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A Low-Order Linear Functional Observer for Time Delay Systems

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Abstract

This paper presents an efficient technique to design low-order state function observers for linear time-delay systems. Assuming the existence of a linear state feedback controller to achieve stability or some control performance criteria of the time-delay system, a design procedure is proposed for reconstruction of the state feedback control action. The procedure involves solving an optimisation problem with the objective to generate a matrix that is as close as possible to the given feedback gain of the required feedback controller. A condition for robust stability of the time-delay system using the observer-based control scheme is given. The attractive features of the proposed design procedure are that the resulted linear functional state observer is of a very low order and it requires information of a small number of outputs. Numerical examples are given to demonstrate the design procedure and its merits.

1 Introduction

Stabilisation and performance enhancement of time-delay systems using linear state feedback control laws have been the subject of considerable research interest in the control literature (see for example, Furukawa and Shimemura (1983), Mori et al. (1983), Petersen and Hollot (1986), Shen et al. (1991), Boyd et al. (1994), Mohimani and Petersen (1997), Wu (2000), Fridman and Shaked (2002), Li and DeCarlo (2003) and many references therein). In most practical cases, either the states of the time-delay system are not physically available for direct measurement or the cost of measurement is prohibitively high. In such cases, a full state feedback control scheme cannot be implemented and a state-estimation scheme may be required. Fortunately, implementation of a state feedback control law does not necessarily require the availability of the complete state vector, x(t). Rather, it requires the feedback control signals, Fx(t), which are linear functions of x(t), to be generated.

Estimating linear functionals of a state vector has been the focus of many researchers over the years. A number of procedures have been proposed to design functional state observers (see eg., Fairman and Gupta (1980), O’Reilly (1983), Aldeen and Trinh (1999), and Trinh and Ha (2000)). In the context of time-delay systems the state estimation problem has also been of great interest (see eg., Salamon (1980), Pearson and Fiagbedzi (1989), Tornambe (1992), Leyva-Ramos and Pearson (1995), and Trinh et al. (1999)). For observer-based feedback control of time-delay systems, estimating the feedback control law instead of the system states has proved to be an interesting problem. In a recent technique proposed by Trinh (1999) to develop linear functional observers for time-delay systems, a condition on the observer order is derived in terms of the number of states and outputs. A question arises however as to whether the restriction on the lower bound of the observer order and the number of outputs can be relaxed so that observers of a lower order can be designed for time-delay systems using only a limited number of outputs.

Assuming a linear state feedback controller has been designed for the time-delay system to meet the stability requirement or some specified control performance objectives, the problem to be addressed is how to design a low-order linear state function observer to reconstruct the given control law. It is shown that the problem can be formulated into a parameter optimisation process. The optimisation objective is to generate a matrix that is maximally close to the given feedback gain of the required feedback controller. A condition is derived to guarantee stability of the closed-loop system under the proposed observer-based control scheme. A step-by-step design algorithm is given. Numerical examples are provided to illustrate the design procedure and salient advantages of the technique.

2 Problem statement

Consider a time-invariant linear system with a time lag, described by

\[ \dot{x}(t) = Ax(t) + A_j x(t - \tau) + B u(t); t \geq 0, \]  
\[ x(t) = \Phi(t), -\tau \leq t \leq 0, \]  
\[ y(t) = C x(t), \]

where \( \tau \) is a positive real number representing the time delay in the state, \( \Phi(t) \) is a continuous function on the interval \([-\tau,0]\); vectors \( x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m \) and \( y(t) \in \mathbb{R}^p \) are respectively the state, input and output; and
matrices \( A \in \mathbb{R}^{m \times n} \), \( A_d \in \mathbb{R}^{m \times n} \), \( B \in \mathbb{R}^{m \times q} \) and \( C \in \mathbb{R}^{p \times n} \) are constant. Without loss of generality, we assume that matrix \( C \) takes the following canonical form:

\[
C = [C_1 \ 0],
\]

(1d)

where \( C_1 \in \mathbb{R}^{p \times r} \) is of full rank. Note that any full rank matrix \( C \) can always be transformed into \((1d)\) using orthogonal similarity transformation.

Let us assume that a linear state feedback control law

\[
u(t) = Fx(t),
\]

(2)

where \( F \in \mathbb{R}^{m \times n} \), has been obtained for system \((1)\) to achieve system stability and/or specified performance objectives.

As there are \( m \) inputs in the system, the system dynamics \((1a)\) can be rewritten in the form

\[
\dot{x}(t) = Ax(t) + A_d x(t - \tau) + \sum_{i=1}^{m} B_i u_i(t); \quad t \geq 0,
\]

(3a)

with the feedback control \((2)\) expressed as

\[
u_i(t) = F_i x(t); \quad i = 1, 2, \ldots, m,
\]

(3b)

where \( B = [B_1 \ldots B_m] \), \( B_j \in \mathbb{R}^{n \times 1} \), and \( F_j \in \mathbb{R}^{m \times n} \).

Implementation of the above feedback control law requires physical measured readings of all the system state variables. To avoid such technical difficulties and high cost associated with direct measurements of all the state variables, a linear functional observer can be used to reconstruct the control actions \((3b)\). Towards the generation of \( \dot{e}_i(t) \) \( (i = 1, 2, \ldots, m) \), let us first decompose the feedback gain matrix \( F_j \) \( (i = 1, 2, \ldots, m) \) as follows

\[
F_j = K_j T_j + W_j C,
\]

(4)

where \( K_j \in \mathbb{R}^{n \times p} \), \( T_j \in \mathbb{R}^{n \times n} \), and \( W_j \in \mathbb{R}^{n \times r} \) are constant matrices to be determined, and \( 1 \leq p < (n - r) \). Using \((4)\), the feedback control signal \( u_i(t) \) \( (i = 1, 2, \ldots, m) \) can be expressed as

\[
u_i(t) = F_j x(t) = K_j T_j x(t) + W_j y(t) = K_j z_i(t) + W_j y(t),
\]

(5)

where

\[
z_i(t) = T_i x(t) \in \mathbb{R}^p; \quad i = 1, 2, \ldots, m.
\]

(6)

The feedback control law \((5)\) can now be implemented by using information of the linear state functions, \( T_i x(t) \) \( (i = 1, 2, \ldots, m) \), and the output, \( y(t) \). The latter is measurable while the formers are to be estimated using observers. Moreover, from the engineering point of view when designing observers, a low order observer using a small number of outputs is most desirable. The problem is thus how to design \( m \) linear function observers, each of dimension \( p \), where \( p \) is kept as small as possible, in order to generate the required \( m \) vector state functions of \( T_i x(t); i = 1, 2, \ldots, m \).

### 3 Main Results

Let us now consider the following \( p \)-th order observer of the form

\[
\dot{z}_i(t) = E_i z_i(t) + T_i B_i u(t) + G_i y(t) + M_i y(t - \tau); \quad t \geq 0,
\]

(7)

with initial conditions

\[
z_i(t) = h_i(t), \quad -\tau \leq t \leq 0,
\]

(8)

where \( E_i \in \mathbb{R}^{p \times p} \) is a stable matrix to be selected, \( G_i \in \mathbb{R}^{p \times q} \) and \( M_i \in \mathbb{R}^{p \times r} \) are constant matrices to be determined, and \( h_i(t) \) is a continuous function on the indicated closed interval.

Let \( e_i(t) \) be defined as the error between the state of system \((7)\), \( z_i(t) \), and its estimate, \( T_i x(t) \), i.e. as

\[
e_i(t) = z_i(t) - T_i x(t); \quad i = 1, 2, \ldots, m.
\]

(9)

Taking derivative of \((9)\) and using \((1a)\) yield

\[
\dot{e}_i(t) = \dot{z}_i(t) - \dot{T}_i x(t)
\]

\[
= E_i z_i(t) + T_i B_i u(t) + G_i y(t) + M_i y(t - \tau) - T_i A x(t) - T_i A_d x(t - \tau) - T_i B u(t)
\]

\[
= E_i [z_i(t) - T_i x(t)] + (G_i C - T_i A) x(t) + (M_i C - T_i A_d) x(t - \tau).
\]

(10)

Given a stable matrix \( E_i \), if matrices \( G_i \), \( T_i \) and \( M_i \) are determined such that

\[
G_i C - T_i A = 0,
\]

(11)

and

\[
M_i C - T_i A_d = 0,
\]

(12)

then the observer error dynamics become

\[
\dot{e}_i(t) = E_i e_i(t).
\]

(13)

Accordingly, \((7)\) can act as a linear functional observer for system \((1)\), provided that matrix \( E_i \) is stable and equations \((4)\), \((11)\) and \((12)\) are satisfied. It is shown in Trinh (1999) that, under the satisfaction of some matrix rank conditions, a linear state function observer of the order \( p \geq \frac{n - r}{2} \) can be derived. The lower bound on the observer order implies that for time-delay systems a functional observer may not exist unless the number of available outputs is greater than half of the number of states \((r > 0.5n)\). This paper seeks to overcome this limitation by proposing an alternative
procedure for solving equations (4), (11) and (12) based on a parameter optimisation process.

Let us first partition matrices $B_i \in \mathbb{R}^{n \times d}$, $F_i \in \mathbb{R}^{p \times d}$, $A \in \mathbb{R}^{n \times n}$, $A_d \in \mathbb{R}^{n \times n}$ and $T \in \mathbb{R}^{p \times n}$ as follow:

\begin{align}
B_i &= \begin{bmatrix} B_{i1} \\ B_{i2} \end{bmatrix}, \\
F_i &= \begin{bmatrix} F_{i1} \\ F_{i2} \end{bmatrix}, \\
A &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \\
A_d &= \begin{bmatrix} A_{d11} & A_{d12} \\ A_{d21} & A_{d22} \end{bmatrix}, \\
T &= \begin{bmatrix} T_{11} \\ T_{12} \end{bmatrix},
\end{align}

and

\begin{align}
&\text{where } B_{i1} \in \mathbb{R}^{n \times d}, B_{i2} \in \mathbb{R}^{(n-r) \times d}, F_{i1} \in \mathbb{R}^{p \times d}, F_{i2} \in \mathbb{R}^{p \times (n-r)}, \\
&A_{11}(A_{11}) \in \mathbb{R}^{n \times n}, A_{12}(A_{12}) \in \mathbb{R}^{n \times (n-r)}, \\
&A_{21}(A_{21}) \in \mathbb{R}^{(n-r) \times n}, A_{22}(A_{22}) \in \mathbb{R}^{(n-r) \times (n-r)}, \\
&\text{and } T \in \mathbb{R}^{p \times n} \text{ are corresponding sub-matrices.}
\end{align}

By incorporating equations (15)-(18) into equations (4), (11), (12) and after some rearranging, the following equations can be obtained:

\begin{align}
F_{i1} &= K_i T_{i1} + W_i C_1, \\
F_{i2} &= K_i T_{i2}, \\
G_i C_1 - (T_{i1} A_{d11} + T_{i2} A_{d12}) + E_i T_{i1} &= 0, \\
E_i T_{i2} - T_{i1} A_{d12} - T_{i2} A_{d22} &= 0, \\
M_i C_1 - T_{i1} A_{d11} - T_{i2} A_{d21} &= 0, \\
T_{i1} A_{d12} + T_{i2} A_{d22} &= 0.
\end{align}

As matrix $E_i$ is selected according to the desired dynamics for the observer to be constructed, there are six unknown matrices (namely $K_i$, $W_i$, $G_i$, $M_i$, $T_{i1}$ and $T_{i2}$) in equations (19)-(24) to be solved for. For the Lyapunov equation (22), the unique solution for $T_{i2}$, matrices $E_i$ and $A_{d22}$ cannot have common eigenvalues (Luenberger (1971)).

Based on the above observations, a new approach for solving equations (20), (22) and (24) is proposed. Instead of trying to solve for matrices $K_i$, $T_{i1}$ and $T_{i2}$ that can satisfy equations (20) and (24) exactly (hence the control law (3b) exactly), our approach here is to solve for matrices $K_i$, $T_{i1}$ and $T_{i2}$, which will produce control feedback signals as close as possible to the given control law (3b) by minimising the following matrix norm:

\begin{equation}
\delta = \| F_{i2} - K_i T_{i2} \| + \| T_{i1} A_{d12} + T_{i2} A_{d22} \|. 
\end{equation}

Remark 3: For the above minimisation, matrix $K_i$ may be chosen according to

\begin{equation}
K_i = F_{i2} T_{i2}^+, 
\end{equation}

where $T_{i2}^+$ is the Moore-Penrose pseudo-inverse of $T_{i2}$.

Examination of equations (20), (22) and (24) reveals that this minimisation problem may be solved by given $E_i$ and searching for matrix $T_{i1}$ such that the solution to the Lyapunov equation (22), i.e. $T_{i2}$, will minimise (28). In order to find $T_{i1}$ such that matrices $K_i T_{i2}$ and $(T_{i1} A_{d12} + T_{i2} A_{d22})$ are as close as possible respectively to $F_{i2}$ and 0, a parameter optimisation technique will be used. All of the elements of $T_{i1}$ are now considered as optimisation parameters of the following optimisation problem:

\begin{align}
\text{Minimise } &\delta(T_{i1}) = \| F_{i2} - F_{i2} T_{i2}^+ T_{i2} \| + \| T_{i1} A_{d12} + T_{i2} A_{d22} \| \\
\text{subject to } &E_i T_{i2} - T_{i1} A_{d12} - T_{i2} A_{d22} = 0.
\end{align}

The above optimisation problem can be solved using MATLAB Optimisation Toolbox. Consequently, matrices $K_i$, $T_{i1}$ and $T_{i2}$ can be obtained and hence matrices $W_i$, $G_i$ and $M_i$ can be computed from equations (25), (26) and (27), respectively.
In the following, a stability condition will be derived for the closed-loop time-delay system using the proposed observer-based feedback.

**Lemma 1**
Consider a linear time-delay system
\[ \dot{w}(t) = Jw(t) + Lw(t-\tau) + \Delta Jw(t) + \Delta Lw(t-\tau), \] (31)
where \( w(t) \) is the state vector, matrices \( J, L \) are known, and \( \Delta J, \Delta L \) are constant but unknown. Let the nominal system (i.e. \( \Delta J = 0 \) and \( \Delta L = 0 \)) be stable, then the perturbed system (31) is asymptotically stable and independent of delay, if the following condition is satisfied:
\[ \|\Delta J\| + \|\Delta L\| < \alpha = \frac{1}{\|(sI - J - e^{-s\tau} L)\|_\infty}, \] (32)
where \( I \) is the unity matrix and \( \|\cdot\|_\infty \) denotes the \( H_\infty \) norm.

**Proposition 1**
Consider the time-delay system (3a) that is asymptotically stable under the feedback control law (3b). Let matrices \( E_i \), of dimension \( p \), where \( p \) is chosen in the interval \( 1 \leq p < (n-\tau) \), be selected to have desired eigenvalues that are different from those of \( A_{22} \). If for any \( i = 1, 2, \ldots, m \), there exist matrices \( T_i \), such that the following condition is satisfied
\[ \mu = \sum_{i=1}^{m} \mu_i < \alpha = \frac{1}{\|(sI - J - e^{-s\tau} L)\|_\infty}, \] (33a)
where
\[ \mu_i = \|B_2(F_i T_{i2}^* T_{i1} - F_{i1})\| + \|T_{i1} A_{d12} + T_{i2} A_{d22}\|, \] (33b)
then systems (7) can be used as low-order linear functional observers to generate as close as possible the control law (3b) and the closed-loop system remains asymptotically stable.

**Proof:**
The development of linear functional observers (7) has been presented above. For any \( i = 1, 2, \ldots, m \), let us first select matrices \( E_i \), of dimension \( p \), where \( p \) can be chosen from the lowest order in the interval \( 1 \leq p < (n-\tau) \), according to the desired dynamics for the observers and Remark 2. Matrices \( T_i \) can be derived from solving the parameter optimisation problem (30). Matrices \( K_i, W_i, G_i \) and \( M_i \) are then determined from \( T_i \) and \( T_{i1} \) in accordance with (29), (25), (26) and (27), respectively. As equations (20) and (24) are not exactly solved, stability of the closed-loop time-delay system using the control signals (5) that are generated by the proposed functional observers may not be guaranteed. Taking this into account, let us define the perturbations in the observer dynamics due to the inexact solutions to equations (20) and (24) as follow:
\[ \Delta M_i = M_i C - T_i A_{d} = 0 - (T_{i1} A_{d12} + T_{i2} A_{d22}); i = 1, 2, \ldots, m. \] (34)
The error dynamic equation (13) thus becomes:
\[ \dot{e}_i(t) = E_i e_i(t) + \Delta M_i x(t-\tau). \] (35)
In addition, by using the proposed functional observer (7) with relevant matrices obtained from the optimisation process (30) the feedback control law (3b) can be rewritten in the form
\[ u_i(t) = \left[ F_{i1} \quad K_i T_{i2} \right] x(t) = (F_i + \Delta F_i) x(t); i = 1, 2, \ldots, m \] (36)
where
\[ \Delta F_i = \left[ K_i T_{i2} - F_{i2} \right] = \left[ 0 \quad F_{i2} T_{i2}^* T_{i1} - F_{i1} \right] \] (37)
is the difference between the new feedback gain \( \left[ F_{i1} \quad K_i T_{i2} \right] \) and the original feedback gain \( \left[ F_i \quad F_{i2} \right] \), which has been minimised in the optimisation process. Equations (36) and (35) can also be put in a compact form as:
\[ u(t) = \left[ F_{11} \quad K T_{2} \right] x(t) = (F + \Delta F) x(t), \] (38)
and
\[ \dot{e}(t) = E e(t) + \Delta M x(t-\tau), \] (39)
where the error vector is defined as \( e(t) = [e_1(t) \ldots e_m(t)]^T \), and the matrices are given by
\[ K = diag(K_i) \in R^{m \times p}; \quad i = 1, 2, \ldots, m, \quad F_i = \left[ F_{i1} \quad \ldots \quad F_{im} \right], \]
\[ T_2 = \begin{bmatrix} T_{12} \\ T_{22} \end{bmatrix}, \quad \Delta F = \begin{bmatrix} \Delta F_1 \\ \Delta F_m \end{bmatrix}, \]
\[ E = diag(E_i) \in R^{mp}; \quad i = 1, 2, \ldots, m, \quad \text{and} \quad \Delta M = \begin{bmatrix} \Delta M_1 \\ \Delta M_m \end{bmatrix}. \] (40)
By substituting equation (38) into equation (1a) and using equation (39), the augmented closed-loop system of the form (31) can be derived, where
\[ w(t) = \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}, \quad J = \begin{bmatrix} A + B F \quad 0 \\ 0 \quad E \end{bmatrix}, \quad L = \begin{bmatrix} A_{d} \\ 0 \end{bmatrix}, \]
\[ \Delta J = \begin{bmatrix} B \Delta F \quad 0 \\ 0 \quad 0 \end{bmatrix}, \quad \text{and} \quad \Delta L = \begin{bmatrix} \Delta M \quad 0 \end{bmatrix}. \] (41)
The nominal part, i.e., \( \dot{x}(t) = Jw(t) + Lw(t - \tau) \), is stable and its eigenvalues are inherently the union of the desired eigenvalues of the control system and of the observer, i.e., the roots of the following equation

\[
\det(sI - (A + BF) - e^{-\tau}A_d) \det(sI_{up} - E) = 0.
\] (42)

The perturbations \( \Delta J \) and \( \Delta L \) in (41) are resulted from the inexact solutions to equations (20) and (24). Note that they have been minimised in the optimisation process (30). Lemma 1 is now applied to derive stability conditions for the combined control and observer system (31). A close examination of the expressions of \( \Delta J \) and \( \Delta L \) reveals that

\[
\| \Delta J \| = \| B\Delta F \| = \sum_{i=1}^{m} B_i \Delta F_i,
\]

and

\[
\| \Delta L \| = \| \Delta M \| = \left[ \begin{array}{c} \Delta M_1 \\ \vdots \\ \Delta M_m \end{array} \right] \leq \sum_{i=1}^{m} \left| T_{i2} A_d t_{i2} + T_{i2} A_d t_{i2} \right|.
\] (43)

Thus, if

\[
\sum_{i=1}^{m} \left( \left| B_i (F_i T_{i2} t_{i2} - F_{i2}) \right| + \left| T_{i1} A_d t_{i1} + T_{i2} A_d t_{i2} \right| \right) < \alpha,
\]

then \( \| \Delta J \| + \| \Delta L \| < \alpha \). According to Lemma 1, the augmented closed-loop system using the proposed linear functional observer is asymptotically stable.

Based on the above development, a design algorithm is given in the following steps:

**Design Algorithm**

1. Determine a suitable state feedback gain matrix \( F \) by using any existing control technique. Partition matrices \( B_i, F_i \) \( (i = 1, 2, \ldots, m) \), \( A \) and \( A_d \) according to equations (14)-(17), respectively.

2. Compute \( \alpha \), where

\[
\alpha = \frac{1}{\|{(sI - J - e^{-\tau}L)}^2\|_{\infty}}.
\]

A method given by Francis (1987) can be used for this step. Set \( j = 0 \).

3. Set the order of the observer (7) as \( p = 1 + j \).

4. For \( i = 1, 2, \ldots, m \), select \( E_i \) in accordance with Remark 2, solve the optimisation problem (30) for \( T_i \), and then derive \( K_i \) according to (29).

5. Check the condition (33), if satisfied, go to step 6; else set \( j = j + 1 \) and go to step 3.

6. Compute matrices \( W_i, G_i \) and \( M_i \) \( (i = 1, 2, \ldots, m) \) respectively from equations (25), (26) and (27).

**Remark 4:** In the above design algorithm, a lowest order \( (p = 1) \) is first assigned for the observer (7). The search for a control law that is as close as possible to the desired feedback law is obtained upon the satisfaction of the stability condition for the overall observer-based system. Otherwise, the observer order can be gradually increased until this condition is met. The procedure is therefore expected to result in a linear functional observer of a low order.

**Remark 5:** In our algorithm, no restriction is imposed on the number of the outputs and hence, on the order of the observer. Reconstruction of the feedback signal may therefore require a small number of outputs and a low order observer may be resulted. This advantage makes the proposed observer-based control scheme feasible and efficient in terms of engineering implementation of controllers for time-delay systems and also for large-scale systems.

**Remark 6:** Note that the proposed control scheme may still be stable without satisfying (33) as it is a sufficient condition. This conservativeness may be relaxed by reformulating the optimisation problem (30) for solving matrices \( T_i \) in one go. In this paper the problem has been broken down into \( m \) sub-optimisation problems for the \( m \) control signals to significantly reduce computational burden required for the parameter optimisation process thanks to a low order of the observer (7) and a small number of involved parameters. The design procedure results in \( m \) functional observers of order \( p \) or equivalently in one \( mp \)-order functional observer.

**Proposition 2**

For the system described in Proposition 1, if the matrix \( A_d \) satisfies the following matching condition:

\[
A_d = PC.
\] (44)

where \( P \in \mathbb{R}^{mc} \), then the optimisation problem (30) can be reduced to

\[
\begin{align*}
\text{Minimise} & \quad \delta(T_{i1}) = \| F_{i2} - F_{i2} T_{i2} T_{i2} \| \\
\text{subject to} & \quad E_i T_{i2} - T_{i1} A_d - T_{i2} A_d = 0,
\end{align*}
\] (45)

and the stability condition (33) becomes

\[
\mu = \sum_{i=1}^{m} \| B_{i2} (F_{i2} T_{i2} T_{i2} - F_{i2}) \| < \alpha = \frac{1}{\|{(sI_{up} - J - e^{-\tau}L)}^2\|_{\infty}}.
\] (46)
Proof:
Under the matching condition (44), equation (12) becomes \( M, C - T, A_d = M, C - T, P, C = 0 \). It can be seen that (12) can be exactly solved to give \( M, T, P \). Matrix \( P \in R^{n \times m} \) can be partitioned as \( P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \), where \( P_1 \in R^{m \times r} \) and \( P_2 \in R^{m \times m} \). Condition (44) is then equivalent to
\[
\begin{align*}
A_{d11} & = P_1 C_1, \\
A_{d21} & = P_2 C_1, \\
A_{d12} & = A_{d22} = 0,
\end{align*}
\]
where \( P_1 \) and \( P_2 \) can be derived as
\[
\begin{align*}
P_1 & = A_{d11} C_1^{-1}, \\
P_2 & = A_{d21} C_1^{-1}.
\end{align*}
\]
From results of Proposition 1 and equations (47) one can easily derive the optimisation problem (45) and the stability condition (46).

Remark 7: Non-delay \( (A_d = 0) \) large-scale systems can be considered as a special case of Proposition 2, where \( P = 0 \) and condition (46) is rewritten as
\[
\mu = \sum_{n=1}^{\infty} \left\| \frac{B_2 (F_{21}^T T_{12} - T_{12})}{A_n} \right\|_\infty < \alpha = \\
\frac{1}{A_1},
\]
(49)

4 Numerical examples

Let us consider first a time-delay system (1) with \( n=5 \) and \( m=1, \) where \( \tau = 2 \text{ sec} \) and \( \Phi(t)=0, \tau \leq t \leq 0, \) and where
\[
A = \begin{bmatrix}
-2 & 0 & 1 & 0 & 0 \\
1 & -3 & 3 & 3 & 4 \\
-4 & 5 & -6 & 5 & 6 \\
-1 & -2 & 3 & 1 & 0 \\
0 & -3 & 3 & 0 & 1
\end{bmatrix},
\]
and
\[
B = \begin{bmatrix}
0 & 2 \\
0 & -1.4 \\
0.6 & 0.6 & -0.2 \\
0 & -0.2 & 0.6 & 0.6 \\
0 & 0.2 & -0.6 & 0 & 0
\end{bmatrix},
\]
(50)
Using the approach proposed by Trinh (1999) this system may require at least \( r=3 \) measured outputs to reconstruct a linear feedback control for the system with a functional observer of an order \( p \geq 2. \) Let us consider the cases where the condition \( r > 0.5n \) is violated, using the approach proposed in this paper. For example, assume that there are now two outputs available for feedback \( (r=2). \) Let matrix \( C \) be given as \( C = [I_2, 0]. \)

First, a full state feedback control law \( u(t) = Fx(t) \) can be designed, e.g. by using the LQR technique, with the feedback gain computed as
\[
F = [-0.3242 -0.1974 -0.1091 -0.1621 -0.2811],
\]
from which \( F_1 = [-0.3242 -0.1974] \) and \( F_2 = [-0.1091 -0.1621 -0.2811] \) are obtained by partitioning. The closed-loop system eigenvalues can be obtained by solving the characteristic equation, equation (48), as \( \lambda = [-0.3632 \pm 1.3918j, -0.4193 \pm 0.8848j, -0.5001 \pm 4.7461j, -0.6903 \pm 4.1567j, -0.9110 \pm 1.8922j, -0.9830 \pm 3.7666j, \ldots], \)
In Step 2, the bound \( \alpha \) in (33) is found to be \( \alpha = 0.2414. \) Let us first try \( p=1 \) in Step 3. Let \( E = -5 \) and initial parameters for \( T_i \) as \([1,1]\). By solving the optimisation problem (30), we obtain \( T_i = [0.2084 -0.0546]^T, T_2 = [0.1417 0.2327 0.3339]^T \) \( 0^\text{th} \), and \( K=7918.3 \) (Step 4). After checking condition (33) is satisfied, \( \mu = 0.0626<\alpha \) (Step 5), one can complete the design procedure (Step 6) by computing matrices \( W, G, M, \) and \( M = [0.4466 0.0569]^T. \)

In summary, the control law \( u(t) = Fx(t) \) with \( F \) given in (51) can be reconstructed by an observer-based signal \( u(t) = K_1 z(t) + M y(t), \) where the output vector is \( y(t) = \left[x_1(t) x_2(t)\right] \), and the observer is of first-order dynamics given by \( \dot{z} = E z + T B u + G y(t) + M y(t - \tau). \)
Choosing a zero initial condition for the observer \( h(t=0, \tau \leq t \leq 0), \) Figure 1 shows the simulation responses of the states \( x_1(t) \) and \( x_2(t) \) when a unit step is applied as the reference to \( x_2(t). \) The results indicate that the required control action can be reconstructed using the proposed observer-based scheme to suppress oscillations due to time-delay.

In the following, we show that our procedure can be applied even for the case where only one output is available \( (r=1), \) e.g. \( C = [I_2 0 0 0 0]. \) It is expected in this case that the order of the functional observer be higher in order to meet the stability condition (33). Indeed, after some unsuccessful trials with \( p=1 \) and \( p=2, \) a third-order functional observer can be obtained with a desired eigenstructure chosen as \([-5,-10,-20]) \), ie. \( E = [-5 0 0 0 -10 0 0 0 -20]. \) Using again the feedback gain (51), the bound in (33) remains at \( \alpha = 0.2414. \) Selecting \([1; 1; 1]\) as the initial parameters for the optimisation process (30) we obtain
\[
T_i = \begin{bmatrix}
-0.3194 \\
-0.2043 \\
-0.7726
\end{bmatrix},
\]
(51)
reconstructed from the new third-order functional observer as the same feedback gain is used for this case.

To illustrate Remark 7, let us now consider a non-delay system comprising of \( n=12 \) states, \( m=3 \) inputs and \( r=6 \) outputs where matrices \( A, B \) and \( C \) are given as below:

\[
A = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \]

\[
B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} I_6 \\ 0 \end{bmatrix} \]

(52)

The open-loop system is unstable with the following eigenvalues: \( \text{eig}(A) = \{-18.8711, -17.0671, -15.1703, 0.1041 \pm 7.8402j, -0.1004 \pm 7.3661j, 0.2956 \pm 4.1155j, -5.9636, -3.3693, -1.5101\} \). For illustrative purpose, matrix \( F \) of the control law \( u(t) = F x(t) \) can be obtained, e.g. from the LQR technique, as

\[
F = \begin{bmatrix} -1.4655 \\ -3.013 \\ 0.104 \\ 0.249 \\ -0.319 \\ 0.379 \end{bmatrix} \]

(53)

The eigenvalues of the closed-loop system are found to be:

\[
\text{eig}(A+BF) = \{-18.8675, -17.0467, -15.0937, -2.3751 \pm 8.0309j, -2.9618 \pm 8.0924j, -3.9593 \pm 5.7773j, -8.0572, -3.0671, -5.3806\} \]

To implement a full state feedback for this system it is required to estimate six inaccessible states or three control inputs. In the following, the above design procedure will be used to derive a low-order observer to reconstruct the control law with the feedback gain given in (53). Note that for state estimation this large-scale system would require a well-known Luenberger’s full order and reduced order observer of a 12th order and a 6th order, respectively. Based on the novel design approach presented in this paper, only a third-order observer will be required for this system.

Indeed, let us start with Step 1 where we will partition \( B, F, A \) and \( A_f \) according to (14)-(17), for example,

\[
B_{12} = \begin{bmatrix} 0; 800; 0; 0; 0 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 0; 0; 900; 0; 0 \end{bmatrix}, \quad B_{32} = \begin{bmatrix} 0; 0; 0; 0; 1000 \end{bmatrix}.
\]

\[ A = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \]

\[ B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} I_6 \\ 0 \end{bmatrix} \]
\[
\begin{bmatrix}
\varepsilon_1
\varepsilon_2
\end{bmatrix} = \begin{bmatrix}
-1.4655 & 46.0533 & 0.3209 & 38.9816 & 0.9681 & 84.4072 \\
-0.2244 & -0.0098 & -0.0281 & -0.0009 & -0.0327 & -0.0009 \\
0.9605 & -11.8578 & -2.3589 & 31.0111 & 1.3179 & 51.5465 \\
0.3720 & -13.9760 & 0.1815 & -8.9625 & -0.5381 & 142.7056
\end{bmatrix},
\]

In Step 2, the stability bound is computed using (47). This gives \( \alpha = 24.1 \). Let us first set \( i = 1 \) (Step 3). In Step 4, by selecting \(-3\) as desired eigenvalues for the observer, \( E_i = -3 \), one can solve the optimisation process (30) to obtain for \( i = 1, 2, 3 \), respectively

\[
T_1 = \begin{bmatrix}
0.0475 \\
0.0075 \\
0.8310 \\
0.0048
\end{bmatrix},
T_2 = \begin{bmatrix}
0.7666 \\
-0.0004 \\
0.5597 \\
0.0003
\end{bmatrix},
T_3 = \begin{bmatrix}
0.3282 \\
-0.0001 \\
0.7458 \\
0.0013
\end{bmatrix},
\]

and \( K_1 = 29.8116 \), \( K_2 = 29.8116 \), \( K_3 = 3.5211 \).

According to (49), \( \mu_i = ||B_i(K_iT_i - F_i)|| \), we obtain \( \mu_1 = 2.96 \), \( \mu_2 = 2.25 \), and \( \mu_3 = 4.3 \). One can easily verify that \( \mu = \mu_1 + \mu_2 + \mu_3 = 9.51 < \alpha \) (Step 5). The final design step is to compute matrices \( W_i \) and \( G_i \) for \( i = 1, 2, 3 \).

According to (25) and (26) we obtain respectively

\[
W_1 = \begin{bmatrix}
-2.8805 & 23.1994 \\
-0.0470 & 20.3170 \\
35.3201 & 311.6038
\end{bmatrix},
G_1 = \begin{bmatrix}
-9.4643 & 38.7363 \\
-0.0004 & 0.9991 \\
-14.6285 & 0.0481
\end{bmatrix},
W_2 = \begin{bmatrix}
-2.5718 \\
3.8797 \\
2.7073
\end{bmatrix},
G_2 = \begin{bmatrix}
-21.5816 \\
61.8476 \\
30.2655
\end{bmatrix},
W_3 = \begin{bmatrix}
-38.7363 \\
22.2850 \\
299.2655
\end{bmatrix},
G_3 = \begin{bmatrix}
-60.3032 \\
288.3924 \\
239.3008
\end{bmatrix},
\]

According to (31), \( X(t) = A^iX(t_i) + Bu(t) + Gy(t) \), where \( K = diag(K_i) \), \( E = diag(E_i); i = 1, 2, 3, \)

\[
T = \begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22} \\
T_{31} & T_{32}
\end{bmatrix},
W = \begin{bmatrix}
W_1 \\
W_2 \\
W_3
\end{bmatrix},
G = \begin{bmatrix}
G_1 \\
G_2 \\
G_3
\end{bmatrix},
\]

One can also verify the stability of the proposed control scheme by calculating the eigenvalues of the augmented system shown in (31), which is rewritten in this case as

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{e}(t)
\end{bmatrix} = \begin{bmatrix}
A + BF + AF & 0 \\
0 & E
\end{bmatrix} \begin{bmatrix}
x(t) \\
e(t)
\end{bmatrix}.
\]

The eigenvalues obtained are the union of the eigenvalues of the system with the new feedback gain and of matrix \( E \): 

\[
\{-24.4244, -21.7510, -21.0140, -2.5191 \pm 8.1544j, -2.1924 \pm 8.0127j, -3.0655 \pm 6.9958j, -5.6373, -4.1751, -2.7231, -3, -3, -3, -3, -3\}.
\]

Figure 2 shows typically the simulation responses of the states \( x_1(t) \) and \( x_2(t) \), when a step external excitation is applied to the system.

In summary, the obtained first-order observers for reconstruction of three control inputs result in a third-order functional observer for the whole system. The observer-based control law takes the form \( u(t) = Kz(t) - 1 - Wy(t) \) with the observer dynamics determined by

\[
\begin{align*}
\dot{z}(t) &= Ez(t) + TBu(t) + Gy(t), \\
E &= diag(E_i); i = 1, 2, 3,
\end{align*}
\]

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e(t)
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\]

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\[
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0 & E
\end{bmatrix} \begin{bmatrix}
x(t) \\
e(t)
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\[
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\]
Assuming that a feedback controller can be designed to achieve stability or some control performance objectives, a parameter optimisation process is then involved in obtaining a feedback gain which is as close as possible to that of the required feedback controller. The design procedure begins with a lowest order and gradually increases the observer order until satisfying a stability test. The resulted functional observer has a remarkably low order and can be applied even when the number of available outputs is limited. The design procedure is illustrated by numerical examples.

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References


