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Handling neutral and absorbent information in aggregation processes

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Abstract

We generalize the concepts of neutral and absorbent elements of aggregation operators. We introduce two types of tuples of values: the neutral tuples and the absorbent tuples. The neutral tuples are useful in situations in which information from different sources, or preferences of several decision makers, cancel each other. Absorbent tuples are useful in situations in which certain decision makers may decide the outcome irrespective of the opinion of the others. We examine the most important classes of aggregation operators in respect to their neutral and absorbent tuples.

Keywords: Aggregation operators, absorbent element, absorbent tuple, neutral element, neutral information, null set.

1 Introduction

Frequently in the aggregation process some pieces of information contradict each other, and need to be canceled out. For example, in group decision making, two members of a five-member jury may be in favor of a decision, and two may be against it. In this case the decision is based solely on the vote of the remaining fifth member. If the members of the jury are allowed to express the strength of their opinion, or have different voting power, their votes can cancel out in more complicated ways. For instance if two members are in favor of a decision, one weakly, the other one strongly, and two others are both moderately against, we still have the fifth member deciding the outcome.

Similarly, in expert systems there may be certain pieces of evidence in favor of a hypothesis, and certain pieces not supporting it, so that in total the hypothesis is neither supported nor rejected. In this case some additional evidence may be sought, which will be decisive. A classical example of an expert system with such behavior is MYCIN [3].

In the above mentioned examples, the “pros” and “cons” cancel each other, and the outcome in some sense is neutral with respect to this information (evidence, opinions). We shall refer to it as neutral information. Within the framework of aggregation operators, the values that cancel each other will be referred to as neutral tuples, and the set made of all the neutral tuples of an aggregation operator will be called its neutral set [1]. It is very interesting to examine known aggregation operators with respect to such cancelative behavior, which helps to determine their applicability.

On the other hand, we also extend the notion of the absorbent element of aggregation operators, a specific value which determines the result of the aggregation whenever any of its arguments takes this value. Our construction involves absorbent tuples - certain inputs combinations that determine the result of aggregation by themselves. Consider the following motivating example. A listed company has an executive board and broader shareholder’s meetings, at both meetings preferences (expressed as votes) are aggregated. However if the executive board is mostly unanimous in its decision, it takes this decision by itself, without other shareholder’s vote. If the executive board were able to take all decisions by itself, we would be talking about oligarchy. In our case we have partial oligarchy - only certain vote combinations result in the outcome determined solely by the board. We call all such combinations a null set [2].

There is a variety of practical situations in which we would like to model neutral/absorbent behavior of the aggregation procedure. We examine the most important families of aggregation operators and establish their neutral/null sets. Existence of neutral/absorbent tuples can be a desired or unwanted property, and the presented analysis will be useful when choosing aggre-
gation operators for specific applications.

**Notation**

Throughout this paper $F$ will denote an aggregation operator, and $f_n, n \geq 1$ will denote an $n$-ary aggregation operator corresponding to $F$. Also $I = [0, 1]$.

To denote the subsets of components of a vector $x \in I^n$ we shall employ the following notation. If $\mathcal{I} = \{I_1, \ldots, I_m\} \subset \{1, \ldots, n\}$ is an index set with cardinality $|\mathcal{I}| = m > 0$, then $x_{\mathcal{I}} = (x_{I_1}, \ldots, x_{I_m})$ will be used to denote the vector obtained from $x$ by selecting the components whose indices are in $\mathcal{I}$ in the order $I_1 < \ldots < I_m$. In addition, if $\mathcal{I} = \{I_1, \ldots, I_{n-m}\}$, with convention $I_1 < \ldots < I_{n-m}$, denotes the complement of $\mathcal{I}$ in $\{1, \ldots, n\}$, then $x_{\mathcal{I}}$ will denote the tuple $(x_{I_1}, \ldots, x_{I_{n-m}})$. For example, if $n = 5$, $\mathcal{I} = \{2, 4, 5\}$ then $x_{\mathcal{I}} = (x_2, x_4, x_5)$, $\mathcal{I} = \{1, 3\}$ and $x_{\mathcal{I}} = (x_1, x_3)$.

### 2 Neutral Information

A neutral element of an aggregation operator $F$ is a value $e \in I$ that can be omitted, without influencing the final output, from any position of any input vector. In order to cope with larger pieces of neutral information, we generalize the standard definition of neutral element to the case of tuples $e = (e_1, \ldots, e_m) \in I^m$, $m \in \mathbb{N} = \{1, 2, \ldots\}$.

**Definition 1** Let $F$ be an aggregation operator and let $\mathcal{I} \subset \{1, \ldots, n\}$, $n > 1$, be an index set such that $0 < |\mathcal{I}| = m$. Then:

- A tuple $e \in I^m$ is neutral for $F$ at level $n$ w.r.t. $\mathcal{I}$ when
  \[ f_n(x) = f_{n-m}(x_{\mathcal{I}}) \tag{1} \]
  holds for all $x \in I^n$ such that $x_{\mathcal{I}} = e$.
- The set made of all the tuples $e \in I^m$ which are neutral for $F$ at level $n$ with respect to (w.r.t.) $\mathcal{I}$ will be denoted by $\mathcal{E}_m(F, n, \mathcal{I})$ and will be called the neutral set of $F$ at level $n$ w.r.t. $\mathcal{I}$.

**Example 1** Let $F$ be an aggregation operator, $n = 3$, $\mathcal{I} = \{2, 3\}$ and $e = (0, 1) \in I^2$. Then $e$ is neutral for $F$ at level 3 w.r.t. $\mathcal{I}$, i.e., $e \in \mathcal{E}_2(F, 3, \mathcal{I})$, if $f_3(t, 0, 1) = f_1(t) = t$ holds for any $t \in I$.

The above definition implies that when aggregating $n$ values with $F$, the information contained in a given tuple $e$, if appearing in the positions indicated by some particular index set $\mathcal{I}$, does not affect the final output. Of course, the same could happen – as it is the case of the standard neutral element – independently of the positions that the components of $e$ occupy in the input vector $x$.

The next definition accommodates this situation. Denote by $\pi_\mathcal{I}$ a permutation of the components of $e$, i.e., if $e = (e_1, \ldots, e_m) \in I^m$, then $e_\pi = (e_{\pi(1)}, \ldots, e_{\pi(m)}) \in I^m$ for some permutation $\pi = (\pi(1), \ldots, \pi(m))$ of $(1, \ldots, m)$.

**Definition 2** Let $F$ be an aggregation operator and let $m, n \in \mathbb{N}$, $m < n$. Then:

- A tuple $e \in I^m$ is neutral for $F$ at level $n$ when, for any index set $\mathcal{I} \subset \{1, \ldots, n\}$ such that $|\mathcal{I}| = m$, and any permutation $\pi$, $e_\pi$ is neutral for $F$ at level $n$ w.r.t. $\mathcal{I}$.
- The set made of all the tuples $e \in I^m$ which are neutral for $F$ at level $n$ will be denoted by $\mathcal{E}_m(F, n)$ and will be called the $m$-neutral set of $F$ at level $n$.

**Example 2** Choosing, as in the previous example, $n = 3$ and $e = (0, 1) \in I^2$, now $e$ is neutral for $F$ at level 3, i.e., $e \in \mathcal{E}_2(F, 3)$, if $f_3(P(t, 0, 1)) = f_1(t) = t$, where $P(t)$ is any permutation of the components of $x$, holds for any $t \in I$.

**Remark 1** If $F$ is an aggregation operator and $m, n \in \mathbb{N}$, $m < n$, then:

1. \[ \mathcal{E}_m(F, n) = \bigcap_{\mathcal{I} \subset \{1, \ldots, n\}, |\mathcal{I}| = m} \mathcal{E}_m(F, n, \mathcal{I}) \]
2. If $F$ is symmetric, then it obviously suffices to have $e \in \mathcal{E}_m(F, n, \mathcal{I})$ for some $\mathcal{I}$ in order to automatically have $e \in \mathcal{E}_m(F, n)$.

Coming back to Definition 1, note now that it refers to just one specific dimension, $n$, of the aggregation operator $F$. Similarly to the way in which the standard neutral element is defined, we could think of tuples $e \in I^m$ remaining neutral for any dimension (as long as such dimension contains the positions given by the index set $\mathcal{I}$):

**Definition 3** Let $F$ be an aggregation operator and let $\mathcal{I} \subset \{1, 2, \ldots\}$ be an index set such that $|\mathcal{I}| = m$. Then:

- A tuple $e \in I^m$ is neutral for $F$ w.r.t. $\mathcal{I}$ when, for any $n \geq \max(|\mathcal{I}| + 1, \max(\mathcal{I}))$, $e$ is neutral for $F$ at level $n$ w.r.t. $\mathcal{I}$.
- The set made of all the tuples $e \in I^m$ which are neutral for $F$ w.r.t. $\mathcal{I}$ will be denoted by $\mathcal{E}_m(F, \mathcal{I})$ and will be called the neutral set of $F$ w.r.t. $\mathcal{I}$.
Example 3 Choosing, as in Example 1, $\mathcal{I} = \{2, 3\}$, then $\varepsilon = (0, 1) \in I^2$ is neutral for $F$ w.r.t. $\mathcal{I}$, i.e., $\varepsilon \in \mathcal{E}_2(F, \mathcal{I})$, if $f_3(t_1, 0, 1) = f_3(t_1) = t_1$ holds for any $t_1 \in I$, $f_3(t_1, 0, 1, t_2) = f_3(t_1, t_2)$ holds for any $t_1, t_2 \in I$, $f_3(t_1, 0, 1, t_2, t_3) = f_3(t_1, t_2, t_3)$ holds for any $t_1, t_2, t_3 \in I$, etc.

Remark 2 If $F$ is an aggregation operator, $\mathcal{I} \subset \{1, 2, \ldots\}$ is an index set such that $|\mathcal{I}| = m$, then:

$$\mathcal{E}_m(F, \mathcal{I}) = \bigcap_{n \geq \max(|\mathcal{I}| + 1, \max(I))} \mathcal{E}_m(F, n, \mathcal{I})$$

Definitions 2 and 3 have been obtained from Definition 1 after independently introducing a stronger demand on two different aspects: the position of the neutral information within the input vector and the dimension of the later, respectively. If these two aspects are taken into account simultaneously, the result can be stated as follows:

Definition 4 Let $F$ be an aggregation operator and let $m \in \mathbb{N}$. Then:

- A tuple $\varepsilon \in I^m$ is neutral for $F$ when, for any $n > m$ and for any index set $\mathcal{I} \subset \{1, \ldots, n\}$ such that $|\mathcal{I}| = m$, and any permutation $\pi$ of $\{1, \ldots, m\}$, $\varepsilon_{\pi}$ is neutral for $F$ at level $n$ w.r.t. $\mathcal{I}$.
- The set made of all the tuples $\varepsilon \in I^m$ which are neutral for $F$ will be denoted by $\mathcal{E}_m(F)$ and will be called the $m$-neutral set of $F$.

Example 4 The tuple $\varepsilon = (0, 1) \in I^2$ is neutral for $F$ if: $\forall n > 2, \forall(x_1, \ldots, x_n) \in I^n$, $f_n(x_1, \ldots, x_n) = f_{n-2}(x_1, \ldots, x_{i-1}, 1, x_{j+1}, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_n)$ whenever there exist $i, j \in \{1, \ldots, n\}$ such that $x_i = x_{j} = 0, x_j = 1$.

Remark 3 If $F$ is an aggregation operator and $m \in \mathbb{N}$, then:

1. $\mathcal{E}_m(F) = \bigcap_{n \geq m, \mathcal{I} \subset \{1, \ldots, n\}, |\mathcal{I}| = m} \mathcal{E}_m(F, n, \mathcal{I})$

2. When choosing $m = 1$, Definition 4 recovers the standard definition of the neutral element, i.e.:

$$\mathcal{E}_1(F) = \begin{cases} \{e\}, & \text{if } F \text{ has neutral element } e \in I \\ \emptyset, & \text{otherwise} \end{cases}$$

Remark 4 The concept of neutral tuple for $F$ could have been alternatively defined using either Definition 2 or Definition 3, that is, the two following statements hold:

1. $\varepsilon \in I^m$ is neutral for $F$ if and only if $\varepsilon$ is neutral for $F$ at level $n$ for any $n > m$, that is:

$$\mathcal{E}_m(F) = \bigcap_{n > m} \mathcal{E}_m(F, n)$$

2. $\varepsilon \in I^m$ is neutral for $F$ if and only if $\varepsilon_{\pi}$ is neutral for $F$ w.r.t. $\mathcal{I}$ for any index set $\mathcal{I} \subset \{1, 2, \ldots\}$ such that $|\mathcal{I}| = m$, and any permutation $\pi$ of $\{1, \ldots, m\}$, that is:

$$\mathcal{E}_m(F) = \bigcap_{\mathcal{I} \subset \{1, 2, \ldots\}, |\mathcal{I}| = m} \mathcal{E}_m(F, \mathcal{I})$$

When referring to the set made of all the tuples, regardless of their dimension, which are neutral for a given aggregation operator $F$, we will use the following:

Definition 5 The neutral set of an aggregation operator $F$, denoted by $\mathcal{E}(F)$, is the set made of all the tuples $\varepsilon \in I^m, m \in \mathbb{N}$, which are neutral for $F$, i.e.,

$$\mathcal{E}(F) = \bigcup_{m \in \mathbb{N}} \mathcal{E}_m(F)$$

From the above definitions it is possible to obtain some basic and interesting properties:

- The concatenation of two neutral tuples provides a new neutral tuple.
- If an aggregation operator has two neutral tuples $\alpha, \tau \in I^m, \alpha \leq \tau$ then any tuple $\xi \in I^m$ such that $\alpha \leq \xi \leq \tau$ is another neutral tuple.
- If an aggregation operator $F$ has two neutral tuples $\alpha, \tau$ then $F(\alpha) = F(\tau)$.
- For any aggregation operator with neutral element $e$, the tuple $(e, \ldots, e)$ is a neutral tuple.

Example 5 For any aggregation operator $F'$ with neutral element $1$ (0) we have $\mathcal{E}_m(F') = \{1, \ldots, 1\}$ ($\mathcal{E}_m(F') = \{0, \ldots, 0\}$).

- Any quasi-linear $T - S$ aggregation operator $F$ has empty neutral sets $\mathcal{E}_m(F, n, \mathcal{I})$. 

• For a representable uninorm with generator \( g \) then 
\[
E_m(F) = \{ x \in (0,1)^m : \sum_{i=1}^m g(x_i) = 0 \}.
\]

**Example 6** 3-\( \Pi \) operator \([4], p.19\), given by
\[
f_n(x_1, x_2, \ldots, x_n) = \prod_{i=1}^n x_i - \prod_{i=1}^n (1 - x_i),
\]
with the convention \( \frac{0}{0} = 0 \) is a representable uninorm with an additive generator \( g(x) = \log(\frac{1}{1-x}) \) and neutral element \( c = \frac{1}{2} \).
The neutral set \( E_m(F) \) containing the tuples \( \varepsilon \in (0,1)^m \) is identified from
\[
\prod_{i=1}^m \varepsilon_i = \prod_{i=1}^m (1 - \varepsilon_i).
\]
In particular, when \( m = 2 \), we have an explicit formula
\[
E_2(F) = \{ \varepsilon \in (0,1)^2 : \varepsilon_1 + \varepsilon_2 = 1 \}.
\]
Thus we also have for \( m > 2 \), by concatenating neutral tuples
\[
\{ \varepsilon \in (0,1)^m : \varepsilon_i + \varepsilon_j = 1, \varepsilon_k = \frac{1}{2}, k \neq i, j \} \subseteq E_m(F),
\]
and also
\[
\{ \varepsilon \in (0,1)^m : \varepsilon_i + \varepsilon_j = 1, \varepsilon_k + \varepsilon_l = 1, \text{all } i, j, k, l \text{ distinct} \}
\subseteq E_m(F), m = 4, \text{ etc.}
\]

**Example 7** \([4, 7]\) Let \( g(t) = t - \frac{1}{2} \) and 
\[
F(x_1, x_2, \ldots, x_n) = g^{-1}(\sum_{i=1}^n g(x_i)).
\]
Then \( F \) is an ordinal sum of Lukasiewicz \( t \)-norm and \( t \)-conorm, given by
\[
F(x) = \max(0, \min(1, \frac{1}{2} + \sum_{i=1}^n (x_i - \frac{1}{2})))
\]
whose neutral set is 
\[
E_m(F) = \{ \varepsilon \in I^m : \sum_{i=1}^m \varepsilon_i = \frac{m}{2} \}.
\]

## 3 Absorben t information

First we recall the definition of an absorbent element of an aggregation operator. A value \( a \in I \) is an absorbent element of \( F \) when it carries the final output to \( a \). In order to cope with larger pieces of absorbent information, we generalize the standard definition of absorbent element to the case of tuples \( \alpha = (\alpha_1, \ldots, \alpha_m) \in I^m, m \in N = \{1, 2, \ldots\} \).

**Definition 6** Let \( F \) be an aggregation operator, \( I \subset \{1, \ldots, n\}, n > 1 \), an index set such that \( |I| = m \).

- A tuple \( \alpha \in I^m \) is absorbent for \( F \) at level \( n \) w.r.t. \( I \) when
\[
f_n(x) = f_m(\alpha)
\]
holds for all \( x \in I^n \) such that \( x_I = \alpha \).
- The set made of all the tuples \( \alpha \in I^m \) which are absorbent for \( F \) at level \( n \) w.r.t. \( I \) will be denoted by \( A_m(F,n,I) \) and will be called the null set of \( F \) at level \( n \) w.r.t. \( I \).

**Example 8** Let \( F \) be an aggregation operator, \( n = 3, I = \{2, 3\} \) and \( \alpha = (1,0) \in I^2 \). Then \( \alpha \) is absorbent for \( F \) at level 3 w.r.t. \( I \), i.e., \( \alpha \in A_2(F,3,I) \), if \( f_3(x_1,1,0) = f_2(1,0) \) holds for any \( x_1 \in I \).

The above definition implies that when aggregating \( n \) values with \( F \), the information contained in a given tuple \( \alpha \), if appearing in the positions indicated by some particular set \( I \), transforms the final output to \( F(x) \). Of course, the same could happen – as it is the case of the standard absorbent element, independently of the positions that the components of \( \alpha \) occupy in the input vector and independently of the dimension of the input vector (as long as it is greater than the dimension of the absorbent tuple). The next definition accommodates this situation.

**Definition 7** Let \( F \) be an aggregation operator and let \( m \in N \). Then:

- A tuple \( \alpha \in I^m \) is absorbent for \( F \) when, for any \( n > m \), for any index set \( I \subset \{1, \ldots, n\} \) such that \( |I| = m \) and for any permutation \( \pi \) of \( (1, \ldots, m) \), \( \alpha_\pi \) is absorbent for \( F \) at level \( n \) w.r.t. \( I \).
- The set made of all the tuples \( \alpha \in I^m \) which are absorbent for \( F \) will be denoted by \( A_m(F) \) and will be called the \( m \)-null set of \( F \).

**Example 9** The tuple \( \alpha = (1,0) \in I^2 \) is absorbent for \( F \) if \( \forall n > 2, \forall x = (x_1, \ldots, x_n) \in I^n \), if there exist \( i, j \in \{1, \ldots, n\} \), such that \( x_i = 0, x_j = 1 \), then
\[
f_n(x) = f_2(1,0).
\]

**Remark 5** If \( F \) is an aggregation operator and \( m \in N \), then:

1. 
\[
F(F,F) = \bigcap_{n=m}^{\infty} A_m(F,n,I)
\]
ample, is absorbent.

\[ F(a) = \begin{cases} \{ a \}, & \text{if } F \text{ has absorbent element } a \\ \emptyset, & \text{otherwise} \end{cases} \]

When referring to the set made of all the tuples, regardless of their dimension, which are absorbent for a given aggregation operator \( F \), we will use the following:

**Definition 8** The null set of an aggregation operator \( F \), denoted by \( \mathcal{A}(F) \), is the set made of all the tuples \( \alpha \in I^m, m \in N \), which are absorbent for \( F \), i.e.,

\[ \mathcal{A}(F) = \bigcup_{m \in N} \mathcal{A}_m(F) \]

Some general properties of absorbent tuples:

- Absorbent tuples have the property that their aggregation, by means of \( F \), always provides the same output: if \( \alpha, \beta \in \mathcal{A}(F) \), then \( F(\alpha) = F(\beta) \).
- If \( F \) is an aggregation operator with absorbent element \( a \in I \), then for any \( \alpha \in \mathcal{A}(F) \), it is \( F(\alpha) = a \).
- For any associative and commutative aggregation operator, if for some tuple \( \alpha, F(\alpha) = b \) (the absorbing element), then \( \alpha \) is an absorbent tuple.
- If \( \alpha \in \mathcal{A}(F) \), then for any tuple \( \alpha^+ \) containing, at least, all the values of \( \alpha \) (in any position), it is \( \alpha^+ \in \mathcal{A}(F) \).
- For any associative and commutative aggregation operator the null set will include, at least, the tuples \((1,0)\) and \((0,1)\).

Observe that in general, if \( a \) is the absorbent element of \( F \), the tuples verifying \( F(\alpha) = a \) are not necessarily absorbent tuples as the reader can see in the next example:

**Example 10** Let \( F \) be an aggregation operator such that each \( f_n, n \geq 2 \), is a possibly different triangular norm (recall that the well-known triangular norms, or \( t \)-norms, are associative and commutative aggregation operator with neutral element 1). Zero is absorbent for any \( t \)-norm, and, therefore, for \( F \). Suppose now that \( f_2(x,y) = T_L(x,y) = \max(0, x + y - 1) \) (the Lukasiewicz \( t \)-norm) and that \( f_3(x,y,z) = \min(x,y,z) \) (the minimum \( t \)-norm). Choosing, for example, \( \alpha = (0.3,0.3) \), it is \( F(\alpha) = f_2(\alpha) = 0 \), but \( \alpha \) is not an absorbent tuple, since we have, for instance, \( F(0.3,0.3,0.2) = 0.2 \neq F(0.3,0.3) \).

It is clear that strictly monotone \( t \)-norms (such as the so-called strict \( t \)-norms, isomorphic to the product \( t \)-norm) have only trivial absorbent tuples (those containing at least one zero). But there are \( t \)-norms (either continuous or non-continuous) with non-trivial absorbent tuples: those having zero divisors, such as the drastic \( t \)-norm, nilpotent \( t \)-norms (isomorphic to the Lukasiewicz \( t \)-norm), the nilpotent minimum, etc.

**Example 11** Nilpotent \( t \)-norms are given by:

\[ T_L(x, y) = \varphi^{-1}(\max(0, \varphi(x) + \varphi(y) - 1)) \]

for all \( x, y \in I, \) where \( \varphi : I \to I \) is an increasing bijection. Then any tuple \((x, y)\) such that \( y \leq \varphi^{-1}(1-\varphi(x)) \) is absorbent for \( T_L \), (the same happens with any tuple containing any two values satisfying the above inequality).

## 4 Conclusion

We have extended the notion of neutral and absorbent element of aggregation operators to neutral and absorbent tuples. We have studied some properties of these generalizations, and have determined the neutral and absorbent tuples of the most important classes of aggregation operators (we include two tables summarizing the main results on this topic). In [1, 2] we have also developed construction procedures allowing to obtain aggregation operators with a predefined neutral and absorbent behavior, respectively.

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**References**


Table 1. Neutral sets of various families of aggregation operators.

<table>
<thead>
<tr>
<th>Aggregation Operator</th>
<th>Neutral set</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangular norms</td>
<td>{1,1,...,1}</td>
</tr>
<tr>
<td>Triangular conorms</td>
<td>{0,0,...,0}</td>
</tr>
<tr>
<td>Uninorms</td>
<td>nonempty and nontrivial neutral sets</td>
</tr>
<tr>
<td>Nullnorms</td>
<td>empty</td>
</tr>
<tr>
<td>Other generated operators</td>
<td>nonempty and nontrivial neutral sets</td>
</tr>
<tr>
<td>Quasi-linear T-S operators</td>
<td>empty</td>
</tr>
<tr>
<td>Means</td>
<td>empty, if all weights are strictly positive</td>
</tr>
<tr>
<td>OWA</td>
<td>empty, if all weights are strictly positive</td>
</tr>
<tr>
<td>Generalized OWA</td>
<td>empty, if all weights are strictly positive</td>
</tr>
<tr>
<td>Symmetrical sums</td>
<td>nonempty if there is neutral element</td>
</tr>
<tr>
<td>Choquoid integrals</td>
<td>nonempty in some cases</td>
</tr>
<tr>
<td>Projection operators</td>
<td>largest neutral set</td>
</tr>
</tbody>
</table>

Table 2. Null sets of various families of aggregation operators.

<table>
<thead>
<tr>
<th>Aggregation Operator</th>
<th>Absorvent tuples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Generated agops</td>
<td>(\sum_{i=1}^{m} g(\alpha_i) \geq g(1) - (n - m) \cdot g(0)) or</td>
</tr>
<tr>
<td></td>
<td>(\sum_{i=1}^{m} g(\alpha_i) \leq g(0) - (n - m) \cdot g(1))</td>
</tr>
<tr>
<td>Means</td>
<td>(A_{m}(F) = {\alpha \in I^{m} : \min \alpha_i = 0})</td>
</tr>
<tr>
<td></td>
<td>if (g(0) = \pm \infty).</td>
</tr>
<tr>
<td></td>
<td>(A_{m}(F) = {\alpha \in I^{m} : \max \alpha_i = 1})</td>
</tr>
<tr>
<td></td>
<td>if (g(1) = \pm \infty).</td>
</tr>
<tr>
<td></td>
<td>(A_{m}(F, n, I, P) = \emptyset).</td>
</tr>
<tr>
<td></td>
<td>otherwise.</td>
</tr>
<tr>
<td>OWA</td>
<td>(A_{m}(F) = {\alpha \in I^{m} : \min \alpha_i = 0})</td>
</tr>
<tr>
<td></td>
<td>if (g(0) = \pm \infty).</td>
</tr>
<tr>
<td></td>
<td>(A_{m}(F) = {\alpha \in I^{m} : \max \alpha_i = 1})</td>
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<tr>
<td></td>
<td>if (g(1) = \pm \infty).</td>
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<td>(A_{m}(F, n, I, P) = \emptyset).</td>
</tr>
<tr>
<td></td>
<td>otherwise.</td>
</tr>
<tr>
<td>Quasi-linear T-S Agops</td>
<td>(T(\alpha) = 0) if (g(0) = \pm \infty).</td>
</tr>
<tr>
<td></td>
<td>(S(\alpha) = 1), if (g(1) = \pm \infty).</td>
</tr>
<tr>
<td></td>
<td>(T(\alpha) = 0, S(\alpha) = 1), if (g(0) \neq \pm \infty, g(1) \neq \pm \infty).</td>
</tr>
</tbody>
</table>