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On migrative means and copulas

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Summary

In this short work we extend the results of J.Fodor and I.J. Rudas [6] characterizing migrative triangular norms, to quasi-arithmetic means. We use idempotisation construction to obtain quasi-arithmetic means migrative with respect to fixed parameter $\alpha$. We also obtain the necessary and sufficient condition for a migrative triangular norm to be a copula.

Keywords: Aggregation operators, migrative property, triangular norms, quasi-arithmetic means, copulas.

1 INTRODUCTION

Recently in [6] Fodor and Rudas provided a construction of $\alpha$-migrative continuous triangular norms based on their additive generators. A binary operation $T$ is called $\alpha$-migrative if $T(\alpha x, y) = T(x, \alpha y)$ for a fixed $\alpha \in [0, 1]$ and for all $x, y \in [0, 1]$. This property was considered for the first time in [10]. However, observe that the term $\alpha$-migrative was introduced by several authors [5] (this work was submitted for publication in 2006).

Further, in [3] Bustince, Mesiar and Montero investigated many classes of aggregation functions and identified those that are migrative for all $\alpha > 0$ (we shall call them simply migrative). In this note we apply the construction in [6] and the method of idempotisation (see, e.g., [4]) to obtain quasi-arithmetic means that are $\alpha$-migrative for a fixed $\alpha$, as well as discuss $\alpha$-migrative copulas.

We consider this property when we want a global evaluation of two individual inputs to be invariant when we increase one input by a factor $k = 1/\alpha$ and at the same time decrease the other input by the same factor, i.e., $T(u, v) = T(u/\alpha, \alpha v)$ for all $u, v \in [0, 1]$ or equivalently $T(\alpha x, y) = T(x, \alpha y)$ where $u = \alpha x$ and $v = y$.

We recall the main results from [6] and [3]. Firstly, a continuous $\alpha$-migrative aggregation function $A$ has 0 as the absorbing element (annihilator) $A(x, 0) = A(0, x) = 0$ for all $x \in [0, 1]$. A migrative aggregation function needs to be symmetric. A continuous $\alpha$-migrative t-norm must be strict. The only continuous migrative t-norm is the product, and the only migrative quasi-arithmetic mean is the geometric mean. There are no migrative t-conorms, uninorms or nullnorms. In [3] the authors establish that to be migrative, an aggregation function must satisfy $A(x, y) = g(xy)$ for some monotone increasing univariate function $g$.

2 IDEMPOTISATION METHOD

It is well-known that for any binary aggregation function $A$ and any strictly increasing bijection $\phi : [0, 1] \rightarrow [0, 1]$, is possible to define a new binary aggregation function (by transformation) $A_{\phi} = \phi \circ A$; it is called the $\phi$-transform of $A$. A special class of binary aggregation functions are the t-norms (see [7, 8]). In this case, we can consider the inverse of the diagonal of a t-norm $(\delta(x) = T(x, x))$ to define the $\phi$-transform of this t-norm, i.e., $T_{\phi} = \delta^{-1} \circ T$, provided $\delta^{-1}$ exists. It is easy to see that the new aggregation function $T_{\phi}$ is idempotent.

An illustrative example is the product t-norm $T(x, y) = xy$ and the geometric mean $G(x, y) = \sqrt{xy}$, with $\phi(x) = \sqrt{x}$.

We will be dealing with strict t-norms, which means that they are also Archimedean. Such t-norms have additive generators, strictly decreasing continuous functions $g : [0, 1] \rightarrow [0, \infty]$ with $g(1) = 1, g(0) = \infty$. 

which are related to $T$ in

$$T(x, y) = g^{-1}(g(x) + g(y)).$$

Idempotisation method applies to strict t-norms. Furthermore, it is not difficult to show that in this case $T_\phi$ is a quasi-arithmetic mean with the same generating function as the additive generator of $T$ (up to a linear transformation).

**Proposition 1** Let $T$ be a strict t-norm, let $g$ be its additive generator, and $\phi = \delta^{-1}$, the inverse of the diagonal of $T$. Then $T_\phi$ is a quasi-arithmetic mean, with a generating function $h(x) = ag(x) + b$ for some $a, b \in \mathbb{R}$, i.e., $T_\phi(x, y) = h^{-1}\left(\frac{h(x) + h(y)}{2}\right)$.

**Proof.** Recall that if $M_1$ and $M_2$ are quasi-arithmetic means with generating functions $g_1$ and $g_2$ respectively, then $M_1 = M_2$ iff $g_1(x) + b$ for some $a, b \in \mathbb{R}$.

From $\delta(x) = T(x, x)$ we get $\delta^{-1}(x) = g^{-1}\left(\frac{2(x)}{2}\right)$. Furthermore, we have $\phi(T(x, y)) = \delta^{-1}(T(x, y)) = g^{-1}\left(\frac{g(x) + g(y)}{2}\right)$, for all $x, y \in [0, 1]$.

Now, if $\phi \circ T$ defines a quasi-arithmetic mean, with a generating function $h$, the following functional equation holds

$$g^{-1}\left(\frac{g(x) + g(y)}{2}\right) = h^{-1}\left(\frac{h(x) + h(y)}{2}\right),$$

or equivalently

$$h \circ g^{-1}\left(\frac{u + v}{2}\right) = \frac{h \circ g^{-1}(u) + h \circ g^{-1}(v)}{2},$$

for all $u, v \in [0, \infty]$. This is the Jensen’s functional equation and its solution is given by $h \circ g^{-1}(u) = a \cdot u + b$ for some $a, b \in \mathbb{R}$. Therefore, $h(x) = ag(x) + b$ for all $x \in [0, 1]$. □

Since the product is the only continuous migrative t-norm, then we have

**Corollary 1** The only migrative quasi-arithmetic mean is the geometric mean.

Next, let us apply the construction method in [6] for $\alpha$-migrative continuous t-norms. Let $\alpha \in [0, 1]$ be fixed and let $t_0 : [\alpha, 1] \rightarrow [0, 1]$ be a strictly decreasing continuous function, $t_0(1) = 0$. The function $g$ is an additive generator of a continuous migrative t-norm iff it is given by

$$g(x) = kt_0(\alpha) + t_0\left(\frac{x}{\alpha^k}\right), \text{ if } x \in [\alpha^{k+1}, \alpha^k],$$

where $k$ is any non-negative integer.

Note that (1) can be rewritten in recursive form as follows. Let $g_0(x) = t_0(x), x \in [\alpha, 1]$. Then $g_1(x) = g_0(\alpha) + g_0\left(\frac{x}{\alpha}\right), x \in [\alpha^2, \alpha^1], and$

$$g_{k+1}(x) = g_k(\alpha) + g_k\left(\frac{x}{\alpha}\right), x \in [\alpha^{k+1}, \alpha^k], k = 2, 3, \ldots. \tag{2}$$

**Corollary 2** A quasi-arithmetic mean is $\alpha$-migrative iff one of its generating functions is given in (1).

### 3 MIGRATIVE COPULAS

It is known that Archimedean copulas are characterized by the convexity of their additive generators [9], see also general references [1, 2, 8]. Let us establish under which conditions an $\alpha$-migrative t-norm $T$ is a copula. Of course, $T$ has to be continuous, hence we apply construction (1).

**Theorem 1** The necessary and sufficient condition for an $\alpha$-migrative t-norm $T$ to be a copula is that its additive generator $g$ be given in (1) with $t_0$ convex on $[\alpha, 1]$ and satisfying

$$\frac{1}{\alpha} t'_0(1) \leq t'_0(\alpha). \tag{3}$$

**Proof.** For $g$ in (1) to be convex, obviously $t_0$ must be convex on $[\alpha, 1]$. A continuous convex function on a compact set is differentiable almost everywhere. Consider left and right derivatives of $g$ at $\alpha$ (which must exist, as $\alpha \in [0, 1] = Dom(g)$). For $g$ to be convex we need $\frac{1}{\alpha} t'_0(1) = g'_0(\alpha) \leq g'_0(\alpha) = t'_0(\alpha)$. For the remaining values $x = \alpha^k, k = 2, 3, \ldots$ we have exactly the same condition of convexity, namely $\frac{1}{\alpha} g'_k(\alpha^k) \leq g_{k+1}'(\alpha^{k+1})$, as can be clearly seen from the recursive form (2), and the above condition follows from (3). □

Consider now piecewise continuous twice differentiable additive generators $g$ in (1), for which we have the following sufficient condition.

**Corollary 3** A sufficient condition for an $\alpha$-migrative t-norm $T$ to be a copula is that its additive generator $g$ be given in (1) and $t_0$ satisfy

$$0 \leq t'_0 \leq \frac{t_0'(1)}{\alpha}. \tag{4}$$

**Proof.** Denote by $f = g'$ (or its left or right derivative at 1 and $\alpha$ respectively). Then we need the condition $f(1) \leq f(\alpha)$. 108
Since $f$ is differentiable on $[\alpha, 1]$, $f(\alpha) \geq f(1) - (1 - \alpha)\sup f'(x)$. Then
\[
f(\alpha) \geq f(1) - \frac{f(1)}{\alpha} + \frac{f(1)}{\alpha} - (1 - \alpha)\sup f'(x) = f(1)\frac{\alpha - 1}{\alpha} - (1 - \alpha)\sup f'(x) + \frac{f(1)}{\alpha}.
\]
Then for condition $\frac{f(1)}{\alpha} \leq f(\alpha)$ to hold we need
\[
f(1)\frac{\alpha - 1}{\alpha} - (1 - \alpha)\sup f'(x) \geq 0
\]
or equivalently,
\[
\sup f'(x) \leq -\frac{f(1)}{\alpha}.
\]
\[\square\]

**Example 1** Let $t_0(x) = 1 - x$. Then $g$ is a continuous piecewise linear function (for any $\alpha \in [0, 1]$), with infinitely many linear pieces joined at $\alpha^k, k = 1, 2 \ldots$. Since $t_0 = 0$, $g$ is convex by Corollary 3.

**Example 2** Let $t_0(x) = x^p - 1, p < 0$. By using Theorem 1, we need to establish
\[
\frac{1}{\alpha}p\lambda^{p-1} \leq p\alpha^{p-1},
\]
or, equivalently,
\[
\alpha^p \leq 1,
\]
which is certainly false for $\alpha \in [0, 1]$, hence migrative t-norms in this example are not copulas.

**Example 3** Let $t_0(x) = \log(2 - x^\lambda), 0 < \lambda \leq \frac{1}{2}$. By using Theorem 1, we need to establish
\[
-\frac{\lambda}{\alpha} \leq -\frac{\lambda \cdot \alpha^{\lambda-1}}{2 - \alpha^{\lambda}},
\]
or, equivalently,
\[
\alpha^\lambda \leq 1,
\]
which is true for $\alpha \in [0, 1]$ and $0 < \lambda \leq \frac{1}{2}$. The migrative t-norm defined by this generator is a copula for $0 < \lambda \leq \frac{1}{2}$ and any $\alpha \in [0, 1]$.

**Example 4** Let $t_0(x) = (1 - \log x)^\lambda - 1$, with $\lambda \in [0, \infty]$. Applying Theorem 1 again, we need to prove that
\[
(1 - \log \alpha)^\lambda - 1 \leq 1,
\]
which is true for $\lambda \leq 1$. The $\alpha$-migrative t-norm defined by this generator is a copula for $0 < \lambda \leq 1$ and any $\alpha \in [0, 1]$.

4 Conclusion

We extended the results of Fodor and Rudas [6] on the characterization of $\alpha$-migrative t-norms to quasi-arithmetic means. In various applications averaging behavior is important, and the described idempotization construction allows one to use the same additive generator as in [6]. Further, we provided verifiable sufficient conditions characterizing $\alpha$-migrative Archimedean copulas, which is an important class of functions with many applications.

ACKNOWLEDGEMENTS

This work was supported by the projects MTM2006-08322 and PR2007-0193 from Ministerio de Educación y Ciencia, Spain.

References