Pointwise construction of Lipschitz aggregation operators

G. Beliakov
Deakin University
Australia
gleb@deakin.edu.au

T. Calvo
Univ. de Alcalá
Spain
tomasa.calvo@uah.es

J. Lázaro
Univ. de Alcalá
Spain
jesus.lazaro@uah.es

Abstract

This paper establishes tight upper and lower bounds on Lipschitz aggregation operators considering their diagonal, opposite diagonal and marginal sections. Also we provide explicit formulae to determine the bounds. These are useful for construction of these type of aggregation operators, especially using interpolation schemata.

Keywords: Aggregation operators, monotone interpolation, 1-Lipschitz aggregation.

1 Introduction

Aggregation operators with a priori known properties are often required for decision support and other systems that use fuzzy logic. The method of pointwise construction of Lipschitz aggregation operators presented in [4, 5] allows one to construct (or identify) the most suitable general aggregation operator that interpolates (or approximates) a given set of desired values. This construction is applicable to n-ary aggregation operators f with a given (or estimated) Lipschitz constant. It works by identifying tight upper and lower bounds on the values of $f(x)$, and then applying the central interpolation algorithm, which delivers the optimal operator.

In many applications there are further requirements on aggregation operators, that allow one to tighten the bounds significantly. For example, M. Grabisch [8] takes the values of the aggregation operator at certain meaningful points (characteristic vectors of binary alternatives) and uses piecewise linear interpolatory method for unipolar and bipolar operators. R. Yager [14] uses the requirement of noble reinforcement (in the context of recommender systems) to build disjunctive aggregation operators, which are bounded by maximum from above for low values of the arguments.

The goal of this work it to develop such bounds explicitly, in a number of prototypical cases. We concentrate on the following conditions.

- Given marginals;
- Given diagonal or opposite diagonal.

For a recent overview of aggregation operators and their properties we refer the reader to [6].

Suppose that we have a set of desired values of the aggregation operator $\mathcal{D} = \{(x^k, y^k)\}_{k=1}^K, x^k \in I^n, y^k \in I, y^k = f(x^k)$, and the Lipschitz condition

$$\exists M \geq 0 : \forall x, z \in I^n,\ |f(x) - f(z)| \leq M||x - z||,$$

where $I = [0, 1]$. The data are consistent with the Lipschitz condition and monotonicity $f \in Lip(M, ||\cdot||) \cap Mon$. Then tight upper and lower bounds on any function from the set $Lip(M, ||\cdot||) \cap Mon$ that interpolate the data are given (see [4, 3]) by $\sigma_l(x) \leq f(x) \leq \sigma_u(x)$, with

$$\begin{aligned} \sigma_u(x) &= \min_k \{y^k + M||(x - x^k)_+||\}, \\ \sigma_l(x) &= \max_k \{y^k - M||(x^k - x)_+||\}, \quad (1) \end{aligned}$$

where z_+ denotes the positive part of vector z: $z_+ = (\bar{z}_1, \ldots, \bar{z}_n)$, with $\bar{z}_i = \max\{z_i, 0\}$.

If the data set is infinite, $\mathcal{D} = \{(t, v(t)) : t \in \Omega \subset I^n, v : \Omega \to I\}$ then the bounds translate into

$$\begin{aligned} \sigma_u(x) &= \inf_{t\in\Omega}\{v(t) + M||(x-t)_+||\}, \\ \sigma_l(x) &= \sup_{t\in\Omega}\{v(t) - M||(t-x)_+||\}. \quad (2) \end{aligned}$$

If the data set is interval-valued $\mathcal{D} = \{(t, [\underline{v}(t), \overline{v}(t)]) : t \in \Omega, \underline{v}, \overline{v} : \Omega \to I, \underline{v} \leq \overline{v}\}$, i.e., for each $t \in \Omega$ we have $\underline{v}(t) \leq f(t) \leq \overline{v}(t)$, the bounds are

$$\begin{aligned} \sigma_u(x) &= \inf_{t\in\Omega}\{\overline{v}(t) + M||(x-t)_+||\}, \\ \sigma_l(x) &= \sup_{t\in\Omega}\{\underline{v}(t) - M||(t-x)_+||\}. \quad (3) \end{aligned}$$

The central algorithm delivers an optimal interpolant g, which minimizes the worst case error

$$g = \arg\min_{h\in\mathcal{F}} \max_{f\in\mathcal{F}} \max_{x\in I^n} |f(x) - g(x)|,$$

where $\mathcal{F} = Lip(M, ||\cdot||) \cap Mon$. It is given by

$$g(x) = \frac{1}{2}(\sigma_l(x) + \sigma_u(x)). \quad (4)$$

The functions $\sigma_l(x)$,$\sigma_u(x)$ and $g(x) \in Lip(M, ||\cdot||) \cap Mon$.

In this paper we translate the above mentioned properties of aggregation operators into tighter bounds

$$B_l(x) \leq f(x) \leq B_u(x),$$

which will be used in conjunction with (1)

$$\begin{aligned} \underline{A}(x) &= \max\{\sigma_l(x), B_l(x)\}, \quad (5) \\ \overline{A}(x) &= \min\{\sigma_u(x), B_u(x)\}, \end{aligned}$$

to deliver an optimal operator

$$g(x) = \frac{1}{2}(\underline{A}(x) + \overline{A}(x)). \quad (6)$$

The next section introduces basic properties of aggregation operators and notation for the rest of the paper. Then we briefly discuss stability of aggregation operators and some known classes and bounds. In Section 4 we discuss the bounds on aggregation operators with a given diagonal or opposite diagonal sections, and in Section 5 we establish the bounds resulting from marginal sections. We also discuss compatibility of given marginals with the Lipschitz constant of the aggregation operator.

2 Preliminaries

An n-ary aggregation operator is a mapping $f : I^n \to I$, which is monotone non-decreasing in all arguments and satisfies $f(\mathbf{0}) = 0, f(\mathbf{1}) = 1$. We will use subscripts to identify components of vectors in $x, y \in I^n$ and will understand vector inequality $x \leq y$ componentwise, i.e., $\forall i \in \{1, \ldots, n\} : x_i \leq y_i$. Thus monotonicity is expressed as $x \leq y \Rightarrow f(x) \leq f(y)$. We will also understand operations of minimum and maximum componentwise, e.g., $\min(x) = \min\{x_1, \ldots, x_n\}$, and apply componentwise the operation $x_+ = (\max\{x_1, 0\}, \max\{x_2, 0\}, \ldots, \max\{x_n, 0\})$.

The set of all monotone non-decreasing functions is denoted by Mon. We will use the terms increasing (decreasing) synonymously with non-decreasing (non-increasing), and will use the terms strictly increasing (strictly decreasing) otherwise.

We will denote by $Lip(M, ||\cdot||)$ the set of all Lipschitz continuous functions on I^n with the Lipschitz constant in the norm $||\cdot||$ smaller or equal to M: $Lip(M, ||\cdot||) =$

$\{f : I^n \to I : \forall x, y \in I^n, \ |f(x) - f(y)| \leq M||x - y||\}$.

The set of all n-ary Lipschitz aggregation operators is expressed as $\mathcal{A}_{M,||\cdot||} = \{f \in Lip(M, ||\cdot||) \cap Mon : f(\mathbf{0}) = 0, f(\mathbf{1}) = 1\}$.

Note that $M \geq ||\mathbf{1}||^{-1}$. If we use a traditional l_p-norm, this translates into $M \geq n^{-1/p}$.

We list the following properties and classes of aggregation operators.

- An aggregation operator f is called 1-Lipschitz if its Lipschitz constant in l_1-norm is one, i.e., $f \in \mathcal{A}_{1,||\cdot||_1}$.

- An aggregation operator is called a quasi-copula, if it is 1-Lipschitz and has neutral element $e = 1$.
- An aggregation operator f is called kernel, if its Lipschitz constant in l_∞-norm is one, i.e., $f \in \mathcal{A}_{1,||\cdot||_\infty}$.

3 Stable aggregation operators

Lipschitz-continuous aggregation operators are very important for applications, because they provide output values stable with respect to small changes of the arguments. Small changes in the arguments may be due to inaccuracies in the data, and one would expect that such inaccuracies do not affect drastically the behavior of the system. The concept of p-stable aggregation operators was proposed in [7]. These are precisely Lipschitz continuous operators whose Lipschitz constant M in l_p norm is one. We can write

$$\mathcal{A}_{p-stable} = \mathcal{A}_{1,||\cdot||_p}.$$

Specific cases include 1-Lipschitz aggregation operators ($p = 1$) and kernel aggregation operators ($p = \infty$).

It is known (see [7]) that the weakest and the strongest p-stable operators are the Yager t-norm and t-conorm

$$\begin{aligned} T_Y(x) &= \max\{0, 1 - ||1 - x||_p\} \\ S_Y(x) &= \min\{1, ||x||_p\} \end{aligned}$$

For kernel aggregation operators we obtain

$$\min(x) \le f(x) \le \max(x), x \in I^n.$$

For 1-Lipschitz aggregation operators we have Lukasiewicz t-norm and t-conorm as the bounds

$$T_L(x) \le f(x) \le S_L(x).$$

Quasi-copulas are bounded by T_L and min, and the upper bound is a consequence of the presence of the neutral element $e = 1$, discussed later.

It is not difficult to check that the above mentioned bounds are a direct consequence of the Eqs.(1), with data $(\mathbf{0}, 0), (\mathbf{1}, 1)$. For an arbitrary M we have

$$\begin{aligned} \sigma_u(x) &= \min\{M||x||_p, 1\}, \\ \sigma_l(x) &= \max\{0, 1 - M||1 - x||_p\} \end{aligned} \quad (7)$$

4 Diagonals and opposite diagonals

We now consider the problem of constructing Lipschitz aggregation operators with a given diagonal or opposite diagonal section. Denote by $\delta(t) = f(t, t, \dots, t)$ the diagonal section of the n-ary aggregation operator f. If $f \in \mathcal{A}_{M,||\cdot||_p}$, then $\delta \in Lip(Mn^{1/p})$. Also $\delta(t)$ is nondecreasing, and $\delta(0) = 0, \delta(1) = 1$. We denote by $\omega(t) = f(t, 1-t)$ the opposite diagonal section of a binary aggregation operator. We note that $\omega \in Lip(M)$.

In the following we assume that the functions $\delta(t), \omega(t)$ are given and they have the required Lipschitz properties. The goal is to determine the upper and lower bounds on Lipschitz aggregation operators with these diagonal and opposite diagonal sections.

4.1 Diagonal section

From (1) it follows that $B_u(x) =$

$$\min_{t \in I}(\delta(t) + M||((x_1 - t)_+, \dots, (x_n - t)_+)||),$$

$B_l(x) =$

$$\max_{t \in I}(\delta(t) - M||((t - x_1)_+, \dots, (t - x_n)_+)||). \quad (8)$$

We remind that these bounds are in addition to (7). For the purposes of computing the values of $B_u(x), B_l(x)$ we need to develop suitable algorithms to solve the optimization problems in (8).

Before we proceed with this general case, we recall the following bounds obtained for bivariate 1-Lipschitz functions in [11].

$$\begin{aligned} B_u(x) &= \max(x_1, x_2) + \min_{t \in [\alpha, \beta]}(\delta(t) - t) \\ B_l(x) &= \min(x_1, x_2) + \max_{t \in [\alpha, \beta]}(\delta(t) - t), \end{aligned} \quad (9)$$

where $\alpha = \min(x_1, x_2)$, $\beta = \max(x_1, x_2)$. Let us show that (9) is a direct consequence of (8).

Consider the upper bound $B_u(x)$ in (8) and $n = 2$. Let

$$B_1 = \min_{\beta\le t\le 1} \delta(t) - M||((t-x_1)_+, (t-x_2)_+)|,$$

$$B_2 = \min_{\alpha\le t\le \beta} \delta(t) - M||((t-x_1)_+, (t-x_2)_+)|,$$

$$B_3 = \min_{0\le t\le \alpha} \delta(t) - M||((t-x_1)_+, (t-x_2)_+)|.$$

Clearly, $B_u(x) = \min\{B_1, B_2, B_3\}$.

For $t \ge \beta$ all the terms $(x_i - t)_+$ are null, hence $B_1 = \delta(\beta)$. On $[\alpha, \beta]$ we have

$$\begin{aligned} B_2 &= \min(\delta(t) + M((\beta - t)^p)^{1/p} \\ &= \min(\delta(t) + M\beta - Mt) \quad (10) \\ &= M\max(x_1, x_2) + \min_{t\in[\alpha,\beta]}(\delta(t) - Mt). \end{aligned}$$

On $[0, \alpha]$ the minimum of the expression in (8) can be achieved inside this interval, depending on the form of $\delta(t)$. However, for the special case $p = 1$, the function

$$\delta(t)+M((x_1-t)+(x_2-t)) = \delta(t)+M(x_1+x_2-2t)$$

is decreasing (remember that the Lipschitz constant of δ is $2M$ in this case), and hence

$$B_3 = \delta(\alpha)+M(x_1+x_2-2\alpha) = \delta(\alpha)+M(\beta-\alpha).$$

When we take $M = 1$,

$$\begin{aligned} B_u(x) = \min\{&\delta(\beta), \beta + \delta(\alpha) - \alpha, \\ &\beta + \min_{t\in[\alpha,\beta]}(\delta(t) - t)\} \\ = \beta + &\min_{t\in[\alpha,\beta]}(\delta(t) - t), \end{aligned}$$

which is expression (9). For $M \ge 1$, $p = 1$ $B_u(x)$ is given by (10). The lower bound for $n = 2, p = 1$ is obtained analogously as

$$B_l(x) = M\min(x_1, x_2) + \max_{t\in[\alpha,\beta]}(\delta(t) - Mt).$$

Interestingly, for $p \to \infty$ a similar formula works for any dimension n. We have

$B_u(x) = \min_{t\in I}(\delta(t) + M\max_i\{(x_i - t)_+\}) =$

$\min_{t\in I}(\delta(t) + M(\max_i\{x_i\} - t)_+)$.

Consider two intervals $[0, \beta]$ and $[\beta, 1]$, where $\beta = \max_i\{x_i\}$. When $t \in [\beta, 1]$, $(\beta - t)_+ = 0$, and the objective function becomes $\delta(t)$. Since it is increasing, the minimum is achieved at $t = \beta$. Therefore

$$\begin{aligned} B_u(x) &= \min_{t\in I}(\delta(t) + M(\beta - t)) \\ &= \min_{t\in[0,\beta]}(\delta(t) + M(\beta - t)) \\ &= M\beta + \min_{t\in[0,\beta]}(\delta(t) - Mt). \end{aligned}$$

Similarly,

$$\begin{aligned} B_l(x) &= \max_{t\in[\alpha,1]}(\delta(t) - M(t - \min_i\{x_i\})) \\ &= M\min_i\{x_i\} + \max_{t\in[\alpha,1]}(\delta(t) - Mt). \end{aligned}$$

Let us return to the general case, in which we need to compute the minimum and maximum in (8). Since the function $\delta(t)$ is fairly arbitrary (we only require $\delta \in Lip(Mn^{1/p}) \cap Mon$), the overall expression may possess a number of local minima. Calculation of the bounds requires the global minimum, and thus we need to use a global optimization technique. Fortunately, for univariate Lipschitz optimization there are a number of efficient deterministic global optimization methods [10]. We shall use Pijavsky-Shubert method [12, 13], which consists in building a sequence of saw-tooth underestimates of the objective function, which converges to it uniformly. The accumulation point of the sequence of global minima of the underestimates is the global minimum of the objective function. Thus we are able to obtain a guaranteed solution with any desired accuracy.

The technique is illustrated on Fig. 1. Let $f(t)$ be the objective function, known to be in $Lip(M)$. Let $\{(t^k, f(t^k))\}, k = 1, \dots, K$ be a sequence of points in the feasible domain with the respective function values. Then the underestimate at iteration K is given by

$$H^K(t) = \max_{k=1,\dots,K}(f(t^k) - M|t - t^k|) \le f(t).$$

The optimization algorithm proceeds by computing the global minimum of $H(t)$, t^*; taking $t^{K+1} = t^*$; adding the point $(t^{K+1}, f(t^{K+1}))$ to the set of function values, and updating the underestimate. The global minimum of H is found by sorting the list of its local minima, which in turn are also organized in a binary tree structure to facilitate updating the underestimate, and this makes the algorithm very efficient numerically. A detailed discussion is provided in [9].

To apply Pijavsky-Shubert algorithm we need an estimate of the Lipschitz constant of the objective function. Since $\delta \in Lip(Mn^{1/p})$ and is increasing, and the function

$$M||(x_1 - t)_+, \ldots, (x_n - t)_+||$$

is in $Lip(Mn^{1/p})$ and is decreasing (we can prove this with the help of the identity $||x||_p \leq n^{1/p}||x||_\infty$), the Lipschitz constant of the sum is $Mn^{1/p}$. Hence we use Pijavsky-Shubert algorithm with this parameter.

4.2 Opposite diagonal

Consider binary aggregation operators with given $\omega(t) = f(t, 1-t)$. The bounds are computed as $B_u(x) =$

$$\min_{t\in I}(\omega(t) + M||((x_1 - t)_+, (t - (1 - x_2))_+)||),$$

$B_l(x) =$

$$\max_{t\in I}(\omega(t) - M||((t - x_1)_+, (1 - x_2 - t)_+)||). \quad (12)$$

We notice that $\omega \in Lip(M)$ and so is the second term in the expression, hence the objective function is in $Lip(2M)$. We apply Pijavski-Shubert method with this Lipschitz parameter to calculate the values of the bounds for any x.

In [11] the following bounds were provided for bivariate 1-Lipschitz increasing functions.

$$\begin{aligned} B_u(x) &= T_L(x) + \min_{t\in[\alpha,\beta]}(\omega(t)) \quad (13) \\ B_l(x) &= S_L(x) - 1 + \max_{t\in[\alpha,\beta]}(\omega(t)), \end{aligned}$$

where $\alpha = \min\{x_1, 1 - x_2\}$, $\beta = \max\{x_1, 1 - x_2\}$. Let us show that these bounds also follow from (12). We have

$$B_u(x) = \min_{t\in I}(\omega(t)+M((x_1-t)_+ + (t-(1-x_2))_+)$$

Let $x_1 \leq 1 - x_2$ and consider three intervals $[0, x_1], [x_1, 1 - x_2]$ and $[1 - x_2, 1]$. On $[0, x_1]$ the objective function becomes $\omega(t) + M(x_1 - t)$. It is decreasing, and the minimum on this interval is achieved at $t = x_1$. On $[1 - x_2, 1]$ the expression becomes $\omega(t) + M(t - (1 - x_2))$. It is increasing, hence the minimum on this interval is achieved at $t = 1 - x_2$. The overall minimum is achieved on $[x_1, 1 - x_2]$: $B_u(x) =$

$$\min_{t\in[x_1,1-x_2]} \omega(t) = MT_L(x) + \min_{t\in[x_1,1-x_2]} \omega(t),$$

since $T_L(x) = \max(0, x_1 + x_2 - 1) = 0$ in this case.

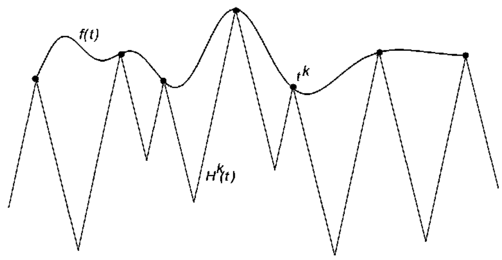

Figure 1: Illustration of the Pijavski-Shubert optimization scheme. The values of the objective function at t^k marked with dots determine the saw-tooth underestimate H^K.

Now let $1 - x_2 \leq x_1$. Again consider three intervals $[0, 1 - x_2]$,$[1 - x_2, x_1]$ and $[x_1, 1]$. On $[0, 1 - x_2]$ and $[x_1, 1]$ the objective function is decreasing and increasing respectively, hence the overall minimum is achieved on $[1 - x_2, x_1]$. On that interval we have $B_u(x) =$

$$\min_{t\in[1-x_2,x_1]}(\omega(t) + M((x_1 - t) + (t - (1 - x_2)))$$

$$= M(x_1 + x_2 - 1) + \min_{t\in[1-x_2,x_1]} \omega(t).$$

Since in this case $x_1 + x_2 - 1 \geq 0$, we can write the bound as

$$B_u(x) = MT_L(x) + \min_{t\in[1-x_2,x_1]} \omega(t),$$

and combining both cases and letting $M = 1$ we indeed obtain (13). The lower bound is obtained in a similar way.

5 Marginals

5.1 Bounds

Now we consider the problem of obtaining the operator f when certain functions are required to be its marginals. There are different aspects of this problem: a) construction of the operator by identifying upper and lower bounds; b) verifying that two or more marginals are compatible with each other; and c) identifying the smallest Lipschitz constant of f such that the marginals are compatible. In this section we will consider $n = 2$ fixed unless otherwise stated.

Consider construction of a Lipschitz aggregation operator f based on a given marginal g, defined on some closed subset Ω, for example $\Omega = \{x = (x_1, x_2) : 0 \le x_1 \le 1, x_2 = 0\}$. Let $g \in Lip(M_g)$. Then obviously the Lipschitz constant of f, $M \ge M_g$. From (2) we obtain

$$\begin{aligned} B_u(x) &= \min_{t\in[0,1]} (g(t) + M||((x_1 - t)_+, x_2)||) \\ &= \min_{t\in[0,x_1]} (g(t) + M||((x_1 - t), x_2)||), \\ B_l(x) &= \max_{t\in[0,1]} (g(t) - M||((t - x_1)_+, 0)||) \\ &= g(x_1). \end{aligned}$$

If the marginal is given on $\Omega = \{x = (x_1, x_2) : 0 \le x_1 \le 1, x_2 = 1\}$, then the bounds are

$$\begin{aligned} B_u(x) &= \min_{t\in[0,1]} (g(t) + M||((x_1 - t)_+, 0)||) \\ &= g(x_1), \\ B_l(x) &= \max_{t\in[0,1]} (g(t) - M||((t - x_1)_+, 1 - x_2)||) \\ &= \max_{t\in[x_1,1]} (g(t) - M||((t - x_1), 1 - x_2)||). \end{aligned}$$

To solve the optimization problem in each case we apply Pijavski-Shubert method with the Lipschitz parameter M.

5.2 Compatibility of the marginals

Consider now the case of two marginals $g_1(t_1)$, $g_2(t_2) \in Lip(M_g)$. We note that $M_g \le M$ and $2^{-1/p} \le M$. We have the following situations:

1. $f(x_1, 0) = g_1(x_1)$, $f(0, x_2) = g_2(x_2)$.
2. $f(x_1, 0) = g_1(x_1)$, $f(x_1, 1) = g_2(x_1)$.
3. $f(x_1, 0) = g_1(x_1)$, $f(1, x_2) = g_2(x_2)$.

By swapping the arguments of f we have three other cases, which are completely analogous to the cases above. We denote the domains on which the first and second marginals are defined by Ω_1 and Ω_2 respectively.

It is incorrect to assume that we can construct an aggregation operator f with the same Lipschitz constant as M_g and both marginals. We refer to this issue as incompatibility of the marginals. For example, consider a kernel aggregation operator with the marginals $g_1(x_1) = f(x_1, 0) = \max\{x_1 - \frac{1}{2}, 0\}$ and $g_2(x_2) = f(1, x_2) = \min\{x_2 + \frac{1}{2}, 1\}$. Clearly $g_1, g_2 \in Lip(1)$, but

$$f(1, \frac{1}{2}) - f(\frac{1}{2}, 0) = 1 > ||(1, \frac{1}{2}) - (\frac{1}{2}, 0)||_\infty = \frac{1}{2}.$$

Hence a kernel aggregation operator is incompatible with these marginals, and the smallest required Lipschitz constant is $M = 2$.

Of course, by choosing a larger M we can always build a suitable $f \in Lip(M)$, but we are interested in the situation $M = M_g$. A monotone Lipschitz function f is compatible with the data it interpolates if and only of the following conditions hold (Proposition 4.1 in [3])

$$\forall x, y \in \Omega_1 \cup \Omega_2 : f(x) - f(y) \le M||(x - y)_+|| \quad (14)$$

Thus a general approach is to verify the above mentioned Lipschitz conditions for all x and y (we only need to check it for x,y not in the same subset Ω_1 or Ω_2). However there are infinitely many points to perform such a test. In what follows, we will obtain a practically computable test.

Consider the following optimization problems

$$\begin{aligned} z_1 &= \min_{x\in\Omega_1, y\in\Omega_2} f(y) - f(x) + M||(x - y)_+||, \\ z_2 &= \min_{x\in\Omega_2, y\in\Omega_1} f(y) - f(x) + M||(x - y)_+||. \end{aligned}$$

Clearly, if $\min\{z_1, z_2\} \ge 0$, the marginals are compatible with $M = M_g$. We shall now consider instances of this problem for the three mentioned choices of Ω_1 and Ω_2.

Case 1. $\Omega_1 = \{(x_1, x_2) : x_1 \in I, x_2 = 0\}$, $\Omega_2 = \{(x_1, x_2) : x_2 \in I, x_1 = 0\}$.

$$\begin{aligned} z_1 &= \min_{t_1,t_2 \in I} \{g_2(t_2) - g_1(t_1) \\ &+ M||((t_1, 0) - (0, t_2))_+||\} \\ &= \min_{t_1,t_2 \in I} \{g_2(t_2) - g_1(t_1) + Mt_1\} \\ &= \min_{t_1 \in I} \{Mt_1 - g_1(t_1)\} = 0, \\ z_2 &= \min_{t_1,t_2 \in I} \{g_1(t_1) - g_2(t_2) \\ &+ M||((0, t_2) - (t_1, 0))_+||\} \\ &= \min_{t_1,t_2 \in I} \{g_1(t_1) - g_2(t_2) + Mt_2\} \\ &= \min_{t_2 \in I} \{Mt_2 - g_2(t_2)\} = 0. \end{aligned}$$

Since $g_1, g_2 \in Lip(M)$, increasing and $g_1(0) = g_2(0) = 0$, the minima are achieved at $t_1 = t_2 = 0$. Therefore in this case, the marginals are compatible for any $M = M_g \geq 2^{-1/p}$.

Case 2. $\Omega_1 = \{(x_1, x_2) : x_1 \in I, x_2 = 0\}$, $\Omega_2 = \{(x_1, x_2) : x_1 \in I, x_2 = 1\}$ (the opposite marginals). We note that $\forall x_2 \geq x_1 : g_1(x_1) \leq g_2(x_2)$.

$$\begin{aligned} z_1 = &\min_{t_1,t_2 \in I} \{g_2(t_2) - g_1(t_1) \\ &+ M||((t_1, 0) - (t_2, 1))_+||\} \\ = &\min_{t_1,t_2 \in I} \{g_2(t_2) - g_1(t_1) + M(t_1 - t_2)_+\} \geq 0, \\ z_2 = &\min_{t_1,t_2 \in I} \{g_1(t_1) - g_2(t_2) \\ &+ M||((t_2, 1) - (t_1, 0))_+||\} \\ = &\min_{t_1,t_2 \in I} \{g_1(t_1) - g_2(t_2) + M(1 + (t_2 - t_1)_+^p)^{1/p}\}. \end{aligned}$$

$z_1 \geq 0$ for any M, whereas the condition $z_2 \geq 0$ has to be verified for $2^{-1/p} \leq M < 1$.

If $M \geq 1$, $z_2 \geq 0$ automatically, since $M(1 + (t_2 - t_1)_+^p)^{1/p} \geq M \geq 1$, and $\min\{g_1(t_1) - g_2(t_2)\} \geq -1$.

Case 3. $\Omega_1 = \{(x_1, x_2) : x_1 \in I, x_2 = 0\}$, $\Omega_2 = \{(x_1, x_2) : x_2 \in I, x_1 = 1\}$. We note $g_1(1) = g_2(0)$, and of course $\forall x_1, x_2 \in I : g_1(x_1) \leq g_2(x_2)$.

$$\begin{aligned} z_1 = &\min_{t_1,t_2 \in I} \{g_2(t_2) - g_1(t_1) \\ &+ M||((t_1, 0) - (1, t_2))_+||\} \geq 0 \\ z_2 = &\min_{t_1,t_2 \in I} \{g_1(t_1) - g_2(t_2) \\ &+ M||((1, t_2) - (t_1, 0))_+||\} \\ = &\min_{t_1,t_2 \in I} \{g_1(t_1) - g_2(t_2) + M((1 - t_1)^p + t_2^p)^{1/p}\}. \end{aligned}$$

Using a change of variables $t = 1 - t_1$ in the second expression, we have

$$z_2 = \min_{t,t_2 \in I} \{g_1(1 - t) - g_2(t_2) + M(t^p + t_2^p)^{1/p}\}.$$

Now, $h_1(t, t_2) = g_1(1 - t) - g_2(t_2)$ is a decreasing function from $Lip(M_g, || \cdot ||_1)$, and hence $h_1 \in Lip(2^{1-1/p} M_g, || \cdot ||_p)$ because of the identity $||x||_1 \leq 2^{1-1/p} ||x||_p, \forall x \in R^2$. Next, $h_2(t, t_2) = M(t^p + t_2^p)^{1/p} = M|| \cdot ||_p = \sup_{h \in Lip(M, ||\cdot||_p : h(0)=0} h(\cdot)$ is increasing in non-negative quadrant, and $h_1(0, 0) = h_2(0, 0)$. The sum $h_1 + h_2$ is guaranteed to be non-negative if $M \geq 2^{1-1/p} M_g$, which is the required condition of compatibility of the marginals.

In summary, in case 1 the marginals are always compatible with $M = M_g$ for any $2^{-1/p} \leq M_g$, in case 2 they are compatible for $1 \leq M = M_g$, and in case 3 they are compatible for $M \geq 2^{1-1/p} M_g$.

If M is smaller than the last value, the marginals may still be compatible, but the value of z_2 has to be found numerically by solving a minimization problem (in two variables). This can be done by using the Cutting Angle deterministic method of global optimization [1, 2], which is a multivariate extension of the Pijavski-Shubert method.

5.3 The optimal Lipschitz constant

By choosing a suitably large M, namely $M \geq 2^{1-1/p} M_g$, we can achieve compatibility of the marginals with f. An interesting question arises: what is the smallest M which guarantees such compatibility of two specific marginals. To answer this question we need to solve the following problem

$$\begin{aligned} &\min M \\ \text{s.t. } z_2 \;=\; &\min_{t_1,t_2 \in I} \{ g_1(t_1) - g_2(t_2) \\ &+ M((1 - t_1)^p + t_2^p)^{1/p} \} \geq 0, \\ &M_g \leq M \leq 2^{1-1/p} M_g. \end{aligned}$$

Since z_2 is a monotone increasing function of M, we can apply the bisection method to solve the equation

$$\min_{t_1,t_2 \in I} \{g_1(t_1) - g_2(t_2) + M((1 - t_1)^p + t_2^p)^{1/p}\} = 0$$

on the interval $[M_g, 2^{1-1/p} M_g]$ with a given tolerance.

6 Summary and Conclusions

Pointwise construction of aggregation operators allows one to fit the desired values while preserving its essential properties. The central interpolation scheme delivers an optimal aggregation operator from a given class, and is based on establishing tight upper and lower bounds on the values of the aggregation operator at all points.

In all cases the bounds are a result of applying general formulae Eqns.(1)-(3). However, the actual computation of the bounds requires solving certain optimization problems, which may be complicated. In this work we found explicit solutions in the cases of a given diagonal, opposite diagonal and marginals, and formulated suitable algorithms which guarantee convergence to the right solution.

Acknowledgements

This work was supported by the projects MTM2004-3175, TIC2002-11942-E and BFM2003-05308 from Ministerio de Educación y Ciencia, Spain, and PRIB-2004-9250, Govern de les Illes Balears.

References

[1] G. Beliakov. Geometry and combinatorics of the cutting angle method. *Optimization*, 52:379–394, 2003.

[2] G. Beliakov. The cutting angle method - a tool for constrained global optimization. *Optimization Methods and Software*, 19:137–151, 2004.

[3] G. Beliakov. Monotonicity preserving approximation of multivariate scattered data. *BIT*, 45:653–677, 2005.

[4] G. Beliakov. Pointwise construction of Lipschitz aggregation operators. Part I – construction methods. *Int. J. of Uncertainty, Fuzziness and Knowledge-Based Systems*, submitted, 2005.

[5] G. Beliakov and T. Calvo. Identification of general and double aggregation operators using monotone smoothing. In E. Montseny and P. Sobrerillo, eds, *5th Conf. of the Europ. Soc. for Fuzzy Logic and Technology*, pp 937–942, Barcelona, Spain, 2005.

[6] T. Calvo, A. Kolesarova, M. Komornikova, and R. Mesiar. Aggregation operators: properties, classes and construction methods. In T. Calvo, G. Mayor, and R. Mesiar, eds, *Aggregation operators. New trends and applications*, pp 3–104. Physica-Verlag, Heidelberg, New York, 2002.

[7] T. Calvo and R. Mesiar. Stability of aggregation operators. In *2nd Conf. of the Europ. Soc. for Fuzzy Logic and Technology*, pp 475–478, Leicester, 2001.

[8] M. Grabisch. The Choquet integral as a linear interpolator. In *10th Int. Conf. IPMU*, pp 373–378, Perugia, Italy, 2004.

[9] P. Hansen and B. Jaumard. Lipschitz optimization. In R.Horst and P. Pardalos, eds, *Handbook of Global Optimization*, pp 407–493. Kluwer, Dordrecht, 1995.

[10] R. Horst, P. Pardalos, and N.V. Thoai, eds. *Introduction to Global Optimization* Kluwer, Dordrecht, 2nd edition, 2000.

[11] E.P. Klement and A. Kolesarova. Extension to copulas and quasi-copulas as special 1-Lipschitz aggregation operators. *Kybernetika*, 41:329–348, 2005.

[12] S.A. Pijavski. An algorithm for finding the absolute extremum of a function. *USSR Comput. Math. and Math. Phys.*, 2:57–67, 1972.

[13] B. Shubert. A sequential method seeking the global maximum of a function. *SIAM J. Numer. Anal.*, 9:379–388, 1972.

[14] R. Yager. Noble reinforcement in disjunctive aggregation operators. *IEEE Trans. on Fuzzy Syst.*, 11:754–767, 2003.