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Modeling the Tensile Strains of Non-uniform Fibers

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Abstract: The maximum strain experienced by the thinnest segment of a non-uniform fiber governs fiber breakage, yet this maximum strain can not be obtained from a normal single fiber test. Only the average strain of the whole fiber specimen can be obtained from a normal single fiber tensile test. This study has examined the relationship between the average strain, the maximum strain and the degree of fiber non-uniformity, expressed in coefficient of variation (CV) of fiber diameters along fiber length. The tensile strain of irregular fibers has been simulated using the finite element method (FEM). Using this method, average and maximum tensile strains of non-uniform fibers were calculated. The results indicate that for irregular fibers such as wool, there is an exponential relationship (i.e. $\frac{\varepsilon_{\text{ave}}}{\varepsilon_{\text{max}}} = a e^{-b \cdot CV}$) between the ratio of average breaking strain and maximum breaking strain ($\frac{\varepsilon_{\text{ave}}}{\varepsilon_{\text{max}}}$) and the along-fiber diameter variation (CV). The strain ratio decreases with the increase of the along-fiber diameter variation.

Keywords: wool, tensile strains, non-uniformity, diameter variation, coefficient of variation (CV)

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Introduction

Geometrical non-uniformity is common in natural fibers such as wool, as exemplified by their diameter variations along length. When such a fiber is extended, the tensile strain experienced by various segments of the fiber differs, with the thinnest segment experiencing the maximum strain [2]. The average breaking extension of a fiber decreases as the specimen gauge length increases [7]. Collins and Chaikin [3-5] have examined the effects of fiber non-uniformity on the tensile properties of non-uniform fibers. They have shown that variations in cross-sectional area along the fiber length markedly affect the stress-strain curves. Very little has been published on the effects of fiber non-uniformity on the average and maximum strains in the fiber under extension. It is important to know the maximum breaking strain of a fiber because it is the maximum strain that governs the fiber breakage. The maximum breaking strain, however, can not be obtained directly from a normal tensile test. It is the average breaking strain that is measured in a tensile test. Banky and Slen [2] have measured the variation along successive sections (5-mm lengths) of a wool fiber and clearly demonstrated that strain varies from section to section and the thin sections in a fiber extend more than the thick ones during the fiber stretching process. Zhang and Wang [10] have established an empirical relationship between the maximum strain ($\varepsilon_{max}$), the average strain ($\varepsilon_{ave}$) and the along-fiber diameter variation (CV), as indicated in Equation 1.

$$\varepsilon_{max} = \varepsilon_{ave} + CV$$

This empirical relationship is verified by the experimental results from Banky and Slen [2], as well as their own experimental results on wool [9]. However the
theoretical basis for Equation 1 has not yet been established. It is also not clear if this relationship applies over a large strain range. The along-fiber diameter variation is a geometrical feature of the fiber and would not change much upon initial stretching, while strain is related to the displacement in the tensile loading process. Under a very small load, the maximum strain in a non-uniform fiber is likely to be well below the CV of along-fiber diameter variation. It is obvious that Equation 1 is not valid under such a condition. It is therefore necessary to examine the relationship between tensile strains and along-fiber diameter variation over a wide strain range.

This paper presents a new model that describes the relationship between the diameter variation along the fiber length and the average and maximum strain experienced by a non-uniform fiber under extension. The finite-element (FE) method is also used to simulate the strain behaviour of an elongated fiber. Results from the two approaches are compared.

Materials and Methods

Assumptions of Fiber Specimen

The variability of fiber internal structures is ignored here and the fiber cross-sectional shape is assumed to be circular along the fiber length. At the same time the fiber diameter variation along length is assumed to be linear and the mean diameter for different diameter variations examined is the same. For wool fiber with a length of 2 mm, the fiber specimen is divided into \( n \) equal length sections (2/\( n \) mm long each) and the force applied to each section during the whole tensioning process is the same. The maximum strain occurs at the section with the thinnest fiber diameter. Relevant wool parameters concerning the fiber specimen are listed in Table 1.
Description of Finite Element Model (FEM)

A non-linear, three-dimensional (3D) finite-element model was developed using ABAQUS (version 6.4.1) software package to analyze the strain behavior of single fibers with different diameter variations along the fiber. A mesh density of 29400-31020 elements per 2 mm of fiber specimen was chosen for different simulation cases. The elements chosen were 8-node quadratic, reduced-integration, continuum (solid) elements (named as C3D8R in ABAQUS). In the analysis, one end of the fiber is fixed and the other end is extended, as illustrated in Figure 1. Intervals of 0.1 mm along the fiber length were used for calculating the strain, as shown in Figure 1 (between 1 and 2 or 2 and 3 etc). The type of simulation used in this analysis was considered to have a quasi-static response [1]. It is noted that the FE model used in this study has been verified to be acceptable for the simulation of single fibers [6].

Equation 2 is used to calculate the strain and Equations 3 and 4 are used to calculate the average strain $\varepsilon_{\text{ave}}$ and maximum strain $\varepsilon_{\text{max}}$:

\[ \varepsilon = \frac{l_1 - l_0}{l_0} \times 100\% \]  

\[ \varepsilon_{\text{ave}} = \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \]  

\[ \varepsilon_{\text{max}} = \text{Max}(\varepsilon_i), \ i = 1, 2, \cdots, n \]

Where $l_0$ represents the initial length and $l_i$ represents the extended length; $n$ is the number of sections along fiber ($n \geq 2$).
Results and Discussion

FEM Simulation

Table 2 shows FEM simulated results from 5, 10 and 20 sections within 2 mm, which represents the section lengths of 0.4 mm, 0.2 mm and 0.1 mm respectively. The average strain and maximum strain listed in Table 2 are at break point.

(Insert table 2 here)

From Table 2 it can be seen that the average strain decreases considerably with the increase of the diameter variation along the fiber length, which agrees with the previous results [6]. For fibers with the same average strain, the diameter CV value changes with n. This suggests that increasing the sampling resolution along the fiber length will result in more accurate CV measurements. On the other hand, the average strain for different section lengths is the same. This can be explained by the fact that the fiber length in each case is equal (2 mm); the average of the strain values for each section length should be the same whether the section length is long or short. At the same time the maximum strain for different cases decreases with the increase of section length which concurs with the weakest-link theory postulated by Peirce [8]. Furthermore Equation 1 does not quite apply to the data in Table 2 and the prediction accuracy drops as the diameter CV increases.

Figure 2 shows the relationship between the ratio of average breaking strain to maximum breaking strain (\( \frac{\varepsilon_{\text{ave}}}{\varepsilon_{\text{max}}} \)) and the diameter variations along the fiber length. It seems that there is an exponential relationship between \( \frac{\varepsilon_{\text{ave}}}{\varepsilon_{\text{max}}} \) and
diameter variations along the fiber length. Taking case \( n = 5 \) as an example, the R-squared value for the fit is up to 0.98. It can be seen from Figure 2 that the ratio of average breaking strain to maximum strain decreases with the increase of the along-fiber diameter variation. As the maximum strain occurs at the thinnest spot and the average breaking strain is derived from all the strain distributions of all small sections along the fiber length, an increase in the number of sections results in a lower \( \frac{\varepsilon_{\text{ave}}}{\varepsilon_{\text{max}}} \) value. If there is no diameter variation along the fiber length (i.e. CV = 0), the value of \( \frac{\varepsilon_{\text{ave}}}{\varepsilon_{\text{max}}} \) equals 1.

(Insert Figure 2 here)

Mathematical deduction and verification

The force for every fiber section is the same at a time of tensile loading. From any two sections we have \( F_1 = F_2 \), or:

\[
\sigma_1 S_1 = \sigma_2 S_2 \tag{5}
\]

Where \( F_1 \) and \( F_2 \) are the extension forces applied to Section 1 and Section 2; \( \sigma_1 \) and \( \sigma_2 \) are the stresses in Section 1 and Section 2; and \( S_1 \) and \( S_2 \) are the mean areas of Section 1 and Section 2.

For wool fiber, the stress is related to strain by a power law [3] so that

\[
\varepsilon = B \sigma^m \tag{6}
\]

Where \( B \) and \( m \) are positive constants, \( \varepsilon \) is the strain and \( \sigma \) is the stress. From Equations 5 and 6, we have

\[
\varepsilon_2 = \left(\frac{S_1}{S_2}\right)^m \times \varepsilon_1 \Rightarrow \varepsilon_1 = \left(\frac{S_1}{S_2}\right)^m \times \varepsilon_1 \tag{7}
\]
Since $S = \pi \times R^2$ and let $R_i$ be the mean fiber diameter of the thinnest fiber section, from Equation 7, Equation 8 can be derived:

$$\frac{\varepsilon_{\text{ave}}}{\varepsilon_{\text{max}}} = \frac{1}{n} \times \sum_{i=1}^{n} \left(\frac{R_i}{R_1}\right)^{2m} \quad \text{..................................................(8)}$$

On the other hand we have Equation 9 for calculating diameter variation,

$$CV = \sqrt{\frac{\sum_{i=1}^{n} (R_i - \overline{R})^2}{(n-1)}} \times 100\% \quad \text{..................................................(9)}$$

Where $\overline{R}$ is the mean fiber diameter of the fiber sections, and $n$ is the number of sections.

Here we discuss the diameter variations in two cases:

Case one is the step form as indicated in Figure 3. It is closer to the experimental analysis method used by Zhang and Wang [10]. In this case there is no diameter variation within a section and the fiber diameter variation is linear, then we have Equation 10.

$$\frac{\varepsilon_{\text{ave}}}{\varepsilon_{\text{max}}} = f(CV) = \frac{1}{n} \times \sum_{i=1}^{n} \left(\frac{2CV \times (i-1)/100}{\sqrt{n(n+1)/3 - CV \times (n-1)/100}} + 1\right)^{-2m} \quad \text{..............................................(10)}$$

(Insert Figure 3 here)

Case two is the cuneiform as indicated in Figure 1. In this case it is the $n+1$ rather than $n$ data points that are needed for $n$ fiber sections in all calculations including the CV. So it may then be shown that,

$$\frac{\varepsilon_{\text{ave}}}{\varepsilon_{\text{max}}} = f(CV) = \frac{1}{n+1} \times \sum_{i=1}^{n+1} \left(\frac{2CV \times (i-1)/100}{\sqrt{(n+2) \times (n+1)/3 - CV \times n/100}} + 1\right)^{-2m} \quad \text{.................(11)}$$
From Figure 4 we can see that there is almost no discrepancy between the two cases. Here the cuneiform case (Case two) is used because in practice, there is diameter variation both between and within small sections. Then, Equation 11 in turn can represent the relationship between $\frac{\varepsilon_{\text{ave}}}{\varepsilon_{\text{max}}}$ and CV on the basis of assumptions made for the fiber specimen (in Materials and Methods). Furthermore the relationship has nothing to do with the value of the mean fiber diameter and the step of arithmetical progression as they have been incorporated into the CV value.

(Insert Figure 4 here)

From Equations 10 and 11, we find that $f(CV)$ is related to the number of sections ($n$) once the sample is selected (hence $m$ is known). However, for a given fiber, $\frac{\varepsilon_{\text{ave}}}{\varepsilon_{\text{max}}}$ should be a constant. As shown in Figure 4, the discrepancy of the $f(CV)$ due to the number of sections can be ignored if the number of sections is large enough. In other words, the effect of the number of sections could be negligible if the number of small sections is greater than 100 although the prediction of $\frac{\varepsilon_{\text{ave}}}{\varepsilon_{\text{max}}}$ will be more accurate with an increase in the number of sections. Because it is difficult to mark a fiber with an interval less than 20 μm (2mm/100) such as in the experiments by Zhang and Wang [10], the accuracy for predicting $\frac{\varepsilon_{\text{ave}}}{\varepsilon_{\text{max}}}$ using CV value will reduce due to the significant reduction in number of sections. From Figure 4 we can also see that the $f(CV)$ will approach different constants for different CV values as the number of sections increases. Figure 5 shows that as the diameter CV
increases the $\frac{\varepsilon_{\text{ave}}}{\varepsilon_{\text{max}}}$ decreases, indicating that fibers with greater along-fiber variation result in smaller average breaking strain.

(Insert Figure 5 here)

As indicated in Figure 5 a relatively simple exponential regression fits very well the calculation from Equation 11. Therefore the new model for correlating $\frac{\varepsilon_{\text{ave}}}{\varepsilon_{\text{max}}}$ and CV could be simplified to Equation 12:

$$\frac{\varepsilon_{\text{ave}}}{\varepsilon_{\text{max}}} = ae^{-b\cdot CV}$$

where $a$ and $b$ are positive constants related to the number of sections along fiber.

In order to verify the simulation results we have carried out a fit analysis for the strain and stress curve of a 22µm wool fiber, and the fit result for Equation 6 is $R^2 = 0.92$ (as shown in Figure 6). The simulation results agree with the calculated results (Equation 11 for Case two) as shown in Figure 7.

(Insert Figure 6 here)

(Insert Figure 7 here)

The maximum strain in a fiber under tensile loading will approach the same true maximum strain if the number of fiber segments is large enough in Case two. The
FEM simulation results show a similar trend in Figure 8 (where the CVs for 100 sections and 200 sections are 13.72 and 13.55 respectively).

(Insert Figure 8 here)

**Comparison of modeling results with experimental data**

To validate the model, experimental data from Zhang and Wang [10] was selected where the fibers are about to break and the fiber diameter variation is approximately linear. The linear correlation coefficient ($R^2$) for fiber segment diameters in table 3 is above 0.9 for all samples.

(Insert Table 3 here)

As the fiber length does not directly affect the relationship (Equation 11), we can assume that each simulation section represents the experimental length. Figure 9 compares the experimental results listed in Table 3 with our modeling results ($n=5$).

For scoured merino wool, the mean squared errors of the $\frac{\varepsilon_{\text{ave}}}{\varepsilon_{\text{max}}}$ from the empirical formula and simulation results (5 sections) are 0.002 and 0.0006 respectively; while for scoured crossbred wool are 0.002 and 0.001 respectively. The mean squared error $E_i$ of an individual program $i$ is evaluated by Equation 13:

$$E_i = \frac{1}{n} \sum_{j=1}^{n} (P_{ij} - T_j)^2 \quad \text{................................................................. (13)}$$
where \( P_{(ij)} \) is the value predicted by the individual program \( i \) for sample case \( j \) (out of \( n \) sample cases); and \( T_j \) is the target value for sample case \( j \). For a perfect fit, \( P_{(ij)} = T_j \) and \( E_i = 0 \). So, the \( E_i \) index ranges from 0 to infinity, with 0 being the ideal. From the definition of the mean squared error it is clear that the model developed in this paper fits experimental data better than the empirical formula from Zhang and Wang [10]. It can be concluded that mathematical deduction can better describe the relationship between maximum strain, average strain and diameter CV along the fiber length.

(Insert Figure 9 here)

As calculation of \( \frac{\varepsilon_{\text{ave}}}{\varepsilon_{\text{max}}} \) is based on the assumptions that the variation in fiber diameter is linear, for other complex distributions of fiber diameter it is necessary to generalise the relationship between the strains and diameter variations. This study is based on a quasi-static condition when the fiber is not subjected to dynamic loading. Further study on this relationship under dynamic conditions is warranted.

**Conclusions**

An exponential relationship exists between the ratio of average and maximum breaking strains (\( \frac{\varepsilon_{\text{ave}}}{\varepsilon_{\text{max}}} \)), and the along-fiber diameter variation (CV) for the non-uniform fibers (i.e. \( \frac{\varepsilon_{\text{ave}}}{\varepsilon_{\text{max}}} = a e^{-b \cdot CV} \)). This ratio (\( \frac{\varepsilon_{\text{ave}}}{\varepsilon_{\text{max}}} \)) decreases with the increase in along-fiber diameter variation (CV). This study also shows that \( \frac{\varepsilon_{\text{ave}}}{\varepsilon_{\text{max}}} \) can be more accurately predicted when the fiber specimen is sub-divided into
a sufficient number of segments (eg. 50 per mm) for calculating the along-fiber diameter variation. Experimental data based on wool fibers have verified the modeling results.

Acknowledgement

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### Table 1. The wool fiber parameters for the FE model

<table>
<thead>
<tr>
<th>Properties</th>
<th>Young’s modulus (MPa)</th>
<th>Poisson’s ratio</th>
<th>Mean fiber diameter (µm)</th>
<th>Maximum stress at break (MP)</th>
<th>Fiber length (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>2000</td>
<td>0.35</td>
<td>22</td>
<td>85</td>
<td>2</td>
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</tbody>
</table>

### Table 2. Simulation results of $\varepsilon_{\text{max}}$, $\varepsilon_{\text{ave}}$ and $CV$

<table>
<thead>
<tr>
<th>n = 5</th>
<th>n = 10</th>
<th>n = 20</th>
</tr>
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<tbody>
<tr>
<td>$\varepsilon_{\text{ave}}$ (%)</td>
<td>$\varepsilon_{\text{max}}$ (%)</td>
<td>CV (%)</td>
</tr>
<tr>
<td>32.30</td>
<td>39.63</td>
<td>5.00</td>
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<tr>
<td>26.37</td>
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<tr>
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<tr>
<td>6.13</td>
<td>26.94</td>
<td>40.00</td>
</tr>
</tbody>
</table>

### Table 3. Related experimental data from Zhang [9]

<table>
<thead>
<tr>
<th>Sample</th>
<th>scoured merino wool diameter (µm)</th>
<th>scoured crossbred wool diameter(µm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fiber</td>
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<td>4</td>
</tr>
<tr>
<td>Section</td>
<td>1</td>
<td>27.25</td>
</tr>
<tr>
<td></td>
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<td>27.67</td>
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<td></td>
<td>5</td>
<td>29.42</td>
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<tr>
<td>Section</td>
<td>CV</td>
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<tr>
<td>Section</td>
<td>R²</td>
<td>0.92</td>
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</table>
Figure 1. Boundary conditions for 3D FEM (Right figure: mesh for a section length of 0.05mm).

Figure 2. Relationships between $\frac{\varepsilon_{av}}{\varepsilon_{max}}$ and the CV.

Figure 3. Step form fiber profile.
Figure 4. Effects of number of sections on $\frac{\varepsilon_{\text{ave}}}{\varepsilon_{\text{max}}}$

Figure 5. Exponential regression between $\frac{\varepsilon_{\text{ave}}}{\varepsilon_{\text{max}}}$ and fiber diameter CV.
Figure 6. Tensile curve of a wool fiber.

Figure 7. Comparisons of simulation and calculation results.
Figure 8. Effects of number of sections on maximum strain.

Figure 9. Comparisons between simulation results and experimental data [9].
References


