This is the published version:


Available from Deakin Research Online:

http://hdl.handle.net/10536/DRO/DU:30029967

Reproduced with the kind permission of the copyright owner.

Copyright: 2010, IEEE.
Identification of Nonlinear Systems Using Hybrid Functions

R. Dosthosseini, F. Sheikholeslam, Member, IEEE, and A. Z. Kouzani, Member, IEEE

Abstract—Most real systems have nonlinear behavior and thus model linearization may not produce an accurate representation of them. This paper presents a method based on hybrid functions to identify the parameters of nonlinear real systems. A hybrid function is a combination of two groups of orthogonal functions: piecewise orthogonal functions (e.g., Block-Pulse) and continuous orthogonal functions (e.g., Legendre polynomials). These functions are completed with an operational matrix of integration and a product matrix. Therefore, it is possible to convert nonlinear differential and integration equations into algebraic equations. After mathematical manipulation, the unknown linear and nonlinear parameters are identified. As an example, a mechanical system with single degree of freedom is simulated using the proposed method and the results are compared against those of an existing approach.

I. INTRODUCTION

NATURALLY, the structures of real systems are nonlinear and have nonlinear dynamics. Although there are some methods to linearize these dynamics, usually parameter identification errors increase in these systems. In such situations, proper methods for nonlinear systems should be developed.

Regression techniques in conjunction with two dimensional orthogonal functions are used in [1] to determine an approximation for a nonlinear one degree of freedom (d.o.f) dynamic model. The model matrix of the system was assumed to be a known mass matrix. Also, this approach was used to handle the special case of chain like multi degree of freedom nonlinear dynamic systems [2].

Chen and Tomlinson [3] presented a parametric identification method. They used time series to identify the dynamical parameters of the system and predict the time response.

There are some methods to identify the parameters of discrete non-linear systems. A wavelet based procedure was developed to identify the mechanical parameters of a discrete nonlinear system [4]. Volterra series [5] is presented to estimate non-linear systems employing multi dimensional kernels.

Recently, orthogonal functions that are a well known method for identification of dynamic and optimal control, are applied in estimating parameters [6-9]. The important characteristic of this approach is the reduction of the differential and integration equations to a system of algebraic equations by introducing the operational matrix of integration and product matrix. Consequently, these approximation algorithms are known as Direct Method [10]. This method is used for identification of linear multi degree of freedom mechanical systems [11-13]. It is also used to estimate the parameters of non-linear mechanical systems [14, 15]. For this, a methodology was developed to identify physical parameters of non-linear systems through orthogonal functions. Also, numerical simulations were used to testify the efficiency of the orthogonal functions and show their applicability for single and multi degree of freedom systems. Although the produced results are reasonable, many bases were used in the simulations taking a large amount of calculations.

Hybrid functions have received considerable attention in dealing with various problems of dynamic systems. The main benefit of using hybrid functions in identification and control problems is that differential equations can be reduced to a set of algebraic equations. In this paper, hybrid functions method is employed to identify the parameters of a nonlinear system. This method involves reducing the optimal control problem to a set of algebraic equations utilized to evaluate unknown coefficients [16]. In order to demonstrate the accuracy of the proposed numerical method and compare it with the results of direct method, the example that is solved in [15] using orthogonal functions is evaluated with the proposed hybrid functions.

This paper is organized as follows. Section II provides an overview of hybrid functions and their properties. Section III describes a nonlinear motion problem using hybrid functions. Section IV explains how an example is solved to demonstrate the accuracy of the presented numerical approach. The results are compared against those reported in [15]. Finally, conclusions are given in Section V.

II. HYBRID FUNCTIONS AND THEIR PROPERTIES

Hybrid functions are based on two groups of orthogonal functions, first block-pulse functions as piecewise orthogonal functions, and second continuous orthogonal polynomials such as Chebyshev or Legendre. They are defined between $[0, t_f]$ as follows [17].
where \( n = 1, 2, ..., N \) is the order of block-pulse functions, \( t \) is the normalized time, and \( P_m(t) \) are orthogonal polynomials such as Legendre or Chebyshev polynomials of order \( m = 1, 2, ..., M - 1 \). \( P_m(t) \) should satisfy the following recursive formula:

\[
\begin{align*}
P_0(t) &= 1, \quad P_1(t) = t, \\
P_m(t) &= 2tP_{m-1}(t) - P_{m-2}(t), \quad t \in [-1, 1]
\end{align*}
\]

for Chebyshev polynomials and

\[
\begin{align*}
P_0(t) &= 1, \quad P_1(t) = t, \\
P_m(t) &= (2m-1)tP_{m-1}(t) - (m-1)P_{m-2}(t), \quad t \in [-1, 1]
\end{align*}
\]

for Legendre polynomials.

Assume \( f(t) \) is a function in the interval \([0, t_f]\). It can be approximated using hybrid functions as follows:

\[
f(t) \approx \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{m} b_{nm} = C^T B
\]

\[
C = [c_{0}, c_{1}, ..., c_{10}, c_{12}, ..., c_{40}, c_{41}, c_{42}]^T,
\]

\[
B = \begin{bmatrix} b_{10}(t), b_{11}(t), b_{12}(t), ..., b_{40}(t), b_{41}(t), b_{42}(t) \end{bmatrix}^T,
\]

and \( \tilde{C} = \text{diagonal matrix}( \tilde{C}_1, \tilde{C}_2, \tilde{C}_3, \tilde{C}_4 ) \)

where \( \tilde{C}_i, i = 1, 2, 3, 4 \) are 3 x 3 matrices defined as follows

\[
\tilde{C}_i = \begin{bmatrix} (1/3) c_{10} & c_{10} + (1/5) c_{12} & (2/3) c_{11} \\ (1/5) c_{12} & (2/5) c_{11} & c_{10} + (2/7) c_{12} \end{bmatrix}
\]

when Legendre polynomials are used and when Chebyshev polynomials are used \( \tilde{C}_i \) is defined as

\[
\tilde{C}_i = \begin{bmatrix} (1/2) c_{10} & c_{10} + (1/2) c_{12} & (1/2) c_{11} \\ (1/2) c_{12} & c_{10} + (1/2) c_{12} & (1/2) c_{11} \end{bmatrix}
\]

III. PROBLEM STATEMENT

The equation of motion of a non-linear system with \( n \) degree of freedom with respect to an external force \( f(t) \) can be described as follows [15]

\[
\ddot{x} + C \dot{x} + Kx + g(x, \dot{x}) = f(t)
\]

where \( M, C, \) and \( K \) are mass, damping and stiffness parameters, \( x \) is the displacement variable, \( g(x, \dot{x}) \) is a function of the displacement and velocity, and \( f(t) \) is the excitation force vector. Although \( g(x, \dot{x}) \) is generally nonlinear, it can be considered as a linear term depending on the nature and magnitude of the nonlinear forces and the vibration level of the system [15]. In this work, \( g(x, \dot{x}) \) is assumed to be a nonlinear function. A general formulation is developed for a d.o.f. system with cubic stiffness and viscous and dry friction damping. Thus

\[
\ddot{x} + C \dot{x} + K_{x}x + f_{d} \text{sign}(\dot{x}) = f(t)
\]

where \( K_{x} \) is the cubic stiffness coefficient, \( f_{d} \) is the dry friction force, and \( \text{sign}(\dot{x}) \) is defined as follows:

\[
\text{sign}(\dot{x}) = \begin{cases} 1, & \text{if } \dot{x} > 0 \\ 0, & \text{if } \dot{x} = 0 \\ -1, & \text{if } \dot{x} < 0 \end{cases}
\]

Integrating (18) twice in the interval \([0, t]\), it becomes:

\[
M(x - x_0 - \dot{x}_0 t) + C \left( \int_{0}^{t} x(t) \, dt - x_0 t \right) + K \int_{0}^{t} \int_{0}^{t} x(t) \, dt^2 + K_{x} \int_{0}^{t} \int_{0}^{t} x(t)^3 \, dt^2 + f_{d} \int_{0}^{t} \int_{0}^{t} \text{sign}(x(t)) \, dt^2
\]

where \( x_0 \) and \( \dot{x}_0 \) are the initial conditions of displacement and velocity, \( x, x^3, \text{sign}(\dot{x}) \), and \( f(t) \) are approximated with hybrid functions as:

\[
\begin{align*}
x &= B^T X \\
x^3 &= B^T X_1 \\
\text{sign}(\dot{x}) &= B^T S \\
f(t) &= B^T F
\end{align*}
\]

where \( X, X_1, S \), and \( F \) are known vectors of order \( MN \times 1 \).
Similarly, \( t, x_0, \) and \( \dot{x}_0 \) can be expanded as follows:

\[
t = B^T T
\]

\[
x_0 = B^T X_0
\]

\[
\dot{x}_0 = B^T X_{\dot{0}}\dot{t}
\]

By substituting (21) and (22) in (20), the one-d.o.f. system with cubic stiffness and viscous and dry friction damping is approximated as:

\[
M(B^T X - B^T X_0 - X_{\dot{0}}B^T T) + C \left( \int_0^t B^T X d\tau - X_{\dot{0}}B^T T \right) + K \int_0^t B^T X d\tau^2 + K_3 \int_0^t B^T \dot{X}_d d\tau^2 + f_d \int_0^t B^T S d\tau^2
\]

\[= \int_0^t B^T F d\tau^2\]  

(23)

Applying the property for the integration of hybrid functions (7) and product operational matrix (11), (23) is converted to

\[
M(X^T B - X^T_0 B - X_{\dot{0}}^T \dot{B}) + C(X^T PB - X^T_0 \dot{B}) + KX^T P^T PB + K_3 X^T_0 \dot{B}^T \dot{P}^T PB + f_d S^T P^T PB
\]

\[= F^T P^T PB\]  

(24)

therefore:

\[
MX^T - MX^T_0 - MX_{\dot{0}}^T \dot{B} + CX^T P - CX^T_0 \dot{B} + KX^T P^T PB + K_3 X^T_0 \dot{B}^T \dot{P}^T PB + f_d S^T P^T PB = F^T P^T PB
\]

(25)

Suppose:

\[
\hat{\mathbf{C}} = [M, -M, -C, -C, K, K_3, f_d]
\]

\[\hat{X} = \begin{bmatrix} X^T, X_0^T, X_{\dot{0}}^T \dot{B}, X^T P, X_0^T \dot{B} \dot{P}, X_{\dot{0}}^T \dot{B} \dot{P}, S^T P^T \end{bmatrix}^T\]

and

\[
\hat{F} = F^T P^T PB
\]

(26)

(27)

(28)

where \( \hat{\mathbf{C}} \) is a 1 \( \times \) 8 vector, \( \hat{\mathbf{X}} \) is a 8 \( \times \) \( MN \) matrix, and \( \hat{\mathbf{F}} \) is a 1 \( \times \) \( MN \) vector. Thus, (25) is converted as:

\[
\hat{\mathbf{C}} \hat{\mathbf{X}} = \hat{\mathbf{F}}
\]

(29)

Using the least-squares method, an estimate of vector \( \hat{\mathbf{C}} \) where its coefficients are unknown is given as follow [15]:

\[
\hat{\mathbf{C}} = \hat{\mathbf{F}} \hat{\mathbf{X}}^T (\hat{\mathbf{X}} \hat{\mathbf{X}}^T)^{-1}
\]

(30)

Equation (30) can be solved using the singular value decomposition method to determine the unknown parameters. If the displacement and velocity initial conditions are unknown, they can be solved with the same rule.

It is important to point out that, in using (30), there is no need for any information about physical or model parameters concerning the mechanical system. However, other identification techniques such as those mentioned in this paper require such information. This means that force identification problems can be studied on a straightforward manner by using our method. Also, it is shown in the following section that the errors for the example solved are less than those reported in [15].

IV. ONE-D.O.F. MECHANICAL SYSTEM WITH MIXED DAMPING

Let

\[
M = 1 \text{ kg, } C = 20 \text{ N s/m, } K = 10000 \text{ N/m, } f_d = 1 \text{ and 3 N}
\]

are the known parameters for a nonlinear one-d.o.f. mechanical system. In this example for simplicity, it is assumed that \( K_3 \) is equal to zero. A swept-sine excitation with \( F_{rms} = 10 \text{ N} \) from 10 to 20 Hz was used [15]. The response, considering \( f_d = 1 \) and 3 N was sampled at a frequency of 1700 Hz using the fourth-order Runge–Kutta method as shown in Fig. 1 and Fig. 2.

![Fig. 1. x(t) of the system using a swept-sine excitation with \( F_{rms} = 10 \text{ N} \)](image)

![Fig. 2. x_d(t) of the system using a swept-sine excitation with \( F_{rms} = 10 \text{ N} \)](image)

The parameters identified using the hybrid functions method and the orthogonal functions [15] are shown in Table I.

Also, Table II represents that when \( f_d/F_{rms} \) is increased, the errors in the identified parameters increases in comparison with the first case. According to Tables I and II, the parameters identified by hybrid functions are more accurate than those identified by orthogonal functions. Also the number of bases used in this approach is less than those reported in [15].

A harmonic excitation is used to identify the unknown parameters. In this case, the excitation force is:

\[
f(t) = F_0 \sin (2\pi f_0 t)
\]

(32)

with \( f_0 = 21 \text{ Hz, } F_{rms} = 10 \text{ N} \), and the corresponding sampling frequency is 10240 [15]. The results are shown in Tables III and IV. Considering these tables, it is clear that the errors using Chebyshev polynomials have increased for both hybrid and orthogonal functions.
In addition, if a parameter is known, the results are considerably improved. If the dry friction force and the stiffness parameters used in the identification procedure are assumed as $1 \, N$ and $9997 \, N/m$, the errors for identified parameters decrease as shown in Table V.

On the other hand, let the dry friction force and the stiffness parameter be $3 \, N$ and $9995 \, N/m$. Table VI presents the identified parameters.

Clearly, Tables V and VI show that when the number of unknown parameters decreases, the results using orthogonal and hybrid functions become close.

Also, if all the parameters in (18) are considered, it can be observe that the magnitude of the errors in the identification of mass, damping and stiffness parameters were similar for all kinds of excitation forces. These parameters are given by

$$M = 1 \, kg, C = 20 \, N \, s/m, K = 10000 \, N/m,$$

$$f_d = 1, \quad \text{and} \quad K_3 = 5 \times 10^9 \, N/m^3 \quad (33)$$

Similarly, a harmonic excitation is used to identify the unknown parameters. In this case, the excitation force is:

$$f(t) = F_0 \sin (2\pi f_0 t) \quad (34)$$
with $f_0 = 21\ Hz$, $F_{rms} = 20\ N$, and the corresponding sampling frequency is 5115 [15]. The results are shown in Tables VII. Since increasing the number of unknown parameters will lead to worse results, in this case to decrease errors occurred, $M = 50$ for orthogonal functions and $M = 6$ for hybrid functions are considered.

### TABLE VI

<table>
<thead>
<tr>
<th>Hybrid functions</th>
<th>$M$ (kg)</th>
<th>$C$ (N s/m)</th>
<th>$f_d$ (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Block-pulse\Legende $N = 4, M = 5$</td>
<td>1.000</td>
<td>20.00</td>
<td>3.004</td>
</tr>
<tr>
<td>Block-pulse\Chebyshev $N = 4, M = 5$</td>
<td>1.006</td>
<td>19.92</td>
<td>2.953</td>
</tr>
</tbody>
</table>

**Orthogonal functions [15]**

| Legendre $M = 33$ | 1.000 | 20.00 | 3.005 |
| Legendre $M = 20$ | 1.007 | 19.88 | 2.941 |
| Block-pulse $M = 512$ | 0.995 | 19.97 | 2.978 |

### TABLE VII

<table>
<thead>
<tr>
<th>Hybrid functions</th>
<th>$M$ (kg)</th>
<th>$C$ (N s/m)</th>
<th>$K_N$ (N/m)</th>
<th>$K_S, 10^{-9}$</th>
<th>$f_d$ (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Block-pulse\Legendre $N = 4, M = 6$</td>
<td>1.000</td>
<td>18.99</td>
<td>10009.7</td>
<td>5.003</td>
<td>1.006</td>
</tr>
<tr>
<td>Block-pulse\Chebyshev $N = 4, M = 6$</td>
<td>0.991</td>
<td>18.81</td>
<td>10011.6</td>
<td>5.003</td>
<td>1.023</td>
</tr>
</tbody>
</table>

**Orthogonal functions [15]**

| Legendre $M = 50$ | 1.000 | 18.94 | 10012 | 5.004 | 1.008 |
| Legendre $M = 50$ | 0.998 | 18.79 | 10013 | 4.997 | 1.031 |
| Block-pulse $M = 512$ | 0.997 | 18.87 | 10012 | 5.002 | 1.013 |

### V. CONCLUSIONS

A numeric approach for parameter identification of one-d.o.f nonlinear system using hybrid functions has been presented. The important property of hybrid functions is their conversion of differential or integral equations to algebraic equations. An example of nonlinear mechanical system has been solved using Chebyshev and Legendre polynomials with a combination of block-pulse bases. The results show that the number of bases used for the estimation has decreased. In addition, the errors become less than those produced by orthogonal functions.

### REFERENCES


