Design of a Common Observer for Two Linear Systems with Unknown Inputs

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Abstract: This paper considers the design of a common linear functional observer for two linear time-invariant systems with unknown inputs. A structure for a common observer which only uses the available output information is proposed. Here, for the proposed structure, we show that the simultaneous functional observation problem of two plants is reduced to a problem of designing two observers: The first is a full-order unknown input observer of one of the two systems; the second observer is a common unknown input observer of a system comprises two-connected systems. In general, the existence conditions for the second observer are very difficult to satisfy. This paper thus concludes that it is indeed very difficult to find a common observer for two linear systems with unknown inputs.

I. INTRODUCTION

For a given set of plants, it is always intrigue to know whether there exists a common observer that can observe the states for each of a given set of plants. Such an observer can be used as a robust observer and/or an observer for nonlinear systems. The simultaneous observation problem has recently been addressed by Moreno [1] for linear time-invariant systems with known inputs. Motivated by this work, this paper considers the problem of designing a common linear functional observer for two linear time-invariant systems with unknown inputs. Here, our objective is to seek a sufficient condition for the existence and design of a common observer for two linear systems that are subject to unknown inputs.

Let us consider two linear time-invariant systems with unknown inputs as described by

\[
\begin{align*}
\Phi_i: \quad \dot{x}_i(t) &= A_i x_i(t) + D_i d_i(t) \\
y_i(t) &= C_i x_i(t), \\
z_i(t) &= L_i x_i(t), \quad i = 1, 2
\end{align*}
\]

where \( x_i(t) \in \mathbb{R}^n \), \( d_i(t) \in \mathbb{R}^n \), \( y_i(t) \in \mathbb{R}^p \) and \( z_i(t) \in \mathbb{R}^r \) are, respectively, the state vector, the input vector, the measured output vector and the vector of signals to be estimated. Matrices \( A_i \), \( D_i \), \( C_i \) and \( L_i \) are known real constant and of appropriate dimensions.

The aim of this paper is to use only the available output information, \( y_i(t) \), to construct a single linear functional observer, called a Common Functional Observer (CFO), such that it is a linear functional observer for every element of the family of LTI systems described by (1). Figure 1 shows the block diagram of a CFO. In Figure 1, \( \Phi_i \), \( i = 1, 2 \) represents the actual plant, \( \Omega_c \) is the CFO and \( \hat{z}_i(t) \) denotes the estimate of \( z_i(t) \) (i.e. \( \hat{z}_i(t) \rightarrow z_i(t) \)).

![Figure 1: Block diagram of a Common Functional Observer.](image)

In the next section, we propose a structure for a CFO. We show that the design of the CFO is reduced to a problem of designing two functional observers for two linear systems with unknown inputs.

II. A COMMON FUNCTIONAL OBSERVER FOR TWO SYSTEMS WITH UNKNOWN INPUTS

Let us now consider the following structure of a CFO:
In Figure 2, the only information that feeds into the CFO is the output information, \( y_i(t) \). The proposed CFO observer for \( \Phi_i \) and \( \Phi_2 \) consists of the interconnection of two unknown input functional observers (UIFOS), namely \( \Omega_i \) and \( \Omega_2 \). In Figure 2, \( \hat{y}_i(t) = \{\hat{y}_i(t) - y_i(t)\} \) and \( v_i(t) \) is defined as \( v_i(t) = \{\hat{z}_i(t) - z_i(t)\} \).

Now, let \( \Omega_i \) be an UIFO of the form:

\[
\begin{align*}
\dot{w}_i(t) &= N_i w_i(t) + F_i y_i(t) \\
\dot{\hat{y}}_i(t) &= M_i w_i(t) + G_i y_i(t) \\
\dot{\hat{z}}_i(t) &= M_i \hat{w}_i(t) + G_i \hat{y}_i(t)
\end{align*}
\]

where \( w_i(t) \in \mathbb{R}^n \), matrices \( N_i \) (Hurwitz), \( F_i \), \( M_i \), \( G_i \), \( M_2 \) and \( G_2 \) are to be determined such that for \( i = 1 \), \( \hat{z}_i(t) \rightarrow z_i(t) \) and \( \hat{y}_i(t) \rightarrow y_i(t) \) for \( \Phi_i \).

Also, let \( \Omega_2 \) be a reduced-order UIFO of the form:

\[
\begin{align*}
\dot{w}_2(t) &= N_2 w_2(t) + F_2 \hat{y}_2(t) \\
\dot{v}_2(t) &= M_2 w_2(t) + G_2 \hat{y}_2(t)
\end{align*}
\]

where \( w_2(t) \in \mathbb{R}^q \), \( q \leq (n_i + n_j - p) \), matrices \( N_2 \) (Hurwitz), \( F_2 \), \( M_2 \) and \( G_2 \) are to be determined such that \( \hat{v}_2(t) \rightarrow v_i(t) \) for the following system:

\[
\begin{align*}
\begin{bmatrix}
\dot{w}_i(t) \\
\dot{x}_i(t)
\end{bmatrix} &=
\begin{bmatrix}
N_i & F_i C_i \\
0 & A_i
\end{bmatrix}
\begin{bmatrix}
w_i(t) \\
x_i(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
D_i
\end{bmatrix} d_i(t) \\
\end{align*}
\]

\[
\Phi_i : \begin{align*}
\hat{y}_i(t) &= \{\hat{y}_i(t) - y_i(t)\} = [M_i \quad \{G_i C_i - C_i\}]
\begin{bmatrix}
w_i(t) \\
x_i(t)
\end{bmatrix} \\
v_i(t) &= \{\hat{z}_i(t) - z_i(t)\} = [M_2 \quad \{G_2 C_i - L_i\}]
\begin{bmatrix}
w_i(t) \\
x_i(t)
\end{bmatrix}.
\end{align*}
\]

The following theorem presents an existence condition for the proposed CFO of Figure 2.

**Theorem 1:** There exists a CFO of the structure depicted in Figure 2 such that \( \hat{z}_i(t) \rightarrow z_i(t) \) (\( i = 1, 2 \)) provided that there exist two UIFOS \( \Omega_i \) and \( \Omega_2 \) for \( \Phi_i \) and \( \Phi_2 \), respectively.

**Proof:** First, consider the case where the information that feeds into \( \Omega_i \) is coming from the first plant, i.e. \( i = 1 \), \( y_i(t) = y_i(t) \). Accordingly, if \( \Omega_i \) for \( \Phi_i \) exists, then \( \Omega_i \) provides the estimates \( \hat{z}_i(t) \) and \( \hat{y}_i(t) \) such that \( \hat{z}_i(t) \rightarrow z_i(t) \) and \( \hat{y}_i(t) \rightarrow y_i(t) \). Since \( \hat{y}_i(t) \rightarrow y_i(t) \), it follows that \( \hat{y}_i(t) \rightarrow 0 \) and therefore provided that \( \Omega_2 \) is internally stable (i.e. \( N_i \) is Hurwitz) then \( \hat{v}_2(t) \rightarrow 0 \). From Figure 2 it is clear that \( \hat{z}_i(t) \rightarrow z_i(t) \). In this case, the CFO provides the estimate \( \hat{z}_i(t) \) of system 1.

Second, consider the case where the information is coming from the second plant, i.e. \( i = 2 \), \( y_i(t) = y_i(t) \). In this case, \( \hat{y}_i(t) = \{\hat{y}_i(t) - y_i(t)\} \rightarrow 0 \) and the interconnection of systems \( \Omega_i \) and \( \Phi_i \) is described by \( \Phi_i \) of equation (4). Accordingly, if \( \Omega_i \) for \( \Phi_i \) exists, then \( \Omega_i \) provides the estimate \( \hat{v}_2(t) \) such that \( \hat{v}_2(t) \rightarrow v_i(t) \). From Figure 2, it is then clear that \( \hat{z}_2(t) = \{\hat{z}_2(t) - \hat{v}_2(t)\} \rightarrow z_2(t) \). In this case, the CFO provides the estimate \( \hat{z}_2(t) \) of system 2. This completes the proof of Theorem 1.

**Remark 1:** From Theorem 1, the design of a CFO now rests with the design of \( \Omega_i \) and \( \Omega_2 \) for \( \Phi_i \) and \( \Phi_2 \), respectively. Now, necessary and sufficient conditions for the existence and design of \( \Omega_i \) for \( \Phi_i \) are well-known and existing observer design methods reported in the literature (see, for example [2]-[6]) can be used. Therefore, the existence of a CFO now depends on the existence of \( \Omega_2 \) for \( \Phi_2 \).

**Existence of \( \Omega_2 \) for \( \Phi_2 \)**

In the following, we consider the necessary and sufficient conditions for the existence of \( \Phi_2 \). We then conclude that it is indeed almost impossible to have a CFO for two linear systems with unknown inputs.

From [3], \( \Omega_2 \) exists if and only if the following conditions are satisfied:

\[
\begin{align*}
\text{for } i = 1, 2, \quad \text{conditions are satisfied:}
\end{align*}
\]
\textbf{Condition 1: } \textit{rank}(\overline{CD}) = \textit{rank}(\overline{D}) \quad (5)

\textbf{Condition 2: } \textit{rank}\begin{bmatrix} sl-A & \overline{D} \\ \overline{C} & 0 \end{bmatrix} = (n_1 + n_2 + m_i)
\forall s \in \mathbf{C}, \text{Re}(s) \geq 0, \quad (6)

where \( \overline{C} = [M_{ii} \quad (G_i C_i - C_i)] \), \( \overline{D} = \begin{bmatrix} 0 \\ D_2 \end{bmatrix} \) and \( \overline{A} = \begin{bmatrix} N_i & F_iC_i \\ 0 & A_i \end{bmatrix} \).

First, let us examine Condition 1: It is clear that when \( G_{ii} = I_r \), matrix \( \overline{C} \) becomes \( \overline{C} = [M_{ii} \quad 0] \) and hence \( \text{rank}(\overline{CD}) \neq \text{rank}(\overline{D}) \). Accordingly, Condition 1 does not hold. Therefore, to ensure that Condition 1 holds, it is necessary that matrix \( (G_i C_i - C_i) \neq 0 \), i.e. \( G_i \neq I_r \). From [2,4], this requirement is met provided that the following condition is satisfied for \( \Omega_i \): \( \text{rank}(C_iD_i) = n_i \) and \( n_i < m_i \). This implies that for system 1, the number of outputs must be greater than the number of unknown inputs. Note that this is a rather restrictive requirement since for normal UIO, \( m_i \leq p \).

Second, let us examine Condition 2: By substituting matrices \( \overline{A} \), \( \overline{C} \) and \( \overline{D} \) into Condition 2, the following can be easily deduced (assuming that matrix \( D_2 \) is of full-column rank):

\[
\text{rank}\begin{bmatrix} (sI_{n_i} - N_i) & -F_iC_i & 0 \\ 0 & (sI_{n_2} - A_i) & D_2 \\ M_{ii} & (G_i C_i - C_i) & 0 \end{bmatrix} = m_2 + \text{rank}\begin{bmatrix} (sI_{n_i} - N_i) & -F_iC_i \\ M_{ii} & (G_i C_i - C_i) \end{bmatrix} \leq (n_i + p) \leq (n_i + n_2) \quad (7)
\]

It is easy to see from (7) that, in general,

\[
\text{rank}\begin{bmatrix} (sI_{n_i} - N_i) & -F_iC_i \\ M_{ii} & (G_i C_i - C_i) \end{bmatrix} \leq (n_i + p) \leq (n_i + n_2) \quad \forall s \in \mathbf{C}, \text{Re}(s) \geq 0. \quad (8)
\]

Thus, we conclude that, Condition 2 is rarely held and therefore it is unfortunate that for the proposed structure of CFO of Figure 2, it is almost impossible to have a CFO for two linear systems with unknown inputs.

\section{Conclusion}

This paper has proposed a structure for a common observer of two linear systems with unknown inputs. It has been shown that the simultaneous functional observation problem of two plants is reduced to a problem of designing two observers for two linear systems with unknown inputs. Unfortunately, we have shown that it is indeed very difficult to find a common observer for two linear systems with unknown inputs. Unlike that of the case for known inputs [1], it is very difficult to have a CFO for two systems with unknown inputs.

\section{References}


