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Absorbent tuples of aggregation operators

G. Beliakov ¹, T. Calvo ² and A. Pradera ³

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¹ School of Engineering and Information Technology, Deakin University, 221 Burwood
   Hwy, Burwood 3125, Australia. email: gleb@deakin.edu.au
² Departamento de Ciencias de la Computación, Universidad de Alcalá
   28871-Alcalá de Henares (Madrid), Spain. email: tomasa.calvo@uah.es
³ Departamento de ATC y CCIA, Universidad Rey Juan Carlos
   28933 Móstoles (Madrid), Spain. email: ana.pradera@urjc.es

Abstract

We generalize the notion of an absorbent element of aggregation
operators. Our construction involves tuples of values that decide the
result of aggregation. Absorbent tuples are useful to model situations
in which certain decision makers may decide the outcome irrespective
of the opinion of the others. We examine the most important classes
of aggregation operators in respect to their absorbent tuples, and also
construct new aggregation operators with predefined sets of absorbent
tuples.

Keywords Aggregation operators, absorbent element, absorbent tuple, null
set.
1 Introduction

When we vote in the elections, our preferences are somehow aggregated to express a compromise decision about the composition of the parliament. Different political systems and jurisdictions require different internally consistent aggregation rules. While a simple jury majority may be required in a civil court, a much different voting scheme is used in criminal courts when considering most serious crimes. Aggregation of preferences is also a key part in business and economics, politics, law and all aspects of decision making ([10,13]).

Combining expert opinions, group decision making and welfare economics are other generic examples in which aggregation procedures with different behaviour are needed ([12, 23]). Such features as the veto rule, oligarchy, abstentions, anonymity and coalitions need to be properly modeled.

Revolutionary developments in information technology have created a new demand for many specials forms of aggregation. Internet search engines determine the relevance of a given web page by aggregating the “votes” of other web pages. Decision support, expert and recommender systems are widely used in many fields, especially e-commerce, and they also require sophisticated methods of combining uncertain information coming from different sources (so called information fusion).

Aggregation operators are used to express in mathematical language various aggregation procedures. In this article we consider aggregation of values (votes, preferences, scores) given on the interval scale, namely as the values from the unit interval. A detailed analysis of various classes of aggregation operators is given in [9,11].

In this work we extend the notion of the absorbent element of aggregation operators, a specific value which determines the result of aggregation whenever any of its arguments takes this value. Our construction involves absorbent tuples – certain tuples (or vote combinations) that determine the result of aggregation by themselves.

Consider the following motivating example. A listed company has an executive board and broader shareholder’s meetings, at both meetings preferences (expressed as votes) are aggregated. Different voters have different powers according to their holdings. If the executive board is mostly unanimous in its decision, it takes this decision by itself, without other shareholder’s vote. If the executive board were able to take all decisions by itself, we would be talking about oligarchy. However, in our case we have partial
oligarchy – only certain vote combinations result in the outcome determined solely by the board. We call all such combinations a null set.

Another example of null sets involves combination of expert opinions, for instance during a competitive grant allocation procedure. A small committee of experts considers a large number of competitive grant applications, and gives each application a numerical score. The applications are also sent to reviewers – experts in a particular field – who also score the applications. At the end of the process all the scores are aggregated. The scores given by committee members weight more, because these experts consider many more applications than the reviewers. Thus several strong votes by committee members may decide the outcome irrespective of the scores of the reviewers, while in case of disagreement reviewers scores will be used. Can we model such situation with a single aggregation operator?

These illustrative examples suggest the variety of practical situations in which we would like to model absorbent behavior of the aggregation procedure. In this article we examine the most important families of aggregation operators and establish their null sets. This will help determine their suitability for particular applications. We also describe a practical method of construction of aggregation operators with a predefined null set.

The rest of the paper is organized as follows. The next two sections give the basic definitions of aggregation operators and absorbent tuples, and provide a number of general results. In section 4 we analyze the most important families of aggregation operators and establish their null sets. In section 5 we solve the inverse problem: how to build an aggregation operator with a predefined null set. We conclude the article with a short summary.

2 Preliminaries

Throughout the paper $I$ will be used to denote the unit interval $[0, 1]$.

**Definition 1** [9] An aggregation operator is a function $F : \bigcup_{n \in \mathbb{N}} ^{n} \rightarrow I$ such that:

(i) $F(x_1, \ldots, x_n) \leq F(y_1, \ldots, y_n)$ whenever $x_i \leq y_i$ for all $i \in \{1, \ldots, n\}$.

(ii) $F(t) = t$ for all $t \in I$. 

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(iii) $F(0, \ldots, 0) = 0$ and $F(1, \ldots, 1) = 1$

Each aggregation operator $F$ can be represented by a family of $n$-ary operators $f_n : I^n \to I$ given by $f_n(x_1, \ldots, x_n) = F(x_1, \ldots, x_n)$. This representation allows one to define most of the properties of aggregation operators:

**Definition 2** Let $F$ be an aggregation operator and $(f_n)_{n \in \mathbb{N}}$ the corresponding family of $n$-ary operations.

(i) $F$ is called symmetric, idempotent, strictly monotone (on the whole domain) or continuous if, for each $n \geq 2$, the $n$-ary operation $f_n$ is symmetric, idempotent, strictly monotone or continuous, respectively.

(ii) $F$ is called associative if $\forall n, m \in \mathbb{N}, \forall x_1, \ldots, x_n, y_1, \ldots, y_m \in I$:

$$F(x_1, \ldots, x_n, y_1, \ldots, y_m) = F(F(x_1, \ldots, x_n), F(y_1, \ldots, y_m))$$

(iii) An element $e \in I$ is called a neutral element and an element $a \in I$ is called an absorbent element (or an annihilator) of $F$ if for each $n \geq 2$, for each $i \in \{1, \ldots, n\}$ and for all $x_1, \ldots, x_n \in I$, we have, respectively:

$$f_n(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) = f_{n-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \text{ whenever } x_i = e;$$

$$f_n(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) = a \text{ whenever } x_i = a.$$  

### 3 Absorbent information

According to Definition 2, an absorbent element of an aggregation operator $F$ is a value $a \in I$ that carries the final output to $a$. In the following, in order to cope with larger pieces of absorbent information, we first generalize this standard definition to the case of tuples $\alpha = (\alpha_1, \ldots, \alpha_m) \in I^m$, $m \in \mathbb{N} = \{1, 2, \ldots\}$ (section 3.1), and then analyze some properties of such absorbent tuples (section 3.2). Before that we introduce the specific notation that will be used in the rest of the paper.

To denote the subsets of components of a vector $x \in I^n$ we shall employ the following notation. If $\mathcal{I} = \{I_1, \ldots, I_m\} \subset \{1, \ldots, n\}$ is an index set
with cardinality $m = |\mathcal{I}|$ and $\mathcal{P} = (\mathcal{P}(1), \ldots, \mathcal{P}(m))$ is a permutation of $(1, \ldots, m)$, then $x_{\mathcal{I}, \mathcal{P}}$ will be used to denote the vector obtained from $x$ by selecting the components whose indices are in $\mathcal{I}$ but in the order given by the permutation $\mathcal{P}$ (and using the convention $\overline{\mathcal{I}}_1 < \ldots < \overline{\mathcal{I}}_m$): that is, we will have $x_{\mathcal{I}, \mathcal{P}} = (x_{\mathcal{I}, \mathcal{P}(1)}, \ldots, x_{\mathcal{I}, \mathcal{P}(m)})$. In addition, if $\overline{\mathcal{I}} = \{\overline{\mathcal{I}}_1, \ldots, \overline{\mathcal{I}}_{n-m}\}$, with convention $\overline{\mathcal{I}}_1 < \ldots < \overline{\mathcal{I}}_{n-m}$, denotes the complement of $\mathcal{I}$ in $\{1, \ldots, n\}$, then $x_{\overline{\mathcal{I}}}$ will denote the tuple $(x_{\overline{\mathcal{I}}_1}, \ldots, x_{\overline{\mathcal{I}}_{n-m}})$. For example, if $n = 5$, $\mathcal{I} = \{2, 4, 5\}$ and $\mathcal{P} = (2, 1, 3)$, then $x_{\mathcal{I}, \mathcal{P}} = (x_4, x_2, x_5)$, $\overline{\mathcal{I}} = \{1, 3\}$ and $x_{\overline{\mathcal{I}}} = (x_1, x_3)$.

### 3.1 Definitions and examples

**Definition 3** Let $F$ be an aggregation operator, $\mathcal{I} \subset \{1, \ldots, n\}$, $n > 1$, an index set such that $|\mathcal{I}| = m$ and $\mathcal{P}$ a permutation of $(1, \ldots, m)$. Then:

- A tuple $\alpha \in I^m$ is **absorbent for $F$ at level $n$ with respect to (w.r.t.) $(\mathcal{I}, \mathcal{P})$** when
  
  $$f_n(x) = f_m(\alpha_{\mathcal{P}})$$  \hspace{1cm} (1)

  holds for all $x \in I^n$ such that $x_{\mathcal{I}, \mathcal{P}} = \alpha$, where $\alpha_{\mathcal{P}} = (\alpha_{\mathcal{P}(1)}, \ldots, \alpha_{\mathcal{P}(m)})$ is the tuple obtained from $\alpha = (\alpha_1, \ldots, \alpha_m)$ by means of the permutation $\mathcal{P}$.

- The set made of all the tuples $\alpha \in I^m$ which are absorbent for $F$ at level $n$ w.r.t. $(\mathcal{I}, \mathcal{P})$ will be denoted by $\mathcal{A}_m(F, n, \mathcal{I}, \mathcal{P})$ and will be called the **null set of $F$ at level $n$ w.r.t. $(\mathcal{I}, \mathcal{P})$**.

**Example 1** Let $F$ be an aggregation operator, $n = 3$, $\mathcal{I} = \{2, 3\}$, $\mathcal{P} = Id$ and $\alpha = (1, 0) \in I^2$. Then $\alpha$ is absorbent for $F$ at level 3 w.r.t. $(\mathcal{I}, \mathcal{P})$, i.e., $\alpha \in \mathcal{A}_2(F, 3, \mathcal{I}, \mathcal{P})$, if $f_3(x_1, 1, 0) = f_2(1, 0)$ holds for any $x_1 \in I$.

**Example 2** Let $F$ be an aggregation operator, $n = 3$, $\mathcal{I} = \{2, 3\}$, $\mathcal{P} = (2, 1)$ and $\alpha = (1, 0) \in I^2$. Then $\alpha$ is absorbent for $F$ at level 3 w.r.t. $(\mathcal{I}, \mathcal{P})$, i.e., $\alpha \in \mathcal{A}_2(F, 3, \mathcal{I}, \mathcal{P})$, if $f_3(x_1, 0, 1) = f_2(0, 1)$ holds for any $x_1 \in I$.

The above definition implies that when aggregating $n$ values with $F$, the information contained in a given tuple $\alpha$, if appearing in the positions indicated by some particular pair $(\mathcal{I}, \mathcal{P})$, transforms the final output to $F(\alpha_{\mathcal{P}})$. Of course, the same could happen – as it is the case of the standard absorbent element – independently of the positions that the components of $\alpha$ occupy in the input vector $x$. The next definition accommodates this situation.
**Definition 4** Let $F$ be an aggregation operator and let $m, n \in \mathbb{N}, m < n$. Then:

- A tuple $\alpha \in I^m$ is absorbent for $F$ at level $n$ when, for any index set $I \subset \{1, \ldots, n\}$ such that $|I| = m$ and any permutation $\mathcal{P}$ of $(1, \ldots, m)$, $\alpha$ is absorbent for $F$ at level $n$ w.r.t. $(I, \mathcal{P})$.
- The set made of all the tuples $\alpha \in I^m$ which are absorbent for $F$ at level $n$ will be denoted by $A_m(F, n)$ and will be called the $m$-null set of $F$ at level $n$.

**Example 3** Choosing, as in the previous examples, $n = 3$ and $\alpha = (1, 0) \in I^2$, now $\alpha$ is absorbent for $F$ at level 3, i.e., $\alpha \in A_2(F, 3)$, if $f_3(x_1, 1, 0) = f_3(1, x_2, 0) = f_3(1, 0, x_3) = f_2(1, 0)$ and $f_3(x_1, 0, 1) = f_3(0, x_2, 1) = f_3(0, 1, x_3) = f_2(0, 1)$ holds for any $x_1, x_2, x_3 \in I$.

**Remark 1** If $F$ is an aggregation operator and $m, n \in \mathbb{N}, m < n$, then:

1. $A_m(F, n) = \bigcap_{I \subset \{1, \ldots, n\}, |I| = m} A_m(F, n, I, \mathcal{P})$.

2. If $F$ is symmetric, then it obviously suffices to have $\alpha \in A_m(F, n, I, \mathcal{P})$ for some $(I, \mathcal{P})$ in order to automatically have $\alpha \in A_m(F, n)$.

3. If $\alpha = (\alpha_1, \ldots, \alpha_m) \in A_m(F, n)$, then $\alpha = (\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(m)}) \in A_m(F, n)$ for any permutation $\sigma = (\sigma(1), \ldots, \sigma(m))$ of $(1, \ldots, m)$.

Coming back to Definition 3, note now that it refers to just one specific dimension, $n$, of the aggregation operator $F$. Similarly to the way in which the standard absorbent element is defined, we could think of tuples $\alpha \in I^m$ remaining absorbent for any dimension (as long as such dimension is greater or equal to the positions given by the index set $I$):

**Definition 5** Let $F$ be an aggregation operator, $I \subset \{1, 2, \ldots\}$ an index set such that $|I| = m$ and $\mathcal{P}$ a permutation of $(1, \ldots, m)$. Then:

- A tuple $\alpha \in I^m$ is absorbent for $F$ w.r.t. $(I, \mathcal{P})$ when, for any $n \geq max(|I| + 1, max(I))$, $\alpha$ is absorbent for $F$ at level $n$ w.r.t. $(I, \mathcal{P})$. 

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The set made of all the tuples \( \alpha \in I^m \) which are absorbent for \( F \) w.r.t. \((I,\mathcal{P})\) will be denoted by \( \mathcal{A}_m(F,I,\mathcal{P}) \) and will be called the null set of \( F \) w.r.t. \((I,\mathcal{P})\).

**Example 4** Choosing, as in Example 2, \( I = \{2,3\} \) with \( \mathcal{P} = (2,1) \), then \( \alpha = (1,0) \in I^2 \) is absorbent for \( F \) w.r.t. \((I,\mathcal{P})\), i.e., \( \alpha \in \mathcal{A}_2(F,I,\mathcal{P}) \), if \( f_3(x_1,0,1) = f_2(0,1) \) holds for any \( t_1 \in I \), \( f_3(x_1,0,1,x_4) = f_2(0,1) \) holds for any \( x_1,x_4 \in I \), \( f_3(x_1,0,1,x_4,x_5) = f_2(0,1) \) holds for any \( x_1,x_4,x_5 \in I \), etc.

**Remark 2** If \( F \) is an aggregation operator, \( I \subset \{1,2,\ldots\} \) is an index set such that \(|I| = m\) and \( \mathcal{P} \) is a permutation of \((1,\ldots,m)\), then:

\[
\mathcal{A}_m(F,I,\mathcal{P}) = \bigcap_{n \geq \max(|I|+1,\max(I))} \mathcal{A}_m(F,n,I,\mathcal{P})
\]

Definitions 4 and 5 have been obtained from Definition 3 after independently introducing a stronger demand on two different aspects: the position of the null information within the input vector and the dimension of the latter, respectively. If these two aspects are taken into account simultaneously, the result can be stated as follows:

**Definition 6** Let \( F \) be an aggregation operator and let \( m \in \mathbb{N} \). Then:

- A tuple \( \alpha \in I^m \) is absorbent for \( F \) when, for any \( n > m \), for any index set \( I \subset \{1,\ldots,n\} \) such that \(|I| = m\) and for any permutation \( \mathcal{P} \) of \((1,\ldots,m)\), \( \alpha \) is absorbent for \( F \) at level \( n \) w.r.t. \((I,\mathcal{P})\).

- The set made of all the tuples \( \alpha \in I^m \) which are neutral for \( F \) will be denoted by \( \mathcal{A}_m(F) \) and will be called the \( m \)-null set of \( F \).

**Example 5** The tuple \( \alpha = (1,0) \in I^2 \) is absorbent for \( F \) if:

\( \forall n > 2, \forall x = (x_1,\ldots,x_n) \in I^n, \)

1. If there exist \( i,j \in \{1,\ldots,n\}, \ i < j, \) such that \( x_i = 0, x_j = 1, \) then

\[
f_n(x) = f_2(0,1).
\]

2. If there exist \( i,j \in \{1,\ldots,n\}, \ i < j, \) such that \( x_i = 1, x_j = 0, \) then

\[
f_n(x) = f_2(1,0).
\]
Remark 3 If $F$ is an aggregation operator and $m \in \mathbb{N}$, then:

1. \[
\mathcal{A}_m(F) = \bigcap_{n>m, \mathcal{I} \subseteq \{1, \ldots, n\}, |\mathcal{I}|=m} \mathcal{A}_m(F, n, \mathcal{I}, \mathcal{P}).
\]

2. When choosing $m = 1$, Definition 6 recovers the standard definition of the absorbent element, i.e.:
   \[
   \mathcal{A}_1(F) = \begin{cases} 
   \{a\}, & \text{if } F \text{ has absorbent element } a \in I, \\
   \emptyset, & \text{otherwise}.
   \end{cases}
   \]

Remark 4 As it can be easily checked, the concept of absorbent tuple for $F$ could have been alternatively defined using either Definition 4 or Definition 5, that is, the two following statements hold:

1. $\alpha \in I^m$ is absorbent for $F$ if and only if $\alpha$ is absorbent for $F$ at level $n$ for any $n > m$, that is:
\[
\mathcal{A}_m(F) = \bigcap_{n>m} \mathcal{A}_m(F, n).
\]

2. $\alpha \in I^m$ is absorbent for $F$ if and only if $\alpha$ is absorbent for $F$ w.r.t. $(\mathcal{I}, \mathcal{P})$ for any index set $\mathcal{I} \subseteq \{1, 2, \ldots\}$ such that $|\mathcal{I}| = m$ and any permutation $\mathcal{P}$ of $(1, \ldots, m)$, that is:
\[
\mathcal{A}_m(F) = \bigcap_{\mathcal{I} \subseteq \{1, 2, \ldots\}, |\mathcal{I}|=m} \mathcal{A}_m(F, \mathcal{I}, \mathcal{P}).
\]

When referring to the set made of all the tuples, regardless of their dimension, which are absorbent for a given aggregation operator $F$, we will use the following:

Definition 7 The null set of an aggregation operator $F$, denoted by $\mathcal{A}(F)$, is the set made of all the tuples $\alpha \in I^m$, $m \in \mathbb{N}$, which are absorbent for $F$
\[
\mathcal{A}(F) = \bigcup_{m \in \mathbb{N}} \mathcal{A}_m(F).
\]
3.2 Basic properties

Let us now discuss some basic properties of absorbent tuples. We will first of all mention a very simple one, that points out the existence of aggregation operators without absorbent tuples:

**Proposition 1** Let $F$ be an aggregation operator and let $m \in \mathbb{N}$. If $F$ is strictly monotone, then $A_m(F, n, \mathcal{I}, \mathcal{P}) = \emptyset$ for any $n > m$, any $\mathcal{I} \subseteq \{1, \ldots, n\}$ such that $|\mathcal{I}| = m$ and any permutation $\mathcal{P}$ of $(1, \ldots, m)$.

**Proof.** Definition 3 clearly shows that $A_m(F, n, \mathcal{I}, \mathcal{P}) \neq \emptyset$ implies that $F$ cannot be strictly monotone.

The arithmetic mean is a well-known example of strictly increasing aggregation operator which, as a consequence, has empty null sets.

We can now notice that absorbent tuples have the property that their aggregation, by means of $F$, always provides the same output:

**Proposition 2** Let $F$ be an aggregation operator. Then for any $\alpha, \beta \in \mathcal{A}(F)$ it is $F(\alpha) = F(\beta)$.

**Proof.** Let us suppose that it is $\alpha = (\alpha_1, \ldots, \alpha_r) \in I^r$ and $\beta = (\beta_1, \ldots, \beta_s) \in I^s$ for some $r, s \in \mathbb{N}$. Then:

(i) $\alpha \in \mathcal{A}(F)$ implies, in particular, $f_{r+s}(\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s) = f_r(\alpha)$

(ii) Similarly, $\beta \in \mathcal{A}(F)$ implies $f_{r+s}(\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s) = f_s(\beta)$

From (i) and (ii) we get $f_r(\alpha) = f_s(\beta)$, i.e., $F(\alpha) = F(\beta)$.

**Remark 5** Observe that the last result allows one to simplify example 5 as follows: the tuple $\alpha = (1, 0) \in I^2$ is absorbent for $F$ if $\forall n > 2, \forall (x_1, \ldots, x_n) \in I^n$, it is $f_n(x) = f_2(0, 1)(= f_2(1, 0))$ whenever there exist $i, j \in \{1, \ldots, n\}$ such that $x_i = 0, x_j = 1$.

Note that choosing $\alpha, \beta \in I$, Proposition 2 recovers the well-known fact which establishes the uniqueness of the standard absorbent element. In addition, the result can be particularized to the case of aggregation operators with standard absorbent element in the following way:
**Corollary 1** Let $F$ be an aggregation operator with absorbent element $a \in I$. Then for any $\alpha \in A(F)$, it is $F(\alpha) = a$.

**Proof.** Since the absorbent element $a \in I$ belongs to $A_1(F)$ (see Remark 3), it suffices to apply the previous result to the particular case $\beta = a \in I$, and then it is $F(\alpha) = F(a) = f_1(a) = a$ for any $\alpha \in A(F)$.

\[ \Box \]

The last result naturally raises the following question: is the converse true, i.e., if $a$ is the absorbent element of $F$, are the tuples verifying $F(\alpha) = a$ necessarily absorbent tuples? The next example proves that the answer to this question is negative:

**Example 6** Let $F$ be an aggregation operator such that each $f_n$, $n \geq 2$, is a possibly different triangular norm (recall that the well-known triangular norms, or t-norms, [16], are associative and symmetric aggregation operators with neutral element 1). Zero is absorbent for any t-norm, and, therefore, for $F$. Suppose now that $f_2(x, y) = T_L(x, y) = \max(0, x + y - 1)$ (the so-called Lukasiewicz t-norm) and that $f_3(x, y, z) = \min(x, y, z)$ (the minimum t-norm). Choosing, for example, $\alpha = (0.3, 0.3)$, it is $F(\alpha) = f_2(\alpha) = 0$, but $\alpha$ is not an absorbent tuple, since we have, for instance, $F(0.3, 0.3, 0.2) = 0.2 \neq F(0.3, 0.3)$.

Therefore, in general tuples whose aggregation coincides with the absorbent element are not necessarily absorbent tuples. Nevertheless, the next result shows that this happens to be true in some specific cases:

**Proposition 3** Let $F$ be an associative and symmetric aggregation operator. Then, for any $\alpha \in I^n$, $m \in \mathbb{N}$, it is $F(\alpha) = F(0, 1)$ if and only if $\alpha \in A_m(F)$.

**Proof.** Recall (see e.g. [18]) that, if $F$ is a symmetric and associative aggregation operator, then $F(0, 1)$ is its absorbent element. Then, we only need to prove that $F(\alpha) = F(0, 1)$ implies $\alpha \in A_m(F)$, since Corollary 1 provides the converse. Therefore, we have to prove that for any $n > m$, any index set $\mathcal{I} \subset \{1, \ldots, n\}$, $|\mathcal{I}| = m$, any permutation $\mathcal{P}$ of $(1, \ldots, m)$ and any $x \in I^n$ such that $x_{\mathcal{I}_\mathcal{P}} = \alpha$, it is $f_n(x) = f_m(\alpha_{\mathcal{P}})$. But $F$ is associative and symmetric, and therefore it is $f_n(x) = f_2(f_m(\alpha_{\mathcal{P}}), f_{n-m}(x_{\bar{\mathcal{P}}}))$, and the latter is equal to $f_m(\alpha_{\mathcal{P}})$, since, by hypothesis, $f_m(\alpha_{\mathcal{P}}) = F(\alpha)$ is the absorbent element.

\[ \Box \]
This last result may be applied, in particular, to triangular norms, triangular conorms, uninorms, and nullnorms, since they all are symmetric and associative aggregation operators with absorbent elements, respectively, 0, 1, 0 or 1, and 0 < F(0, 1) < 1 (see, e.g., [9]). Proposition 3 proves the existence of aggregation operators having absorbent tuples: indeed, it shows that the null set of any symmetric and associative aggregation operator will include, at least, the tuples (0, 1) and (1, 0) (in addition to the absorbent element F(0, 1)). Moreover, Proposition 3 also points out the existence of operators – nullnorms – having absorbent tuples – (0, 1) and (1, 0) – that do not contain their corresponding absorbent element. In the next section we shall see that nullnorms, as well as certain classes of triangular norms, triangular conorms and uninorms, have other non-trivial absorbent tuples.

We shall now see that given an absorbent tuple, any tuple containing it (in whatever positions) is also an absorbent tuple:

**Proposition 4** Let F be an aggregation operator and let α ∈ A(F). Then for any tuple α+ containing, at least, all the values of α (in any position), it is α+ ∈ A(F).

**Proof.** Let us suppose that α ∈ I’ and α+ ∈ Is, r ≤ s. Then we have to prove that for any n > s, any index set I ⊂ {1, . . . , n}, |I| = s, any permutation P of (1, . . . , s) and any x ∈ In such that xIP = α+, it is fn(x) = fs(α+P). But:

1. On one hand, xIP = α+ and the fact that α is contained in α+ implies the existence of an index set J ⊂ {1, . . . , n}, |J| = r and a permutation Q of (1, . . . , r) such that xJQ = α, then, since by hypothesis it is α ∈ A(F), we get fn(x) = fP(αQ).

2. On the other hand, the fact that α is contained in α+ (and therefore in α+P), alone, implies the existence of an index set K ⊂ {1, . . . , s}, |K| = r and a permutation R of (1, . . . , r) such that (α+P)KR = α, then, since by hypothesis it is α ∈ A(F), we get fs(α+P) = fR(αR).

Now, from 1. and 2. and the fact that fP(αQ) = fR(αR) (see Remark 1 and Proposition 2), we get fn(x) = fs(α+P).

Proposition 4 allows one to enlarge the class of aggregation operators having non-empty null sets with operators not necessarily symmetric or associative:
Corollary 2 Let $F$ be an aggregation operator with absorbent element $a \in I$ and let $\alpha = (\alpha_1, \ldots, \alpha_m) \in I^m$, $m \in \mathbb{N}$, verify $\alpha_i = a$ for some $i \in \{1, \ldots, m\}$. Then $\alpha \in A_m(F)$.

Proof. Since $a \in A_1(F)$ (see Remark 3), the result is easily obtained from Proposition 4.

Thus we have established that the null sets of aggregation operators with a standard absorbent element will include, at least, any tuple containing the absorbent element. In the particular case of associative and symmetric aggregation operators, Proposition 4 also entails the following:

Corollary 3 Let $F$ be an associative and symmetric aggregation operator and let $\alpha \in I^m$, $m \geq 2$, be a tuple containing at least one 0 and one 1. Then $\alpha \in A_m(F)$.

Proof. Proposition 3 shows that $\alpha = (0, 1)$ is an absorbent tuple for $F$, and it then suffices to apply Proposition 4.

Recall also (see for example [9]) that given an aggregation operator $F$ and a monotone bijection $\varphi : I \to I$, the operator $F_\varphi : \bigcup_{n \in \mathbb{N}} I^n \to I$, defined as $(f_n)_\varphi(x_1, \ldots, x_n) = \varphi^{-1}(f_n(\varphi(x_1), \ldots, \varphi(x_n)))$, is in turn an aggregation operator, usually known as the $\varphi$-transform of $F$. The relationship between the null information associated to $F$ and the one related to its $\varphi$-transform $F_\varphi$ is described in the next proposition.

Proposition 5 Let $F$ be an aggregation operator and let $\varphi : I \to I$ be a monotone bijection. If $\mathcal{I} \subset \{1, \ldots, n\}$, $n > 1$, is an index set such that $|\mathcal{I}| = m$ and $\mathcal{P}$ is a permutation of $(1, \ldots, m)$, then for any $\alpha \in I^m$:

$$\alpha \in A_m(F, n, \mathcal{I}, \mathcal{P})$$

if and only if

$$\varphi^{-1}(\alpha) \in A_m(F_\varphi, n, \mathcal{I}, \mathcal{P})$$

where, if $\alpha$ represents the vector $(\alpha_1, \ldots, \alpha_m)$, then $\varphi^{-1}(\alpha)$ denotes the vector $(\varphi^{-1}(\alpha_1), \ldots, \varphi^{-1}(\alpha_m))$.

Proof. It suffices to take into account that $\phi(x_{\mathcal{I}_\mathcal{P}}) = \phi(x)_{\mathcal{I}_\mathcal{P}}$ and $\phi(\alpha_{\mathcal{P}}) = \phi(\alpha)_{\mathcal{P}}$ for any monotone bijection $\phi : I \to I$.

\[\Box\]
When choosing $\varphi$ as the duality transformation $\varphi(x) = \varphi_d(x) = 1 - x$ for any $x \in I$, the $\varphi_d$-transform of a given aggregation operator $F$, $F_d$, is known as the dual of $F$, and it is easy to verify (see e.g. [9]) that if $F$ has absorbent element $a \in I$, then $1 - a$ is the absorbent element of its dual operator. Proposition 5 is a generalization of this result, since it states that the absorbent tuples of a given operator may be directly obtained from the ones of its dual operator.

**Remark 6** Note that the results obtained previously show the existence of tuples which are absorbent at the same time for an aggregation operator and its dual. Indeed:

- If $F$ is an associative and symmetric aggregation operator, an immediate consequence of Propositions 3 and 5 is that a tuple is absorbent both for $F$ and its dual $F_d$ if and only if it is $F(\alpha) = F(1 - \alpha) = F(1,0)$ (indeed, $F_d$ is obviously associative and symmetric and has $1 - F(1,0)$ as absorbent element). In addition, Corollary 3 shows that this is the case, in particular, of any tuple containing the values 0 and 1. Note that when dealing with nullnorms, such tuples do not necessarily contain the absorbent element of either $F$ or $F_d$.

- If $F$ is an aggregation operator with absorbent element $a \in I$, an immediate consequence of Corollary 2 is that any tuple containing the values $a$ and $1 - a$ is absorbent both for $F$ and for its dual operator.

4 Null sets of aggregation operators

4.1 Conjunctive and disjunctive operators

*Conjunctive aggregation operators*, i.e., those verifying $F \leq \min$, constitute an important class of operators that includes widely-used *triangular norms* (t-norms) and *copulas* (see e.g. [16,21]). The following general result may be established regarding their null sets:

**Proposition 6** If $F$ is a conjunctive aggregation operator, then any tuple $\alpha \in I^m$, $m \in \mathbb{N}$, containing at least one 0 satisfies $\alpha \in A_m(F)$.

**Proof.** Since zero is clearly the absorbent element of any conjunctive operator, it suffices to apply Corollary 2. \qed
Proposition 7 For any t–norm $T$ and any $\alpha \in I^m$ it is $\alpha \in A_m(T)$ if and only if $T(\alpha) = 0$.

Proof. It suffices to apply Proposition 3 taking into account that $T(0,1) = 0$ for any t-norm $T$. \hfill \Box

It is clear that the t-norms which are strictly monotone in $[0,1]^n$ (such as strict t-norms, isomorphic to the product t-norm) have only trivial absorbent tuples (those containing at least one zero). But there are t-norms (either continuous or non-continuous) with non-trivial absorbent tuples: those having zero divisors, such as the drastic t-norm, nilpotent t-norms (isomorphic to the Lukasiewicz t-norm), the nilpotent minimum, etc ( [16]).

Example 7 Nilpotent t-norms are given by:

$$T_{L\varphi}(x_1,\ldots,x_n) = \varphi^{-1}\left[\max\left(0,\sum_{i=1}^{n}\varphi(x_i) - (n-1)\right)\right]$$

for all $x_1,\ldots,x_n \in I$, where $\varphi : I \rightarrow I$ is a strictly increasing bijection. Then a tuple $\alpha \in I^m$ is absorbent for $T_{L\varphi}$ if and only if it satisfies the inequality $\sum_{i=1}^{m}\varphi(\alpha_i) \leq m-1$.

Similar results may be obtained, by duality (see Proposition 5), for the case of disjunctive operators ($F \geq \max$), for which 1 is an absorbent element:

Proposition 8 If $F$ is a disjunctive aggregation operator, then any tuple $\alpha \in I^m$, $m \in \mathbb{N}$, containing at least one 1 satisfies $\alpha \in A_m(F)$.

Proposition 9 For any t–conorm $S$ and any $\alpha \in I^m$ it is $\alpha \in A_m(S)$ if and only if $S(\alpha) = 1$.

Similarly to t-norms, it is possible to find t-conorms (those with one divisors) that appear to have non-trivial absorbent tuples:

Example 8 Nilpotent t-conorms are given by:

$$S_{L\varphi}(x_1,\ldots,x_n) = \varphi^{-1}\left[\min\left(1,\sum_{i=1}^{n}\varphi(x_i)\right)\right]$$

for all $x_1,\ldots,x_n \in I$, where $\varphi : I \rightarrow I$ is a strictly increasing bijection. Then a tuple $\alpha \in I^m$ is absorbent for $S_{L\varphi}$ if and only if it satisfies the inequality $\sum_{i=1}^{m}\varphi(\alpha_i) \geq 1$. 

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Remark 7 Following Remark 6, note that examples 7 and 8 provide tuples which are absorbent both for a nilpotent t-norm $T_L$ and for its dual t-conorm, $S_L$: those $\alpha \in I^m$ verifying $1 \leq \sum_{i=1}^m \varphi(\alpha_i) \leq m - 1$.

4.2 Quasi-linear T-S operators

Consider now quasi-linear T-S aggregation operators \cite{22}, which are constructed from a t-norm $T$, a t-conorm $S$, a parameter $\lambda \in ]0,1[$ and a continuous and strictly monotone function $g : [0,1] \rightarrow [-\infty, \infty]$ such that $\{g(0),g(1)\} \neq \{-\infty, +\infty\}$ as follows:

$$QL_{T,S,\lambda,g}(x) = g^{-1}[(1 - \lambda)g(T(x)) + \lambda g(S(x))] .$$

This class includes the well-known linear and exponential convex T-S operators \cite{9, 20}, which may be obtained by choosing, respectively, $g = Id$ and $g = log$, and that are given by:

$$L_{T,S,\lambda}(x) = (1 - \lambda)T(x) + \lambda S(x) \quad \text{and} \quad E_{T,S,\lambda}(x) = T(x)^{1-\lambda}S(x)^{\lambda}$$

As it was shown in \cite{22}, quasi-linear T-S aggregation operators with generating function $g$ verifying $g(0) = \pm \infty$ (such as exponential convex T-S operators) have absorbent element $\alpha = 0$, whereas their dual operators (those such that $g(1) = \pm \infty$) have absorbent element $\alpha = 1$. This means, by Corollary 2, that these operators have absorbent tuples of any dimension. The next result provides the general characterization of the null set of any quasi-linear T-S operator:

Proposition 10 Let $QL_{T,S,\lambda,g}$ be a quasi-linear T-S aggregation operator. A tuple $\alpha \in I^m$ is absorbent for $QL_{T,S,\lambda,g}$ if and only if one of the three following conditions is satisfied:

1. $g(0) = \pm \infty$ and $T(\alpha) = 0$.
2. $g(1) = \pm \infty$ and $S(\alpha) = 1$.
3. $g(0) \neq \pm \infty$, $g(1) \neq \pm \infty$, $T(\alpha) = 0$ and $S(\alpha) = 1$. 

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Proof. Let us suppose that $\alpha \in I^m$ is absorbent for $QL_{T,S,\lambda,g}$, that is, for any $n > m$, any index set $I \subset \{1, \ldots, n\}$, $|I| = m$, any permutation $P$ of $(1, \ldots, m)$ and any $x \in I^n$ such that $x_{I_P} = \alpha$, it is $f_n(x) = f_m(\alpha_P)$, i.e.,

$$(1 - \lambda)g(T(x)) + \lambda g(S(x)) = (1 - \lambda)g(T(\alpha_P)) + \lambda g(S(\alpha_P))$$

(2)

Now we can split the proof in three cases depending on $\text{Ran}(g)$:

1. If $g(0) = \pm \infty$, substituting $x_T = (0, \ldots, 0)$ in (2) we have $(1 - \lambda)g(0) = (1 - \lambda)g(T(\alpha_P))$, from which, since $\lambda \neq 1$, it is $T(\alpha_P) = 0$ (note that Proposition 7 shows that this is equivalent to $\alpha \in A_m(T)$). On the other hand, it is easy to see that this last result is also a sufficient condition for this case.

2. If $g(1) = \pm \infty$, putting $x_T = (1, \ldots, 1)$ in (2) we have $\lambda g(1) = \lambda g(S(\alpha_P))$ from which, since $\lambda \neq 0$, it is $S(\alpha_P) = 1$ (or, according to Proposition 9, $\alpha \in A_m(S)$). Coming back to equation (2), the above result is also a sufficient condition.

3. If $g(1) \neq \pm \infty$ and $g(0) \neq \pm \infty$, putting $x_T = (1, \ldots, 1)$ (or $x_T = (0, \ldots, 0)$) in (2) we obtain $S(\alpha_P) = 1$ ($T(\alpha_P) = 0$), but coming back to equation (2) it follows that $T(x) = T(\alpha_P)$, which is equivalent, thanks to Proposition 7, to $T(\alpha) = 0$ (or $S(x) = S(\alpha_P)$ and then, by Proposition 9, $S(\alpha) = 1$).

Observe that Proposition 10 entails that quasi-linear $T$-$S$ aggregation operators always have non-empty null sets. Indeed:

1. If $g(0) = \pm \infty$ ($g(1) = \pm \infty$), any tuple containing the absorbent element zero (one) is absorbent for $QL_{T,S,\lambda,g}$.

2. Otherwise, any tuple $\alpha$ containing the values 0 and 1 clearly verifies $T(\alpha) = 0$ and $S(\alpha) = 1$, and is, as a consequence, absorbent for $QL_{T,S,\lambda,g}$. Note also that in this case we have a family of aggregation operators having absorbent tuples but not absorbent element.

Of course, these null sets may be larger in the case of operators built by means of t-norms and t-conorms having, respectively, zero and one divisors:

Example 9
• Consider the class of exponential convex $T - S$ operators ($g = \log$) built upon a nilpotent $t$-norm $T_{L\psi}$. Then any tuple $\alpha \in I^m$ such that $\sum_{i=1}^{m} \varphi(\alpha_i) \leq m - 1$ is absorbent for $T_{L\psi}$ (see example 7) and then also for $QL_{T_{L\psi}, S, \lambda}$.

• Consider the class of quasi-linear $T - S$ operators with generating function verifying $g(x) = \log(1 - x)$ and built upon a nilpotent $t$-conorm $S_{L\psi}$. Then any tuple $\alpha \in I^m$ such that $\sum_{i=1}^{m} \varphi(\alpha_i) \geq 1$ is absorbent for $S_{L\psi}$ (see example 8) and then also for $QL_{T_{L\psi}, S_{L\psi}, \lambda}$.

• Consider the class of linear convex $T - S$ operators ($g = \text{Id}$) built upon a nilpotent $t$–norm $T_{L\psi}$ and its dual nilpotent $t$–conorm $S_{L\psi}$. Then (see remark 7) the tuples $\alpha \in I^m$ such that $1 \leq \sum_{i=1}^{m} \varphi(\alpha_i) \leq m - 1$ are absorbent both for $T_{L\psi}$ and $S_{L\psi}$, and, therefore, for $QL_{T_{L\psi}, S_{L\psi}, \lambda}$.

4.3 Uninorms and Nullnorms

Uninorms ([14,26]) are associative and symmetric aggregation operators $U$ with the neutral element $e \in [0,1]$. When $e = 1$ or $e = 0$ they coincide, respectively, with a $t$-norm or a $t$-conorm. They act as a $t$-norm in $[0,e]$ and as a $t$-conorm in $[e,1]$, that is, for each uninorm $U$ there exist a $t$-norm $T$ and a $t$-conorm $S$ such that:

$$U(x_1, \ldots, x_n) = \begin{cases} e \cdot T \left( \frac{x_1}{e}, \ldots, \frac{x_n}{e} \right) & \text{if } \max x_i \leq e, \\ e + (1 - e) \cdot S \left( \frac{x_1 - e}{1 - e}, \ldots, \frac{x_n - e}{1 - e} \right) & \text{if } \min x_i \geq e \end{cases}$$

Otherwise (that is, when $\min x_i < e < \max x_i$), uninorms verify $\min x_i \leq U(x_1, \ldots, x_n) \leq \max x_i$. Recall in addition that for any uninorm it is $U(0,1) \in \{0,1\}$, and this allows one to classify them into two different classes: those such that $U(0,1) = 0$, which have absorbent element 0 and are known as conjunctive uninorms, and those with absorbent element $U(0,1) = 1$, known as disjunctive uninorms.

Regarding absorbent information, we have the following result:

**Proposition 11** Let $U$ be a uninorm with neutral element $e \in [0,1]$, underlying $t$-norm $T$ and underlying $t$-conorm $S$, and let $\alpha \in I^m$, $m \in \mathbb{N}$.

1. If $U$ is conjunctive, then $\alpha$ is absorbent for $U$ if and only if one of the two following conditions is satisfied:
• \( \min \alpha_i = 0 \), or
• \( \max \alpha_i \leq e \) and \( T(\frac{\alpha_1}{e}, \ldots, \frac{\alpha_m}{e}) = 0 \).

2. If \( U \) is disjunctive, then \( \alpha \) is absorbent for \( U \) if and only if one of the two following conditions is satisfied:

• \( \max \alpha_i = 1 \), or
• \( \min \alpha_i \geq e \) and \( S(\frac{\alpha_1-e}{1-e}, \ldots, \frac{\alpha_m-e}{1-e}) = 1 \)

**Proof.** Since conjunctive (disjunctive) uninorms are associative and symmetric aggregation operators such that \( U(0, 1) = 0 \) (\( U(0, 1) = 1 \)), Proposition 3 establishes that \( \alpha \) is absorbent for \( U \) if and only if it is \( U(\alpha) = 0 \) (\( U(\alpha) = 1 \)). It then suffices to solve these equations taking into account the structure of \( U \) that was recalled above.

Note therefore that a conjunctive (disjunctive) uninorm has non-trivial absorbent tuples if and only if its underlying t-norm (t-conorm) has zero divisors (one divisors).

**Example 10**

• Consider the conjunctive uninorm with neutral element \( e \in ]0, 1[ \) and with the nilpotent minimum (drastic product) as underlying t–norm. Then the tuples \( \alpha = (x, y) : x + y \leq e \) (\( \alpha = (x, y) \in [0, e[^2 \) ) are absorbent for \( U \).

• Consider the disjunctive uninorm with neutral element \( e \in ]0, 1[ \) and with the nilpotent maximum (drastic sum) as underlying t–conorm. Then the tuples \( \alpha = (x, y) : x + y \geq 1 + e \) (\( \alpha = (x, y) \in ]e, 1[^2 \) ) are absorbent for \( U \).

• The so-called representable uninorms (see e.g. [14, 15]), which are defined with the help of an additive generator, have strict underlying t–norms and t–conorms. Therefore, their null sets are limited to the tuples containing the value 0 (if \( U \) is conjunctive) or the value 1 (if \( U \) is disjunctive).
Nullnorms ([8, 17]) are symmetric and associative aggregation operators with an absorbent element \( a \in [0, 1] \). They coincide with a t-norm or a t-conorm when \( a = 0 \) or \( a = 1 \), respectively, and may be built by means of a t-norm \( T \) and a t-conorm \( S \) in the following way:

\[
F(x_1, \ldots, x_n) = \begin{cases} 
  a \cdot S\left(\frac{x_1}{a}, \ldots, \frac{x_n}{a}\right) & \text{if } \max x_i \leq a, \\
  a + (1-a) \cdot T\left(\frac{x_1-a}{1-a}, \ldots, \frac{x_n-a}{1-a}\right) & \text{if } \min x_i \geq a, \\
  a & \text{otherwise.}
\end{cases}
\]

**Proposition 12** Let \( F \) be a nullnorm with absorbent element \( a \in [0, 1] \), underlying t-norm \( T \) and underlying t-conorm \( S \). Then \( \alpha \in I^m \) is absorbent for \( F \) if and only if one of the following conditions is satisfied:

- \( \min \alpha_i < a < \max \alpha_i \), or
- \( \max \alpha_i \leq a \) and \( S\left(\frac{\alpha_1}{a}, \ldots, \frac{\alpha_m}{a}\right) = 1 \), or
- \( \min \alpha_i \geq a \) and \( T\left(\frac{\alpha_1-a}{1-a}, \ldots, \frac{\alpha_m-a}{1-a}\right) = 0 \).

**Proof.** Since nullnorms are associative and symmetric aggregation operators with absorbent element \( F(0, 1) \), Proposition 3 establishes that \( \alpha \) is absorbent for \( F \) if and only if it is \( F(\alpha) = F(0, 1) = a \). It then suffices to solve this equation taking into account the structure of nullnorms that has just been recalled.

As a consequence, nullnorms always have some fixed absorbent tuples (those \( \alpha \in I^m \) such that \( \min \alpha_i \leq a \leq \max \alpha_i \)) but may also have additional ones, depending on whether the underlying t-norm and t-conorm have, respectively, zero or one divisors:

**Example 11** Let \( F \) be a nullnorm with absorbent element \( a \in [0, 1] \), underlying t-norm \( T \) and underlying t-conorm \( S \). Then:

- If both \( T \) and \( S \) are strictly increasing (in \([0, 1]^n \) and \([0, 1]^n \), respectively), then the only tuples which are absorbent for \( F \) are those verifying \( \min \alpha_i \leq a \leq \max \alpha_i \).
- If \( T \) is a nilpotent t–norm \( T_L \varphi \) (see example 7), then the tuples \( \alpha \in [a, 1]^m \) such that \( \sum_{i=1}^{m} \varphi\left(\frac{\alpha_i-a}{1-a}\right) \leq m - 1 \) are absorbent for \( F \).
- If \( S \) is a nilpotent t–conorm \( S_L \varphi \) (see example 8), then the tuples \( \alpha \in [0, a]^m \) such that \( \sum_{i=1}^{m} \varphi\left(\frac{\alpha_i}{a}\right) \geq 1 \) are absorbent for \( F \).
4.4 Generated aggregation operators

Consider now the following class of generated aggregation operators ([19,20]). Let \( g : I \to [c, d], -\infty < c < d < \infty \), be a monotone increasing continuous function with zero \( e \in I \). For any \( n \in \mathbb{N} \), define

\[
    f_n(x_1, \ldots, x_n) = g^{-1}(g(x_1) + \ldots + g(x_n)), \tag{3}
\]

where \( g^{-1} \) denotes the pseudoinverse of \( g \). The function (3) is continuous on \( I^n \), but it is not associative. Further, on \( [e, 1]^n \) it coincides with a (scaled) t-conorm and on \( [0, e]^n \) it coincides with a (scaled) t-norm. As with uninorms, \( e \) is its neutral element, and when \( e = 1 \) or \( e = 0 \) we obtain t-norms and t-conorms as limiting cases.

**Proposition 13** Let \( F \) be a generated aggregation operator given by (3), with additive generator \( g \), and let \( m, n \in \mathbb{N} \) such that \( m < n \). Then \( \alpha \in I^m \) is absorbent for \( F \) at level \( n \) (i.e., \( \alpha \in A_m(F, n) \)) if and only if it is

\[
    \sum_{i=1}^{m} g(\alpha_i) \geq g(1) - (n - m) \cdot g(0) \quad \text{or} \quad \sum_{i=1}^{m} g(\alpha_i) \leq g(0) - (n - m) \cdot g(1)
\]

**Proof.** Since \( F \) is a symmetric operator, a tuple \( \alpha \in I^m \) is, by definition, absorbent for \( F \) at level \( n \) if and only if for any \( x \in I^{n-m} \) it is

\[
    g^{-1}\left( \sum_{i=1}^{m} g(\alpha_i) + \sum_{i=1}^{n-m} g(x_i) \right) = g^{-1}\left( \sum_{i=1}^{m} g(\alpha_i) \right)
\]

Then the three following situations may be distinguished, depending on the value of \( \sum_{i=1}^{m} g(\alpha_i) \):

1. If it is \( \sum_{i=1}^{m} g(\alpha_i) \geq g(1) \) then equation (4) becomes

\[
    g^{-1}\left( \sum_{i=1}^{m} g(\alpha_i) + \sum_{i=1}^{n-m} g(x_i) \right) = 1
\]

which may only happen if \( \sum_{i=1}^{m} g(\alpha_i) + \sum_{i=1}^{n-m} g(x_i) \geq g(1) \) holds for any \( x \in I^{n-m} \), or, equivalently, if it is \( \sum_{i=1}^{m} g(\alpha_i) \geq g(1) - (n - m) \cdot g(0) \).
2. Similarly, if it is \( \sum_{i=1}^{m} g(\alpha_i) \leq g(0) \) then equation (4) translates into
\[
g^{(−1)} \left( \sum_{i=1}^{m} g(\alpha_i) + \sum_{i=1}^{n−m} g(x_i) \right) = 0
\]
for any \( x \in I^{n−m} \), which is equivalent to \( \sum_{i=1}^{m} g(\alpha_i) \leq g(0) − (n−m) \cdot g(1) \).

3. Finally, let us suppose that it is \( g(0) < \sum_{i=1}^{m} g(\alpha_i) < g(1) \) and see that, in this case, \( \alpha \) cannot be absorbent for \( F \). Indeed, equation (4) becomes
\[
g^{(−1)} \left( \sum_{i=1}^{m} g(\alpha_i) + \sum_{i=1}^{n−m} g(x_i) \right) = g^{−1} \left( \sum_{i=1}^{m} g(\alpha_i) \right)
\]
and it is not difficult to show that there are \( x \in I^{n−m} \) that do not satisfy this equality. Take, for example, \( x = (\varepsilon, e, \ldots, e) \) where \( \varepsilon \neq e \) is chosen as close to \( e \) as needed in order to get \( g(0) < \sum_{i=1}^{m} g(\alpha_i) + g(\varepsilon) < g(1) \). Then (5) becomes \( g^{−1} (\sum_{i=1}^{m} g(\alpha_i) + g(\varepsilon)) = g^{−1} (\sum_{i=1}^{m} g(\alpha_i)) \), which implies \( g(\varepsilon) = 0 \), and the latter is contradictory with \( \varepsilon \neq e \).

The last proposition shows the existence of absorbent tuples of dimension \( m \in \mathbb{N} \) at some specific levels \( n > m \), but the characterization found clearly shows that the value of \( n \) cannot be arbitrary. Consequently:

**Corollary 4** Let \( F \) be a generated aggregation operator given by (3). Then \( \mathcal{A}_m(F) = \emptyset \) for any \( m \in \mathbb{N} \).

**Example 12** [9, 19, 20] Let \( F \) be a generated aggregation operator given by (3), with additive generator \( g \) defined as \( g(t) = t − \frac{1}{2} \). Then \( F \) is an ordinal sum of Lukasiewicz t-norm and t-conorm, given by
\[
F(x) = \max \left( 0, \min \left( 1, \frac{1}{2} + \sum_{i=1}^{n} \left( x_i - \frac{1}{2} \right) \right) \right),
\]
and for any \( n > m \) the corresponding null set \( \mathcal{A}_m(F, n) \) is given by
\[
\mathcal{A}_m(F, n) = \left\{ \alpha \in I^m : \sum_{i=1}^{m} \alpha_i \geq \frac{n+1}{2} \right\} \cup \left\{ \alpha \in I^m : \sum_{i=1}^{m} \alpha_i \leq m - \frac{n+1}{2} \right\}
\]
4.5 Averaging operators

We recall that an aggregation operator is called an averaging if it is bounded by minimum and maximum. Averaging aggregation operators are idempotent. A distinguished class of averaging operators is the one made of the so-called weighted quasi-arithmetic means ([1], [9]):

**Definition 8** An aggregation operator is a weighted quasi-arithmetic mean if, for each \( n \in \mathbb{N} \), it can be written as

\[
    f_n(x) = g^{-1}\left(\sum_{i=1}^{n} w_i g(x_i)\right)
\]

where \( g : [0, 1] \to [\pm \infty, +\infty] \) is a continuous strictly monotone function and \( w_n = (w_{1n}, \ldots, w_{nn}) \in I^n \) verifies \( \sum_{i=1}^{n} w_i = 1 \).

The above definition includes two important classes of commonly used operators: weighted arithmetic means (obtained when choosing \( g(t) = t \)) and quasi-arithmetic means (obtained when taking the weights \( w_{in} = 1/n \) for all \( n \in \mathbb{N}, i \in \{1, \ldots, n\} \)).

If we concentrate on weighted quasi-arithmetic means with strictly positive weights (such as quasi-arithmetic means) we get the following result:

**Proposition 14** Let \( F \) be a weighted quasi-arithmetic mean with generating function \( g \) and such that \( w_{in} > 0 \) for any \( n \in \mathbb{N} \) and any \( i \in \{1, \ldots, n\} \). Then:

1. If \( g \) is such that \( g(0) = \pm \infty \), then for any \( m \in \mathbb{N} \) it is

\[
    \mathcal{A}_m(F) = \{ \alpha \in I^m : \min \alpha_i = 0 \}.
\]

2. If \( g \) is such that \( g(1) = \pm \infty \), then for any \( m \in \mathbb{N} \) it is

\[
    \mathcal{A}_m(F) = \{ \alpha \in I^m : \max \alpha_i = 1 \}.
\]

3. Otherwise, \( \mathcal{A}_m(F, n, I, \mathcal{P}) = \emptyset \) for any \( n, m \in \mathbb{N}, m < n \) and for any index set \( I \subset \{1, \ldots, n\} \) and any permutation \( \mathcal{P} \).

**Proof.**
1. If $g$ is such that $g(0) = \pm \infty$, it is known (see e.g. [9]) that 0 is the absorbent element whenever all the weights are strictly positive. Then Corollary 2 shows that any tuple containing 0 (i.e., any tuple such that $\min \alpha_i = 0$) is an absorbent tuple. To prove the converse, note that $\alpha \in A_m(F)$ means, by definition, that for any $n > m$, any index set $I \subset \{1, \ldots, n\}$, $|I| = m$, any permutation $P$ of $(1, \ldots, m)$ and any $x \in I^n$ such that $x_{I_P} = \alpha$, it is $f_n(x) = f_m(\alpha_P)$, i.e.,

$$\sum_{i=1}^{m} w_{I,n} g(\alpha_{P(i)}) + \sum_{i=1}^{n-m} w_{I,n} g(x_{I_i}) = \sum_{i=1}^{m} w_{im} g(\alpha_{P(i)}) \quad (6)$$

Now, if $x$ is chosen such that $x_{I_j} = 0$ for some $j \in \{1, \ldots, m\}$, then equation (6) becomes

$$\sum_{i=1}^{m} w_{im} g(\alpha_{P(i)}) = \pm \infty,$$

and the latter may only happen if $\min \alpha_i = 0$.

2. If $g$ is such that $g(1) = \pm \infty$, then 1 is the absorbent element, and the result is obtained by duality.

3. Otherwise, we are dealing with weighted quasi-arithmetic means such that $\text{Ran}(g) \subset \mathbb{R}$ and having strictly positive weights. Such operators are strictly monotone (see e.g. [9]) and therefore (Proposition 1) their null sets are necessarily empty.

We will now deal with another important class of averaging operators:

**Definition 9** [9,25] An aggregation operator is a *generalized Ordered Weighted Averaging (OWA)* if, for each $n \in \mathbb{N}$, it can be written as

$$f_n(x) = g^{-1}\left(\sum_{i=1}^{n} w_{in} g(x_{(i)})\right)$$

where $g : [0, 1] \rightarrow [-\infty, +\infty]$ is a continuous strictly monotone function, $w_n = (w_{1n}, \ldots, w_{nn}) \in I^n$ verifies $\sum_{i=1}^{n} w_{in} = 1$ and $(x_{(1)}, \ldots, x_{(n)})$ is a vector obtained from $x$ by arranging its components in a non-decreasing order.
Generalized OWA operators are nothing else than symmetrized weighted quasi-arithmetic means ([9]), and it is not difficult to see that their behavior regarding absorbent information, in the case of operators defined with strictly positive weights, is the same:

**Proposition 15** Let $F$ be a generalized OWA with generating function $g$ and such that $w_{in} > 0$ for any $n \in \mathbb{N}$ and any $i \in \{1, \ldots, n\}$. Then:

1. If $g$ is such that $g(0) = \pm \infty$, then for any $m \in \mathbb{N}$ it is
   \[ \mathcal{A}_m(F) = \{ \alpha \in I^m : \min \alpha_i = 0 \}. \]

2. If $g$ is such that $g(1) = \pm \infty$, then for any $m \in \mathbb{N}$ it is
   \[ \mathcal{A}_m(F) = \{ \alpha \in I^m : \max \alpha_i = 1 \}. \]

3. Otherwise, $\mathcal{A}_m(F, n, I, P) = \emptyset$ for any $n, m \in \mathbb{N}$, $m < n$ and for any index set $I \subset \{1, \ldots, n\}$ and any permutation $P$.

**Proof.** Similar to the proof of Proposition 14.

\[ \square \]

5 Construction methods

From the previous sections we see that different classes of aggregation operators have quite distinct null sets, ranging from empty sets to finite and infinite null sets. If we have a given aggregation operator $F$, we can characterize its null set explicitly. The goal of this section is to solve the opposite problem: given a desired null set, design an aggregation operator which will have this null set. We concentrate on building an $n$-ary Lipschitz aggregation operator $f_n(x)$ of a fixed dimension $n$, with a null set $\mathcal{A}_m(F, n, I, P)$ w.r.t $I, P$.

We note that in general the value $f_n(x) = f_m(\alpha_P)$ need not be a constant function (cf. Proposition 2, in which we used absorbent tuples from $\mathcal{A}(F)$, in that case we have a constant function indeed). Thus we will use the following ingredients for our construction: given $n, m$, index set and permutation $I, P$, the null set $\mathcal{A}_m(F, n, I, P)$, and a given function $h : \mathcal{A}_m(F, n, I, P) \to [0, 1] : h(\alpha) = f_m(\alpha_P)$. 

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In the following we will assume that the aggregation operator is Lipschitz-
continuous, with a Lipschitz constant \( M \) in some norm \( \| \cdot \|_p \):
\[
\exists M \geq 0 : \forall x, y \in I^n, |f_n(x) - f_n(y)| \leq M \| x - y \|_p,
\]
(7)
The smallest such number \( M \) is called the Lipschitz constant of \( f_n \). Such
aggregation operators are of significant practical interest, as they provide
stable output with respect to inaccuracies in the values of arguments. \( p \)-
stable, 1-Lipschitz, kernel aggregation operators, copulas and quasi-copulas
are special classes of Lipschitz aggregation operators with Lipschitz constant
\( M = 1 \). In this section we construct the largest and the smallest Lipschitz
aggregation operators with the desired null set, and will also identify the
optimal operator.

The method of construction of Lipschitz aggregation operators with the
desired values at a given subset is based on the results on optimal interpola-
tion from [4–7]. We denote by \( Lip(M) \) the set of functions with the Lipschitz
condition no greater than \( M \), and denote by \( Mon \) the set of functions mono-
tone increasing in each argument.

Suppose that we have a set of desired values of the aggregation opera-
tor \( f_n \), \( D = \{ (z, h(z)), z \in \Omega, h(z) \in I, f_n(z) = h(z) \} \), and the Lipschitz
condition (7). The data are consistent with the Lipschitz condition and monotonocity \( h \in Lip(M) \cap Mon \). Then tight upper and lower bounds on
any function from the set \( Lip(M) \cap Mon \) that interpolate the data are given
(see [4]) by \( \sigma_l(x) \leq f_n(x) \leq \sigma_u(x) \), with
\[
\sigma_u(x) = \inf_{z \in \Omega} \{ h(z) + M \| (x - z)_+ \| \},
\sigma_l(x) = \sup_{z \in \Omega} \{ h(z) - M \| (z - x)_+ \| \},
\]
(8)
where \( x_+ \) denotes the positive part of vector \( x \): \( x_+ = (x_1 \lor 0, \ldots, x_n \lor 0) \).

For example, if the data set \( D \) consists of two data \( D = \{ ((0, \ldots, 0), 0), ((1, \ldots, 1), 1) \} \),
which is true for any aggregation operator, we obtain the general bounds
\[
\sigma_{ul}^{gen}(x) = \min \{ 1, M \| x \| \},
\sigma_{ul}^{gen}(x) = \max \{ 0, 1 - M \| 1 - x \| \}.
\]
(9)
The optimal aggregation operator is the one that minimizes the approxi-
mation error in the worst case scenario,
\[
\min_{g \in Lip(M) \cap Mon} \max_{f \in Lip(M) \cap Mon} \max_{x \in I^n} |f(x) - g(x)|
\]
is given by the central interpolation scheme [24]

\[ g(x) = \frac{1}{2}(\sigma_l(x) + \sigma_u(x)). \]  

(10)

Our construction method consists in identifying the data set \( \mathcal{D} \) and then applying Eqns. (8) and (10). Practical application of this method involves solving an optimization problem in (8), for which we indicate suitable numerical algorithms. Consider a Lipschitz aggregation operator of dimension \( n \), and a given null set \( \mathcal{A}_m(F,n,\mathcal{I},\mathcal{P}) \) and function \( h(\alpha) = f_m(\alpha P) \). We concentrate on the case of a closed null set. Then we have

\[ \sigma_u(x) = \min_{z \in \Omega} \{ f_m(z_{\mathcal{I}_p}) + M||x - z|| \}, \]
\[ \sigma_l(x) = \max_{z \in \Omega} \{ f_m(z_{\mathcal{I}_p}) - M||x - z|| \}, \]

(11)

where \( \Omega \subset [0,1]^n \) is an extension of \( \mathcal{A}_m(F,n,\mathcal{I},\mathcal{P}) \) to \( n \) variables:

\[ \Omega = \{ x \in [0,1]^n | x_{\mathcal{I}_p} \in \mathcal{A}_m(F,n,\mathcal{I},\mathcal{P}), x_I \in [0,1]^{n-m} \}. \]

We note that the bounds given by (11) are in addition to those given by (9), as any aggregation operator interpolates \(((0,\ldots,0),0),((1,\ldots,1),1)\), and also in addition to any other bounds arising from other considerations.

Let us now discuss the optimization problems (11) in detail and consider their numerical solution. First of all notice that the minimum and maximum are achieved at points \( z \) such that \( \forall j \in \mathcal{I} : z_j = x_j \). What this means is that

\[ \sigma_u(x) = \sigma_u(x_{\mathcal{I}_p}) = \min_{z \in \mathcal{A}_m(F,n,\mathcal{I},\mathcal{P})} \{ f_m(z) + M||x_{\mathcal{I}_p} - z|| \}, \]
\[ \sigma_l(x) = \sigma_l(x_{\mathcal{I}_p}) = \max_{z \in \mathcal{A}_m(F,n,\mathcal{I},\mathcal{P})} \{ f_m(z) - M||z - x_{\mathcal{I}_p}|| \}, \]

(12)

i.e., functions \( \sigma_u, \sigma_l \) do not depend on the components \( x_{\mathcal{T}} \). However, this does not imply that the aggregation operator \( f_n \) itself does not depend on these components for any \( x \) (that would mean that the null set \( \mathcal{A}_m(F,n,\mathcal{I},\mathcal{P}) = I^m \)). Indeed, in addition to (12) we always have other bounds, at least those given by (9), which depend on all components of \( x \).

Next, since we are interested in locating optima in (12) for \( x_{\mathcal{I}_p} \notin \mathcal{A}_m(F,n,\mathcal{I},\mathcal{P}) \), it is sufficient to compute the optima over the boundary of \( \mathcal{A}_m(F,n,\mathcal{I},\mathcal{P}) \) rather than the entire \( \mathcal{A}_m(F,n,\mathcal{I},\mathcal{P}) \).

We also notice that the function \( f_m \in Lip(M) \), which means that the expression under minimum and maximum is a function from \( Lip(2M) \). In
general this function is not convex, it may have multiple local extrema. For this reason the numerical optimization process must be a global method (i.e., descent-based local optimization methods, such as quasi-Newton methods, are not suitable). Multivariate global optimization is a notoriously difficult NP-hard problem. For Lipschitz functions, a recently developed Cutting Angle method of deterministic global optimization \[2,3\] will be an appropriate approach.

In the special case when \(f_m(\alpha_p) = \text{const}\) for all \(\alpha \in \mathcal{A}_m(F, n, \mathcal{I}, \mathcal{P})\), the optimization problems simplify significantly, and become

\[
\sigma_u(x) = \sigma_u(x_{I_P}) = f_m(\alpha_p) + M \min_{z \in \mathcal{A}_m(F, n, \mathcal{I}, \mathcal{P})} ||(x_{I_P} - z)_+||,
\]

\[
\sigma_l(x) = \sigma_l(x_{I_P}) = f_m(\alpha_p) - M \max_{z \in \mathcal{A}_m(F, n, \mathcal{I}, \mathcal{P})} ||(z - x_{I_P})_+||. \tag{13}
\]

The objective functions are convex, and if the null set \(\mathcal{A}_m(F, n, \mathcal{I}, \mathcal{P})\) is also convex, we obtain problems of convex optimization, that can be solved using standard local methods.

6 Conclusion

We have extended the notion of the absorbent element of aggregation operators to absorbent tuples. It turns out that many important classes of aggregation operators have non-trivial absorbent tuples, which appear even in the absence of the standard absorbent element. We studied some properties of this generalization, and have determined the absorbent tuples of the most important classes of aggregation operators. Further, we have also developed a construction procedure allowing to obtain aggregation operators with a predefined absorbent behavior. This procedure is useful when existing aggregation operators do not possess the desired null sets.

Applications of the proposed notion of absorbent tuples can be found in various instances of multicriteria and multiperson decision making, as was illustrated in the introduction. Existence of absorbent tuples can be a desired or unwanted property, and the presented analysis will be useful when choosing aggregation operators for specific applications.
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References


