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Robust Filtering for Uncertain Discrete-Time Systems with Uncertain Noise Covariance and Uncertain Observations

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***Abstract*— The use of Kalman filtering is very common in state estimation problems. The problem with Kalman filters is that they require full prior knowledge about the system modeling. It is also assumed that all the observations are fully received. In real applications, the previous assumptions are not true all the time. It is hard to obtain the exact system model and the observations may be lost due to communication problems. In this paper, we consider the design of a robust Kalman filter for systems subject to uncertainties in the state and white noise covariances. The systems under consideration suffer from random interruptions in the measurements process. An upper bound for the estimation error covariance is proposed. The proposed upper bound is further minimized by selection of optimal filter parameters. Simulation example shows the effectiveness of the proposed filter.**

I. Introduction

Kalman filters are widely used in state estimation problems [1]. However, the use of Kalman filters requires prior knowledge about the system under consideration and if these are not appropriately known, the estimation process will not be optimal and it may even diverge. In addition , it is assumed that all the observations are available during estimation, since Kalman filters are sensitive to incomplete or missing measurements [12],[13],[15]. In real-world applications, these assumptions are not always accurate; since there are applications for which the exact system model is hard to obtain and only approximations to the real model are available. Moreover, the observations are not guaranteed to include the signal of interest; instead, the observations may contain only noise in a random manner, and in this case, they are called *false alarms*.

In this paper, the problem of state estimation is addressed for systems suffering from two classes of uncertainties: uncertainty in the modeling parameters and uncertainty in the observation process. Uncertainty in the modeling parameters refers to the problem of not knowing the exact model parameters that describe the input and the output processes. Such uncertainty is common in industries, such as the minerals and materials industry [2]. The uncertainties can be due to several reasons, such as linearization, unmodeled dynamics, or model reduction [16]. For such cases, the classical Kalman filters are not suitable and there is no guarantee that the filters will provide the optimal parameter estimation. The uncertainties in the modeling parameters in this paper will be represented as norm-bounded uncertainties. We adopt this approach so that to prove the the estimation error covariance is guaranteed to fall below a certain upper bound. Although the estimation in this case is not guaranteed to be optimal, in many applications such as tracking, it is the goal to guarantee estimation robustness rather than optimality criteria [11].

The problem of interest in this paper has recently gained much of interest in research [2]-[8]. The problem has been addressed for continuous-time and discrete-time systems and for time-invariant and time-varying systems. The uncertainties were assumed to be either time-invariant or time-variant. Petersen and McFarlane addressed continuous time-invariant systems with uncertainties in the state matrix to get the optimal guaranteed-cost estimation filter [2]. Later, they provided the optimal guaranteed-cost estimator for continuous time-invariant systems with norm-bounded uncertainties in the state and output matrices [3]. They addressed the discrete time-invariant systems with norm-bounded uncertainties in the state matrix in [4]. Xie et al. in [5] and [6] have presented a robust Kalman filter with a guaranteed bound on the estimation error for linear continuous and discrete time-invariant systems, respectively, with time-varying norm-bounded uncertainties in the state and output matrices.In [7], X. Zhu et al. found the robust filters in the finite and infinite horizon cases where they assumed the uncertainty is in the state and output matrices. Z.Dong and Z. You in [8] have presented a robust finite-horizon Kalman filter for linear discrete time-varying systems with time-varying norm-bounded uncertainties in the state, output and white noise covariance matrices. The aforementioned work have assumed that all the observations are available at the time of estimation. The main difference between this work and the literature reviewed is that we consider systems subject to uncertainty in the state, output and white noise covariance matrices combined with the possibility of

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missing measurements which is the more general case.

The second type of uncertainty is the uncertainty in the observation process. In this type, there is a nonzero probability that the observed signal contains only noise without the signal of interest. This problem may rise in tracking applications when the object being tracked has high maneuverability, and also in the case of failures in the measuring sensors, high noise environments, and poor communication resources [13]. The problem of observation uncertainty has been investigated by many researchers. It was first addressed for a class of linear filters by Nahi [9] who obtained the optimal state estimator with uncertain observations. The uncertainty in the observations was assumed to be independent and identically distributed (i.i.d.). The work in [10] has generalized the work of Nahi where the uncertainty is not necessarily i.i.d.

Fewer papers were published on the problem of uncertain observations combined with the norm-bounded uncertainties in the modeling parameters. In [14] and [15], the problem of robust filtering in the case of missing measurements was studied using a jump Riccati equation approach. Robust Kalman filtering for systems suffering from missing measurements was developed by Zidong Wang et al. [17] in the infinite-horizon case. They solved the problem for the finite-horizon case in [18] where a robust finite-horizon filtering for linear discrete time-varying systems with time-varying norm-bounded uncertainty in the state matrix and the possibility of missing measurements was provided. In this work, the approach of [18] and [8] is adopted to derive the robust finite-horizon Kalman filter for the discrete time-varying systems with time-varying norm-bounded uncertainties in the state and the white noise covariance matrices in the case of missing measurements. Our objective is to provide the robust filtering for systems with uncertain block in the modeling parameters and the possibility of missing measurements in a recursive form to be suitable for online applications and the resulting filter does not include the uncertain block. Simulation example is provided to illustrate the effectiveness of the proposed filter.

The rest of the paper is organized as follows: Section II provides the problem formulation and necessary assumptions, Section III derives the robust finite-horizon Kalman filter, and Section IV provides a simulation example. Section V concludes this work.

II. Problem Formulation

The systems under consideration in this paper are defined as

$$x_{k+1} = (A_k + \Delta A_k)x_k + (B_k + \Delta B_k)w_k \qquad (1)$$

Where $x_k \in \Re^n$ is the state vector and $w_k \in \Re^m$ is white Gaussian noise sequence with zero mean and covariance $R_k > 0$. The initial state x_0 has mean value $\bar{x}_0$ and initial covariance value P_0. A_k,B_k are known real time-varying matrices with appropriate dimensions. The matrices ΔA_k and ΔB_k are the uncertainties in the state and process noise matrices, respectively, and

$$\begin{aligned} \Delta A_k &= H_{1,k}F_kE_{1,k} \\ \Delta B_k &= H_{1,k}F_kE_{2,k} \end{aligned}$$

Where $F_k \in \Re^{rs}$ is the norm-bounded time-varying uncertainty, i.e.

$$F_k^T F_k < I$$

The matrices $H_{1,k}$,$E_{1,k}$ and $E_{2,k}$ are known matrices with appropriate dimensions and they represent how the system will be affected by the norm-bounded uncertainty and I is the identity matrix with appropriate dimension. The observation process with the possibility of missing measurements will have the form

$$y_k = \gamma_k C_k x_k + (D_k + \Delta D_k)v_k \qquad (2)$$

Where the variable $\gamma_k \in \Re$ is a Bernoulli distributed white sequence taking values 0 and 1 randomly with

$$\begin{aligned} P(\gamma_k = 1) &= \mu_k \\ P(\gamma_k = 0) &= 1 - \mu_k \end{aligned}$$

Where $\mu_k \in \Re$ is the percentage of successful arrival of measurements and it can be obtained by test sessions. $y_k \in \Re^p$ is the measurement output vector, v_k is the measurements noise which is assumed white Gaussian sequence with mean zero and covariance $V_k > 0$. C_k is a known time-varying matrix of appropriate dimension.

The matrix $\Delta D_k = H_{2,k}F_kE_{2,k}$ will represent the uncertainty in the output noise covariance. It is assumed that γ_k,w_k,v_k and x_0 are mutually uncorrelated.

Consider the following filter for the uncertain system (1),(2)

$$\hat{x}_{k+1} = \hat{A}_k x_k + \hat{K}_k[y_k - \mu_k C_k \hat{x}_k] \qquad (3)$$

Where $k \in [0, N]$, $\hat{x}_k \in \Re^n$ is the estimated state value, $\hat{A}_k$ and $\hat{K}_k$ are the filter parameters to be determined, assume $\hat{x}_0 = x_0$. In the next section we will show that the estimation error covariance can be upper-bounded and we will determine a candidate upper bound.

III. Robust Filter Design

In this section, our goal is to design the robust Kalman filter that guarantees an upper bound on the estimation error covariance and this upper bound is guaranteed to be minimal. The first step is to find the possible upper bound and the second step will be to derive the optimal filter that leads to this bounded estimation.

A. Upper Bound on the Estimation Error Covariance

To obtain a possible upper bound for the estimation error covariance, we will use the following lemma

Lemma 1 [8] For a given set of matrices A,H,E and F where $FF^T \leq I$, X is a positive definite matrix. If there exist arbitrary $\alpha > 0$ that satisfy $\alpha^{-1}I - EXE^T > 0$ then we have

$$\begin{aligned} (A + HFE)X(A &+ HFE)^T \leq AXA^T \\ &+ AXE^T(\alpha^{-1}I - EXE^T)^{-1}EXA^T \\ &+ \alpha^{-1}HH^T \end{aligned}$$

Our next step will be to formulate the problem to be similar in structure as in lemma 1 so that we can conclude the existence of an upper bound on the estimation error covariance.

Using the approach in [16] and taking into consideration the addition of norm-bounded uncertainties in the process and measurement white noise covariance matrices, we formulate the augmented state-space model combining the system (1)-(2) and the filter (3). Define the state vector as

$$\widetilde{x}_k = \begin{bmatrix} x_k \\ \hat{x}_k \end{bmatrix}$$

And the augmented state-space model

$$\begin{aligned} \widetilde{x}_{k+1} &= (\widetilde{A}_k + \widetilde{H}_{1,k} F_k \widetilde{E}_{1,k})\widetilde{x}_k + \widetilde{A}_{ek}\widetilde{x}_k \\ &\quad +(\widetilde{B}_k + \widetilde{H}_{2,k}\widetilde{F}_k\widetilde{E}_{2,k})\widetilde{w}_k \end{aligned} \tag{4}$$

where

$$\begin{aligned} \widetilde{A}_k &= \begin{bmatrix} \hat{A}_k & 0 \\ \mu_k \hat{K}_k C_k & \mu_k \hat{K}_k C_k - \hat{A}_k \end{bmatrix} \\ \widetilde{A}_{e,k} &= \begin{bmatrix} 0 & 0 \\ (\gamma_k - \mu_k)\hat{K}_k C_k & 0 \end{bmatrix} \\ \widetilde{E}_{1,k} &= \begin{bmatrix} E_{1,k} & 0 \end{bmatrix}, \widetilde{E}_{2,k} = \begin{bmatrix} E_{2,k} & 0 \\ 0 & E_{2,k} \end{bmatrix} \\ \widetilde{H}_{1,k} &= \begin{bmatrix} H_{1,k} \\ 0 \end{bmatrix}, \widetilde{H}_{2,k} = \begin{bmatrix} H_{,k} & 0 \\ 0 & \hat{K}_k H_{2,k} \end{bmatrix} \\ \widetilde{B}_k &= \begin{bmatrix} B_k & 0 \\ 0 & \hat{K}_k D_k \end{bmatrix}, \widetilde{w}_k = \begin{bmatrix} w_k \\ v_k \end{bmatrix} \\ \widetilde{F}_k &= \begin{bmatrix} F_k & 0 \\ 0 & F_k \end{bmatrix} \end{aligned}$$

The covariance matrix of the state vector (4) is

$$\widetilde{\Sigma}_k = \mathbf{E}[\widetilde{x}_k \widetilde{x}_k^T]$$

The Lyapunov equation that governs its evolution is defined as

$$\begin{aligned} \widetilde{\Sigma}_{k+1} &= (\widetilde{A}_k + \widetilde{H}_{1,k} F_k \widetilde{E}_{1,k})\widetilde{\Sigma}_k(\widetilde{A}_k + \widetilde{H}_{1,k} F_k \widetilde{E}_{1,k})^T \\ &\quad +\Psi_k + (\widetilde{B}_k + \widetilde{H}_{2,k}\widetilde{F}_k\widetilde{E}_{2,k}) \\ &\quad .\widetilde{W}_k(\widetilde{B}_k + \widetilde{H}_{2,k}\widetilde{F}_k\widetilde{E}_{2,k})^T \end{aligned} \tag{5}$$

And since Ψ_k has mean zero

$$\Psi_k = (1-\mu_k)\mu_k \begin{bmatrix} 0 & 0 \\ \hat{K}_k C_k & 0 \end{bmatrix} \widetilde{\Sigma}_k \begin{bmatrix} 0 & 0 \\ \hat{K}_k C_k & 0 \end{bmatrix}^T \tag{6}$$

Now we can see that the error covariance of the augmented system (5) is similar to the structure in lemma 1, we can get the following result:

Theorem 1: If there exist arbitrary $\alpha > 0$ such that $\alpha^{-1}I - \widetilde{E}_{1,k}\widetilde{\Sigma}_k\widetilde{E}_{1,k}^T > 0$ and there exist arbitrary $\beta > 0$ such that $\beta^{-1}I - \widetilde{E}_{2,k}\widetilde{W}_k\widetilde{E}_{2,k}^T > 0$, we get

$$\begin{aligned} \widetilde{\Sigma}_{k+1} \leq \widetilde{A}_k\widetilde{\Sigma}_k\widetilde{A}_k^T &+ \widetilde{A}_k\widetilde{\Sigma}_k\widetilde{E}_k^T(\alpha^{-1}I - \widetilde{E}_{1,k}\widetilde{\Sigma}_k\widetilde{E}_{1,k}^T)^{-1} \\ &\times \widetilde{E}_{1,k}\widetilde{\Sigma}_k\widetilde{A}_k^T + \alpha^{-1}\widetilde{H}_{1,k}\widetilde{H}_{1,k}^T + \Psi_k \\ &+ \widetilde{B}_k\widetilde{W}_k\widetilde{B}_k^T \\ &+ \widetilde{B}_k\widetilde{W}_k\widetilde{E}_k^T(\beta^{-1}I - \widetilde{E}_{2,k}\widetilde{W}_k\widetilde{E}_k^T)^{-1} \\ &\times \widetilde{E}_k\widetilde{W}_k\widetilde{B}_k^T + \beta^{-1}\widetilde{H}_{2,k}\widetilde{H}_{2,k}^T \end{aligned} \tag{7}$$

With initial value

$$\widetilde{\Sigma}_0 = \begin{bmatrix} P_0 & 0 \\ 0 & 0 \end{bmatrix}$$

and if we have Σ_{k+1} where

$$\begin{aligned} \Sigma_{k+1} = \widetilde{A}_k\Sigma_k\widetilde{A}_k^T &+ \widetilde{A}_k\Sigma_k\widetilde{E}_k^T(\alpha^{-1}I - \widetilde{E}_{1,k}\Sigma_k\widetilde{E}_{1,k}^T)^{-1} \\ &\times \widetilde{E}_{1,k}\Sigma_k\widetilde{A}_k^T + \alpha^{-1}\widetilde{H}_{1,k}\widetilde{H}_{1,k}^T + \Psi_k \\ &+ \widetilde{B}_k\widetilde{W}_k\widetilde{B}_k^T \\ &+ \widetilde{B}_k\widetilde{W}_k\widetilde{E}_k^T(\beta^{-1}I - \widetilde{E}_{2,k}\widetilde{W}_k\widetilde{E}_k^T)^{-1} \\ &\times \widetilde{E}_k\widetilde{W}_k\widetilde{B}_k^T + \beta^{-1}\widetilde{H}_{2,k}\widetilde{H}_{2,k}^T \end{aligned} \tag{8}$$

where

$$\alpha^{-1}I - \widetilde{E}_{1,k}\Sigma_k\widetilde{E}_{1,k}^T > 0 \tag{9}$$

$$\beta^{-1}I - \widetilde{E}_{2,k}\widetilde{W}_k\widetilde{E}_{2,k}^T > 0 \tag{10}$$

Then $\widetilde{\Sigma}_k \leq \Sigma_k$ when Σ_k satisfies (8) and Σ_k will be a possible upper-bound for the estimation error covariance. The proof is obvious since Σ_k and $\widetilde{\Sigma}_k$ have the same initial value

$$\Sigma_0 = \widetilde{\Sigma}_0$$

and the use of the structure of *Lemma 1*.

B. Optimal Filter Design

Deriving the robust finite-horizon Kalman filter (3) leads to the following result

Theorem 2: The optimal filter parameters $\hat{A}_k$ and $\hat{K}_k$ will be

$$\begin{aligned} \hat{A}_k &= A_k + (A_k - \mu_k\hat{K}_k C_k)\bar{\Sigma}_k \\ &\quad E_{1,k}^T(\alpha_k^{-1}I - E_{1,k}\bar{\Sigma}_k E_{1,k}^T)^{-1}E_{1,k} \end{aligned} \tag{11}$$

And

$$\begin{aligned} \hat{K}_k &= \beta_k A_k(\bar{\Sigma}_k - \alpha_k E_{1,k}^T E_{1,k})^{-1} \\ &\quad .C_k^T[C_k S_k C_k^T + D_k T_k D_k^T \\ &\quad +(\alpha_k^{-1} + \beta_k^{-1}H_{2,k}H_{2,k}^T)]^{-1} \end{aligned} \tag{12}$$

Where

$$S_k = \bar{\Sigma}_k + \bar{\Sigma}_k E_{1,k}^T(\alpha_k^{-1}I - E_{1,k}\bar{\Sigma}_k E_{1,k}^T)^{-1}E_{1,k}\bar{\Sigma}_k \tag{13}$$

And

$$T_k = R_k + R_k E_{1,k}^T(\beta_k^{-1}I - E_{1,k}R_k E_{1,k}^T)^{-1}E_{1,k}R_k \tag{14}$$

The state and error covariance will be, respectively

$$\begin{aligned} P_{k+1} &= A_k(\bar{\Sigma}_k^{-1} - \alpha_k E_{1,k}^T E_{1,k})^{-1} A_k^T \\ &+ B_k(R^{-1} - \beta_k E_{2,k}^T E_{2,k})^{-1} B_k^T \\ &+ (\alpha_k^{-1} + \beta_k^{-1}) H_{1,k} H_{1,k}^T \end{aligned} \quad (15)$$

$$\begin{aligned} \bar{\Sigma}_{k+1} &= -\mu_k^2 A_k(\bar{\Sigma}_k^{-1} - \alpha_k E_{1,k}^T E_{1,k})^{-1} C_k^T [C_k S_k C_k^T \\ &+ D_k T_k D_k^T + (\alpha_k^{-1} + \beta_k^{-1}) H_{2,k} H_{2,k}^T]^{-1} \\ &. C_k(\bar{\Sigma}_k^{-1} - \alpha_k E_{1,k}^T E_{1,k})^{-1} A_k^T \\ &+ A_k(\bar{\Sigma}_k^{-1} - \alpha_k E_{1,k}^T E_{1,k})^{-1} A_k^T \\ &+ B_k(R_k^{-1} - \beta_k E_{2,k}^T E_{2,k})^{-1} B_k^T \\ &+ (\alpha_k^{-1} + \beta_k^{-1}) H_{1,k} H_{1,k}^T \end{aligned} \quad (16)$$

proof: Assume the solution to (8) is of the form

$$\Sigma_k = \begin{bmatrix} \Sigma_{1,k} & \Sigma_{2,k} \\ \Sigma_{2,k} & \Sigma_{2,k} \end{bmatrix} \quad (17)$$

And its initial value is $\Sigma_0 = \begin{bmatrix} P_0 & 0 \\ 0 & 0 \end{bmatrix}$

To prove that it is a solution to (8), we will use induction. We can see that they are equivalent in the initial case. Assume the argument is valid at time k. We will show that the argument is still valid at time $k+1$

Let

$$\Sigma_{k+1} = \begin{bmatrix} \Sigma_{1,k+1} & \Sigma_{12,k+1} \\ \Sigma_{21,k+1} & \Sigma_{2,k+1} \end{bmatrix} \quad (18)$$

and substituting (18) into (8) we get

$$\begin{aligned} \Sigma_{1,k} &= A_k(\Sigma_k^{-1} - \alpha_k E_{1,k}^T E_{1,k})^{-1} A_k^T \\ &+ (\alpha_k^{-1} + \beta_k^{-1}) H_{1,k} H_{1,k}^T \\ &+ B_k(R_k^{-1} - \alpha_k E_{2,k}^T E_{2,k})^{-1} B_k^T \\ \Sigma_{12,k} &= A_k(\Sigma_k^{-1} - \alpha_k E_{1,k}^T E_{1,k})^{-1} C_k^T \hat{K}_k^T \\ &+ B_k(R_k^{-1} - \alpha_k E_{2,k}^T E_{2,k})^{-1} B_k^T \\ \Sigma_{21,k} &= \Sigma_{12,k}^T \\ \Sigma_{2,k} &= (\hat{A}_k - \mu_k \hat{K}_k C_k)(\Sigma_k^{-1} - \alpha_k E_{1,k}^T E_{1,k})^{-1} \\ &. (\hat{A}_k - \mu_k \hat{K}_k C_k)^T \\ &+ \mu_k^2 \hat{K}_k C_k (\Sigma_k^{-1} - \alpha_k E_{1,k}^T E_{1,k})^{-1} C_k^T \hat{K}_k^T \\ &+ (\alpha_k^{-1} + \beta_k^{-1}) \hat{K}_k H_{2,k} H_{2,k}^T \hat{K}_k^T \\ &+ \hat{K}_k D_k (V_k^{-1} - \beta_k E_{2,k}^T E_{2,k})^{-1} D_k^T \hat{K}_k^T \end{aligned} \quad (19)$$

It can be shown that $\Sigma_{12,k+1} = \Sigma_{2,k+1}$ and (17) will be a solution to (8).

To derive the optimal filter parameters, let $\bar{\Sigma}_k = \begin{bmatrix} I & -I \end{bmatrix} \Sigma_k \begin{bmatrix} I & -I \end{bmatrix}^T$, then

$$\mathbf{E}[(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T] \leq tr(\bar{\Sigma}_k) \quad (20)$$

The function $tr(.)$ refers to the trace of a matrix.

$$\begin{aligned} \bar{\Sigma}_{k+1} &= \alpha_k^{-1}(H_{1,k} - \hat{K}_k H_{2,k})(H_{1,k} - \hat{K}_k H_{2,k})^T \\ &+ B_k(R_k^{-1} - \alpha_k E_{2,k} E_{2,k}^T)^{-1} B_k^T \\ &+ \beta_k^{-1} H_{1,k} H_{1,k}^T + (\alpha_k^{-1} + \beta_k^{-1}) \hat{K}_k H_{2,k} H_{2,k}^T \hat{K}_k^T \\ &+ \hat{K}_k D_k (V_k^{-1} - \beta_k E_{2,k} E_{2,k}^T)^{-1} D_k^T \hat{K}_k^T \\ &+ \mu_k(1-\mu_k) \hat{K}_k C_k \Sigma_{1,k} C_k^T \hat{K}_k^T \\ &+ \begin{bmatrix} A_k - \mu_k \hat{K}_k C_k & \mu_k \hat{K}_k C_k - \hat{A}_k \end{bmatrix} \\ &. (\Sigma_k^{-1} - \alpha_k \widetilde{E}_{1,k}^T \widetilde{E}_{1,k})^{-1} \\ &\begin{bmatrix} A_k - \mu_k \hat{K}_k C_k & \mu_k \hat{K}_k C_k - \hat{A}_k \end{bmatrix}^T \end{aligned} \quad (21)$$

In order to find the optimal filter parameters $\hat{A}_k$ and $\hat{K}_k$ that minimize $\bar{\Sigma}_{k+1}$, we take the first variation to (21) with respect to $\hat{A}_k$ and $\hat{K}_k$ and obtain

$$\begin{aligned} \frac{\partial \bar{\Sigma}_{k+1}}{\partial \hat{A}_k} &= \begin{bmatrix} A_k - \mu_k \hat{K}_k C_k & \mu_k \hat{K}_k C_k - \hat{A}_k \end{bmatrix} \\ &. (\Sigma_k^{-1} - \alpha_k \widetilde{E}_{1,k}^T \widetilde{E}_{1,k})^{-1} \begin{bmatrix} 0 & -I \end{bmatrix}^T \\ &= 0 \end{aligned} \quad (22)$$

and

$$\begin{aligned} \frac{\partial \bar{\Sigma}_{k+1}}{\partial \hat{K}_k} &= \alpha_k^{-1}(H_{1,k} - \hat{K}_k H_{2,k})(-H_{2,k})^T \\ &+ (\alpha_k^{-1} + \beta_k^{-1}) \hat{K}_k H_{2,k} H_{2,k}^T \\ &+ \hat{K}_k D_k (V_k^{-1} - \beta_k E_{2,k}^T E_{2,k})^{-1} D_k^T \\ &+ \mu_k(1-\mu_k) \hat{K}_k C_k \Sigma_{1,k} C_k^T \\ &+ \begin{bmatrix} A_k - \mu_k \hat{K}_k C_k & \mu_k \hat{K}_k C_k - \hat{A}_k \end{bmatrix} \\ &. (\Sigma_k^{-1} - \alpha_k \widetilde{E}_{1,k}^T \widetilde{E}_{1,k})^{-1} \\ &\begin{bmatrix} -\mu_k C_k & \mu_k C_k \end{bmatrix}^T = 0 \end{aligned} \quad (23)$$

Following some algebraic manipulations, the filter parameters $\hat{A}_k$ and $\hat{K}_k$ will be of the form

$$\begin{aligned} \hat{A}_k &= A_k + (A_k - \mu_k \hat{K}_k C_k) \bar{\Sigma}_k \\ &. E_{1,k}^T (\alpha_k^{-1} I - E_{1,k} \bar{\Sigma}_k E_{1,k}^T)^{-1} E_{1,k} \end{aligned} \quad (24)$$

and

$$\hat{K}_k = (\mu_k A_k S_k C_k^T + \alpha_k^{-1} H_{1,k} H_{1,k}^T) \hat{R}_k^{-1} \quad (25)$$

where

$$\begin{aligned} \hat{R}_k &= C_k S_k C_k^T + D_k T_k D_k^T + (\alpha_k^{-1} + \beta_k^{-1}) H_{2,k} H_{2,k}^T \\ &+ \mu_k(1-\mu_k) C_k \bar{\Sigma}_k C_k^T \\ S_k &= (\Sigma_k^{-1} - \alpha_k E_{1,k}^T E_{1,k})^{-1} \quad (26) \\ T_k &= (V_k^{-1} - \alpha_k E_{2,k}^T E_{2,k})^{-1} \quad (27) \end{aligned}$$

These filter parameters guarantee that the estimation error covariance will be upper bounded by the proposed upper bound $\bar{\Sigma}_k$ and this ends the proof.

IV. Simulation Results

In this section, we will demonstrate the effectiveness of the proposed filter in the state estimation of systems with uncertainties in the process and noise covariance matrices combined with the possibility of missing measurements. We will use the system used in [18] and add the uncertain block to the process and output noise covariances. The parameters for the simulation of system (1),(2) are:

$$
\begin{aligned}
A_k &= \begin{bmatrix} 0 & 0.sin(6k) \\ 0.2 & 0.3 \end{bmatrix} \\
C_k &= \begin{bmatrix} 0.5 + 0.3sin(6k) & 1 \end{bmatrix} \\
B_k &= \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, D_k = 2.3 \\
H_{1,k} &= \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, E_{1,k} = \begin{bmatrix} 0.2 & 0.1 \end{bmatrix} \\
H_{2,k} &= 4, E_{2,k} = -0.7 \\
\alpha_k &= 3, \beta_k = 1, \mu_k = 0.8 \\
x_0 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, F_k = sin(0.6k) \\
P_0 &= 2I, \bar{\Sigma}_0 = I \\
\mathbf{E}[\gamma_k] &= 0.8
\end{aligned}
$$

The simulation shows in Figs. 1 and 2 that the proposed robust Kalman filter is bounded by the proposed upper bound while the conventional Kalman filter exceeds it. It also shows that The robust Kalman filter outperforms the conventional Kalman filter in the case of possible missing measurements and the existence of uncertainty blocks in the modeling parameters. A comparison between the proposed robust Kalman filter which considers the possible uncertainty in the noise covariances and the robust Kalman filter presented in [18] is made in figs. 3 and 4. By seeing that the proposed robust Kalman filter has lower error variance, figs. 3 and 4 demonstrate the effectiveness of considering the existence of uncertainties in the noise covariance matrices.

V. Conclusion

In this paper, a robust finite-horizon Kalman filter was presented for systems suffering from norm-bounded uncertainty blocks in the state and the noise covariances. The systems under consideration also suffer from random missing measurements. The filter was obtained in a recursive form to be suitable for online applications and it does not include the uncertain block. The upper bound on the estimation error covariance was obtained. The upper bound was guaranteed to be minimal by selection of the optimal filter parameters. If the system parameters are known precisely and all the measurements are guaranteed to arrive to the estimation point, then the proposed robust Kalman filter will be equivalent to the conventional Kalman filter.

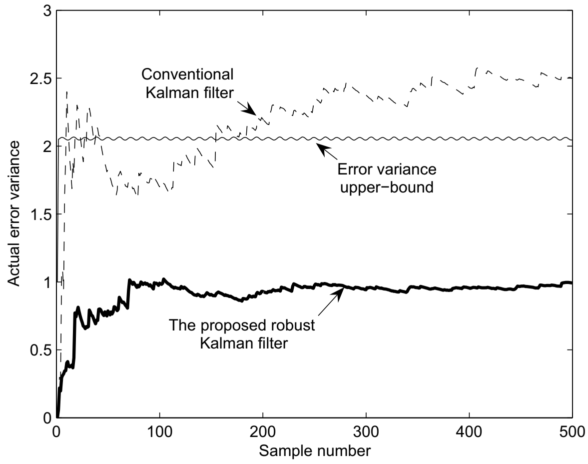

Fig. 1. Estimation error covariance for the first state using the conventional Kalman filtering and the proposed robust Kalman filter relative to the proposed upper-bound

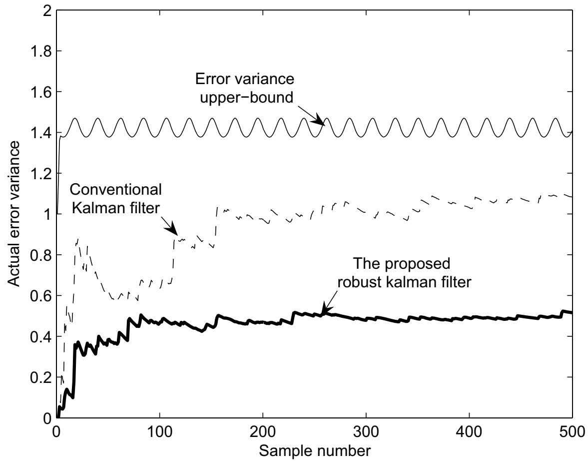

Fig. 2. Estimation error covariance for the second state using the conventional Kalman filtering and the proposed robust Kalman filter relative to the proposed upper-bound

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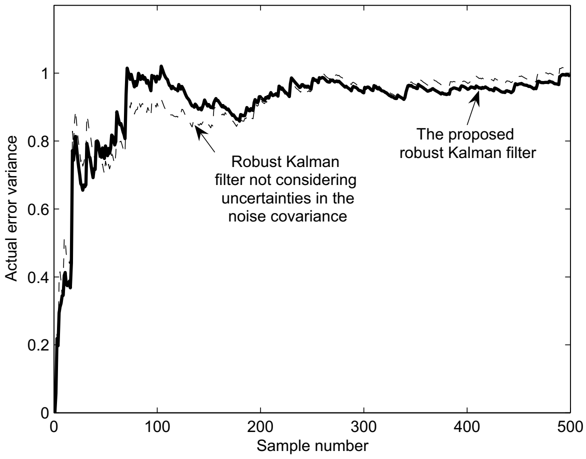

Fig. 3. Estimation error covariance comparing the proposed robust Kalman filter considering the uncertainties in the noise covariances and the robust Kalman filter not considering these uncertainties in [18] for the first state.

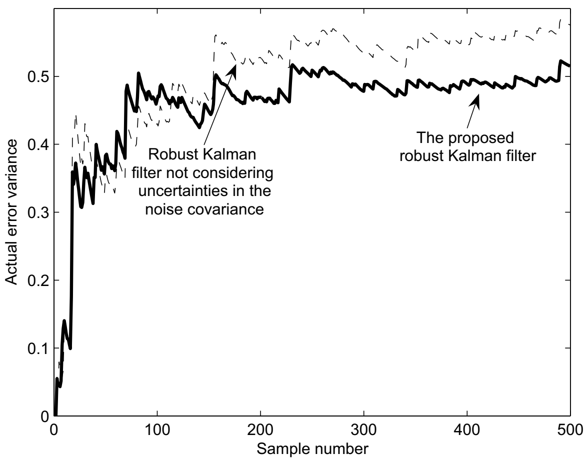

Fig. 4. Estimation error covariance comparing the proposed robust Kalman filter considering the uncertainties in the noise covariances and the robust Kalman filter not considering these uncertainties in [18] for the second state.

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