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On Lipschitz properties of generated aggregation functions

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Abstract

This article discusses Lipschitz properties of generated aggregation functions. Such generated functions include triangular norms and conorms, quasi-arithmetic means, uninorms, nullnorms and continuous generated functions with a neutral element. The Lipschitz property guarantees stability of aggregation operations with respect to input inaccuracies, and is important for applications. We provide verifiable sufficient conditions to determine when a generated aggregation function holds the k-Lipschitz property, and calculate the Lipschitz constants of power means. We also establish sufficient conditions which guarantee that a generated aggregation function is not Lipschitz. We found the only 1-Lipschitz generated function with a neutral element $e \in]0, 1[$.

Keywords Aggregation functions, generated aggregation functions, k-Lipschitz aggregation functions, triangular norms, quasi-arithmetic means, uninorms, nullnorms, stability.

1 Introduction

Aggregation of several input values into a single output value is an indispensable tool in many disciplines and applications such as decision making [22], pattern recognition, expert and decision support systems, information retrieval, etc [6]. There is a wide range of aggregation functions which provide flexibility to the modeling process, including weighted quasi-arithmetic means, ordered weighted averaging functions, Choquet and Sugeno integrals, triangular norms and conorms, uninorms, nullnorms, and many others. There are several recent books that provide details of many aggregation methods [1, 2, 4, 8, 20].

For applications it is important to design aggregation functions that are stable with respect to small perturbations of inputs (e.g., due to input inaccuracies). Such aggregation functions need not only be continuous, but also Lipschitz continuous [5]. Two classes, kernel and 1-Lipschitz aggregation functions, have been studied in [7, 9, 10]. It is known, for instance, that 1-Lipschitz triangular norms are copulas [1, 8, 18]. More recently, k-Lipschitz t-norms and t-conorms were studied [15–17]. These functions have the Lipschitz constant k in 1-norm (see below). k-Lipschitz t-norms do not increase the perturbation of inputs due to inaccuracies by more than a factor of k, which is suitable for many applications.

Many aggregation functions are constructed with the help of univariate generating functions, called additive generators. Notable examples are Archimedean triangular norms and conorms, as well as representable uninorms. In this article we examine such generated functions and establish conditions on their generators which guarantee that their Lipschitz constant is not greater than k. We also establish conditions under which generated functions are not Lipschitz.

The structure of this paper is as follows. In Section 2 we recall basic notions and the main classes of aggregation functions. We recall the results concerning k-Lipschitz triangular norms and conorms in Section 3. Section 4 contains our main results concerning quasi-arithmetic means. Section 5 deals with continuous generated functions with a neutral element. We finish the paper with conclusions.

2 Preliminaries

We restrict ourselves to aggregation functions defined on $[0, 1]^n$. A number or a letter in boldface will denote a vector in $[0, 1]^n$.

Definition 1 (Aggregation function) *A function* $f : [0,1]^n \to [0,1]$ *is called an aggregation function if it is monotone non-decreasing in each variable and satisfies* $f(0,0,\ldots,0) = 0,\ f(1,1,\ldots,1) = 1$.

Well known classes of aggregation functions and their properties are discussed in detail in the books [1, 2, 4, 8, 20]. We will concentrate on continuous Archimedean triangular norms and conorms (t–norms and t–conorms), quasi-arithmetic means and generated aggregation functions with a neutral element. We recall that the dual of a t-norm is a t-conorm, and vice versa, and that the classes of uninorms, nullnorms and quasi-arithmetic means are closed under duality. We recall the following representation theorem.

Theorem 1 *[12] Any continuous Archimedean t-norm* T *admits the following representation*

$$T(x,y) = t^{(-1)}(t(x) + t(y)),$$

for all $x, y \in [0,1]$ *where* $t : [0,1] \to [0,\infty]$ *with* $t(1) = 0$ *is a continuous, strictly decreasing function and* $t^{(-1)}$ *is the pseudo inverse of* t,

$$t^{(-1)}(x) = \begin{cases} t^{-1}(x), & \text{if } 0 \le x < t(0), \\ 0, & \text{if } t(0) \le x \le +\infty \end{cases}$$

The function t is called an additive generator of T. Strict t-norms are characterized by additive generators satisfying $t(0) = \infty$, whereas additive generators of nilpotent t-norms satisfy $t(0) < \infty$. The following construction provides a general class of aggregation functions built by using additive generators.

Definition 2 (Generated function) *Let* $g_1, \ldots, g_n : [0,1] \to [-\infty, +\infty]$ *be a family of continuous non-decreasing functions and let* $h : \sum_{i=1}^{n} Ran(g_i) \to [0,1]$ *be a continuous non-decreasing surjection*[1]. *The function* $f : [0,1]^n \to [0,1]$ *given by*

$$f(x_1, \ldots, x_n) = h\left(g_1(x_1) + \ldots + g_n(x_n)\right)$$

is called a generated function, and $(\{g_i\}_{i \in \{1,\ldots,n\}}, h)$ *is called a generating system.*

[1] That is, $Ran(h) = [0,1]$.

The monotonicity of functions that form the generating system, along with the fact that h is surjective, ensure that every generated function is an aggregation function. Some properties of generated functions were studied in [11], among which we note the following: generated functions such that $Dom(h) \neq [-\infty, +\infty]$ are always continuous on the whole domain. However the condition $Dom(h) = [-\infty, +\infty]$ by itself does not necessarily entail lack of continuity.

We concentrate on the case where $g_1 = g_2 = \ldots = g_n = g$ and $h = g^{-1}$ (or $h = g^{(-1)}$ if required), and later on extend it to $g_i(x) = w_i g(x), w_i \geq 0, \sum w_i = 1$.

Let $g : [0,1] \to [-\infty, \infty]$ be a continuous strictly monotone function, called a *generating function* or generator. Of course, g is invertible, but it is not necessarily a bijection (its range may be $Ran(g) \subset [-\infty, \infty]$). The examples of generated functions are continuous Archimedean t-norms and t-conorms, and representable uninorms. With respect to uninorms, we recall that a) representable uninorms are *almost continuous* (i.e., continuous on $[0,1]^2 \backslash \{(0,1),(1,0)\}$), and b) the underlying t-norm and t-conorm of a representable uninorm are necessarily *strict*.

In this paper our main focus is on the following two classes of generated aggregation functions.

Definition 3 (Weighted quasi-arithmetic mean) *For a given generating function g, and a weighting vector $\mathbf{w}$, such that $w_i \geq 0$, $\sum w_i = 1$, the weighted quasi-arithmetic mean is the function*

$$M_{\mathbf{w},g}(x_1, \ldots, x_n) = g^{-1}\left(\sum_{i=1}^{n} w_i g(x_i)\right). \tag{1}$$

When all the weights are equal ($w_i = \frac{1}{n}$), we call this function simply a quasi-arithmetic mean and denote it by M_g. Quasi-arithmetic means do not have a neutral element, and they can have an absorbing element when $g(0) = \pm\infty$ or $g(1) = \pm\infty$. When $g(x) = x^p, p \neq 0$, the quasi-arithmetic mean is called the *power mean* and is denoted by $M_{[p]}$. The limiting cases $p \to -\infty$, $p \to \infty$ and $p \to 0$ correspond to the minimum, maximum and the geometric mean respectively.

Definition 4 (Generated aggregation functions with neutral element) *A function $f : [0,1]^n \to [0,1]$ defined by*

$$f(x_1, \ldots, x_n) = g^{(-1)}\left(g(x_1) + \ldots + g(x_n)\right) \tag{2}$$

with $g : [0,1] \to [-\infty, +\infty]$ a continuous strictly increasing function such that $g(e) = 0, g^{(-1)}$ its pseudo-inverse,

$$g^{(-1)}(x) = g^{-1}(\min(g(1), \max(g(0), x))).$$

and $Ran(g) \subsetneq [-\infty + \infty]$, is called a continuous generated function with the neutral element e [13, 14].

Now, we consider the central concept of this work:

Definition 5 (Lipschitz continuity) *An aggregation function f is called Lipschitz continuous if there is a positive number k, such that for any two vectors $\mathbf{x}, \mathbf{y}$ in the domain of definition of f:*

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq kd(\mathbf{x}, \mathbf{y}), \tag{3}$$

where $d(\mathbf{x}, \mathbf{y})$ is a distance between $\mathbf{x}$ and $\mathbf{y}$. The smallest such number k is called the Lipschitz constant of f (in the distance d).

Typically the distance is chosen as a p-norm $d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}||_p$, with $||\mathbf{x}||_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$, for finite p, and $||\mathbf{x}||_\infty = \max_{i=1,\ldots,n} |x_i|$.

Definition 6 (Locally Lipschitz functions) *A function f is called locally Lipschitz continuous on Ω if for every $x \in \Omega$ there exists a neighborhood $D(x)$ such that f restricted to $D(x)$ is Lipschitz.*

Definition 7 (k–Lipschitz aggregation functions) *An aggregation function f is called k-Lipschitz if for all $\mathbf{x}, \mathbf{y} \in [0,1]^n$ the following holds*

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq k||\mathbf{x} - \mathbf{y}||_1.$$

Of course, duality w.r.t. standard negation preserves the Lipschitz property (and the Lipschitz constant). It is easy to see that if an aggregation function f is k–Lipschitz, it is also m–Lipschitz for any $m \geq k$. Also any convex combination of $k-$Lipschitz aggregation functions $f = \alpha f_1 + \beta f_2$, $\alpha + \beta = 1, \alpha, \beta \geq 0$, is $k-$Lipschitz. If f is continuously differentiable, the Lipschitz constant k in 1-norm is simply the least upper bound on the partial derivatives of f. For example, if f is a product, then $\frac{\partial f}{\partial x_i} = x_1 \ldots x_{i-1} x_{i+1} \ldots x_n$, and the maximum of this expression is reached at 1, hence the product is 1-Lipschitz. We also remind that for univariate functions, the Lipschitz condition implies differentiability almost everywhere, which implies the existence of left- and right- derivatives. The supremum of the left- and right-derivatives is then the Lipschitz constant.

3 k-Lipschitz triangular norms

The class of k–Lipschitz t–norms, whenever $k > 1$, has been characterized in [15]. 1–Lipschitz t–norms are copulas, see [1, 2, 8]. 1–Lipschitz Archimedean t–norms are characterized by convex additive generators g.

The k–Lipschitz property implies continuity of the t–norm. Characterization of all k–Lipschitz t–norms is reduced to the problem of characterization of all Archimedean k–Lipschitz t–norms.

Definition 8 (k–convex function) *Let $g : [0,1] \to [0,\infty]$ be a strictly monotone function and let $k \in]0,\infty[$ be a real constant. Then g will be called k–convex if*

$$g(x + k\varepsilon) - g(x) \leq g(y + \varepsilon) - g(y)$$

holds for all $x \in [0,1[, y \in]0,1[$, with $x \leq y$ and $\varepsilon \in]0, \min(1-y, \frac{1-x}{k})]$.

Obviously, if $k = 1$, the function g is convex. Observe that, if a strictly monotone function is k–convex then it is also a continuous function on $]0,1[$. A decreasing function g can be k–convex only for $k \geq 1$. Moreover, when a decreasing function g is k–convex, it is also m–convex for all $m \geq k$. In the case of a strictly increasing function $g^\star$, it can be k-convex only for $k \leq 1$. Moreover, when $g^\star$ is k–convex, it is m–convex for all $m \leq k$.

Considering $k \geq 1$ and a strictly decreasing function g, we provide the following characterization given in [15].

Proposition 1 *Let $T : [0,1]^2 \to [0,1]$ be a continuous Archimedean t–norm and let $g : [0,1] \to [0,\infty], g(1) = 0$ be an additive generator of T. Then T is k–Lipschitz if and only if g is k–convex.*

Corollary 1 *[19] Let $g : [0,1] \to [0,\infty]$ be an additive generator of a t–norm T which is differentiable on $]0,1[$ and let $g'(x) < 0$ for $0 < x < 1$. Then T is k–Lipschitz if and only if $g'(y) \geq kg'(x)$ whenever $0 < x < y < 1$.*

Corollary 2 *[15] Let $T : [0,1]^2 \to [0,1]$ be an Archimedean t–norm and let $g : [0,1] \to [0,\infty]$ be an additive generator of T such that g is differentiable on $]0,1[\setminus\mathcal{S}$, where $\mathcal{S} \subset [0,1]$ is a discrete set. Then T is k–Lipschitz if and only if $kg'(x) \leq g'(y)$ for all $x, y \in [0,1], x \leq y$ such that $g'(x)$ and $g'(y)$ exist.*

The following useful results follow from Corollary 1. They help to determine whether a given piecewise differentiable t–norm is k–Lipschitz.

Corollary 3 *Let* $T : [0,1]^2 \to [0,1]$ *be an Archimedean t–norm and let* $g : [0,1] \to [0,\infty]$ *be its additive generator differentiable on* $]0,1[$*, and* $g'(t) < 0$ *on* $]0,1[$*. If*

$$\inf_{t \in]x,1[} g'(t) \geq k \sup_{t \in]0,x[} g'(t)$$

holds for every $x \in]0,1[$ *then* T *is* k*–Lipschitz.*

Corollary 4 *Let* $g : [0,1] \to [0,\infty]$ *be a strictly decreasing function, differentiable on* $]0,a[\cup]a,1[$*. If* g *is k-convex on* $[0,a[$ *and on* $]a,1]$*, and if*

$$\inf_{t \in]a,1[} g'(t) \geq k \sup_{t \in]0,a[} g'(t),$$

then g *is* k*–convex on* $[0,1]$*.*

Consider two related classes of aggregation functions.

Remark 1 *Clearly, generated uninorms are not k-Lipschitz, since they are discontinuous at* $(0,1)$ *and* $(1,0)$ *(in the bivariate case).*

Remark 2 *Nullnorms are not generated functions, nevertheless they are closely related to t-norms and t-conorms. Recall that the values of a nullnorm* V *coincide with those of a (scaled) t-conorm* S *on* $[0,a]^2$*, with values of a (scaled) t-norm* T *on* $[a,1]^2$*, and are constant* $V(x,y) = a$ *elsewhere, where* a *is the absorbing element. Thus, each nullnorm univocally defines a t-norm and a t-conorm and vice versa.*

Then it is clear that a nullnorm V *is k-Lipschitz if and only if the underlying t-norm and t-conorm* T_V *and* S_V *are k-Lipschitz. The Lipschitz constant of* V *is the maximum of the Lipschitz constants of* T_V *and* S_V*.*

4 Quasi-arithmetic means

Quasi-arithmetic means, given in Definition 3, are a class of generated aggregation functions. We start with bivariate quasi-arithmetic means. We recall that quasi-arithmetic means are continuous if and only if $Ran(g) \neq [-\infty, \infty]$

[11], and that their generators are not defined uniquely, i.e., if $g(t)$ is a generating function of some weighted quasi-arithmetic mean, then $ag(t) + b$, $a, b \in \Re$, $a \neq 0$ is also a generating function of the same mean provided $Ran(g) \neq [-\infty, \infty]$. For example, $g_1(t) = t^p$ and $g_2(t) = -t^p + 1$ generate the same power mean $M_{[p]}$. For this reason, one can assume that g is monotone increasing, as otherwise we can simply take $-g$.

We shall consider two cases: I) $g(0) = 0$, $g(1) = 1$, and II) $g(0) = -\infty, g(1) = Const < \infty$. Of course, by duality we also cover the case $g(1) = \infty, g(0) > -\infty$, and by using appropriate linear transformations, all generators can be reduced to the mentioned cases.

Remark 3 *As opposed to the case of convex additive generators of t-norms, where the resulting t-norms are 1-Lipschitz, convexity of the generator g does not play any role by itself for quasi-arithmetic means. Since both g and $-g$ are generators of the same mean, and obviously when g is convex $-g$ is concave, convexity of g by itself does not lead to the Lipschitz condition. Also note that $g(x) = -\ln(x)$ is a convex generator of the geometric mean $G(x, y) = \sqrt{xy}$, which is not Lipschitz.*

Further, even if g is convex and increasing, or convex and decreasing, this does not imply the Lipschitz condition either: note that $g_d(x) = 1 - g(1 - x)$ is a generator of a quasi-arithmetic mean dual to the one generated by g, and that the Lipschitz condition is preserved under duality. If g is convex increasing, then g_d is convex decreasing and vice versa. Thus a different condition guaranteeing Lipschitz properties is needed.

4.1 Case of finite generators

We start with the case I) of g finite. We show that M_g is $k-$Lipschitz $\Longrightarrow$ g is Lipschitz $\Longrightarrow$ g has left- and right-derivatives on $[0, 1]$. Then characterization of $k-$Lipschitz quasi-arithmetic means is expressed as (5) below for smooth generators, and as (6) for non-smooth generators.

First, let us show that g must be Lipschitz on [0,1].

Lemma 1 *Let g be finite, continuously differentiable and locally Lipschitz except at a point $a \in [0, 1]$. Then M_g is not k-Lipschitz for any k.*

Proof. Suppose that M_g is k-Lipschitz, which means it is differentiable almost everywhere in its domain (Rademacher's theorem, e.g., [21]), and we must have

$$\frac{\partial M_g}{\partial x}(x,y) \leq k,\ x,y \neq a$$

whenever such a derivative exists, and similarly for the other partial derivative. Since M_g is symmetric, only the derivative with respect to x is needed.

$$\frac{\partial M_g}{\partial x} = \frac{1}{g'\left(M_g(x,y)\right)} \cdot \frac{1}{2} g'(x) \leq k.$$

Since g is strictly increasing we must have

$\frac{1}{2}g'(x) \leq kg'(M_g(x,y))$ for all $x, y \in [0,1], x, y \neq a, M_g(x,y) \neq a$ or

$$\frac{1}{2}g'(x) \leq k \cdot \inf_{y\in[0,1]} g'(M_g(x,y)),\ x \neq a. \tag{4}$$

Since g is finite, M_g does not have an absorbing element. Let $\lim_{x\to a} g'(x) = \infty$.

$\exists y \neq a : z = M_g(a,y) \neq a$ such that $g'(z) \leq N < \infty$ (because g is locally Lipschitz). Then inequality (4) fails, because we can always choose such $x \neq a$, that $g'(x) > 2kN$, which would give us

$$kN < \frac{1}{2}g'(x) \leq kg'(z) \leq kN,$$

which is false. Then $\frac{\partial M_g}{\partial x} > k$, hence M_g is not Lipschitz. □

Theorem 2 *Let M_g be a quasi-arithmetic mean with a finite increasing generator g. The necessary and sufficient conditions for M_g to be k-Lipschitz are:*

- *If g is continuously differentiable, for all $x \in [0,1]$:*

$$\frac{1}{2}g'(x) \leq k \cdot \min_{y\in[0,1]} g'(M_g(x,y)) \tag{5}$$

- *If g is not differentiable, using left- and right- derivatives g'_-, g'_+:*

$$\frac{1}{2}g'_-(x) \leq k \inf_{z\in[M(x,0),M(x,1)]} g'_-(z) \tag{6}$$

$$\frac{1}{2}g'_+(x) \leq k \inf_{z\in[M(x,0),M(x,1)]} g'_+(z)$$

for all $x \in]0,1[$, and only one of the above inequalities for $x = 0$ and $x = 1$.

Proof. By Lemma 1, g must be locally Lipschitz on $[0, 1]$ (and hence Lipschitz; g needs not be continuously differentiable). It follows that g is differentiable almost everywhere, which means that the left- and right-derivatives exist in $[0, 1]$. We start with the case of g differentiable on $[0, 1]$. Let M_g be k-Lipschitz. Then partial derivatives of M_g exist and we must have

$$\frac{\partial M_g}{\partial x}(x, y) \leq k.$$

Following the same procedure as in Lemma 1,

$$\frac{\partial M_g}{\partial x} = \frac{1}{g'\left(M_g(x, y)\right)} \cdot \frac{1}{2}g'(x) \leq k,$$

from which we obtain condition (5). If g is not differentiable, then we adapt (5) by using left- and right-derivatives.

Sufficiency is straightforward, because the bounds on the partial derivatives of M_g (resp. left- and right- partial derivatives) on the whole domain imply k-Lipschitz. □

Remark 1 *If g is finite and concave increasing, then it is sufficient to check*

$$\frac{1}{2}g'(x) \leq kg'(M(x, 1)) = k(g' \circ g^{-1})\left(\frac{g(x)}{2} + \frac{1}{2}\right),$$

(and similarly for left- and right-derivatives if g is not smooth). If g is finite and convex increasing, it is sufficient to check

$$\frac{1}{2}g'(x) \leq kg'(M(x, 0)) = k(g' \circ g^{-1})\left(\frac{g(x)}{2}\right).$$

Let us provide some examples of Lipschitz and non-Lipschitz quasi-arithmetic means.

Example 1 *If g is linear (M_g is the arithmetic mean), $g'(x) = const$, and M_g is k-Lipschitz for $k = \frac{1}{2}$.*

Example 2 *If $g(x) = x^p, p > 1$ (M_g is a power mean $M_{[p]}$), $g'(x) = px^{p-1}$, and M_g is k-Lipschitz for $k = \left(\frac{1}{2}\right)^{\frac{1}{p}}$. It follows from*

$$\frac{1}{2}px^{p-1} \leq kp\left(\frac{x^p}{2}\right)^{\frac{p-1}{p}} = kpx^{p-1}\left(\frac{1}{2}\right)^{\frac{p-1}{p}}.$$

Example 3 *If $g(x) = x^p, 0 < p < 1$, $M_{[p]}$ is not Lipschitz by Lemma 1.*

4.2 Case of infinite generators

Now we turn to case II), $g(0) = -\infty$, which entails that 0 is the absorbing element of M_g. We have an analogue of Lemma 1. The proof is similar, except that it fails for $a = 0$, hence the modification.

Lemma 2 *Let g be continuously differentiable on $]0, 1]$ and locally Lipschitz except at a point $a \in]0, 1]$. Then M_g is not k-Lipschitz for any k.*

Then the conditions which ensure M_g is $k-$Lipschitz are:

1. Condition (6) for $x \in]0, 1]$, and
2. For any fixed $y \in]0, 1]$

$$\frac{1}{2} \lim_{x \to 0^+} \frac{g'_+(x)}{g'_+(M_g(x, y))} \leq k. \tag{7}$$

Condition (7) may or may not be satisfied depending on the rate at which $M(x, y) \to 0$ as $x \to 0$. The choice of $y > 0$ is irrelevant as $g(y)$ is finite and disappears under the limit.

Example 4 *If $g(x) = -x^p, p < -1$ (M_g is a power mean $M_{[p]}$), $g'(x) = -px^{p-1}$, and M_g is k-Lipschitz for $k = \left(\frac{1}{2}\right)^{\frac{1}{p}}$. Differentiating M_g*

$$\begin{aligned}
\frac{\partial M_g}{\partial x} &= \frac{1}{p} \left(\frac{x^p + y^p}{2} \right)^{\frac{1}{p}-1} \frac{p}{2} x^{p-1} \\
&= \left(\frac{1}{2}\right)^{\frac{1}{p}} x^{-p(\frac{1}{p}-1)} (x^p + y^p)^{\frac{1}{p}-1} \\
&= \left(\frac{1}{2}\right)^{\frac{1}{p}} (1 + x^{-p} y^p)^{\frac{1}{p}-1}.
\end{aligned}$$

Given $p < -1$,

$$k = \left(\frac{1}{2}\right)^{\frac{1}{p}} \lim_{x \to 0} \left(1 + x^{-p} y^p\right)^{\frac{1}{p}-1} = \left(\frac{1}{2}\right)^{\frac{1}{p}}.$$

Example 5 *Let $M_{[p]}$, $-1 < p < 0$ be a power mean with a generator given by $g(x) = -x^p = -x^{-\frac{1}{q}}$, $q > 1$. The Lipschitz constant will be $k = \sup \frac{\partial M}{\partial x} = 2^q = \frac{1}{2}^{\frac{1}{p}}$.*

To see this

$$\frac{\partial M_g}{\partial x} = \frac{-q}{2}\left(\frac{x^{-\frac{1}{q}} + y^{-\frac{1}{q}}}{2}\right)^{-q-1} \cdot \left(-\frac{1}{q}x^{-\frac{1}{q}-1}\right)$$
$$= 2^q x^{\frac{1}{q}(-q-1)}(x^{-\frac{1}{q}} + y^{-\frac{1}{q}})^{-q-1}$$
$$= 2^q \left(1 + \left(\frac{x}{y}\right)^{\frac{1}{q}}\right)^{-q-1}.$$
$$k = \sup\left\{2^q \left(1 + \left(\frac{x}{y}\right)^{\frac{1}{q}}\right)^{-q-1}\right\} = 2^q = \left(\frac{1}{2}\right)^{\frac{1}{p}}.$$

Condition (7) deals with the asymptotic behavior of the additive generators near 0. Its direct verification for a given g may be difficult. In the remainder of this section we will establish two sufficient conditions that guarantee that a quasi-arithmetic mean *is not* Lipschitz (although it is continuous). These conditions are easier to verify, and they provide a tool for a quick screening of additive generators with respect to their suitability for applications.

One sufficient condition involves an inequality on the derivatives of the inverse of an additive generator. The other condition is that a decreasing additive generator cannot decrease slower than a certain rate (1/polynomial) when $x \to 0$. We will express this rate through the growth of an auxiliary function $\frac{1}{g^{-1}}$, for which the growth is expressed in traditional terms (e.g., polynomial) when $x \to \infty$. First, two simple auxiliary results.

Lemma 3 *If two functions f, g are continuous and differentiable at $x = 0$ and, $f(0) = g(0)$ and $f(x) \geq g(x)$ for $x > 0$, then $f'(0) \geq g'(0)$.*

Proof. Follows directly from the definition of the derivative. □

The next result is a well-known condition for comparability of quasi-arithmetic means, see, e.g., [3].

Theorem 3 *Let g_1, g_2 be the generators of quasi-arithmetic means M_{g_1} and M_{g_2}, and g_1 decreasing. Then $M_{g_1} \leq M_{g_2}$ if and only if $g_1 \circ g_2^{-1}$ is convex.*

Theorem 4 *Let g be an increasing (decreasing) twice continuously differentiable on $]0, 1]$ generator of a quasi-arithmetic mean M_g where $g^{-1} = h$, and $\lim_{x\to 0} g(x) = -\infty$ ($\lim_{x\to 0} g(x) = +\infty$). If $h'^2 - hh'' \geq 0$ then M_g is not Lipschitz.*

Proof. We will show that $M_{[p]} \leq M_g$ for any $-1 < p < 0$, and hence by Lemma 3 is not Lipschitz. If $x^p \circ g^{-1}$ is convex, for $-1 < p < 0$ by Theorem 3, with $g_1(x) = x^p$, $M_{[p]} \leq M_g$. Let us show that $(x^p \circ h)'' \geq 0$.

$$\begin{aligned}(x^p \circ h)' &= ph^{p-1}h' \\ (x^p \circ h)'' &= p(p-1)h^{p-2}h'^2 + ph^{p-1}h'' \\ &= ph^{p-2}\left((p-1)h'^2 + hh''\right) \geq 0.\end{aligned}$$

Given $ph^{p-2} < 0$ for $p < 0, h > 0$, convexity will hold if for all $p < 0$

$$(1-p)h'^2 - hh'' \geq 0. \tag{8}$$

Therefore $h'^2 - hh'' \geq 0$ implies $(x^p \circ h)'' \geq 0$ and $M_{[p]} \leq M_g$, and by Lemma 3 the Lipschitz constant of M_g is greater than that of $M_{[p]}$, which is $2^{-\frac{1}{p}}$, and $p \to 0^-$. □

Remark 2 *The generator g can be either increasing or decreasing. Clearly when changing g to $-g$, we change $h(x)$ to $h(-x)$. Then h' changes the sign but h'' does not, hence the inequality in Theorem 4 is the same for either increasing or decreasing generators.*

Example 6 *Using the geometric mean M_g, take $g(x) = \ln x$ with $h(x) = h'(x) = h''(x) = e^x$. Then $(h'^2 - hh'')(x) = e^{2x} - e^{2x} = 0$. Therefore M_g is not Lipschitz.*

For the sake of convenience, we will formulate our next result for *decreasing* additive generators satisfying $g(0) = \infty$. To obtain the respective condition on the increasing generators, we simply invert the sign of g.

Theorem 5 *Let $h = g^{-1}$ be the inverse of a decreasing generator g of a quasi-arithmetic mean M_g. If the function $\frac{1}{h}$ grows faster than any power $x^q, q > 0$, then M_g is not Lipschitz.*

Proof. Fix y so that $g(y) = h^{-1}(y) = 0$, which is always possible (we remind that g is defined up to an arbitrary linear transformation).

$$\begin{aligned}
\lim_{x\to 0}\frac{\partial M_g(x,y)}{\partial x} &= \lim_{x\to 0}\frac{dh\left(\frac{h^{-1}(x)}{2}\right)}{dx},\\
&= \lim_{x\to 0}\frac{1}{2}h'\left(\frac{h^{-1}(x)}{2}\right)(h^{-1})'(x)\\
&= \frac{1}{2}\lim_{x\to 0}h'\left(\frac{h^{-1}(x)}{2}\right)\frac{1}{h'(h^{-1}(x))}.
\end{aligned}$$

Let $z = \frac{h^{-1}(x)}{2}$. Then

$$\lim_{x\to 0}\frac{\partial M_g(x,y)}{\partial x} = \frac{1}{2}\lim_{z\to\infty}\frac{h'(z)}{h'(2z)}.$$

Since h decreases faster than the power function $p(z) = Cz^r$, l'Hôpital's rule gives

$$0 = \lim_{z\to\infty}\frac{h(z)}{p(z)} = \lim_{z\to\infty}\frac{h'(z)}{p'(z)} = \lim_{z\to\infty}\frac{h'(2z)}{p'(2z)}.$$

For convenience of notation take p such that $p'(z) = \frac{1}{z^q}$. Then $p'(z) = p'(2z)2^q$.

$$\lim_{x\to 0}\frac{\partial M_g(x,y)}{\partial x} = \frac{1}{2}\lim_{z\to\infty}\frac{h'(z)}{h'(2z)} = \frac{1}{2}\lim_{z\to\infty}\frac{h'(z)}{h'(2z)}\frac{p'(2z)2^q}{p'(z)} = 2^{q-1}.$$

Since q can be arbitrarily large, the derivative is unbounded and M_g is not Lipschitz. □

Example 7 *Let the generator be* $g(x) = -\ln x$, *cf. Example 6. Clearly its inverse is* $\exp(-x)$, *and the auxiliary function* $\frac{1}{h(x)} = \exp(x)$, *which grows faster than any polynomial, hence the corresponding geometric mean is not Lipschitz.*

Further take any power of the logarithm $g(x) = (-\ln x)^r, r > 1$. *The auxiliary function* $\frac{1}{h(x)} = \exp(x^{\frac{1}{r}})$, *it grows faster than a polynomial, hence the resulting mean is not Lipschitz either. Note that this quasi-arithmetic mean is related to the Aczél-Alsina family of t-norms [2, 8] by the equation*

$$M_g = (T_r^{AA})^{\frac{1}{\sqrt[r]{2}}},$$

which shows directly that M_g *is not Lipschitz* ($f(x) = M_g(x,1) = T_r^{AA}(x,1)^{\frac{1}{\sqrt[r]{2}}} = x^{\frac{1}{\sqrt[r]{2}}}$ *is not Lipschitz).*

Example 8 *Consider the generator* $g(x) = (-\ln x)^2$, $\frac{1}{h(x)} = e^{\sqrt{x}}$. *From the previous example,* $r = 2$ *and we know the resulting mean is not Lipschitz, however this would not have been apparent from the application of Theorem 4,*

$$(h'^2 - hh'')(x) = \frac{1}{4x}e^{-2\sqrt{x}} - \frac{1}{4x}e^{-2\sqrt{x}} - \frac{1}{4\sqrt{x^3}}e^{-2\sqrt{x}} = -\frac{1}{4\sqrt{x^3}}e^{-2\sqrt{x}} < 0.$$

4.3 Weighted quasi-arithmetic means

Consider weighted quasi-arithmetic means $M_{\mathbf{w},g}$. We recompute conditions (6) and (7) for the case of unequal weights. For this we take partial derivatives with respect to all arguments. The Lipschitz constant is the largest, hence we have conditions

$$\begin{aligned} g'_-(x) &\leq \frac{k}{\max w_i} \inf_z g'_-(z) \\ g'_+(x) &\leq \frac{k}{\max w_i} \inf_z g'_+(z) \end{aligned} \tag{9}$$

where the minimum for z is over $[M(x, 0, \ldots, 0), M(x, 1, \ldots, 1)]$, and

$$\lim_{x \to 0^+} \frac{g'_+(x)}{g'_+(M(x, c, \ldots, c))} \leq \frac{k}{\max w_i} \tag{10}$$

with $c \in]0, 1]$.

Conditions (9) and (10) can also be used for symmetric means in the multivariate case, where $\max w_i = \frac{1}{n}$. It is clearly seen that the higher the number of variables, the smaller is the Lipschitz constant, if it exists.

5 Continuous generated functions with neutral element

Consider generated aggregation functions in Definition 4. f has neutral element e and is continuous on $[0, 1]^n$ [11]. Similarly to uninorms (see Section 2), f acts on $[0, e]^n$ as a continuous scaled t-norm T_f built from the additive generator $g_T(t) = -g(et)$ (underlying t–norm), and f acts on $[e, 1]^n$ as a continuous scaled t-conorm S_f built from the additive generator $g_S(t) = g(e+(1-e)t)$ (underlying t–conorm). Note that, either T_f, or S_f, or both are necessarily nilpotent (if both are strict, $Ran(g) = [-\infty + \infty]$ and we obtain a representable uninorm, which is discontinuous).

Conversely, given a value $e \in]0, 1[$, a continuous t-norm T with an additive generator g_T and a continuous t-conorm S with an additive generator g_S, the mapping $g : [0, 1] \to [-\infty, +\infty]$ given by

$$g(t) = \begin{cases} -g_T\left(\frac{t}{e}\right), & \text{if } t \in [0, e], \\ g_S\left(\frac{t-e}{1-e}\right), & \text{if } t \in]e, 1], \end{cases}$$

defines a generated aggregation function with the neutral element e, with the help of (2). If either T or S or both are nilpotent, and both are continuous, then f is continuous.

If T_f is strict, 0 is the absorbing element of f, and if S_f is strict, 1 is the absorbing element of f. In these cases we note that f cannot be $k-$Lipschitz for $k < 1/e$ (or $k < 1/(1-e)$ respectively), since $f(e, 1) = 1, f(0, 1) = 0$ (or $f(0, e) = 0, f(0, 1) = 1$ respectively).

Remark 4 *Since the additive generators of T_f and S_f are defined up to an arbitrary positive multiplier, it is possible to use different g_T and g_S which produce the same t-norm and t-conorm on $[0, e]^n$ and $[e, 1]^n$, but different values on the rest of the domain. Thus we can use*

$$g(t) = \begin{cases} -ag_T\left(\frac{t}{e}\right), & \textit{if } t \in [0, e], \\ bg_S\left(\frac{t-e}{1-e}\right), & \textit{if } t \in]e, 1], \end{cases}$$

with arbitrary $a, b > 0$.

Let us now concentrate on k-Lipschitz generated functions. First, consider the case of both T_f and S_f being nilpotent (both $g(0)$ and $g(1)$ are finite). By applying the results of Section 4 we have the conditions analogous to (5) (or (6)), except that the factor $\frac{1}{2}$ is not present, namely:

$$g'(x) \le k \cdot \min_{y \in [0,1]} g'(f(x, y)). \tag{11}$$

Note that $f(x, y)$ ranges over $[0, f(x, 1)]$ for $x \le e$ and over $[f(x, 0), 1]$ for $x \ge e$ in (11). This implies $k-$concavity of g on $[0, e]$ and $\frac{1}{k}$-convexity on $[e, 1]$). A function g is called $k-$concave if $-g$ is $k-$convex (we remind that g is increasing in (2), as opposed to the case of t–norms in Section 3). This is not surprising, given that f is $k-$Lipschitz only if the underlying t-norm and t-conorm are both $k-$Lipschitz. However, condition (11) is obviously stronger.

Example 9 *One interesting result of this is that the only 1-Lipschitz generated aggregation function with the neutral element* $e \in]0,1[$ *is the one that has underlying Łukasiewicz t–norm and t–conorm* T_f *and* S_f*. Indeed,* g *must be concave on* $[0,e]$ *and convex on* $[e,1]$ *for* T_f *and* S_f *to be 1-Lipschitz, but also* g *must be concave on* $[0, f(x,1)]$ *for all* $x \in [0,e]$ *for condition (11) to hold, and similarly must be convex on* $[f(x,0),1]$ *for all* $x \in [e,1]$*. Taking* $x = e$*,* g *must be both convex and concave on* $[0,1]$*, which implies* g *is linear on* $[0,1]$*. Then* $g(x) = c(x-e)$*, and since it is defined up to an arbitrary multiplier we can simply take* $g(x) = x - e$*. This 1-Lipschitz generated function is given by*

$$f(x,y) = \min(1, \max(0, x+y-e)).$$

Consider now the case of one of T_f, S_f being strict, let it be T_f. Then we also need the limit condition analogous to (7),

$$\lim_{x \to 0^+} \frac{g'_+(x)}{g'_+(f(x,y))} \leq k. \tag{12}$$

In this case 0 is the absorbing element of f. However, it is easy to check that both x and $f(x,y)$ tend to 0 at the same rate for any y, and the limit in (12) is 1. Hence condition (12) is always satisfied. This implies that Theorems 4 and 5 are no longer true for this type of aggregation functions.

Example 10 *Let* $e = \frac{1}{2}$*,* $g(x) = \ln 2x$ *and* $g^{(-1)}(x) = \min(1, \frac{1}{2}\exp(x))$*. Then* $f(x,y) = \min(1, 2xy)$*, which is 2-Lipschitz.*

6 Conclusion

k-Lipschitz aggregation functions are important for applications because they limit the changes in the outputs due to input inaccuracies, to a fixed factor of k. k-Lipschitz triangular norms and conorms have been already characterized by k-convex additive generators, however no analogous results were available for other aggregation functions such as quasi-arithmetic means and generated functions with a neutral element. In this work we have established: a) verifiable conditions which guarantee that an aggregation function is $k-$Lipschitz for a given k, and b) alternative sufficient conditions which guarantee that an aggregation function is not Lipschitz. We also presented various examples of both Lipschitz and non-Lipschitz aggregation functions.

Interestingly, we found that the only 1-Lipschitz generated aggregation function with a neutral element is the one which has the Łukasiewicz underlying t-norm and t-conorm. Our results will benefit those who design aggregation functions for practical applications, as they allow one to make an informed choice on suitability of specific functions for these applications.

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References

[1] C. Alsina, M.J. Frank, and B. Schweizer. *Associative Functions: Triangular Norms And Copulas.* World Scientific, Singapore, 2006.

[2] G. Beliakov, A. Pradera, and T. Calvo. *Aggregation Functions: A Guide for Practitioners.* Springer, Heidelberg, Berlin, New York, 2007.

[3] P.S. Bullen. *Handbook of Means and Their Inequalities.* Kluwer, Dordrecht, 2003.

[4] T. Calvo, A. Kolesárová, M. Komorníková, and R. Mesiar. Aggregation operators: properties, classes and construction methods. In T. Calvo, G. Mayor, and R. Mesiar, editors, *Aggregation Operators. New Trends and Applications*, pages 3–104. Physica-Verlag, Heidelberg, New York, 2002.

[5] T. Calvo and R. Mesiar. Stability of aggregation operators. In *2nd Conf. of the Europ. Soc. for Fuzzy Logic and Technology*, pages 475–478, Leicester, 2001.

[6] D. Dubois and H. Prade. On the use of aggregation operations in information fusion processes. *Fuzzy Sets and Systems*, 142:143–161, 2004.

[7] E.P. Klement and A. Kolesárová. Extension to copulas and quasi-copulas as special 1-Lipschitz aggregation operators. *Kybernetika*, 41:329–348, 2005.

[8] E.P. Klement, R. Mesiar, and E. Pap. *Triangular Norms.* Kluwer, Dordrecht, 2000.

[9] A. Kolesárová. 1-Lipschitz aggregation operators and quasi-copulas. *Kybernetika*, 39:615–629, 2003.

[10] A. Kolesárová, E. Muel, and J. Mordelová. Construction of kernel aggregation operators from marginal values. *Int. J. of Uncertainty, Fuzziness and Knowledge-Based Systems*, 10 Suppl.:37–49, 2002.

[11] M. Komorníková. Aggregation operators and additive generators. *Int. J. of Uncertainty, Fuzziness and Knowledge-Based Systems*, 9:205–215, 2001.

[12] C.M. Ling. Representation of associative functions. *Publ. Math. Debrecen*, 12:189–212, 1965.

[13] R. Mesiar. Compensatory operators based on triangular norms. In H.-J. Zimmermann, editor, *EUFIT*, pages 131–135, Aachen, 1995.

[14] R. Mesiar and M. Komorníková. Triangular norm-based aggregation of evidence under fuzziness. In B. Bouchon-Meunier, editor, *Aggregation and Fusion of Imperfect Information*, pages 11–35. Physica-Verlag, Heidelberg, 1998.

[15] A. Mesiarová. *Special Classes of Triangular Norms.* PhD thesis, Slovak Technical University, 2005.

[16] A. Mesiarová. Extremal k-Lipschitz t-conorms. *Int. J. of Uncertainty, Fuzziness and Knowledge-Based Systems*, 14:247–257, 2006.

[17] A. Mesiarová. Approximation of k-Lipschitz t-norms by strict and nilpotent k-Lipschitz t-norms. *Int. J. General Syst*, 36:205 – 218, 2007.

[18] B. Schweizer and A. Sklar. *Probabilistic Metric Spaces.* North-Holland, New York, 1983.

[19] Y.-H. Shyu. *Absolute continuity in the τ-operations.* PhD thesis, Illinois Institute of Technology, 1984.

[20] Y. Torra, V. Narukawa. *Modeling Decisions. Information Fusion and Aggregation Operators.* Springer, Berlin, Heidelberg, 2007.

[21] N. Weaver. *Lipschitz Algebras*. World Scientific, Singapore, 1999.

[22] H.-J. Zimmermann and P. Zysno. Latent connectives in human decision making. *Fuzzy Sets and Systems*, 4:37–51, 1980.