pair-work, which was sometimes of an exploratory nature. Where exploratory activity was included, it was generally at the end of a topic for the purpose of reinforcing new ideas. During pair-work, Mrs Milano moved around the classroom assisting individual students or groups of students particularly when they requested her attention.

Leon reported in his interviews (after Lesson 12 and Lesson 13) that mathematics learning was a process of adapting what was previously learnt to new situations. This was an indicator of optimism along the Permanent-Temporary dimension showing that he perceived ‘not knowing’ a solution method (‘Failure’) as temporary (Seligman, 1995). I have previously provided detailed analysis of Leon’s optimistic orientation (Williams, 2003). The way he described himself as learning mathematics did not place a heavy reliance upon assistance from the teacher because he perceived success as resulting from his own effort (‘Success’ as ‘Personal’). There were many instances during the research period where Leon demonstrated he continually tried to make his own sense of the mathematics upon which the class focused. This independence of thought could explain why he (unlike many other students), did not discuss the quality of his teacher in his interview; he was not so reliant upon her for his learning. Observation during Lesson 12 suggested Leon spent the majority of his time teasing the female students seated around him. His post-lesson interview provided evidence of deep mathematical thinking about ideas that interested him.

‘LOOKING-IN’: THE CONTEXT

Early in Lesson 12, Mrs Milano discussed the properties of squares and rectangles in an attempt to convince class members that a square was a rectangle. Leon displayed intense interest in the properties of these figures by answering softly to himself each time Mrs Milano asked whether a certain property was present. Later in the lesson, Mrs Milano placed three large coloured triangles on the board (see Figure 1) and pairs of students were allocated a triangle and asked to find its area (without using a rule). The positioning of this exploratory activity at the beginning of a topic sequence differed from the usual lesson sequencing in this class because the exploration occurred prior to the introduction of the rule.

Leon and Pepe were meant to work together but Pepe wanted to count squares and Leon wanted to find a general method to find areas of triangles. Pepe did the
writing and drawing for the pair and Leon answered Pepe’s queries, and provided unsolicited advice to the female student pair (Elina and Serina) working beside him. Leon did not write or draw anything during the lesson. Pepe worked on Triangle 1 (see Figure 1) and Leon utilised the time he had manoeuvred (by letting Pepe do the written work) to think more broadly about the three triangles on the board and their respective areas. He had manoeuvred so that he was not obliged to negotiate with others about how he would proceed with his exploratory activity; he had ‘cognitive autonomy’ (Williams, 2005).

LEON’S EXPLORATORY ACTIVITY

Leon structured his exploration by attending simultaneously to the three images on the board and asking himself “which triangle was easiest”.

![Figure 1. Triangles placed on the board as part of Task C2](image)

This focus was evident from his interview after Lesson 12 and his interactions with the pair seated beside him: “[to Elina] You're doing two? Two's like the easiest one there. It's easy!” iii Leon reconstructed his thinking about how to find the area of a right-angled triangle (identified herein as Method A) in his post-lesson interview after Lesson 12:

> All you have got to do is figure out what a rectangle is that has those two um (pause) lengths- length and width and ... then you can just halve it

Leon did not make explicit whether he recognized that right-angled triangles could be formed by cutting rectangles in half, or that juxtaposing two right-angled triangles would make a rectangle. He also did not discuss whether he used the properties of rectangles to justify the nature of the shape produced or whether he relied solely upon visual images. Leon’s intense interest in properties of rectangles earlier in the lesson and the method he developed to find areas of acute-angled triangles (described below) suggested he juxtaposed right-angled triangles and used the properties of rectangles to justify the nature of the shape formed. Leon’s initial thinking about finding areas of acute-angled triangles (Method B) involved juxtaposing two acute-angled triangles to make a parallelogram. In his interview after Lesson 13, he sketched this juxtaposition process and was not satisfied with his sketch. His accompanying comments showed he knew a parallelogram was formed and that the properties of the triangles justified that shape:
That’s a parallelogram ... mmh pretend they are the same length and width and everything.

Leon’s juxtaposition of two acute-angled triangles and his awareness of the properties of the figure produced, supported my interpretation that he used these ideas (juxtaposition, properties of shape formed) in Method A. He sectioned the parallelogram in Method B (see Figure 2) to find the acute-angled triangle’s area.

Figure 2. Interpretation of Leon’s Method B

Figure out what it would be if it were a parallelogram and then halve it ... figure out what it would be if it was four because you could just trial whatever it was if it was four

Figure 2 shows the parallelogram sectioned into four right-angled triangles “figure out what it would be if it was four” and the rectangles formed by the juxtaposition of another congruent right-angled triangles with each of these four right angled triangles (multiple use of Method A as part of Method B) to form four rectangles “just trial whatever it was if it was four”. The areas of these rectangles were found and totalled and the total halved in accordance with the halving of each rectangle in Method A “and then halve it”. The separate parts in Method B probably made it difficult for Leon to keep all of these ideas in his head at once. The fragility of Leon’s ideas was expressed in his interview as he discussed Method B:

I understood it- I didn’t understand it then I understood it then I didn’t understand it
Looking-In Activity Associated with Goal Change (Method C begins to Emerge)

While he was still considering Method B, Leon’s Looking-In activity occurred and he began to realise there could be a more elegant way to proceed. In his interview, he described Looking-In activity that leads to goal change:

When you look around the classroom and see how everyone else is doing it and you are doing it a completely different way- ... and you think ooh! [soft] maybe my method isn't the best and ... you think about everyone’s ... and then you think about your own and they all sort of piece together and you just sort of go oh! and it pops into your head [LI3-SI-Leon].

It was as Leon ‘Looked-in’ on what was happening around him, and considered this in conjunction with the ideas he had been developing himself that his goal changed from finishing the work to understanding the work. By Looking-In, he extracted a big idea that he had not previously been aware of: “triangles come in rectangles”. He was excited by what he had found because it opened possibilities for a simpler way to proceed. He might no longer need to section up parallelograms and laboriously apply Method A to each of the four parts, total the areas, and halve the sum. There could be a more ‘elegant’ way. Krutetskii (1976) emphasised a quest for elegance as crucial aspect of the problem solving activity of students with high mathematical ability. Through his quest for elegance, Leon set up the conditions for flow (discovering complexity)—a spontaneous intellectual challenge (how could these enclosing rectangles help?) requiring development of mathematics beyond his present understanding. The nature of Leon’s Looking-In activity is discussed in detail later. For the present, the remainder of Leon’s exploratory activity is reported to emphasise the significance of Leon’s Looking-In activity to the success of his exploration.

By Lesson 13, Leon knew ‘triangles come in rectangles’ and knew that the area of the triangle was half that of the enclosing rectangle (see Table I, Row 3, Column 2) but he did not know why. This was what he was puzzling to understand. When Mrs Milano held a large pink enclosing rectangle behind the red acute-angled triangle in Lesson 13, Leon softly exclaimed “Oh!” [L13, 20:00 Mins]. He reconstructed his thinking in his interview:

I sort of- sort of thought a little bit about why it was happening [triangle area half rectangle area] ... that it was um- if you take one part out like a triangle that’s set at an angle if you take both parts out and put them together it equals the rectangle. ... I was sort of looking at them and then I just realised, like I (pause) sort of just in my head I pulled it apart and put them together so that they equalled the same.

Figure 3 includes the mental images Leon reported generating as he suddenly realised why the area of the triangle was half the area of the rectangle. He had removed two right-angled sections of the acute angled triangle (shaded parts) out of the larger rectangle. The triangles congruent to each of these shaded sections were then juxtaposed with the appropriate shaded triangle to make two smaller
rectangles: "if you take one part out like a triangle that’s set at an angle if you take both parts out and put them together it equals the rectangle". He had simultaneously drawn upon what he knew from Method A and shifted pieces around in his head to justify that the area of the acute angled triangle was always half the area of the enclosing rectangle.

![Figure 3](image)

*Figure 3 Leon applied Method A twice within his elegant Method*

Less than a minute later, Leon clapped his hand against his cheek as he realised he could use his novel concept for another purpose. His comments in class showed what he had realised:

Leon That’s half of the rectangle as well
Leon That would be three hundred and thirty centimetres squared
T Three hundred and thirty. Why?
Leon It would be exactly the same as the first one [Triangle 2]
T Why?
Leon Because the green one is half of the rectangle too (see Figure 1)

Leon had realised that triangles enclosed in the same rectangle have the same area because each has an area of half the area of the rectangle.

The exercise set towards the end of Lesson 13 included finding areas of triangles in different orientations. Leon (unlike other members of the class) could do this exercise because he was aware of the significance of the perpendicular height of the triangle and how to recognize it in any orientation. In her interview, Mrs Milano expressed surprise at the difficulty other students had encountered:

I assumed ... that they knew what the base and height of a triangle ... is ... and how to recognise it ... it just sort of (pause) was made very obvious that ... they don’t understand- ...

Leon had subsumed the attributes of the rectangle into equivalent attributes of triangles (length as base, width as perpendicular height) so he was able to operate with attributes of triangles to find their areas. This enabled him to calculate areas of
triangles that other students could not calculate because they had not identified the critical rectangle and triangle attributes that were equivalent. They knew the rule but not why it worked (instrumental understanding, Skemp, 1976) so were not able to ‘see’ the perpendicular height with triangles in varying orientations. Wertheimer (1959) identified similar problems with perpendicular heights in parallelograms.

**WHAT DID LEON LOOK-IN UPON?**

What did Leon Look-in upon and why did this lead to a change in his goal? By analysing the activity of students seated around Leon in Lesson 12, asking Mrs Milano what she saw, and observing that other students did not volunteer general methods for finding areas of acute-angled triangles in Lesson 13, it was evident that Leon was the only student who developed methods (other than counting squares) for finding areas of acute-angled triangles. The only pair who found an area without counting squares worked with the right-angled triangle (Triangle 2). Thus, whatever Leon Looked-in upon, it was not work associated with finding the area of an acute-angled triangle by calculating the area of the enclosing rectangle.

To find what Leon Looked-in upon, an enriched transcript was developed for intervals of time over which this Looking-In activity could have occurred (see Table 1). I developed these enriched transcripts to enable data from different sources to be considered simultaneously over time intervals of interest (Williams, 2004). Table 1 shows the time interval [Column 1], Leon’s interview comments related to that time interval [Column 2], the sketches he made during the interview to illustrate the internal images he was generating [Column 3], the images in the classroom that Leon had seen [Column 4], those parts of Leon’s Lesson 12 talk that were relevant to this analysis [Column 5], Pepe’s activity [Column 6], and other talk relevant to this analysis [Column 7]. The dotted horizontal line in Table 1 separates two intervals of time found relevant to this analysis.

Table 1 shows that at 36:14 Leon knew Methods A and B because he compared the ease of two methods. Between 38:21 and 38:57, he Looked-in on Elina and Serina’s diagram and Pepe’s diagram. This was evident from Leon’s comment to Mrs Milano that Elina and Serina could use shorter ways [see Columns 5, 4], and Leon’s interview reflections about quicker ways for Pepe [see Columns 2, 4]. Leon’s reconstruction of his Looking-In activity in his interview after Lesson 13 confirms what he saw, and illuminates how he attended to it idiosyncratically:

People were drawing the actual rectangles around it- I don’t know whether they knew they [triangles] were coming from rectangles ... the way they were drawing it made it look like they did ... made me think about it.

Leon Looked-in on dynamic visual displays [Table 1, Column 4] that were being generated by other students. His comment “I don’t know whether they knew they [triangles] were coming from rectangles” is supported by the lesson video; students were generating displays to count squares rather than to calculate. They used rectangles to rule grids to make squares.
Table 1. Enriched transcript identifying interval in the lesson surrounding Leon’s Looking-In activity

<table>
<thead>
<tr>
<th>Time in LI2</th>
<th>Leon’s Interview Reconstruction</th>
<th>‘Internal’ images (Leon)</th>
<th>Images in Classroom</th>
<th>Leon’s Talk and Actions in Class</th>
<th>Pepe’s Activity</th>
<th>Others</th>
</tr>
</thead>
<tbody>
<tr>
<td>36:14-36:22</td>
<td>… I thought I could just figure out what it would be if it were a parallelogram and then halve it … figure out what it would be if it was four because you could just trial whatever it was if it was four. … I understood it- I didn’t understand it then I understood it then I didn’t understand it.</td>
<td>Asked to explain more, Leon prefaced his sketch with ‘Um I … think it was … like that … [drew 2 triangles] should have the same length and width …’</td>
<td>Elina’s and Serina’s Diagram generated by others (including Pepe)</td>
<td>Leon: [to T] I think I know. [Nodded] Except if I was doing Triangle 2-2’s the easiest one there. [Leon did not respond to Pepe’s actions]</td>
<td>Pepe: Bip bip [drove a pen over Leon’s arm]</td>
<td>T: [to Leon] Do you?</td>
</tr>
<tr>
<td>38:21-38:57</td>
<td>… people were drawing the actual rectangles around it- I don’t know whether they knew they [triangles] were coming from rectangles … the way they were drawing it made it look like they did … made me think about it. He’s [Pepe] drawn up the grid and … I thought … you could do it quicker … figure out what the area would be fully and then halve it.</td>
<td></td>
<td>Leon: [to T] Aren’t they silly? There’s so many other shorter ways. [to Pepe] [Laughs] You are doing a great job Pepe.</td>
<td></td>
<td>T: [to Elina and Serina] … You’ve resorted to drawing the grid?</td>
<td></td>
</tr>
</tbody>
</table>

Key. --- Break between time intervals displayed
As a result of his Looking-In activity, Leon changed his focus from finishing the work to wanting to understand why the area of a triangle was half the area of the enclosing rectangle. Table 1, Row 3 indicates Leon was aware that the area was half the area of the rectangle at the end of Lesson 12. Leon’s gasp in Lesson 13 and his explanation of this gasp in his post-lesson interview showed Leon did not know why the area of the acute-angled triangle was half that of the enclosing rectangle in Lesson 12.

Leon’s Looking-In activity changed his focus from laborious calculation to searching for an elegant method. This led to his recognition of relationships between attributes of triangles and enclosing rectangles. Such realisation was unlikely to have occurred if he had continued with Method B. Leon’s Looking-In activity compensated for his lack of appropriate cognitive artefacts; enabling him to autonomous access mathematics to progress his exploration.

THE ROLE OF LOOKING-IN IN CREATIVE MATHEMATICAL THINKING

The “Space to Think” model emerged from my analysis for students discovering complexity (Williams, 2005, pp. 365-371). It is an activity space that was manoeuvred by each of the five students including Leon (in Australia and the USA) who spontaneously undertook creative mathematical thinking (were discovering complexity). Six activities were found common to these students as they spontaneously engineered flow situations associated with mathematical explorations (Williams, 2002a):
- ‘inclining to explore’,
- ‘spontaneously identifying a mathematical complexity’,
- ‘manoeuvring cognitive autonomy’,
- ‘autonomously accessing mathematics’,
- ‘spontaneously pursuing an exploration’,
- ‘asking questions to structure future exploration’.

These activities were found necessary (but not necessarily sufficient) for creative mathematical activity. Figure 4 summarises the cognitive activity Leon undertook. This figure is used to focus discussion of activity in the Space to Think with particular emphasis on the role of Looking-In as support to ‘autonomously accessing mathematics’. Figure 4 contains: questions Leon asked himself [Column 1], Leon’s solution process [Column 2], and novel ideas and mathematical structures Leon developed [Column 3]. Leon’s cognitive activity was ‘observable’ through cognitive elements in his classroom talk (Hershkowitz, Schwarz, & Dreyfus, 2001), and his interview reconstruction of his thinking.

Enacting optimism or ‘inclining to explore’ is evidenced in Figure 4, Columns 1, 2. By considering that he would be able to adapt the mathematics he knew to find a rule for finding areas of triangles Leon enacted success as pervasive and personal. He enacted failure as temporary by searching for ways to circumvent the difficulties he encountered, and enacted failure as specific by examining ‘failures’ to identify what he could change (Column 2).
<table>
<thead>
<tr>
<th>Exploration Questions</th>
<th>Cognitive elements (RBC) observed, inferred or reconstructed</th>
<th>Novel Mathematical Ideas and Structures</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. *# Which of the three types of triangles are easiest to find areas of? * What happens if I put two right-angled triangles together?</td>
<td>Method A: (RB) <img src="image" alt="Diagram of Method A" /></td>
<td>*# Area of a right-angled triangle is half the area of a rectangle. I know how to find the area of a rectangle. * Visual imagery supported by analysis of properties of this shape, and rectangles.</td>
</tr>
<tr>
<td>2. * Can I use the same idea (putting two triangles together) to find the area of an acute-angled triangle?</td>
<td>Method B: (RB) <img src="image" alt="Diagram of Method B" /></td>
<td># Two acute-angled triangles placed together make a parallelogram (opposite sides are equal but sides are angled in so it is not a rectangle). Can Leon find area of parallelogram?</td>
</tr>
<tr>
<td>3. *# &quot;Oh! Maybe my method isn’t the best.&quot;</td>
<td><img src="image" alt="Diagram of Method C" /></td>
<td># Triangles can be enclosed by rectangles. *# The right-angled triangle is half the area of this enclosing rectangle.</td>
</tr>
<tr>
<td>4. *# If an acute-angled triangle can be enclosed by a rectangle, can this help find the area of that triangle?</td>
<td><img src="image" alt="Diagram of Method C" /></td>
<td>^ Areas of triangles are half the length of the base multiplied by the length of the perpendicular line from the apex to the base. This perpendicular line may be dotted (imagined) rather than a triangle side. This is because the area of the triangle is half the area of the enclosing rectangle and these two lengths are also the base and height of the enclosing rectangle. For this reason, two different triangles enclosed by the same rectangle have the same area.</td>
</tr>
<tr>
<td>5. Why is it half?</td>
<td><img src="image" alt="Diagram of Method C" /></td>
<td>^ &quot;If you take one part out like a triangle that’s set at an angle [N say] if you take both parts out [M &amp; N] and put them together it equals the rectangle.&quot;</td>
</tr>
</tbody>
</table>

Key. Inferred from: * other evidence, video activity; # interview; ^ lesson dialogue.
Black arrows: progression of thinking (RBC; observable cognitive elements)
Cognitive Elements: C, Constructing (synthesising, evaluating); B, Building-with (novel analysing, novel synthetic analysing, novel evaluative-analysing); R, Recognizing (analysing, comprehending)

Figure 4. Cognitive activity during Leon’s development of novel conceptual understanding
For example, when the juxtaposition of acute-angled triangles was not very successful, Leon focused attention on rectangles instead (because that had also previously been successful). Instead of responding to failure as though it were a pervasive attribute of himself: "I can't do this, I'm stupid", he examined the situation and identified what could be changed (Failure as Specific). The first complexity Leon spontaneously identified was that the three triangles on the board might differ in their relative ease for finding areas (see Figure 4, Column I, Row 1). Another complexity he discovered was that triangles have enclosing rectangles.

Leon manoeuvred cognitive autonomy by working alone not with Pepe who wanted to count squares. In doing so, he manoeuvred time to think without needing to take into account the way Pepe wanted to proceed. This is evident from Table 4, Column 1; Leon responded to his own questions and focused his own attention on external stimuli. Cognitive autonomy was also enabled by Leon's perception that he was under no obligation to undertake the task set by the teacher.

Leon autonomously accessed mathematics by assembling cognitive artefacts he possessed (e.g., properties of rectangles, area of rectangle formula, Method A as part of Method B and Method C). When he did not possess appropriate cognitive artefacts (necessary to develop an elegant solution pathway), he autonomously accessed mathematics by Looking-In on the dynamic visual displays generated by others and used these displays along with his own previous thinking to develop a new understanding—'triangles always come in rectangles' (see Figure 4, Row 3). Through this Looking-In activity Leon developed this cognitive artefact that was crucial to his construction of new knowledge.

Cifarelli (1999) identified students asking specific mathematical questions that structured their future exploration (structuring questions). My study illuminates a more general type of structuring question. As a result of Looking-In activity Leon asked: "Oh! Maybe my method isn't the best?" (Figure 4, Row 3) and this structured spontaneous pursuit of a new direction of exploration (see Figure 4, Row 4). This type of question is not specific to the mathematical context in which it occurred; there are many situations in which such a question could be appropriate. Spontaneous pursuit involves using increasingly complex thinking whilst trying to answer the structuring question asked, and developing novel mathematical concepts during this process. Leon's activity elaborates the nature of structuring questions.

In this study, Leon recognized (e.g., the rectangle could be used to find areas of right triangles) and built-with previously known mathematical ideas (e.g., used what he had learnt for right-triangles to calculate areas of acute-angled triangles), and constructed a novel mathematical structure thus eliminating the need to explicitly use rectangles to find areas of triangles. From least to most complex, the categories of thinking Leon used during his spontaneous pursuit were recognizing (R), building-with (analysis, synthetic-analysis, evaluative-analysis) (B), and constructing (synthesis, evaluation) (C); new conceptual understanding developed. My previous work (Williams, 2002b) integrated Krutetskii's (1976) 'mental activities' into Hershkowitz, Schwarz, and Dreyfus's (2001) observable cognitive elements of the process of abstracting to add further subcategories to assist in
studying increasingly more complex thinking. Examples of each of these types of thinking are contained in Figure 4. The words italicised in the following descriptions include key aspects of each thinking category. Leon’s spontaneous question at the start of Row 3 (Figure 4) involved evaluative-analysis because he simultaneously considered two ways to proceed (synthetic-analysis) for the purpose of making a judgment about the respective elegance of these solution pathways. Leon finally synthesised his previous mathematical ideas with the new cognitive artefact he had developed ‘triangles come in rectangles’ when he answered his question, “Why is it half?” (Figure 4, Row 4). Leon used procedures he had developed early in his exploratory activity in different ways throughout his exploration (see Figure 4, Column 2, e.g., Method A). His final constructing relied upon Method A and cognitive artefacts he developed through earlier Looking-In. Soon after, Leon recognized the usefulness of his new ideas in another context (evaluating); triangles enclosed by the same rectangle have the same area.

Without Leon’s Looking-In activity, he would not have accessed the mathematics, necessary to pursue his exploration and would not have experienced the positive affect associated with flow; an optimism-building experience (Seligman, 1995). His realisation that ‘angles come in rectangles’ and why the area was half were accompanied by positive affect. Looking-In activity was found to involve:

- A generator of a dynamic visual display
- An extractor of mathematics implicit within this display

The generator was sometimes aware of the mathematical significance the extractor perceived and sometimes (as in this case) they were not. Looking-In activity was identified in Australian classrooms where exploratory tasks were set, opportunities to talk to other students existed, and students found time to explore idiosyncratically. This study highlights the need for further research into the process of Looking-In and situations in which it is likely to occur. Such activity could support the development of relational understanding for students with limited access to cognitive artefacts, and contribute to an increase in the frequency of creative mathematical thinking in class.

NOTES

i Key to symbols used in quotes from student interviews in this chapter:

... Omitted text that does not affect meaning
(pause) A pause
[L12-S1-Leon] Lesson number, student interview, name of student
italics Italicised text indicates emphatic speech

ii The text in square brackets is additional information provided by the transcriber.

iii [L12, 24 mins, 16 secs into the lesson]

iv The text in square brackets is additional information provided by the transcriber.

v The text in brackets is additional information provided by the author.

vi [L13-T1-Mrs Milano] Teacher interview post-Lesson 13

vii 36 minutes and fourteen seconds into Lesson 12
ACKNOWLEDGEMENTS

This analysis was undertaken at The University of Melbourne as part of my PhD studies under the supervision of Professor David Clarke whose expertise as a supervisor was much appreciated. My thanks to ICCR team for their technical assistance, and to the LPS teams for their collegiality and encouragement. I appreciate the perceptive comments made by Fritjof Sahlström on two previous drafts of this chapter, and Seah Lay Hoon’s thoughtful comments, which assisted my elaboration of key constructs.

REFERENCES


Williams, G. (2003). Associations between student pursuit of novel mathematical ideas and resilience. In L. Bragg, C. Campbell, G. Herbert & J. Mousley (Eds.), Mathematical education research:

Gaye Williams
Faculty of Education
Deakin University
Australia