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This paper describes an approach to pointwise construction of general aggregation operators, based on monotone Lipschitz approximation. The aggregation operators are constructed from a set of desired values at certain points, or from empirically collected data. It establishes tight upper and lower bounds on Lipschitz aggregation operators with a number of different properties, as well as the optimal aggregation operator, consistent with the given values. We consider conjunctive, disjunctive and idempotent n-ary aggregation operators; p-stable aggregation operators; various choices of the neutral element and annihilator; diagonal, opposite diagonal and marginal sections; bipolar and double aggregation operators. In all cases we provide either explicit formulas or deterministic numerical procedures to determine the bounds. The findings of this paper are useful for construction of aggregation operators with specified properties, especially using interpolation schemata.

Keywords: Aggregation operators, monotone interpolation, 1-Lipschitz aggregation, quasi-copulas.

1. Introduction

Aggregation operators with certain a priori known properties are often required for decision support and other systems that use fuzzy logic. There exist a large number of families of aggregation operators, with a wide range of properties, and the choice of the operator suitable for a particular system is not simple. Overviews of the most important families of aggregation operators can be found in [1].
Construction methods typically depart from the existing classes of operators, or from certain theoretical properties (such as functional equations [1]). This paper considers construction based on the desired values of the operator at certain points (or subsets). This approach has recently attracted attention of a number of researchers. For instance, in [2, 3] the authors consider construction of aggregation operators based on their marginal values and diagonal sections. M. Grabisch [4] proposed piecewise linear interpolatory method for unipolar and bipolar operators, which recovers Choquet and Sugeno integrals. The idea is to take the values of an aggregation operator at certain meaningful points, and determine the aggregation operator on the whole of its domain by interpolation.

A closely related problem is that of identification of the most suitable aggregation operator from empirical data, or fitting operators to data. This study was pioneered by Zimmermann and Zysno [5]. Here the data are collected in an experiment, by questioning the experts (or lay people) about suitable membership values of objects in a number of fuzzy sets. This approach was continued in [6, 7] where the authors identified the weights of OWA operators from empirical data, and by the present author in [8, 9]. Identification of generated aggregation operators was treated in [10], and identification of fuzzy measure in Choquet integral-based operators was discussed in [11, 12]. In statistical literature an issue of identifying copulas (which can be viewed as special cases of aggregation operators) was dealt with in [13], and recently in [14].

Both tasks, construction of aggregation operators from the desired values, and identification of operators from empirical data, are essentially the same, and will be treated here in the same framework. One distinction is that empirical data are usually noisy and need smoothing, whereas the desired values are typically exact, although it may happen that they are incompatible with other desired properties of the operator, and hence also need adjustment.

Thus we consider the following problem. Given a set (possibly uncountable) of values of an aggregation operator \( f \), construct (identify, approximate) \( f \), subject to a number of properties, which we discuss in the sequel. We will apply pointwise construction method, which results in an algorithm whose output is a value of the aggregation operator \( f \) at a given point \( x \in [0, 1]^n \).

In this paper we will treat general \( n \)-ary aggregation operators, which are monotone functions \( f : [0, 1]^n \rightarrow [0, 1] \), satisfying boundary conditions \( f(0) = 0, f(1) = 1 \). We will first provide a general framework and generic formulae for pointwise construction of these aggregation operators, and then will consider aggregation operators with specific properties and will treat a number of special cases, including popular 1-Lipschitz and kernel aggregation operators, and quasi-copulas.

Lipschitz-continuous aggregation operators are of our particular interest. As noted in [1, 15, 16] Lipschitz aggregation operators are very important for applications, as small errors in the input do not drastically affect the behavior of the system. The concept of \( p \)-stable aggregation operators was proposed in [15]. These are precisely Lipschitz continuous operators whose Lipschitz constant \( M \) in \( l_p \) norm
is one. We will not restrict the Lipschitz constant of the aggregation operator (besides an obvious restriction $M \geq n^{-1/p}$), and later obtain $p$-stable operators as special cases.

The key part of the approach used in this study is a monotone Lipschitz interpolation technique based on the central algorithm. This method delivers an optimal interpolant to multivariate monotone scattered data by minimizing the worst approximation error. This technique allows one to construct not just one but all possible aggregation operators with given properties that fit the data. This family turns out to be a compact convex set. We will identify the boundary elements of this set and its center. The central element of this set is the optimal operator, the one which minimizes the approximation error in the worst case scenario.

In many applications, besides the set of numerical values to be fitted, there are other requirements which restrict the choice of aggregation operators. For example, R. Yager [17] uses the requirement of noble reinforcement (in the context of recommender systems) to build disjunctive aggregation operators, which are bounded by maximum from above for low values of the arguments. The presence of a neutral element also tightens the bounds for conjunctive disjunctive and mixed aggregation operators.

Our goal is to develop such improved bounds explicitly, in a number of prototypical cases. We concentrate on the following requirements.

- Conjunctive and disjunctive behavior;
- Idempotency;
- Symmetry;
- Neutral element and annihilator;
- Given marginals;
- Given diagonal and opposite diagonal;
- Various combinations of the above.

The next section presents the method of pointwise construction of aggregation operators and provides mathematical formulation of the problem. It discusses the method of optimal monotone interpolation and the optimal central algorithm and presents the key equations that will be the basis for our construction algorithm. Section 3 instantiates the problem of monotone Lipschitz interpolation for aggregation operators, discusses some of their properties and presents $p$-stable aggregation operators. Section 4 discusses construction of conjunctive, disjunctive and idempotent aggregation operators using their bounds. Section 5 establishes tight bounds when the aggregation operator possesses a neutral element or annihilator. In Section 6 we discuss the bounds on aggregation operators with a given diagonal or opposite diagonal sections, and in Section 7 we establish the bounds resulting from marginal sections. We also discuss compatibility of given marginals with the Lipschitz constant of the aggregation operator. In Section 8 we treat bipolar aggregation operators, in which positive and negative values of the arguments are treated dif-
ferently. Finally section 9 illustrates our methods on some numerical examples. The last section provides a summary of all cases.

2. Pointwise construction of aggregation operators

2.1. Problem formulation

In the development of fuzzy systems one is frequently faced with the task of selecting an aggregation operator most suitable for a particular problem. Application-specific properties of the aggregation procedure do not recover a single operator, but an infinite family, for instance a family of triangular norms or means. Evidently, different members of such a family will result in different behavior of the system, which may or may not be acceptable for the user.

Frequently the user can provide the desired values of the aggregation procedure at certain key points. For instance, the decision maker may require certain values at the characteristic vectors that correspond to the vertices of the hypercube \( I^n \):= \([0,1]^n\), (or even a larger set of key points in the case of bipolar aggregation, see discussion in [4]). Further, such values may be specified at subsets of \( I^n \), such as diagonals, opposite diagonals and marginals [2, 3]. The existence of the neutral element or annihilator also translates into specified values. Then the task is to choose an aggregation operator from a given family, consistent with the desired values.

One approach is to fix the class of aggregation operators, and fit its parameters. For example, if the desired class is OWA operators (or more generally Choquet integral-based operators), the goal is to fit the OWA weights (or the coefficients of the fuzzy measure), and obtain the aggregation operator explicitly. This approach was studied in [6–9,11,18].

The pointwise construction presented here does not identify the algebraic form of the aggregation operator (at least it does not fix the form a priori), but provides an algorithm capable of computing the values of \( f(x) \) at any \( x \in I^n \), such that \( f \) satisfies all the desired conditions. For application purposes this is as good as having an algebraic form. From the point of view of modelling, this delivers much greater flexibility than fixing the algebraic form. Furthermore, for very general classes of aggregation operators, the presented approach is capable to identify not one, but the whole family of aggregation operators consistent with the data and given conditions. For each \( x \in I^n \), the algorithm returns tight upper and lower bounds on \( f \) from a given family consistent with the data. The optimal value, which maximizes the accuracy of this construction, is the center of the interval.

We now proceed to the mathematical formulation of the problem. Suppose we have a data set \( D = \{(x^k, y^k)\}_{k=1}^K \subseteq I^n \times I \), \( y^k = f(x^k) \), generated by some unknown aggregation operator \( f \in \mathcal{F} \) that we want to model. We are looking for an aggregation operator \( g \approx f \), consistent with \( f \) at given data and sharing known properties of \( f \), i.e. \( g \in \mathcal{F} \). Except some special cases, there would be many aggregation operators from the same class \( \mathcal{F} \) interpolating the data, and the available information does not allow us to distinguish between them. Our goal is to find the
best representative of this class, such that the error
\[
\max_{f \in F} \max_{x \in I} |f(x) - g(x)|
\]
is as small as possible. This is the error in the worst case scenario. Thus we solve the interpolation problem

**Problem 1**

\[
\min_{g \in F} \max_{f \in F} \max_{x \in I} |f(x) - g(x)|
\]
\[
\text{s.t. } g(x_k) = f(x_k), k = 1, \ldots, K.
\]

2.2. Optimal interpolation

We will outline the basics of the method of optimal monotone Lipschitz approximation developed in [19, 20]. Consider the space of continuous functions on \( X \subset \mathbb{R}^n \), \( C(X) \), with the supremum norm. A function \( f : X \to R \) is called Lipschitz on \( X \) if \( \exists M \geq 0 \) such that \( \forall x, z \in X \), \( |f(x) - f(z)| \leq M||x - z|| \).

We call the smallest such number \( M \), the Lipschitz constant of \( f \) in the given norm \( || \cdot || \).

We denote the class of functions with the Lipschitz constant smaller than or equal to \( M \) by \( Lip(M, || \cdot ||) \), or \( Lip(M) \). We will subsequently assume that \( X \) is compact. The Lipschitz seminorm is defined by

\[
\forall f \in C(X) : \rho(f) = \inf\{M : |f(x) - f(z)| \leq M||x - z||, \forall x, z \in X\}
\]

We set \( \rho(f) = \infty \) for functions that are not Lipschitz.

We use the Lipschitz seminorm to define a nonlinear restriction on \( C(X) \) by means of the condition \( \rho(f) \leq M \), in the framework of Golomb – Weinberger theory [21]. In this case the set of possible values of \( f(x) \) is a closed interval \([\sigma_l(x), \sigma_u(x)]\), and its midpoint yields the optimal approximation to \( f(x) \) at any \( x \in X \). We can construct the interpolant explicitly by recovering the upper and lower bounds on \( f(x) \), \( \sigma_l(x) \leq f(x) \leq \sigma_u(x) \).

Suppose we have an algorithm \( A \), which allows us compute an approximation to \( f(x) \). Such an algorithm will produce an error \( E(A) \), which is no smaller than the intrinsic error of the problem, \( E_{int} = \inf_A E(A) \). The problem of optimal interpolation is to determine an optimal algorithm whose error is precisely \( E_{int} \). Using the bounds on \( f \), an optimal approximation to \( f(x) \) is given by

\[
g(x) = \frac{1}{2} (\sigma_l(x) + \sigma_u(x)).
\]

(1)

Such an interpolation scheme is called the central scheme [22–24]. It provides the smallest possible error in the worst case scenario, i.e., solves Problem 1. The error of approximation is (see [23, 24])

\[
E_{int} = \max_{f \in Lip(M)} \|f - g\|_{C(X)} = M \max_{x \in X} \min_{k=1,\ldots,K} ||x - x_k||.
\]
Interestingly, the error bounds cannot be improved by restricting \( \text{Lip}(M) \) by the assumptions of differentiability or analyticity of \( f \). This is due to the fact that \( C^{(m)} \) are dense in the space of continuous functions, and that we can always find \( h \in C^{(m)} \), which is Lipschitz, whose values are arbitrarily close to \( \sigma_l \) or \( \sigma_u \). In other words, \( \sigma_l \) and \( \sigma_u \) remain tight lower and upper bounds even if \( f \in \text{Lip}(M) \cap C^{(m)} \), and \( g \) in (1) is still the optimal interpolant, even though \( g \notin C^{(m)} \).

### 2.3. Monotone Lipschitz Interpolation

Let us now instantiate the optimal interpolation problem for aggregation operators. Denote by \( \text{Mon} \) the set of monotone nondecreasing functions on \( I^n \). An \( n \)-ary aggregation operator is a mapping \( f : I^n \to I \), monotone in all arguments \( x \leq y \Rightarrow f(x) \leq f(y) \), satisfying \( f(0) = 0, f(1) = 1 \). We will use subscripts to identify components of vectors in \( R^n \). Vector inequality \( x \leq y \) is understood componentwise \( \forall i \in \{1, \ldots, n\} : x_i \leq y_i \). We will use the terms increasing (decreasing) synonymously with non-decreasing (non-increasing), and will use the terms strictly increasing (strictly decreasing) otherwise.

Then the set of general Lipschitz \( n \)-ary aggregation operators can be characterized as

\[
A_{M,||·||} = \{ f \in \text{Lip}(M,||·||) \cap \text{Mon} : f(0) = 0, f(1) = 1 \}
\]

The problem of constructing an optimal general aggregation operator from the data set \( D \) is then

**Problem 2**

\[
\min_{g \in A_{M,||·||}} \max_{f \in A_{M,||·||}} \max_{x \in I^n} |f(x) - g(x)|
\]

s.t. \( g(x^k) = y^k, k = 1, \ldots, K \).

We assume that the data set \( D \) is consistent with the class \( A_{M,||·||} \), i.e. \( A_{M,||·||} \cap V_D \neq \emptyset \), where \( V_D \) denotes the set of all functions interpolating the data from \( D \). The following theorem establishes tight upper and lower bounds on monotone Lipschitz functions interpolating \( D \) [19]

**Theorem 1.** Let \( D \) be a data set compatible with the conditions \( f \in \text{Lip}(M) \cap \text{Mon} \). Then for any \( x \in X \), the values \( f(x) \) are bounded by \( \sigma_l(x) \leq f(x) \leq \sigma_u(x) \), with

\[
\sigma_u(x) = \min_k \{ y^k + M \| (x - x^k)_+ \| \},
\]

\[
\sigma_l(x) = \max_k \{ y^k - M \| (x^k - x)_+ \| \},
\]

(2)

where \( z_+ \) denotes the positive part of vector \( z \): \( z_+ = (z_1, \ldots, z_n) \), with

\[
\bar{z}_i = \max \{ z_i, 0 \}.
\]
Note that these bounds appear in a number of publications (e.g., [3, 13, 25, 26]) in the special cases of 1-Lipschitz aggregation operators. If the data set is infinite, \( D = \{(t, v(t)) : t \in \Omega \subset I^n, v : \Omega \to I\} \) then the bounds translate into

\[
\begin{align*}
\sigma_u(x) &= \inf_{t \in \Omega} \{v(t) + M||x - t|| + ||\}, \\
\sigma_l(x) &= \sup_{t \in \Omega} \{v(t) - M||(t - x)|| + ||\}.
\end{align*}
\] (3)

If the data set is interval-valued \( D = \{(t, [\nu(t), \tau(t)]) : t \in \Omega, \nu, \tau : \Omega \to I, \nu \leq \tau\} \), i.e., for each \( t \in \Omega \) we have \( \nu(t) \leq f(t) \leq \tau(t) \), the bounds are

\[
\begin{align*}
\sigma_u(x) &= \inf_{t \in \Omega} \{\tau(t) + M||x - t|| + ||\}, \\
\sigma_l(x) &= \sup_{t \in \Omega} \{\nu(t) - M||(t - x)|| + ||\}.
\end{align*}
\] (4)

The optimal interpolant \( g \) is given by (1).

Finally we need to ensure that the boundary conditions \( f(0) = 0, f(1) = 1 \) are satisfied. This is best done by adding the data \( \{(0, 0), (1, 1)\} \) to the data set \( D \), as formulae (1), (2) ensure that all the data are interpolated. However, in some cases we need to treat these conditions separately. This situation arises when the rest of the data set \( D \) is noisy. In this case, as we shall see in the sequel, one can smoothen the data set, to make it compatible with the desired Lipschitz conditions. However boundary conditions \( f(0) = 0, f(1) = 1 \) are not negotiable, and hence should be explicitly excluded from the smoothing process.

Equations (2) provide tight bounds on any general aggregation operator interpolating the data. We recover the whole set of values of possible aggregation operators, and take the central interpolant as the best representative of this set.

### 2.4. Smoothing and identification of Lipschitz constant

When the data comes from an experiment, it is not always possible to fit a function with given properties. The presence of noise means that the data set will no longer be compatible with our assumptions about \( f \), namely \( f \in Lip(M) \cap Mon \). We need to smoothen the data so that it becomes compatible with the required conditions.

The method of Lipschitz approximation from [19] works in two steps. First we construct the smoothened data set \( \hat{D} \), which is compatible with Lipschitz and monotonicity conditions, and then we use Eqns. (1) and (2) to define the approximation \( g(x) \). The data smoothing procedure in [19] is formulated in the framework of constrained least squares and least absolute deviation problems, which translates into standard quadratic and linear programming problems, solved by using proven techniques.

Let us denote the residuals by \( r_k = \hat{y}^k - y^k \). For the least squares approximation we obtain

\[
\begin{align*}
\min & \sum_{k=1}^K r_k^2, \\
\text{s.t.} & \quad r_k - r_j \leq y^j - y^k + M||(x^k - x^j)||^+,
\end{align*}
\] (5)
∀ j, k ∈ \{1, \ldots, K\}.

This is a quadratic programming problem, with a sparse matrix. There is a number of standard methods to solve such a problem [27].

In the case of the least absolute deviation smoothing we obtain

\[
\min \sum_{k=1}^{K} |r_k|,
\]

\[
s.t. \; r_k - r_j \leq y^j - y^k + M\|(x^k - x^j)\|, \forall j, k \in \{1, \ldots, K\}.
\]

We convert it to a linear programming problem by splitting \( r_k \) into positive and negative parts, \( r_k = r_k^+ - r_k^- \), \( r_k^+, r_k^- \geq 0 \), so that \( |r_k| = r_k^+ + r_k^- \). This problem is solved by the standard simplex method.

Until this point we assumed that the Lipschitz properties of \( f \) were specified a priori in the form of the Lipschitz constant \( M \). In some cases this information is available (e.g., \( p \)-stable aggregation operators), but often it is not, in which case the Lipschitz constant should be estimated from the data set \( D \). If the data is noiseless, \( M \) can be found by direct computation, solving

\[
\min M,
\]

\[
s.t. \; y^j - y^i \leq M\|(x^i - x^j)\|, \forall i, j \in \{1, \ldots, K\},
\]

\[M \geq 0.\]

For noisy data, this approach is not applicable, as it will result in overestimation of \( M \), and consequently in undesired interpolation of noisy data. The method developed in [19] estimates the Lipschitz constant by using two standard techniques of sample splitting and cross-validation.

3. Fitting aggregation operators

3.1. General approach

Equations (1),(2) and the methods of data smoothing outlined in the previous section are applicable to construction of general \( n \)-ary aggregation operators. General Lipschitz aggregation operators provide great flexibility and are capable to fit virtually any data. However, the broader is the class of functions to choose from, the wider are the error margins. If there are any other desired properties of the aggregation operator, they further restrict the class of allowable functions, and translate into tighter upper and lower bounds on the values \( f(x) \). These properties should be included into the construction process, as they reduce ambiguity and error margins. Our goal is to determine these tighter bounds

\[B_l(x) \leq f(x) \leq B_u(x),\]

and use them in conjunction with (2)

\[\Delta(x) = \max\{\sigma_l(x), B_l(x)\}, \; \Delta_u(x) = \min\{\sigma_u(x), B_u(x)\},\]

(8)
to deliver an optimal aggregation operator

\[ g(x) = \frac{1}{2}(A(x) + \overline{A}(x)). \]  

(9)

In this section we list a number of important properties of aggregation operators, that will be subsequently used to determine tighter bounds \( A(x), \overline{A}(x) \).

3.2. Properties of aggregation operators

Let \( t, e, a \in I \). \( e(t, i) \) will denote the vector whose components are all \( e \) except the \( i \)-th component: \( e(t, i) = (e, \ldots, e, t, e, \ldots, e) \). \( a(x, i) \) will denote the vector whose \( i \)-th component is \( a \):

\[ a(x, i) = (x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n). \]

• An aggregation operator \( f \) is said to have conjunctive behavior on \( I^n \) if \( e = 1 \), and \( \forall t \in I, i \in \{1, \ldots, n\} : f(e(t, i)) \leq t \). This implies (because of monotonicity) \( f(a(x, i)) = 0 \) with \( a = 0 \).

• Similarly, an aggregation operator has disjunctive behavior on \( I^n \) if \( e = 0 \), and \( \forall t \in I, i \in \{1, \ldots, n\} : f(e(t, i)) \geq t \). This implies \( f(a(x, i)) = 1 \) with \( a = 1 \).

• A binary aggregation operator has a neutral element \( e \in I \) if \( \forall t \in I, f(t, e) = f(e, t) = t \). Let us have a family of \( n \)-ary aggregation operators, \( f_n(x) n = 2, 3, \ldots \). Then this family has a neutral element \( e \) if for any \( n > 2 \)

\[ f_n(x_1, \ldots, x_{i-1}, e, x_{i+1}, \ldots, x_n) = f_{n-1}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \]

for \( e \) at any position within the vector \( x \). It is not difficult to check that applying this formula iteratively, one obtains the following

\[ \forall t \in I, i \in \{1, \ldots, n\} : f(e(t, i)) = t. \]  

(10)

• An aggregation operator has annihilator \( a \in I \) if \( \forall x \in I^n : f(a(x, i)) = a \) for all \( i \).

• An aggregation operator is idempotent if \( \forall t \in I : f(t, t, \ldots, t) = t \).

• An aggregation operator \( f \) is called 1-Lipschitz if its Lipschitz constant in \( l_1 \)-norm is one, i.e., \( f \in A_{1,||\cdot||_1} \).

• An aggregation operator is called a quasi-copula, if it is 1-Lipschitz and has neutral element \( e = 1 \).

• An aggregation operator \( f \) is called kernel, if its Lipschitz constant in \( l_{\infty} \)-norm is one, i.e., \( f \in A_{1,||\cdot||_\infty} \).

3.3. Stable aggregation operators

Lipschitz-continuous aggregation operators are very important for applications, because they provide output values stable with respect to small changes of the arguments. Small changes in the arguments may be due to inaccuracies in the data, and one would expect that such inaccuracies do not affect drastically the behavior
of the system. The concept of $p$-stable aggregation operators was proposed in [15]. These are precisely Lipschitz continuous operators whose Lipschitz constant $M$ in $l_p$ norm is one,

$$A_{\text{stable}} = A_{1,||\cdot||_p}.$$ 

Specific cases include 1-Lipschitz aggregation operators ($p = 1$) and kernel aggregation operators ($p = \infty$). Quasi-copulas arise as a special case of 1-Lipschitz operators, when the neutral element $e = 1$.

It is known (see [15]) that the weakest and the strongest $p$-stable operators are the Yager t-norm and t-conorm

$$T_Y(x) = \max\{0, 1 - ||1 - x||_p\},$$

$$S_Y(x) = \min\{1, ||x||_p\}.$$ 

For kernel aggregation operators we obtain

$$\min(x) \leq f(x) \leq \max(x), x \in I^n.$$ 

For 1-Lipschitz aggregation operators we have Lukasiewicz t-norm and t-conorm as the bounds

$$T_L(x) \leq f(x) \leq S_L(x).$$

Quasi-copulas are bounded by min and $T_L$, and the upper bound is a consequence of the presence of the neutral element $e = 1$, discussed later.

Copulas [13, 25] are defined using the property of $n$-increasingness, they constitute a proper subclass of quasi-copulas. In the bi-variate case, 2-increasingness implies monotonicity and 1-Lipschitz property. The bounds on bi-variate copulas are exactly the same as those on quasi-copulas, called Fréchet-Hoeffding bounds. However, our method of pointwise construction (9) is not applicable to copulas, because the class of copulas is not closed under pointwise minimum or maximum operation. Recent results on construction of bi-variate copulas are presented in [13, 14, 28].

It is not difficult to check that the above mentioned bounds are a direct consequence of the Eqs.(2), with data $(0, 0), (1, 1)$:

$$\sigma_u(x) = \min\{0 + ||(x - 0)||_p, 1 + ||(x - 1)||_p\} = \min\{||x||_p, 1\},$$

$$\sigma_l(x) = \max\{0 - ||(0 - x)||_p, 1 - ||(1 - x)||_p\} = \max\{0, 1 - ||1 - x||_p\}. \quad (11)$$

For an arbitrary $M \geq n^{-1/p}$ we have

$$\sigma_u(x) = \min\{M||x||_p, 1\},$$

$$\sigma_l(x) = \max\{0, 1 - M||1 - x||_p\}. \quad (12)$$

4. Conjunctive, disjunctive and idempotent operators

When choosing a particular aggregation operator for some application, it is important to identify the character of this operator and distinguish between conjunctive,
disjunctive, averaging and mixed behaviour. Yager also calls this pessimistic or optimistic character of the aggregation operator [17]. For example, in the context of recommender systems (systems used to recommend online customers products such as movies, music or books, based on identifying customer’s preferences), aggregation is optimistic (disjunctive), as matching any feature (justification) triggers a recommendation, and matching more features reinforces recommendation. Yager also calls it upward reinforcement. In other applications the situation is different: a small value of only one argument pulls the aggregated value down (downward reinforcement, or conjunctive behaviour). When small and large values of arguments are allowed to compensate each other, we talk about averaging behaviour. Yager [17] defines what he calls generalized OR, generalized AND and generalized MEAN operators (GENOR, GENAND and GENMEAN) based on a number of properties, among which the key property is the neutral element ($e = 0$ for GENOR, $e = 1$ for GENAND) and self-identity (for GENMEAN). It is easy to check that conjunctive aggregation operators (like GENAND) are bounded from above by the minimum, disjunctive operators (GENOR) are bounded from below by the maximum, and averaging operators (GENMEAN) are idempotent, and consequently are bounded by the minimum and maximum (see [1,29]).

We consider specific neutral elements in the next section, and now we concentrate on the weakest form of conjunctive, disjunctive and averaging behaviour. We have the following well known restrictions on $[0, 1]^n$.

- Conjunctive behavior implies $f \leq \min$.
- Disjunctive behavior implies $f \geq \max$.
- Idempotency implies $\min \leq f \leq \max$.

The above mentioned bounds, together with the data, immediately translate into the optimal aggregation operator given by (9), where $\mathcal{A}, \overline{\mathcal{A}}$ are given by

- Conjunctive operator
  $$\mathcal{A}(x) = \max\{\sigma_l(x), 1-M||1-x||, 0\}, \quad \overline{\mathcal{A}}(x) = \min\{\sigma_u(x), \min(x)\}$$
  and $M \geq 1$.
- Disjunctive operator
  $$\mathcal{A}(x) = \max\{\sigma_l(x), \max(x)\}, \quad \overline{\mathcal{A}}(x) = \min\{\sigma_u(x), M||x||, 1\}$$
  and $M \geq 1$.
- Idempotent operator
  $$\mathcal{A}(x) = \max\{\sigma_l(x), 1-M||1-x||, \min(x)\}, \quad \overline{\mathcal{A}}(x) = \min\{\sigma_u(x), M||x||, \max(x)\}.$$

Note that we do not require symmetry (commutativity), which is indeed useful in some applications. If required, the symmetry can be imposed in a straightforward manner by ordering the arguments, as follows.

Consider the simplex $S = \{x \in I^n| x_1 \geq x_2 \geq \ldots \geq x_n\}$ and a function $\tilde{f} : S \to I$. The function $f : I^n \to I$ defined by $f(x) = \tilde{f}(x_{\downarrow})$ is symmetric ($x_{\downarrow}$ denotes the
vector obtained from $x$ by arranging its components in non-increasing order). Then in order to construct a symmetric $f$, it is sufficient to construct $\tilde{f}$.

To build $\tilde{f}$ we simply apply Eq. (8), with the bounds $\sigma_u, \sigma_l$ modified as

$$
\sigma_u(x) = \min_k \{ y^k + M ||(x - x^k_0) + || \},
$$

$$
\sigma_l(x) = \max_k \{ y^k - M ||(x^k_0 - x) + || \},
$$

i.e., we order the abscissae of each datum in non-increasing order. There is no need to modify any of the subsequent formulae for $B_u, B_l$, as long as the conditions which define these bounds are consistent with the symmetry themselves ($B_u, B_l$ will be automatically symmetric).

5. Neutral element and annihilator

The existence of a neutral element is a stronger condition than conjunctive/disjunctive behavior, and consequently the bounds on the values of $f$ are tighter. We shall see that calculation of these bounds depends on the Lipschitz constant and the norm used, and frequently requires solving an optimization problem.

The presence of a neutral element restricts the range for allowable Lipschitz constants to $1 \leq M$. Consider condition (10) for all $i$. The bounds implied by this condition are

$$
\forall x \in I^n : \quad B_l(x) \leq f(x) \leq B_u(x),
$$

where

$$
B_u(x) = \min_{i=1,...,n} B^i_u(x),
$$

$$
B_l(x) = \max_{i=1,...,n} B^i_l(x),
$$

(13)

where for a fixed $i$ the bounds are

$$
B^i_u(x) = \min_{t \in I} (t + M ||(x - e(t, i)) + ||),
$$

$$
B^i_l(x) = \max_{t \in I} (t - M ||(e(t, i) - x) + ||).
$$

(14)

Of course, we need translate these bounds into practically computable values, for which we need to find the minimum/maximum with respect to $t$. Since any norm is a convex function of its arguments, the expression we minimize (maximize) is also convex (concave), and hence the minimum (maximum) is unique. The following proposition establishes these optima.

**Proposition 1.** Given $e \in [0,1], x \in [0,1]^n, i \in \{1,\ldots,n\}, M \geq 1, p \geq 1, \text{ and } || \cdot || \text{ a } l_p\text{-norm, let}$

$$
f_{x,e}(t) = t + M ||((x_1 - e)_+, \ldots, (x_{i-1} - e)_+, (x_i - t)_+, (x_{i+1} - e)_+, \ldots, (x_n - e)_+)||
$$

The minimum of $f_{x,e}(t)$ is achieved at
\[ t^* = 0, \text{ if } M = 1; \]
\[ t^* = x_i, \text{ if } p = 1 \text{ and } M > 1; \]
\[ t^* = \text{med} \left\{ 0, x_i - \left( \frac{c(i)}{M^{\frac{1}{p}} - 1} \right)^\frac{1}{\beta}, x_i \right\} \text{ otherwise,} \]

and its value is
\[
\min f_{x,e}(t) = \begin{cases} 
M(c(i) + x_i^p)^{\frac{1}{p}}, & \text{if } t^* = 0, \\
x_i + (M^{\frac{1}{p}} - 1)^{\frac{1}{p}} c(i)^{\frac{1}{p}}, & \text{if } t^* = x_i - \left( \frac{c(i)}{M^{\frac{1}{p}} - 1} \right)^{\frac{1}{\beta}}, \\
x_i + Mc(i)^{\frac{1}{p}}, & \text{if } t^* = x_i, 
\end{cases}
\]
\label{eq:15}

where \( c(i) = \sum_{j \neq i} (x_j - c)^p \).

\textbf{Proof.} If \( p = 1 \), then
\[ f_{x,e}(t) = t + M \sum_{j \neq i} (x_j - c)_+ + (x_i - t)_+ = t + Mc(i) + M(x_i - t)_+. \]
It is clear that when \( t \geq x_i \) the function \( f_{x,e}(t) = t + Mc(i) \) is strictly increasing with respect to \( t \) and therefore its minimum is obtained at \( t^* = x_i \). On the other hand, if \( t \leq x_i \) then
\[ f_{x,e}(t) = t(1 - M) + Mc(i) + M x_i \]
and since \( M > 1 \), \( f_{x,e}(t) \) would be decreasing and the minimum would be located at \( t^* = x_i \); both cases provide
\[ \min f_{x,e}(t) = f_{x,e}(x_i) = x_i + Mc(i) = x_i + Mc(i). \]

For the case \( p > 1 \), note that for all \( t \geq x_i \), \( f_{x,e}(t) \) is again strictly increasing in \( t \) (and it will have its minimum at \( t^* = x_i \)) so we only have to find the minimum of \( f_{x,e}(t) \) on \([0, x_i]\). The possible minimizers are the endpoints of this interval and the points fulfilling \( \frac{df_{x,e}(t)}{dt} = 0 \). The derivative is
\[ \frac{df_{x,e}(t)}{dt} = 1 - M \left( \frac{(x_i - t)^p}{c(i) + (x_i - t)^p} \right)^{\frac{\beta}{p}}. \]
In the special case \( M = 1 \), if \( c(i) = 0 \) then \( f_{x,e}(t) = t + (x_i - t) = x_i \) and \( \min f_{x,e}(t) = f_{x,e}(0) \). If \( c(i) > 0 \), \( f_{x,e}(t) \) is increasing, and the minimum is achieved also at \( t = 0 \).

For \( M > 1 \), the critical points are \( t = 0, t = x_i - \left( \frac{c(i)}{M^{\frac{1}{p}} - 1} \right)^{\frac{1}{\beta}} \), and \( t = x_i \). Now, since \( f_{x,e}(t) \) is a convex function it is clear that its minimum in \([0, x_i]\) is achieved at \( \text{med} \left\{ 0, x_i - \left( \frac{c(i)}{M^{\frac{1}{p}} - 1} \right)^{\frac{1}{\beta}}, x_i \right\} \). The value of the minimum is easily obtained by substituting \( t \) in \( f_{x,e}(t) \) by these values. \( \square \)
Corollary 1. The upper bound on an aggregation operator with neutral element $e \in [0,1]$ and Lipschitz constant $M$ takes the value

$$
\overline{A}(x) = \min_{i=1,...,n} \{ B_i^{u}(x), \sigma_u(x) \},
$$

where $B_i^{u}(x) = f_{x,e}(t^*)$ is given by (15) and $\sigma_u(x)$ is given by (12).

Proposition 2. Given $e \in [0,1]$, $x \in [0,1]^n$, $i \in \{1,\ldots,n\}$, $M \geq 1$, $p \geq 1$, and $|| \cdot ||$ a $l_p$-norm, let

$$
g_{x,e}(t) = t - M \left( (e - x_1)_+, \ldots, (e - x_{i-1})_+, (t - x_i)_+, (t - x_{i+1})_+, \ldots, (e - x_n)_+ \right)
$$

The maximum of $g_{x,e}(t)$ is achieved at

- $t^* = 1$, if $M = 1$;
- $t^* = x_i$, if $p = 1$ and $M > 1$, or
- $t^* = \text{med} \{ x_i, x_i + \left( \frac{\tilde{c}(i)}{M^{\frac{1}{p} - 1}} \right)^\frac{1}{p}, 1 \}$ otherwise,

and its value is

$$
\max g_{x,e}(t) = \begin{cases} 
  x_i - M \tilde{c}(i)^\frac{1}{p}, & \text{if } t^* = x_i, \\
  x_i - (M^{\frac{1}{p} - 1} - 1)^\frac{1}{p} \tilde{c}(i)^\frac{1}{p}, & \text{if } t^* = x_i + \left( \frac{\tilde{c}(i)}{M^{\frac{1}{p} - 1}} \right)^\frac{1}{p}, \\
  1 - M(\tilde{c}(i) + (1 - x_i)^p)^\frac{1}{p}, & \text{if } t^* = 1,
\end{cases}
$$

(16)

where $\tilde{c}(i) = \sum_{j \neq i} (e - x_j)^p$.

Proof. This proof is obtained directly from Proposition 1 by noticing that

$$
f_{x,e}(t) = 1 - g_{1-x,1-e}(1-t).
$$

Corollary 2. The lower bound on an aggregation operator with neutral element $e \in [0,1]$ and Lipschitz constant $M$ takes the value

$$
\underline{A}(x) = \max_{i=1,...,n} \{ B_i^{l}(x), \sigma_l(x) \}
$$

where $B_i^{l}(x) = g_{x,e}(t^*)$ is given by (16) and $\sigma_l(x)$ is given by (12).

Consider now a few special cases.

Corollary 3. Let $M = 1$ and $e = 1$. Then the bounds are

$$
\overline{A}(x) = \min \{ x \},
$$

$$
\underline{A}(x) = \max \{ 0, 1 - ||1 - x|| \}.
$$
Proof. For $\overline{A}(x)$, $e = 1$ implies $c(i) = 0$ for all $i$, and with $M = 1$ it is clear that $\min f_i(t) = x_i$; this generates $B_u(x) = \min \{ \min \{ x \}, \min \{ ||x||, 1 \} \} = \min \{ x \}$.

With respect to $\overline{A}(x)$, we only need to notice that when $M = 1$,

$$1 - ||(1 - x_1, \ldots, 1 - x_n)|| \geq x_i - ||(1 - x_1, \ldots, 0_i, \ldots, 1 - x_n)||$$

for all $i \in \{1, \ldots, n\}$, and since $\sigma_l(x) \geq B_l(x)$, we obtain $\overline{A}(x) = \max \{ 0, 1 - ||1 - x|| \}$ by applying (12).

For the special case $p = 1$, i.e., 1-Lipschitz operators with the neutral element $e = 1$, which are quasi-copulas, we obtain well known Fréchet-Hoeffding bounds

$$\max(0, \sum_{i=1}^{n} x_i - (n - 1)) \leq f(x) \leq \min(x).$$

Corollary 4. Let $M = 1$ and $e = 0$. Then the bounds are

$$\overline{A}(x) = \min \{ 1, ||x|| \},$$

$$\underline{A}(x) = \max \{ x \}.$$  

Proof. Now, having $e = 0$ implies $c(i) = \sum_{j \neq i} x_j^p$ for $B_u^i(x)$, and then

$$B_u^i(x) = \left( \sum_{j \neq i} x_j^p + x_i^p \right)^{\frac{1}{p}} = ||x||, i = 1, \ldots, n,$$

and taking $\min \{ B_u^i, \sigma_u \}$ results in $\overline{A}(x) = \min \{ 1, ||x|| \}$.

On the other hand, the condition $B_l(x) = \max \{ x \}$ is easily deduced as $\max g_i(t) = x_i$, and $\max \{ x \} \geq \max \{ 0, 1 - ||1 - x|| \}$ provides $\underline{A}(x) = \max \{ x \}$.

For $p = 1$ we obtain known bounds on dual quasi-copulas.

Consider now an aggregation operator which has an annihilator $a \in [0,1]$. The existence of an annihilator does not imply conjunctive or disjunctive behaviour on any part of the domain, but together with monotonicity, it implies $f(x) = a$ on $[a,1] \times [0,a]$ and $[0,a] \times [a,1]$ (and their multivariate extensions).

Such restrictions are easily incorporated into the bounds by using

$$\max_i B_l^i(x) \leq f(x) \leq \min_i B_u^i(x),$$

$$B_l^i(x) = a - M(a - x_i)_+, \quad B_u^i(x) = a + M(x_i - a)_+.$$  

(17)
6. Diagonals and opposite diagonals

We now consider the problem of constructing Lipschitz aggregation operators with a given diagonal or opposite diagonal section. Denote by \( \delta(t) = f(t, t, \ldots, t) \) the diagonal section of the \( n \)-ary aggregation operator \( f \). If \( f \in \mathcal{A}_M, \|\cdot\|_p \), then \( \delta \in Lip(Mn^{1/p}) \). Also \( \delta(t) \) is nondecreasing, and \( \delta(0) = 0, \delta(1) = 1 \). We denote by \( \omega(t) = f(t, 1-t) \) the opposite diagonal section of a binary aggregation operator. We note that \( \omega \in Lip(M) \).

In the following we assume that the functions \( \delta(t), \omega(t) \) are given and they have the required Lipschitz properties. The goal is to determine the upper and lower bounds on Lipschitz aggregation operators with these diagonal and opposite diagonal sections.

6.1. Diagonal section

From (2) it follows that

\[
B_u(x) = \min_{t \in I} (\delta(t) + M \|((x_1 - t)_+, \ldots, (x_n - t)_+)\|),
\]

\[
B_l(x) = \max_{t \in I} (\delta(t) - M \|((t - x_1)_+, \ldots, (t - x_n)_+)\|). \tag{18}
\]

We remind that these bounds are in addition to (12). For the purposes of computing the values of \( B_u(x), B_l(x) \) we need to develop suitable algorithms to solve the optimization problems in (18).

Before we proceed with this general case, we recall the following bounds obtained for bivariate 1-Lipschitz functions (i.e., \( n = 2, p = 1, M = 1 \)) in [3].

\[
B_u(x) = \max(x_1, x_2) + \min_{t \in [\alpha, \beta]} (\delta(t) - t),
\]

\[
B_l(x) = \min(x_1, x_2) + \max_{t \in [\alpha, \beta]} (\delta(t) - t), \tag{19}
\]

where \( \alpha = \min(x_1, x_2), \beta = \max(x_1, x_2) \). Let us show that (19) are a direct consequence of (18).

Consider the upper bound \( B_u(x) \) in (18) and \( n = 2 \). Take three intervals \([0, \alpha], [\alpha, \beta] \) and \((\beta, 1]\). We show that the minimum cannot be achieved on \([0, \alpha) \) or \((\beta, 1]\). For \( t \in (\beta, 1] \) all terms \((x_i - t)_+\) are null, what remains from the expression is \( \delta(t) \), which is increasing. Hence the expression under the minimum in (18) is increasing on this interval, therefore the interval \((\beta, 1]\) does not contain the minimum. In general, the minimum of the expression in (18) can be achieved on \([0, \alpha) \), depending on the form of \( \delta(t) \). However, for the special case \( p = 1 \), the function

\[
\delta(t) + M((x_1 - t) + (x_2 - t)) = \delta(t) + M(x_1 + x_2 - 2t)
\]

is decreasing (remember that the Lipschitz constant of \( \delta \) is 2\( M \) in this case), and hence the minimum cannot be achieved on \([0, \alpha) \). On \([\alpha, \beta] \) we have

\[
B_u(x) = \min_{t \in [\alpha, \beta]} (\delta(t) + M((\beta - t)^p)^{1/p}) = \min_{t \in [\alpha, \beta]} (\delta(t) + M\beta - Mt)
\]

\[
= M \max(x_1, x_2) + \min_{t \in [\alpha, \beta]} (\delta(t) - Mt), \tag{20}
\]

\[
M_n(\delta), \quad M_n(\omega),
\]

\[
\omega(t) \in Lip(M),
\]

\[
\omega(0) = 0, \omega(1) = 1.
\]
which converts into (19) if we take $M = 1$. Hence we obtain the upper bound (20), of which (19) is a special case. The lower bound for $n = 2, p = 1$ is obtained analogously as

$$B_l(x) = M \min(x_1, x_2) + \max_{t \in [a, b]} (\delta(t) - Mt).$$  \(21\)

Interestingly, for $p \to \infty$ a similar formula works for any dimension $n$. We have

$$\min(\delta(t) + M \max_i \{x_i - t\}) = \min(\delta(t) + M \max_i \{x_i\} - t).$$

For $t > \max x_i$ the expression in the brackets is null, and $\delta$ is increasing, hence the minimum is achieved at some $t \leq \max x_i$. Then

$$B_u(x) = \min_{t \in [0, b]} (\delta(t) + M \max_i \{x_i\} - t) = M \max_i \{x_i\} + \min_{t \in [0, b]} (\delta(t) - Mt).$$  \(22\)

Similarly,

$$B_l(x) = \max_{t \in [a, 1]} (\delta(t) - M(t - \min_i \{x_i\})) = M \min_i \{x_i\} + \max_{t \in [a, 1]} (\delta(t) - Mt).$$  \(23\)

Let us now return to the general case, in which we need to compute the minimum and maximum in (18). Since the function $\delta(t)$ is fairly arbitrary (we only require $\delta \in \text{Lip}(M^{n^{1/p}}) \cap \text{Mon}$), the overall expression may possess a number of local minima. Calculation of the bounds require the global minimum, and thus we need to use a global optimization technique. Fortunately, for univariate Lipschitz optimization there are a number of efficient deterministic global optimization methods [30]. We shall use Pijavsky-Shubert method [31,32], which consists in building a sequence of saw-tooth underestimates of the objective function, which converges to it uniformly. The accumulation point of the sequence of global minima of the underestimates converges to the global minimum of the objective function. Thus we are able to obtain a guaranteed solution with any desired accuracy.

The technique is illustrated on Fig. 1. Let $f(t)$ be the objective function, known to be in $\text{Lip}(M)$. Let $\{(t^k, f(t^k))\}, k = 1, \ldots, K$ be a sequence of points in the feasible domain with the respective function values. Then the underestimate at iteration $K$ is given by

$$H^K(t) = \max_{k=1, \ldots, K} (f(t^k) - M |t - t^k|) \leq f(t).$$

The optimization algorithm proceeds by computing the global minimum of $H(t)$, $t^*$, taking $t^{K+1} = t^*$, adding the point $(t^{K+1}, f(t^{K+1}))$ to the set of function values, and updating the underestimate. The global minimum of $H$ is found by sorting the list of its local minima, which in turn are also organized in a binary tree structure to facilitate updating the underestimate, and this makes the algorithm very efficient numerically. A detailed discussion is provided in [33].

To apply Pijavsky-Shubert algorithm we need an estimate of the Lipschitz constant of the objective function. Since $\delta \in \text{Lip}(M^{n^{1/p}})$ and is increasing, and the function

$$M ||(x_1 - t)_+ + \ldots, (x_n - t)_+||
Fig. 1. Illustration of the Pijavski-Shubert optimization scheme. The values of the objective function at \( t^k \) marked with dots determine the saw-tooth underestimate \( H^K \). The global minimizer of \( H^K \) determines the next iteration.

is in \( \text{Lip}(M n^{1/p}) \) and is decreasing (we can prove this with the help of the identity \( \|x\|_p \leq n^{1/p} \|x\|_\infty \)), the Lipschitz constant of the sum is \( M n^{1/p} \). Hence we use Pijavsky-Shubert algorithm with this parameter.

6.2. **Opposite diagonal**

Consider binary aggregation operators with given \( \omega(t) = f(t, 1 - t) \). The bounds are computed as

\[
B_u(x) = \min_{t \in I} (\omega(t) + M \|((x_1 - t)_+ + (1 - x_2)_+)\|), \\
B_l(x) = \max_{t \in I} (\omega(t) - M \|((t - x_1)_+ + (1 - x_2)_+)\|). 
\]

We notice that \( \omega \in \text{Lip}(M) \) and so is the second term in the expression, hence the objective function is in \( \text{Lip}(2M) \). We apply Pijavsky-Shubert method with this Lipschitz parameter to calculate the values of the bounds for any \( x \).

In [3] the following bounds were provided for bivariate 1-Lipschitz increasing functions.

\[
B_u(x) = T_L(x) + \min_{t \in [\alpha, \beta]} (\omega(t)), \\
B_l(x) = S_L(x) - 1 + \max_{t \in [\alpha, \beta]} (\omega(t)), 
\]

where \( \alpha = \min\{x_1, 1 - x_2\} \), \( \beta = \max\{x_1, 1 - x_2\} \).

Let us show that these bounds also follow from (24). We have

\[
B_u(x) = \min_{t \in I} (\omega(t) + M ((x_1 - t)_+ + (1 - x_2)_+)\). 
\]
Let $x_1 \leq 1 - x_2$. Then on $[0, x_1)$ the objective function becomes $\omega(t) + M(x_1 - t)$. It is decreasing, and the minimum is not achieved in this interval. On $(1 - x_2, 1]$ the expression becomes $\omega(t) + M(t - (1 - x_2))$. It is increasing, hence the minimum is not achieved in this interval either. On $[x_1, 1 - x_2]$ we have

$$B_u(x) = \min_{t \in [x_1, 1 - x_2]} \omega(t) = MT_L(x) + \min_{t \in [x_1, 1 - x_2]} \omega(t),$$

since $T_L(x) = \max(0, x_1 + x_2 - 1) = 0$ in this case.

Now let $1 - x_2 \leq x_1$. On $[0, 1 - x_2)$ and $(x_1, 1]$ the objective function is either decreasing or increasing, hence the minimum is achieved on $[1 - x_2, x_1]$. On that interval we have

$$B_u(x) = \min_{t \in [1 - x_2, x_1]} (\omega(t) + M((x_1 - t) + (t - (1 - x_2)))) = M(x_1 + x_2 - 1) + \min_{t \in [1 - x_2, x_1]} \omega(t).$$

Since in this case $x_1 + x_2 - 1 \geq 0$, we can write the bound as

$$B_u(x) = MT_L(x) + \min_{t \in [1 - x_2, x_1]} \omega(t),$$

and combining both cases and letting $M = 1$ we obtain (25). The lower bound is obtained in a similar way.

7. Marginals

7.1. Bounds

Now we consider the problem of obtaining the operator $f$ when certain functions are required to be its marginals. There are different aspects of this problem: a) construction of the operator by identifying upper and lower bounds; b) verifying that two or more marginals are compatible with each other; and c) identifying the smallest Lipschitz constant of $f$ such that the marginals are compatible. In this section we will consider $n = 2$ fixed unless otherwise stated.

Consider construction of a Lipschitz aggregation operator $f$ based on a given marginal $g$, defined on some closed subset $\Omega$, for example $\Omega = \{x = (x_1, x_2) : 0 \leq x_1 \leq 1, x_2 = 0\}$. Let $g \in Lip(M_g)$. Then obviously the Lipschitz constant of $f$, $M \geq M_g$. From (3) we obtain

$$B_u(x) = \min_{t \in [0, 1]} (g(t) + M||((x_1 - t)_+, x_2)||) = \min_{t \in [0, x_1]} (g(t) + M||((x_1 - t), x_2)||),$$

$$B_l(x) = \max_{t \in [0, 1]} (g(t) - M||((t - x_1)_+, 0)||) = g(x_1). \quad (26)$$

If the marginal is given on $\Omega = \{x = (x_1, x_2) : 0 \leq x_1 \leq 1, x_2 = 1\}$, then the bounds are

$$B_u(x) = \min_{t \in [0, 1]} (g(t) + M||((x_1 - t)_+, 0)||) = g(x_1),$$

$$B_l(x) = \max_{t \in [0, 1]} (g(t) - M||((t - x_1)_+, 0 - x_2)||)$$

$$= \max_{t \in [x_1, 1]} (g(t) - M||((t - x_1)_+, 1 - x_2)||). \quad (27)$$
To solve the optimization problem in each case we apply Pijavski-Shubert method with the Lipschitz parameter $M$.

### 7.2. Compatibility of the marginals

Consider now the case of two marginals $g_1(t_1), g_2(t_2) \in Lip(M_g)$. We note that $M_g \leq M$ and $2^{-1/p} \leq M$. We have the following situations:

1. $f(x_1, 0) = g_1(x_1), f(0, x_2) = g_2(x_2)$.
2. $f(x_1, 0) = g_1(x_1), f(x_1, 1) = g_2(x_1)$.
3. $f(x_1, 0) = g_1(x_1), f(1, x_2) = g_2(x_2)$.

By swapping the arguments of $f$ we have three other cases, which are completely analogous to the cases above. We denote the domains on which the first and second marginals are defined by $\Omega_1$ and $\Omega_2$ respectively.

It is incorrect to assume that we can construct an aggregation operator $f$ with the same Lipschitz constant as $M_g$ and both marginals. We refer to this issue as incompatibility of the marginals. For example, consider a kernel aggregation operator with the marginals $g_1(x_1) = f(x_1, 0) = \max\{x_1 - \frac{1}{2}, 0\}$ and $g_2(x_2) = f(1, x_2) = \min\{x_2 + \frac{1}{2}, 1\}$. Clearly $g_1, g_2 \in Lip(1)$, but

$$f(1, \frac{1}{2}) - f(\frac{1}{2}, 0) = 1 > 1 \cdot ||(1, \frac{1}{2}) - (\frac{1}{2}, 0)||_{\infty} = \frac{1}{2}.$$ 

Hence a kernel aggregation operator is incompatible with these marginals, and the smallest required Lipschitz constant is 2.

Of course, by choosing a larger $M$ we can always build a suitable $f \in Lip(M)$, but we are interested in the situation $M = M_g$. A monotone Lipschitz function $f$ is compatible with the data it interpolates if and only if the following conditions hold (Proposition 4.1 in [19])

$$\forall x, y \in \Omega_1 \cup \Omega_2 : f(x) - f(y) \leq M ||(x - y)_+||.$$ 

Thus a general approach is to verify the above mentioned Lipschitz conditions for all $x$ and $y$ (we only need to check it for $x, y$ not in the same subset $\Omega_1$ or $\Omega_2$). However there are infinitely many points for which we need to perform such a test. In what follows, we will obtain a practically computable test.

Consider the following optimization problems

$$z_1 = \min_{x \in \Omega_1, y \in \Omega_2} f(y) - f(x) + M ||(x - y)_+||,$$

$$z_2 = \min_{x \in \Omega_2, y \in \Omega_1} f(y) - f(x) + M ||(x - y)_+||.$$ 

Clearly, if $\min\{z_1, z_2\} \geq 0$, the marginals are compatible with $M = M_g$. We shall now consider instances of this problem for the three mentioned choices of $\Omega_1$ and $\Omega_2$. 
Case 1. $\Omega_1 = \{(x_1, x_2) : x_1 \in I, x_2 = 0\}$, $\Omega_2 = \{(x_1, x_2) : x_2 \in I, x_1 = 0\}$.

\[
\begin{align*}
z_1 &= \min_{x_1, x_2 \in I} \{g_2(x_2) - g_1(x_1) + M||((x_1, 0) - (x_2, 1))_+||\} \\
&= \min_{x_1, x_2 \in I} \{g_2(x_2) - g_1(x_1) + Mx_1 - g_1(x_1)\} = 0, \\
z_2 &= \min_{x_1, x_2 \in I} \{g_1(x_1) - g_2(x_2) + M||((0, x_2) - (x_1, 0))_+||\} \\
&= \min_{x_1, x_2 \in I} \{g_1(x_1) - g_2(x_2) + Mx_2 - g_2(x_2)\} = 0.
\end{align*}
\]

Since $g_1, g_2 \in Lip(M)$, increasing and $g_1(0) = g_2(0) = 0$, the minima are achieved at $x_1 = x_2 = 0$. Therefore in this case, the marginals are compatible for any $M = M_g \geq 2^{-1/p}$.

Case 2. $\Omega_1 = \{(x_1, x_2) : x_1 \in I, x_2 = 0\}$, $\Omega_2 = \{(x_1, x_2) : x_1 \in I, x_2 = 1\}$ (the opposite marginals). We note that $\forall x_2 \geq x_1 : g_1(x_1) \leq g_2(x_2)$.

\[
\begin{align*}
z_1 &= \min_{x_1, x_2 \in I} \{g_2(x_2) - g_1(x_1) + M||((x_1, 0) - (x_2, 1))_+||\} \\
&= \min_{x_1, x_2 \in I} \{g_2(x_2) - g_1(x_1) + Mx_1 - x_2\} \geq 0, \\
z_2 &= \min_{x_1, x_2 \in I} \{g_1(x_1) - g_2(x_2) + M||((1, x_2) - (x_1, 0))_+||\} \\
&= \min_{x_1, x_2 \in I} \{g_1(x_1) - g_2(x_2) + M(1 + (x_2 - x_1)^p)_+^{1/p}\}.
\end{align*}
\]

$z_1 \geq 0$ for any $M$, whereas the condition $z_2 \geq 0$ has to be verified for $2^{-1/p} \leq M < 1$. This can be done by solving the minimization problem using Pijavski-Shubert method with the Lipschitz parameter $2M$.

If $M \geq 1$, $z_2 \geq 0$ automatically, since $M(1 + (x_2 - x_1)^p)_+^{1/p} \geq M \geq 1$, and $\min\{g_1(x_1) - g_2(x_2)\} \geq -1$.

Case 3. $\Omega_1 = \{(x_1, x_2) : x_1 = 0, x_2 \in I\}$, $\Omega_2 = \{(x_1, x_2) : x_2 = 1, x_1 \in I\}$. We note $g_1(1) = g_2(0)$, and of course $\forall x_1, x_2 \in I : g_1(x_1) \leq g_2(x_2)$.

\[
\begin{align*}
z_1 &= \min_{x_1, x_2 \in I} \{g_2(x_2) - g_1(x_1) + M||((x_1, 0) - (1, x_2))_+||\} \geq 0, \\
z_2 &= \min_{x_1, x_2 \in I} \{g_1(x_1) - g_2(x_2) + M||((1, x_2) - (x_1, 0))_+||\} \\
&= \min_{x_1, x_2 \in I} \{g_1(x_1) - g_2(x_2) + M(1 - x_1)^p + x_2^p_+^{1/p}\}.
\end{align*}
\]

Using a change of variables $t = 1 - x_1$ in the second expression, we have

\[
z_2 = \min_{t, x_2 \in I} \{g_1(1 - t) - g_2(x_2) + M(t^p + x_2^p)_+^{1/p}\}.
\]

Now, $h_1(t, x_2) = g_1(1 - t) - g_2(x_2)$ is a decreasing function from $Lip(M_g, || \cdot ||_1)$, and hence $h_1 \in Lip(2^{-1/p}M_g, || \cdot ||_p)$ because of the identity $||x||_1 \leq 2^{-1/p}||x||_p, \forall x \in R^2$. Next, $h_2(t, x_2) = M(t^p + x_2^p)_+^{1/p} = M|| \cdot ||_p = \sup_{h \in Lip(M, || \cdot ||_p)} h(0) = h(0)$ is increasing in non-negative quadrant, and $h_1(0, 0) = h_2(0, 0)$. The sum $h_1 + h_2$ is guaranteed to be non-negative if $M \geq 2^{-1/p}M_g$, which is the required condition of compatibility of the marginals.
In summary, in case 1 the marginals are always compatible with $M = M_g$ for any $2^{-1/p} \leq M_g$, in case 2 they are compatible for $1 \leq M = M_g$, and in case 3 they are compatible for $M \geq 2^{1-1/p} M_g$.

If $M$ is smaller than the last value, the marginals may still be compatible, but the value of $z_2$ has to be found numerically by solving a corresponding minimization problem (in two variables). This can be done by using the Cutting Angle deterministic method of global optimization [34,35], which is a multivariate extension of the Pijavski-Shubert method.

### 7.3. The optimal Lipschitz constant

We saw in this section that two marginals can be sometimes incompatible with the chosen Lipschitz constant of the aggregation operator $f$, even though each marginal possesses the required Lipschitz properties. By choosing a suitably large $M$, namely $M \geq 2^{1-1/p} M_g$, we can achieve compatibility of the marginals with $f$. An interesting question arises: what is the smallest $M$ which guarantees such compatibility of two specific marginals.

To answer this question we need to solve the following problem

$$\min M$$

s.t. $z_2 = \min_{x_1, x_2 \in I} \left\{ g_1(x_1) - g_2(x_2) + M ((1 - x_1)^p + x_2^p)^{1/p} \right\} \geq 0,$$

$$M_g \leq M \leq 2^{1-1/p} M_g.$$

Since $z_2$ is a monotone increasing function of $M$, we can apply the bisection method to solve the equation

$$\min_{x_1, x_2 \in I} \left\{ g_1(x_1) - g_2(x_2) + M ((1 - x_1)^p + x_2^p)^{1/p} \right\} = 0$$

on the interval $[M_g, 2^{1-1/p} M_g]$ with a given tolerance.

### 8. Bipolar operators

When dealing with aggregation of pieces of information that may be in favor or against, it is customary to employ bipolar aggregation operators, which are functions $f : [-1,1]^n \rightarrow [-1,1]$, monotone increasing in each argument and satisfying $f(-1) = -1, f(1) = 1$. Negative values of the arguments are often referred to as negative information, whereas positive values are referred to as positive information.

Of course there is an isomorphism between unipolar and bipolar scales (and aggregation operators), and one can easily construct a bipolar aggregation operator $f$ from a unipolar one $\tilde{f}$ by taking, e.g., $f(x) = (2\tilde{f}(x+1) - 1)$. Thus previously discussed methods are equally applicable.

However, bipolar operators sometimes offer a more intuitive interpretation of the desired properties, and also allow one to refine certain conditions on the aggregation operator. When positive and negative information are considered to be of
different nature, positive and negative arguments of the aggregation operator should be handled in different ways. For example, one may require mutual reinforcement of negative information, but averaging behaviour for positive information. Then the aggregation operator should have conjunctive behaviour for negative values and be idempotent for positive values.

In this section we develop bounds specific for this type of heterogeneous operators. Consider the following cases.

I. \( f \) is conjunctive for negative \( x \) and disjunctive for positive \( x \);
II. \( f \) is disjunctive for negative \( x \) and conjunctive for positive \( x \);
III. \( f \) is disjunctive for negative \( x \) and idempotent for positive \( x \);
IV. \( f \) is conjunctive for negative \( x \) and idempotent for positive \( x \).

We will not require specific behaviour for the values of \( x \) with components with mixed signs. The reason is that in most cases the restrictions on that part of the domain will follow automatically.

In the first case is the behaviour of the aggregation operator is similar to a (rescaled to \([-1,1]\)) uninorm, with no associativity. Yager [17] calls such class of operators generalized uninorm GENUNI. However we should note that our conditions are weaker, since we did not require commutativity nor the neutral element \( e = 0 \). If we did require \( e = 0 \), the operator would have bounds given by Corollaries 1 and 2 in section 5, after transforming that bound from unipolar to bipolar scale.

Thus on \([-1,0]^n \) \( f \) is bounded from above by minimum, and on \([0,1]^n \) it is bounded from below by maximum. This implies \( M \geq 1 \). Examine the bounds on the rest of the domain. Consider the lower bound. The bounds on \([-1,0]^n \) imply a trivial bound \(-1 \leq f(x) \) elsewhere. However, since on \([0,1]^n \) \( f(x) \geq \max(x) \), this implies (see (4))

\[
f(x) \geq \max_{z \in [0,1]^n} (g(z) - M||z - x||) = \max_{z \in [0,1]^n} (\max(z_i) - M||z - x||).
\]

After some technical calculations we obtain

\[
f(x) \geq \max_{x_k \leq \cdots \leq x_1} \left( t - M||\max\{0,-x_1\}, \ldots, \right.
\]

\[
\left. \max\{0,-x_{k-1}\}, (t-x_k), \max\{0,-x_{k+1}\}, \ldots, \max\{0,-x_n\}|| \right) \right) \right), \quad (30)
\]

where \( x_k = \max_{i=1,\ldots,n} \{ x_i \} \).

Applying Proposition 2, the point of maximum

- \( t^* = 1 \), if \( M = 1 \);
- \( t^* = x_k \), if \( p = 1 \) and \( M > 1 \), or
- \( t^* = \text{med} \left\{ x_k, x_k + \left( \frac{K}{M - 1} \right)^{\frac{1}{p}} \right\} \) otherwise,
with \( K = \sum_{i \neq k} \max\{0, -x_i\}^p \). Thus the lower bound \( B_l(x) \) is

\[
B_l(x) = \begin{cases} 
  x_k - MK^\frac{1}{p}, & \text{if } t^* = x_k \\
  x_k - (M \frac{p}{p+1} - 1) \frac{p+1}{p} K^\frac{1}{p}, & \text{if } t^* = x_k + \left( \frac{K}{M \frac{p}{p+1} - 1} \right)^\frac{1}{p} \\
  1 - M(K + (1 - x_k)^p)^\frac{1}{p}, & \text{if } t^* = 1.
\end{cases} \tag{31}
\]

Similarly, the fact that \( f \) is bounded from above by minimum on \([-1, 0]^n\) implies the following upper bound on the rest of the domain

\[
f(x) \leq \min_{z \in [-1, 0]^n} (\overline{\tau}(z) + M M||(x - z)||) = \min_{z \in [-1, 0]^n} (\min_i (z_i) + M M||(x - z)||),
\]

which translates into

\[
f(x) \leq \min_{-1 \leq i \leq x_j} \{ (t + M M||(\max\{0, x_1\}, \ldots, \\
  \max\{0, x_{j-1}\}, (x_j - t), \max\{0, x_{j+1}\}, \ldots, \max\{0, x_n\})||), \}
\]

where \( x_j = \min_{i=1, \ldots, n} \{x_i\} \). By applying Proposition 1 the minimizer is given by

- \( t^* = -1 \), if \( M = 1 \);
- \( t^* = x_j \), if \( p = 1 \) and \( M > 1 \);
- \( t^* = \text{med}\left\{ -1, x_j - \left( \frac{K}{M \frac{p}{p+1} - 1} \right)^\frac{1}{p}, x_j \right\} \) otherwise,

and the upper bound is

\[
B_u(x) = \begin{cases} 
  M(K + (1 + x_j)^p)^\frac{1}{p}, & \text{if } t^* = -1, \\
  x_j - (M \frac{p}{p+1} - 1) \frac{p+1}{p} K^\frac{1}{p}, & \text{if } t^* = x_j - \left( \frac{K}{M \frac{p}{p+1} - 1} \right)^\frac{1}{p}, \\
  x_j + MK^\frac{1}{p}, & \text{if } t^* = x_j,
\end{cases} \tag{33}
\]

where \( K = \sum_{i \neq j} \max\{0, x_i\}^p \).

Summarizing, for a bipolar aggregation operator with conjunctive behaviour for negative \( x \) and disjunctive behaviour for positive \( x \), the bounds are

\[
-1 \leq f(x) \leq \min(x), \quad \text{if } x \in [-1, 0]^n \\
\max(x) \leq f(x) \leq 1, \quad \text{if } x \in [0, 1]^n \\
B_l(x) \leq f(x) \leq B_u(x) \quad \text{elsewhere}, \tag{34}
\]

with \( B_l, B_u \) given by (31) and (33).

In the case II, \( f \geq \max \) on \([-1, 0]^n\) and \( f \leq \min \) on \([0, 1]\). We immediately obtain that \( f \) has zero as the annihilator, i.e., \( \forall x \in I^n, i \in \{1, \ldots, n\} : f(0(x, i)) = 0 \). Such an operator has a similar structure to nullnorms, but need not be associative. It follows that \( f(x) = 0 \) for all vectors whose components have different signs. Thus the bounds are

\[
\max(x) \leq f(x) \leq 0, \quad \text{if } x \in [-1, 0]^n
\]
\[0 \leq f(x) \leq \min(x), \quad \text{if } x \in [0, 1]^n\]
\[f(x) = 0, \quad \text{elsewhere.} \tag{35}\]

In the case III, \(f\) is bounded by maximum from below for negative \(x\), and is bounded by minimum and maximum for positive \(x\). This implies that 0 is the lower bound for all \(x\) which have at least one negative and one positive component. At the same time, since \(f\) is bounded from above by maximum for all positive \(x\), it will have the same bound for \(x\) with mixed components due to monotonicity. The bounds are

\[\max(x) \leq f(x) \leq 0, \quad \text{if } x \in [-1,0]^n,\]
\[\min(x) \leq f(x) \leq \max(x), \quad \text{if } x \in [0,1]^n,\]
\[0 \leq f(x) \leq \max(x), \quad \text{elsewhere.} \tag{36}\]

In the case IV we obtain the bounds

\[-1 \leq f(x) \leq \min(x), \quad \text{if } x \in [-1,0]^n,\]
\[\min(x) \leq f(x) \leq \max(x), \quad \text{if } x \in [0,1]^n,\]
\[B_l(x) \leq f(x) \leq B_u(x), \quad \text{elsewhere}, \tag{37}\]

where \(B_u\) is given as the minimum of \(\max(x)\) and Eq. (33), and \(B_l\) is given as

\[B_l(x) = \max_{t \in [0,1]} (t - M ||((t_1 - x_1)_+ + \ldots + (t_n - x_n)_+)||) = M ||(-x)_+||.\]

9. Computation and examples

Computation of the optimal aggregation operator from a given dataset consistent with the specified properties involves two main steps: 1) computation of tight upper and lower bounds \(B_u, B_l\) in (9), and 2) computation of the bounds \(\sigma_u, \sigma_l\) from the data using (2). The latter step is straightforward, it involves \(O(Kn)\) arithmetical operations, and typically the number of data \(K\) is not extremely large. Thus the bulk of the computations involves the bounds \(B_u, B_l\).

Computation of these bounds involves solution to an optimization problem. In some cases, namely for conjunctive, disjunctive and idempotent aggregation operators, operators with a given neutral element and annihilator we found explicit solutions, hence their computation is trivial. In the cases of given diagonal and opposite diagonal sections, as well as univariate marginals, the optimization problem is univariate, and we used Pijavski-Shubert method to find the globally optimal solution. This method is numerically efficient, its accuracy and speed depends on the Lipschitz constant of the objective function, and its worst case performance is \(O(M^2 \epsilon)\) objective function evaluations, where \(M\) is the Lipschitz constant and \(\epsilon\) is the desired accuracy.

The two-variate global optimization problem needed to establish the compatibility of marginals in section 7.2 is more challenging, however the Cutting Angle method (CAM) [34,35] is sufficiently fast in this case, its running time is typically
1-2 minutes on a modern Pentium IV workstation. Note that this optimization problem needs to be solved just once, and that CAM not only finds but also proves the global optimum.

Fig. 2. The optimal aggregation operator which interpolates the data from [5] (with parameters $p = 1, M = 2$ and no additional requirements).

Fig. 3. The optimal aggregation operator with the diagonal section $\delta(t) = t^{1.2}$ which interpolates the data from [5].
Let us now present some examples of constructing aggregation operators from empirical data, with given properties, such as diagonal and opposite diagonal. As the data set we used the data from [5]. Figures 2-5 illustrate different the aggregation operators. The running time to generate the graph by computing 2500 function values (i.e., to solve 5000 univariate optimization problems for the bounds $B_u, B_l$) on a Pentium IV 2 GHz workstation was below 3 sec in all these cases. The data are marked with circles.

We note that the choice of the diagonal and opposite diagonal in our examples does not reflect the properties of the data, they were chosen fairly arbitrary for the sake of an example illustrating efficiency of the algorithm. We also note that we did not perform any smoothing of the data, consequently the graph of $f$ exhibits some sharp rises and flat spots. For comparison, Fig. 5 shows a 1-Lipschitz aggregation operator with a given opposite diagonal section and no empirical data.

Fig. 4. The optimal aggregation operator with the opposite diagonal section $\omega(t) = -t^2 + t + 0.25$ which interpolates the data from [5].

10. Summary and conclusion

Pointwise construction of aggregation operators allows one to fit the desired values while preserving its essential properties, such as conjunctive or disjunctive behaviour, existence of a neutral element, commutativity and so forth. The central interpolation scheme delivers an optimal aggregation operator from a given class, and is based on establishing tight upper and lower bounds on the values of the aggregation operator at all points.

This paper examines in detail the bounds on aggregation operators from several
classes and with distinct properties. In all cases the bounds are a result of applying general formulae Eqns.(2)-(4). However, the actual computation of the bounds requires solving certain optimization problems, which may be complicated. In this work we found explicit solutions in certain cases, while in others we formulated suitable algorithms which guarantee convergence to the right solution.

Table 1 summarizes our findings.

We note that the bounds we established are applicable to any interpolation scheme, not only to the central interpolant given by (9). These bounds should always be included into construction process, and thus will be useful in other construction methods. An attractive feature of the central interpolant is that it delivers an optimal aggregation operator with respect to the largest possible error, and incorporation of the bounds is straightforward.

Acknowledgements

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Table 1. The lower and upper bounds $B_l, B_u$ on Lipschitz aggregation operators with listed properties.

<table>
<thead>
<tr>
<th>Properties</th>
<th>$B_l(x)$</th>
<th>$B_u(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conjunctive operator, $M \geq 1$</td>
<td>$\max{1 - M||1 - x|, 0}$</td>
<td>$\min(x)$</td>
</tr>
<tr>
<td>Disjunctive operator, $M \geq 1$</td>
<td>$\max(x)$</td>
<td>$\min{M|x|, 1}$</td>
</tr>
<tr>
<td>Idempotent operator</td>
<td>$\max{1 - M||1 - x|, \min(x)}$</td>
<td>$\min(M|x|, \max(x))$</td>
</tr>
<tr>
<td>Neutreal element $e \in [0,1], M \geq 1$</td>
<td>$\max_{i=1,\ldots,n}{(16)}$</td>
<td>$\min_{i=1,\ldots,n}{(15)}$</td>
</tr>
<tr>
<td>Neutreal element $e = 1, M = 1$</td>
<td>$\max(1 -</td>
<td></td>
</tr>
<tr>
<td>Neutreal element $e = 0, M = 1$</td>
<td>$\max(x)$</td>
<td>$\min{</td>
</tr>
<tr>
<td>Diagonal section $\delta(t) = f(t, t)$</td>
<td>given by (18)</td>
<td></td>
</tr>
<tr>
<td>$p = 1$, Diag. section $\delta(t) = f(t, t)$, $n = 2$</td>
<td>$M \min{x_1, x_2} + $</td>
<td>$M \max{x_1, x_2} + $</td>
</tr>
<tr>
<td></td>
<td>$\max_{t \in [a, b]} (\delta(t) - M t)$</td>
<td>$\min_{t \in [a, b]} (\delta(t) - M t)$</td>
</tr>
<tr>
<td>$\alpha = \min{x_1, x_2}, \beta = \max{x_1, x_2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$p \to \infty$, Diag. section $\delta(t) = f(t, \ldots, t)$</td>
<td>$M \min(x) +$</td>
<td>$M \max(x) +$</td>
</tr>
<tr>
<td></td>
<td>$\max_{t \in [a, b]} (\delta(t) - M t)$</td>
<td>$\min_{t \in [0, b]} (\delta(t) - M t)$</td>
</tr>
<tr>
<td>$\alpha = \min{x_1, x_2}, \beta = \max{x_1, x_2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Opposite diagonal $\omega(t) = f(t, 1 - t)$</td>
<td>given by (24)</td>
<td></td>
</tr>
<tr>
<td>$p = 1$, Opposite diag. $\omega(t) = f(t, 1 - t)$</td>
<td>$MS_L - M + $</td>
<td>$MT_L(x) + $</td>
</tr>
<tr>
<td></td>
<td>$\max_{t \in [a, b]} \omega(t)$</td>
<td>$\min_{t \in [a, b]} \omega(t)$</td>
</tr>
<tr>
<td>$\alpha = \min{x_1, 1 - x_2}, \beta = \max{x_1, 1 - x_2}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Marginal $g = f(x)_{x \in \Omega}$, where $\Omega$ is the domain of the marginal</td>
<td>$\max{g(t) -$</td>
<td>$\min{g(t) +$</td>
</tr>
<tr>
<td></td>
<td>$\in \Omega ; M||t - x| + ||}$</td>
<td>$\in \Omega ; M||x - t| + ||$</td>
</tr>
<tr>
<td>Bipolar operator conjunctive in $[-1,0]^n$, disjunctive in $[0,1]^n$</td>
<td>$-1$, if $x \in [-1,0]^n$</td>
<td>$\min{x}$, if $x \in [-1,0]^n$</td>
</tr>
<tr>
<td></td>
<td>$\max(x)$, if $x \in [0,1]^n$</td>
<td>$1$, if $x \in [0,1]^n$</td>
</tr>
<tr>
<td></td>
<td>(31), elsewhere</td>
<td>(33), elsewhere</td>
</tr>
<tr>
<td>Bipolar operator disjunctive in $[-1,0]^n$, conjunctive in $[0,1]^n$</td>
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<td>$\min(x)$, if $x \in [0,1]^n$</td>
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<tr>
<td></td>
<td>$0$, if $x \in [0,1]^n$</td>
<td>$0$, elsewhere</td>
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<tr>
<td></td>
<td>$0$, elsewhere</td>
<td></td>
</tr>
<tr>
<td>Bipolar operator disjunctive in $[-1,0]^n$, idempotent in $[0,1]^n$</td>
<td>$\max(x)$, if $x \in [-1,0]^n$</td>
<td>$\max(x)$, if $x \in [0,1]^n$</td>
</tr>
<tr>
<td></td>
<td>$\min(x)$, if $x \in [0,1]^n$</td>
<td>$\max(x)$, if $x \in [0,1]^n$</td>
</tr>
<tr>
<td></td>
<td>$0$, elsewhere</td>
<td>$\max(x)$, elsewhere</td>
</tr>
<tr>
<td>Bipolar operator conjunctive in $[-1,0]^n$, idempotent in $[0,1]^n$</td>
<td>$-1$, if $x \in [-1,0]^n$</td>
<td>$\min{x}$, if $x \in [-1,0]^n$</td>
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<tr>
<td></td>
<td>$\max(x)$, if $x \in [0,1]^n$</td>
<td>$\max(x)$, if $x \in [0,1]^n$</td>
</tr>
<tr>
<td></td>
<td>$M||(-x)|$, elsewhere</td>
<td>$\min{\max(x), (33)}$, elsewhere</td>
</tr>
</tbody>
</table>
References