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Contingent Prices and Money*
(Shortened title: Contingent Prices)

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Price posting with directed search is a widely-used trading mechanism. Coles and Eeckhout (2003) showed that if sellers are allowed to post prices contingent on realized demand instead of one price, then there is real market indeterminacy. In this paper we fit this contingent price posting protocol into a monetary economy. We show that, as long as holding money is costly, there exists a unique equilibrium rather than a continuum. In this equilibrium sellers post a low price for when the buyer is alone, a high price for when several buyers show up, and buyers randomize between sellers and money holdings.

1 Introduction

Price posting with directed search is a pricing mechanism in which sellers advertise terms of trade, buyers observe those terms of trade and decide which seller to visit. While originally

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developed in the context of the goods market (e.g. Butters, 1977; Peters, 1984), it was popularized by several influential studies of the job market (e.g. Montgomery, 1991; Moen, 1997) where it fits nicely: employers advertise wages, workers observe wages and decide where to apply.

The central friction in this pricing protocol is the ex ante probability that both buyers and sellers can be rationed. If two or more buyers show up at the same seller, one buyer will leave the market empty handed. If no buyer shows up at one seller, the seller will leave empty handed. Given those contingencies, a key aspect of directed search is the commitment by sellers to not act opportunistically ex post. In particular, one may wonder why sellers do not advertise auctions instead of fixed prices in order to take advantage of possible competition among buyers. Such criticism has raised doubts about the opportunity to use price posting with directed search, arguing that it does not describe equilibrium behavior.

Coles and Eeckhout (2003) proposed the following answer. They build a model of directed search in which sellers, rather than advertising a fixed price, are allowed to post prices contingent on realized demand. For instance a seller might advertise a higher price should more than one buyer desire the good. Specifically, each seller posts a price pair \((p_1, p_2)\) where \(p_1\) is the (pairwise) price charged if only one buyer shows up (who gets the good with certainty), while \(p_2\) is the (multilateral) price charged if two or more buyers visit (and the good is then randomly allocated to one of the buyers). They show that in equilibrium sellers post a unique price for when only one buyer arrives (the pairwise price \(p_1\)), but the price that applies if both buyers visit (the multilateral price \(p_2\)) can be anywhere between the seller's reservation value and the buyer's reservation value: there is real market indeterminacy. It follows that posting a fixed price or an auction both describe equilibrium behavior. This indeterminacy result is often used to support the use of price posting with directed search as a trading mechanism (e.g. Rogerson, Shimer and Wright, 2005).

Our goal in this paper is to study how robust Coles and Eeckhout (2003)'s robustness test is. In particular buyers and sellers do not trade in a nominal vacuum. Sellers advertise nominal
prices and accordingly buyers bring money to pay for the goods. One suspects that monetary factors, especially the cost of using money, may limit the range of possibilities for sellers in their pricing decisions. Our strategy is then the following. Assume that sellers post contingent prices. In addition to which seller they should visit, buyers must now choose how much money to bring to the seller. As in Coles and Eeckhout (2003), buyers will weigh the price pairs against the probability to trade. But now, buyers will also have to weigh the return on bringing more money (which allows them to trade in multilateral meetings) against the cost of holding that extra money (the foregone interests for instance). Basically, we are asking sellers to endogenize buyers' budget constraint while deciding upon prices.

Our result comes in sharp contrast to Coles and Eeckhout (2003): if money holdings are costly for buyers, then rather than a continuum of equilibria there exists a unique equilibrium in contingent prices. In this equilibrium sellers post a low price for when the buyer is alone, a high price for when several buyers show up, and buyers randomize between sellers and money holdings. This result implies that posting an auction or posting a fixed price are no longer different equilibria of a more general contingent-price posting mechanism. In a monetary economy, when using money is costly, there is just one price pair that enables sellers to price discriminate on the basis of realized demand. Put it another way, money (or should we say costly money) destroys the mapping from contingent prices into auctions and fixed-price posting.

The intuition behind this result comes from the internalization by sellers of buyers' liquidity constraints. In the non-monetary version, advertising more buyer surplus in some states and offering less in others, that is posting $p_2 \neq p_1$, can leave expected buyer and seller surplus unchanged. Those changes, however, change the demand elasticities of buyers and therefore the best-response correspondence of competing sellers, hence the continuum of equilibria. In such non-monetary economy buyers are assumed to be able to hold any amount of money for free so that monetary strategies are irrelevant. But if money is costly, because of inflation for instance, buyers weigh the cost of holding more money against the augmented trading
opportunities that it offers. For a given nominal interest rate, holding \( p_2 \) in monetary units will be preferred to holding \( p_1 \) if \( p_2 \) is only slightly greater than \( p_1 \) since the marginal cost is small compared to the additional trading opportunities. But a buyer will be better off holding \( p_1 \) in monetary units if \( p_2 \) is significantly greater than \( p_1 \) since the marginal cost more than cancels the marginal benefit. Because buyer's expected net surplus decreases monotonically as \( p_2 \) rises above \( p_1 \), there exists only one multilateral price \( p_2 \) within the former continuum that leaves buyers indifferent between holding \( p_2 \) or \( p_1 \). As a result, sellers post a low price for when the buyer is alone, a high price for when several buyers show up, buyers randomize between sellers and money holdings, producing unique non-degenerate distributions of money holdings and prices.

We believe this result is important because it should change the way we think of directed search with price posting. The standard assumption in directed search models that sellers commit to a single, non-contingent price is often defended by Coles and Eeckhout's result that this behavior is an equilibrium. Our paper suggests that this result is not robust: given a small perturbation in the environment, the indeterminacy vanishes and a unique equilibrium emerges.\(^2\)

In a second part of the paper we investigate the impact of changing the cost of liquidity on the equilibrium terms of trade by changing the growth rate of the money supply, resulting in

\(^2\)Green and Zhou (1998) also examine indeterminacy in search-based monetary models. They show that in a large economy with price posting and random matching there exists a continuum of distinct, stationary, single-price monetary equilibria. Related works in the line of this paper include Zhou (1999, 2003), Kamiya and Sato (2004), Kamiya, Morishita and Shimizu (2005), Kamiya and Shimizu (2006, 2007a, 2007b). Recently, Jean, Rabinovich and Wright (2010) studied the multiplicity of monetary equilibria and show that introducing a round of centralized trading into Green and Zhou (1998), as in Lagos and Wright (2005), preserves the indeterminacy while the resulting model is easier to manipulate and understand. Also, Geromichalos (2010) extends Coles and Eeckhout (2003) by allowing sellers to choose price schedules contingent on ex post realized demand. He shows that indeterminacy of equilibria arises in small markets but vanishes as the number of traders in the market approaches infinity.
shifts in anticipated inflation and then the nominal interest rate. We first show that uniqueness of equilibrium holds for the entire range of nominal interest rates compatible with existence of a monetary equilibrium (above a certain threshold buyers simply discard money). In particular, one may have expected the indeterminacy to come back at low nominal interest rates as the low cost of holding money could make it profitable for buyers to always hold the high amount of money. It turns out sellers extract this low cost of liquidity by charging buyers a multilateral price that approaches the buyer's reservation value as the nominal interest rate approaches zero, thereby supporting uniqueness. Second, we show that for small deviations from the Friedman rule, sellers increase the pairwise price they charge in real terms, forcing buyers who pick the low money holding to bring more money. As inflation rises it becomes proportionally more costly to hold the large amount of money. Sellers extract this incentive to shift to the low amount of money by raising the pairwise price. As a result price dispersion—measured by the distance between the pairwise price and the multilateral price—falls as inflation increases. While most studies find that price dispersion increases with inflation, some have documented falling dispersion, e.g. Reinsdorf (1994). Our paper provides theoretical backing to such finding.

The paper is organized as follows. Section 2 describes the players, stages, strategies and payoffs. Section 3 characterizes the unique symmetric equilibrium. Section 4 examines how inflation influences strategic price posting by sellers. Section 5 concludes.

2 The model

2.1 Players and Strategies

There are two sellers indexed by \( y \in \{1, 2\} \) and two buyers indexed by \( x \in \{1, 2\} \). As in Lagos and Wright (2005), they all meet first in a Walrasian market where they can produce, trade and

\(^3\)Our results generalize to an \( n \)-buyer, \( n \)-seller economy. A 2-by-2 economy highlights the interdependence between strategies and facilitates comparison with Coles and Eckhout (2003). We study the \( n \)-buyer, \( n \)-seller economy in Appendix A.4 and contrast the allocation with that of a 2-by-2 economy in section 4.
consume any quantity of a first good called the *general good*. Then they enter a second market in which each seller is endowed with one indivisible unit of a second good, called the *search good*, that buyers with identical preferences wish to buy. We call the first market the centralized Walrasian market (CWM) and the second market the directed search market (DSM). Agents discount at rate $\beta$ between the CWM and the DSM but not between periods.

The sequence of events is as follows. First, the CWM opens and buyers receive a lump sum money injection from the central bank, which is common knowledge. Then sellers simultaneously and publicly advertise terms of trade contingent on realized demand for the coming DSM. The terms of trade consist of one unit of the search good and a price schedule: $p_1$ is the price posted by seller 2 that applies in case only one buyer shows up (pairwise match), and $p_2$ is the price posted by seller 2 that applies if both buyers show up (multilateral match), respectively $p'_1$ and $p'_2$ for seller 1. After observing all the price offers, the two buyers simultaneously choose both which seller to visit and how much money to carry to the coming DSM. Money is acquired by producing and selling some general good in exchange for money on the CWM. Finally, the CWM closes and the DSM opens, where buyers visit the seller of their choosing. If a buyer is alone he pays the pairwise price and consumes the good. If the two buyers visit the same seller and both can afford the multilateral price, one buyer is chosen randomly, pays the multilateral price and consumes the good. If only one buyer can afford the multilateral price then he wins the good for sure and pays the multilateral price. If none of the buyers can afford to pay the high price, no trade takes place. The sequence of events is represented on Figure 1.4

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4 Note that if none of the buyers holds the multilateral price, then a seller can actually be made worse off if two buyers visit than if only one buyer visits since none is able to pay. This peculiar feature of the model is a by-product of casting the contingent-price economy into a monetary environment. It does not drive our uniqueness result, however. As will be clear, it is the ex ante indifference condition between money holdings internalized by sellers that brings uniqueness of equilibrium. In a similar environment in which sellers post auctions instead of contingent prices, Galenianos and Kircher (2008) show that the equilibrium is unique as well, with unique distributions of prices and money holdings as in here.
2.2 Money

To create a role for money in this economy, we make three assumptions. First, as in Rocheteau and Wright (2005), we assume that buyers cannot produce in the DSM so that they cannot compensate sellers on the spot by producing in exchange for the search good. Second, we assume that both goods perish at the end of their subperiod and therefore cannot be used as commodity money. Finally, the cost of setting up a public record-keeping system is extremely high, implying that each agent’s trading history is private information. These assumptions make money essential for trade on the DSM (Kocherlakota, 1998; Wallace, 2001).\footnote{Keeping track of each other trading history is probably not that difficult in a 2-by-2 economy and our last assumption may sound extreme in that regard. Correlated equilibria can also be an issue in a small economy like ours since buyers may infer from the equilibrium price of money other buyers’ choices of money holding. Our interest in focusing on the small economy is expositional as it enables to spell out the interdependence in agents’ strategies. See Appendix A.4 for details on the large economy where such assumptions are natural.}

To produce the general good in the CWM buyers and sellers can exert effort \( h \) to produce \( h \) units of that good. We assume linearity in effort on the CWM as it erases the influence of trading histories on money demand (see Lagos and Wright, 2005). Typically, buyers use the
CWM to replenish their money holdings by selling some of their general good output to sellers. Symmetrically, because sellers do not need money in the DSM, they use the CWM to spend any balances accumulated in the previous DSM.

The price of money in terms of the general good on the CWM is denoted \( \phi \), that is 1 unit of money buys \( \phi \) units of the general good. It adjusts every period to equate supply of money from sellers \( M^S \) to demand of money coming from buyers \( M^D \). The quantity of money grows at rate \( \tau \) via lump sum transfers to buyers by the central bank at the beginning of each CWM. Inflation (noted \( \pi \)) is perfectly forecasted. Therefore, if the money supply increases at rate \( \tau \), so do prices and \( \pi = \tau \). Since \( \beta = 1/(1+r) \) the Fisher equation \((1+i) = (1+r)(1+\pi)\) enables to write the nominal interest rate as \( i = (1 - \beta + \tau) / \beta \). Finally, we assume that money holdings are buyers' private information.

2.3 The Payoffs

If a buyer pays \( p \) for the search good in the DSM, he consumes it immediately, which provides him with utility \( Q \). The seller’s utility for consuming her endowed search good in this market is normalized to zero. In the CWM we denote \( X \) consumption of the general good and \( U(X) \) the corresponding utility with \( U' > 0 \) and \( U'' < 0 \). The instantaneous utility function for a buyer is then given by

\[
U(X) = h + \beta Q. \tag{1}
\]

Since sellers are endowed with the search good, their instantaneous utility function is simply

\[
U(X) = h. \tag{2}
\]

Because decisions are related through time via the choice of money holdings, the model is one of dynamic optimization. We will focus on stationary allocations, however, where aggregate
real variables are constant, including the aggregate real money supply. This implies $\phi_{t+1}M_{t+1} = \phi_t M_t = (1 + \tau) \phi_{t+1}M_t$ and therefore $\phi_{t+1} = \phi_t/(1 + \tau)$. From now on we will suppress time subscripts and use the subscript $+1$ to denote the value of a variable in the next period.

Let $V^{bz}(m)$ be the value function for buyer $x$ holding $m$ units of money when entering the DSM and $W^{bz}(m)$ the value function when entering the CWM. We have

\begin{equation}
W^{bz}(m) = \max_{X,h,m} \left\{ U(X) - h + \beta V^{bz}(\hat{m}) \right\}
\end{equation}

\begin{equation}
\text{s.t. } \phi\hat{m} + X = h + \phi(m + T).
\end{equation}

A buyer chooses how much of the general good to consume, $X$, how much effort to exert, $h$, and how much money to bring to the DSM, $\hat{m}$. The budget constraint equalizes resources, $h + \phi(m + T)$, to demand, $\phi\hat{m} + X$, where $T$ is how many units of money per buyer are injected by the central bank at the beginning of the CWM. Substituting for $h$, the program for a buyer in the CWM can be rewritten

\begin{equation}
W^{bz}(m) = U(X^*) - X^* + \phi(m + T) + \max_{m} \left\{ -\phi\hat{m} + \beta V^{bz}(\hat{m}) \right\},
\end{equation}

where $X^*$ is such that $U''(X^*) = 1$.

Since sellers have no reason to carry money into the DSM and only buyers receive a lump sum money transfer, the program for seller 2 is

\begin{equation}
W^{sz}(m) = \max_{X,h,p_1,p_2} \left\{ U(X) - h + \beta V^{sz} \right\}
\end{equation}

\begin{equation}
\text{s.t. } X = h + \phi m.
\end{equation}

Seller 2's problem is to choose consumption in the Walrasian market $X$, effort $h$ and a pair of
prices \((p_1, p_2)\) for the DSM. Substituting for \(h\) this implies

\[
W^{s_2}(m) = U(X^*) - X^* + \phi m + \max_{p_1, p_2} \beta V^{s_2}.
\]

A similar exercise yields

\[
W^{s_1}(m) = U(X^*) - X^* + \phi m + \max_{p_1, p_2} \beta V^{s_1}
\]

for seller 1.

3 The Equilibrium

We define a *monetary strategy* as the probability with which a buyer chooses to carry a particular amount of money. Notice that since each seller advertises only two prices, \((p'_1, p'_2)\) for seller 1 and \((p_1, p_2)\) for seller 2, if a buyer decides to visit seller 1 for instance, then bringing an amount of money equal to either \(p'_1\) or \(p'_2\) strictly dominates bringing any other amount. To see this assume buyer 1 visits seller 1 and brings \(p_1\). If he is alone, either \(p_1 > p'_1\) in which case he brought too much money, or \(p_1 < p'_1\) in which case he did not bring enough. The same is true if the other buyer is there: either \(p_1 > p'_2\) in which case he brought too much money or \(p_1 < p'_2\) in which case he did not bring enough. As a result, given there can be four prices at most and idle balances are costly as soon as \(i > 0\), money holdings can take four values: \(p_1, p_2, p'_1\) or \(p'_2\).

Correspondingly, we define \(\theta'_x\) as the probability with which buyer \(x \in \{1, 2\}\) chooses to hold \(p'_1\) given that he has decided to visit seller 1. Then \(1 - \theta'_x\) is the probability that buyer \(x\) chooses to hold \(p'_2\) given that he has decided to visit seller 1. Similarly, \(\theta_x\) denotes the probability that buyer \(x\) chooses to hold \(p_1\) given that he visits seller 2 and \(1 - \theta_x\) the probability that buyer \(x\) chooses to hold \(p_2\) given that he visits seller 2.

We define a *visit strategy* for buyer \(x\) as the probability with which a buyer chooses to visit a particular seller. We denote by \(\sigma_x\) the probability that buyer \(x \in \{1, 2\}\) chooses to visit seller
1. With probability $1 - \sigma_x$ he chooses to visit seller 2.

Finally, we define a *pricing strategy* for seller 1 as posting a price pair $(p'_1, p'_2)$, and a pricing strategy for seller 2 as posting a price pair $(p_1, p_2)$. We restrict our attention to sellers using pure pricing strategies and strictly positive prices.

The resulting game is a two-stage dynamic game of complete but imperfect information due to the simultaneity of moves for sellers and for buyers. The equilibrium is subgame perfect Nash equilibrium. First, taking prices and visit strategies as given, we characterize buyers' best response in terms of money holdings. Second, given the equilibrium monetary strategies and the prices posted by sellers, we solve for the equilibrium visit strategies. Finally, given the equilibrium visit and monetary strategies, we solve for the Nash equilibrium in the price posting game between the two sellers. As for the price of money on the CWM, $\phi$, it equalizes money supply and money demand consistent with equilibrium strategies.

**Definition 1.** A subgame perfect Nash equilibrium is a list of prices $(p_1, p_2, p'_1, p'_2)$ and a list of visit and monetary strategies $(\sigma_1, \sigma_2, \theta'_1, \theta'_2, \theta_1, \theta_2)$ such that:

1. **Best response by buyers:** given $(p_1, p_2, p'_1, p'_2)$, $\sigma_x, \theta'_x$ and $\theta_x$ describe the Nash equilibrium in visit and monetary strategies for each buyer $x$;

2. **Best response by sellers:** given the subgame visit and monetary strategies $\sigma_x, \theta'_x$ and $\theta_x$, $(p_1, p_2)$ and $(p'_1, p'_2)$ describe a Nash equilibrium in pricing strategies for the two sellers;

3. **Individual rationality:** $-\phi p_j + \beta V^{bs}(p_j) \geq \beta W^{bs}(0)$ and $-\phi p'_j + \beta V^{bs}(p'_j) \geq \beta W^{bs}(0)$, $\forall j \in \{1, 2\}$, $\forall x \in \{1, 2\}$;

4. **Money market clearing and stationarity:** $M^D = M^S$ and $\phi_{t+1} = \phi/(1 + \tau)$.

The inequalities in point 3 of the definition guarantee that buyers' expected return from holding money through the frictional market is no smaller than proceeding directly to the centralized market with no money. Note that there also exists a non-monetary equilibrium in which money is not valued and no economic activity takes place in the DSM. We will focus on the monetary equilibria in which money has a positive price, that is $\phi > 0$. 

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Since our focus is on uniqueness of symmetric equilibrium, we concentrate on buyers playing
the same strategies (no coordination). We also impose $-i\phi_{+1} p'_1 + Q - \phi_{+1} p'_1 \geq -i\phi_{+1} p_2 + (Q - \phi_{+1} p_2)/2$ and $-i\phi_{+1} p_1 + Q - \phi_{+1} p_1 \geq -i\phi_{+1} p'_2 + (Q - \phi_{+1} p'_2)/2$ (buyers would rather be alone at either seller) ruling out $\sigma_x = 0, 1$.\textsuperscript{6} We also concentrate on the case where $p_2 > p_1$. It is easy to show that no equilibrium exists when $p_2 < p_1$ as soon as $i > 0$.

### 3.1 Monetary strategies

Consider buyer 1’s choice, taking as given both the prices posted by sellers and buyer 2’s visit
and monetary strategies. If he visits seller 1 who posts $(p'_1, p'_2)$ holding a money level of $p'_1$, buyer 1’s payoff is $-\phi p'_1 + \beta V^{b1}(p'_1)$ with

$$
V^{b1}(p'_1) = (1 - \sigma_2) \left[ Q + W^{b1}_{+1} (p'_1, p'_2) \right] + \sigma_2 W^{b1}_{+1} (p'_1).
$$

Equation (10) says that if buyer 1 is alone, which happens with probability $1 - \sigma_2$, he purchases the good, enjoys utility $Q$ and proceeds to the centralized market with no money. If he is not alone, he cannot purchase the good since he does not have enough money.

Holding $p'_2$ his payoff is $-\phi p'_2 + \beta V^{b1}(p'_2)$ with

$$
V^{b1}(p'_2) = \sigma_2 \left( \theta'_2 + \frac{1 - \theta'_2}{2} \right) \left[ Q + W^{b1}_{+1} (p'_2, p'_2) \right] + (1 - \sigma_2) \left[ Q + W^{b1}_{+1} (p'_2, p'_1) \right] + \sigma_2 \frac{1 - \theta'_2}{2} W^{b1}_{+1} (p'_2).
$$

With probability $\sigma_2$ buyer 1 is not alone, in which case he purchases and consumes the good
if buyer 2 holds the low amount of money, which happens with probability $\theta'_2$, or if buyer 2

\textsuperscript{6}If $\sigma_x = 0$, then $\theta_x = 0$ since it is a dominant strategy to bring the high amount of money. While $(\sigma_x, \theta_x) = (0, 0)$ is an equilibrium of the second-stage game, as is $(\sigma_x, \theta'_x) = (1, 0)$, it is not subgame perfect since it is not a seller’s best response to let the two buyers visit his competitor. These two conditions are always satisfied when $p_2 > p_1$, which will be true in equilibrium.
holds the high amount of money yet buyer 1 wins the draw, which happens with probability 
\((1 - \theta_2)/2\). If buyer 1 is alone, which happens with probability \(1 - \sigma_2\), he consumes the good 
but pays only \(p'_1\). Finally, with probability \(\sigma_2(1 - \theta_2)/2\) buyer 2 is there too holding the high 
amount of money, but buyer 1 does not win the draw and then proceeds to the centralized 
market with the same amount of money as when he entered the DSM.

Lemma 1 If a symmetric equilibrium exists, then buyers randomize over their money holdings.

Proof. See Appendix. ■

Taking as given the visit and money decisions of his competitor, a buyer's best response 
upon visiting a seller is to take the low amount of money if he anticipates that the other buyer 
takes the high amount of money and vice versa. Both equilibria require coordination, however. 
The only other Nash equilibrium is for both buyers to randomize over their money holding.

3.2 Visit strategies

Lemma 2 If a symmetric equilibrium exists, then buyers randomize over sellers with probability 
\(\sigma\) where

\[
\sigma = \sigma(p'_1, p_1) = \frac{i\phi_{+1}(p_1 - p'_1) + Q - \phi_{+1}p'_1}{(Q - \phi_{+1}p'_1) + (Q - \phi_{+1}p_1)}.
\]

Proof. See Appendix. ■

When deciding which seller to visit, a buyer internalizes the other buyer's indifference be-
tween the low and the high money holding (Lemma 1). This has two effects. First, neither \(p_2\) 
or \(p'_2\) appear as an argument for \(\sigma\). And second, a buyer's best response depends on the other 
buyer's visit strategy but it does not depend on how much money the other buyer has decided 
to bring, see equations (30) and (31) in the Appendix. Consequently, taking as given the visit 
and money holding decisions of his competitor, the only uncoordinated best-response for both 
buyers is to randomize between sellers. As for the opportunity cost of money (measured by
it can be seen from (12) that it influences buyers' visit strategy only if sellers post different price pairs. Should they post identical price pairs, then $\sigma = 1/2$.

Combining Lemmas 1 and 2, we have a last lemma.

Lemma 3 Let us note $j'$ and $\bar{j}$ two lower bounds on the nominal interest rate, and $\bar{i}'$ and $\bar{i}$ two upper bounds. Given $p'_1 < p'_2$, $p_1 < p_2$ and $\sigma$ defined by (12), buyers randomize over money holdings with probabilities:

\begin{align}
\theta_1' &= \theta_2' = \theta' = \frac{2\phi_{+1}(p'_2 - p'_1)}{\sigma(Q - \phi_{+1}p'_2)} - 1 \quad \text{if } i \in [j', \bar{i}]', \\
\theta_1 &= \theta_2 = \theta = \frac{2\phi_{+1}(p_2 - p_1)}{(1-\sigma)(Q - \phi_{+1}p_2)} - 1 \quad \text{if } i \in [j, \bar{i}],
\end{align}

where $j = \sigma (Q - \phi_{+1}p_2) / [2\phi_{+1}(p_2 - p_1)]$, $j' = \sigma (Q - \phi_{+1}p'_2) / [2\phi_{+1}(p'_2 - p'_1)]$, $\bar{i} = 2j$ and $\bar{i}' = 2j'$.

The proof is straightforward. Inserting $\sigma_1 = \sigma_2 = \sigma$ from Lemma 2 into the expressions for buyers' monetary strategies in (24) and (25) and into (26) and (27) in the Appendix yields $\theta_1' = \theta_2' = \theta'$ and $\theta_1 = \theta_2 = \theta$ as in (13) and (14).

From Lemma 1 buyers randomize over how much money they bring to the seller they are visiting. Given the expressions for $\theta'$ and $\theta$ in (13) and (14) this imposes bounds on the value of the nominal interest rate, i.e. $i \in [i', \bar{i}]'$ and $i \in [i, \bar{i}]$ respectively. As a matter of fact, for low enough $i$ for instance, we would have $\theta < 0$ and $\theta' < 0$, suggesting that buyers always opt for the high money holding, destroying the equilibrium in contingent prices. Note that those bounds are endogenous, however, and they turn out to adjust to shifts in $i$ in a way that ensures $i$ stays within the intervals $[i', \bar{i}]'$ and $[i, \bar{i}]$ that support the mixed monetary strategies (see Proof of Proposition 1). Finally, note that symmetric price posting by sellers ($p'_1 = p_1$ and $p'_2 = p_2$) would imply $\theta' = \theta$, $i = i'$ and $\bar{i} = \bar{i}'$ in addition to $\sigma = 1/2$. 

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3.3 Pricing strategies

Now, using the equilibrium strategy profile \((\sigma, \theta', \theta)\) of the second-stage game for given \((p_1, p_2)\) and \((p'_1, p'_2)\), we derive sellers' best responses. Seller 1's value function upon entering the DSM is given by

\[
V^{x_1} = 2\sigma (1 - \sigma) W_{+1}^{x_1}(p'_1) + \sigma^2 \left[ 1 - (\theta')^2 \right] W_{+1}^{x_1}(p'_2) + \left[ \sigma^2 (\theta')^2 + (1 - \sigma)^2 \right] W_{+1}^{x_1}(0).
\]

Equation (15) says that if buyer 1 is the only buyer, which happens with probability \(\sigma (1 - \sigma)\), or if buyer 2 is the only buyer, which also happens with probability \(\sigma (1 - \sigma)\), then seller 1 proceeds to the CWM with \(p'_1\). If both buyers are present and at least one of them holds the high amount of money, which happens with probability \(\sigma^2 \left[ 1 - (\theta')^2 \right]\), then seller 1 enters the next CWM with \(p'_2\). In all other circumstances no sale takes place. Similarly, for seller 2 we have

\[
V^{x_2} = 2\sigma (1 - \sigma) W_{+1}^{x_2}(p_1) + (1 - \sigma)^2 (1 - \theta^2) W_{+1}^{x_2}(p_2) + \left[ (1 - \sigma)^2 \theta^2 + \sigma^2 \right] W_{+1}^{x_2}(0).
\]

Using the linearity of the \(W\) function and getting rid of the constant \(W_{+1}^{x_2}(0)\) terms, the two value functions simplify into the following expressions

\[
\Pi^1 = 2\sigma (1 - \sigma) \phi_{+1}p'_1 + \sigma^2 \left[ 1 - (\theta')^2 \right] \phi_{+1}p'_2
\]

\[
\Pi^2 = 2\sigma (1 - \sigma) \phi_{+1}p_1 + (1 - \sigma)^2 (1 - \theta^2) \phi_{+1}p_2
\]

which we call sellers' profits.

We are now in a position to characterize the equilibrium. As indicated, we focus on the symmetric price posting equilibrium as is natural when the economy becomes large. Note however that symmetry in visit \((\sigma_1 = \sigma_2 = \sigma)\) and monetary strategies \((\theta'_1 = \theta'_2 = \theta)\) and \(\theta_1 = \theta_2 = \theta\) were found to be the unique uncoordinated equilibrium of the second-stage game.
without assuming symmetric price posting.

**Proposition 1** Denote $i_a$ an upper bound on the nominal interest rate. For $i \in (0, i_a]$, there exists a unique symmetric subgame Nash equilibrium where each seller posts $p^*_2 > p^*_1$, and in the resulting subgame each buyer visits either seller with probability $\sigma^* = 1/2$ and chooses to hold the low amount of each $\theta^* = \theta^* = 4i\phi_i(y)(p^*_2 - p^*_1)/(Q - \phi_i(y)p^*_2) - 1$. For $i = 0$ there exists a continuum of symmetric equilibria indexed by $\alpha \in (0, Q]$ in which each seller charges $p^*_1 = Q/2$ in pairwise meetings and $p^*_2 = \alpha$ in multilateral meetings (cf Coles and Eeckhout 2003). For $i > i_a$ there is no monetary equilibrium.

**Proof.** See Appendix.

The continuum of symmetric equilibria in Coles and Eeckhout (2003) is characterized by sellers posting $p_1 = Q/2$ and $p_2 = \alpha \in (0, Q]$ in their $2 \times 2$ game. In the monetary version of their economy, when deciding upon prices, sellers must now internalize buyers' ex ante indifference condition between the low and the high money holding. For instance, when the nominal interest rate $i$ is close to zero, then sellers will still post a pairwise price $p_1 = Q/2$ as in Coles and Eeckhout (2003). But there is now a unique multilateral price within the former continuum that makes holding $p_1$ in monetary units equivalent to holding $p_2$ once the opportunity costs of holding $p_1$ and holding $p_2$ are taken into account. This additional constraint destroys the degree of freedom in choosing two prices that sellers had in the non-monetary version of the model. The indeterminacy found in directed search with contingent prices does not hold once buyers trade using costly money. At the Friedman rule, however, since holding money is free, a (weakly) dominant strategy is to hold $p_2$ and we are back to a dynamic version of Coles and Eeckhout's economy: there exists a continuum of symmetric equilibria indexed by $\alpha \in (0, Q]$ in which each seller charges $p_1 = Q/2$ in pairwise meetings and $p_2 = \alpha$ in multilateral meetings.

One way to understand this result is to look at Figure 2. In the non-monetary version of this model buyers are assumed to be able to pay any posted price so that monetary strategies are irrelevant. For a given $p_1$, when money is costly, holding $p_2$ will be preferred to holding
$p_1$ if $p_2$ is only slightly greater than $p_1$ (the marginal cost is small compared to the additional trading opportunities). By contrast a buyer is clearly better off holding $p_1$ if $p_2$ is significantly greater than $p_1$ (the marginal cost more than cancels the marginal benefit). Since the value of holding $p_2$ decreases linearly with $p_2$ there exists only one multilateral price that makes a buyer indifferent between holding the low or the high amount of money. Although Fig. 2 takes $p_1$ as given, note that it is the pair $(p_1,p_2)$ that is uniquely determined in equilibrium.

This result is important. The standard assumption in directed search models that sellers commit to a single, non-contingent price is often defended by Coles and Eeckhout's result that this behavior is an equilibrium. Our paper suggests that this result is not robust: if agents trade using money, unless the central bank runs the Friedman rule, the indeterminacy vanishes and a unique price-posting scheme emerges with the property that $p_1 \neq p_2$. Models with money have then different implications than models without money. In the next section we explore further the impact of liquidity costs on strategic price posting decisions by sellers.\footnote{Note that profit is higher when sellers post contingent prices than when they post a fixed price, regardless of the nominal interest rate. From Burdett, Shi and Wright (2001), profit in the symmetric equilibrium of an economy in which sellers post a single price is $\frac{\beta}{\phi}$. In our model, when $i \to 0$, profit is $\frac{1}{2} > \frac{3}{2}$. It is easy to show}
Timing is important for our result. So far we have assumed that sellers post prices first and then buyers select a seller and a money holding simultaneously. Assume now that sellers post prices after buyers have chosen a money holding and a seller. First, note that buyers are now ignorant of prices when deciding upon a seller to visit. This has two consequences: buyers randomize between sellers so that the model becomes one of random matching and posting; and sellers no longer compete against each other by way of prices since buyers are already committed to visiting a seller. As a result, the seller is able to extract maximum surplus by posting the monopoly price, as in Diamond (1971). This monopoly price takes into account the cost of holding money for the buyer, i.e. each seller posts a unique monopoly price $p_m$ given by $-i\phi_{+1}p_m + (1 - \sigma + \sigma/2) [Q - \phi_{+1}p_m] = 0$. Buyers anticipate the monopoly price and carry $p_m$. Uniqueness is then robust to a change in timing, but the contingent price posting equilibrium unravels as sellers are now posting a unique price.\footnote{This last result is reminiscent of a result in Jean, Rabinovich and Wright's (2010) model of random matching and posting. There are a couple of differences, however. First, sellers are allowed to post contingent prices here, and second, Jean et al. (2010) examine two different timing assumptions: either buyers and sellers move at the same time, in which case there is a continuum of prices, or sellers move first in which case there is a unique equilibrium where sellers post the monopoly price. The version of our model in which sellers post last is simply a version of their model in which buyers move first and sellers are allowed to post contingent prices, yet they end up posting a unique price since they have no use of a multilateral price.}

4 Inflation and Strategic Price Posting

In this section we examine how shifts in the inflation rate impact on the price pair posted by sellers and the distribution of money holdings. Recall from the proof of Proposition 1 that in
the symmetric equilibrium real prices are given by:

\[
(19) \quad \phi_{p1} = \phi_{p2} = Q \left(1 + \theta \right) \frac{2i - \theta (1 + 6i) - \theta^2}{2i \left(1 + 2\theta + 8i\theta + \theta^2\right)},
\]

\[
(20) \quad \phi_{p1} = \phi_{p2} = \frac{Q(1 - \theta^2)}{1 + \theta (2 + 8i + \theta)}.
\]

As an example, setting \(Q = 1\) and \(i = 0.1\) yields \(\phi_{p1} = 0.518\) and \(\phi_{p2} = 0.867\) and \(\theta' = 0.052\).

As for the individual rationality constraints in Definition 1, they transform into \(i < i_s\) with \(i_s \approx 63\%\). Figure 3 represents the equilibrium real values of \(P_1\) and \(P_2\) as the nominal interest rate increases away from the Friedman rule. The blue lines represent the price pair in the 2-by-2 game while the red lines represent the price pair in the \(n\)-by-\(n\) game. 9

First, one may have expected the indeterminacy to survive for low values of the nominal interest rate. As the opportunity cost of holding money becomes small, holding even the high amount of money does not cost much yet it allows to trade in multilateral matches. Actually, sellers take advantage of buyers' greater incentive to hold large amounts of money by charging a real price \(\phi_{p2}\) that gets closer to \(Q\) (the buyer's reservation value) as the nominal interest rate approaches zero. Because of this high price in multilateral meetings, buyers still have an incentive to carry the low amount of money despite the low opportunity cost of money.

Second, it turns out that for small deviations from the Friedman rule sellers actually increase the pairwise price they charge in real terms, forcing buyers who pick the low money holding to bring more money. As inflation rises it becomes proportionally more costly to hold the large amount of money. Sellers extract this incentive for buyers to shift to the low money holding by raising the price they charge in pairwise meetings. That is, higher inflation allows sellers to raise the real price they charge in the unfavorable event of receiving the visit of only one buyer, despite greater inflation and a low effective bargaining power. Note that while the pairwise

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9 We assume \(n\) very large and use properties of the urn-ball matching function in the limit in which only the ratio of buyers to sellers, \(\theta = \frac{n}{s} = 1\), matters. Details on the \(n\)-by-\(n\) game are given in Appendix A.4.
price rises with small increases in inflation, the relationship between average money holdings and inflation is still negative.

Third, as can be seen from Figure 3, the model predicts that price dispersion decreases as inflation rises. Higher inflation reduces average real balances, and then sellers' ability to discriminate on the basis of realized demand. In addition, as noted above, sellers raise the pairwise price but reduce the multilateral price. As a result higher inflation means less dispersion in prices. While most empirical studies point to price dispersion increasing with inflation, some have documented an inverse relationship as in our model. See for instance Reinsdorf (1994).

5 Conclusion

Coles and Eeckhout (2003) have shown that if sellers are allowed to post prices contingent on realized demand, then there exists a continuum of equilibria in which posting a fixed price or posting an auction both describe equilibrium behavior. This result is often called to support the use of price posting with directed search. In this research we have shown that it is not robust to adding costly money holdings for buyers. If trading with money is costly, and it is
at least because of inflation, then there exists a unique equilibrium. In this equilibrium sellers post a price pair that exploits ex post competition between buyers yet leaves them ex ante indifferent between the corresponding money holdings. The key ingredient for uniqueness is the endogeneity of the buyer's liquidity constraint at the point of sale. This result should change the way we think about directed search: it is not one equilibrium among the continuum of equilibria in the contingent-price posting game. This result also highlights the importance of incorporating money in the study of trading mechanisms.
Appendix


Using the linearity of the \( W \) function, that is \( W_{+1}^b (m-p) = W_{+1}^b (m) - \phi_{+1} p_1 \), rewrite (10) as

\[
V^b_1(p'_1) = (1 - \sigma_2) [Q - \phi_{+1} p'_1] + W_{+1}^b (p'_1),
\]

and (11) as

\[
V^b_2(p'_2) = \sigma_2 \left( \beta'_2 + \frac{1 - \beta'_2}{2} \right) [Q - \phi_{+1} p'_2] + (1 - \sigma_2) [Q - \phi_{+1} p'_1] + W_{+1}^b (p'_2).
\]

When visiting seller 1, buyer 1 strictly prefers holding \( p'_1 \) to \( p'_2 \) if \( -\phi p'_1 + \beta V^b_1(p'_1) > -\phi p'_2 + \beta V^b_2(p'_2) \). Plugging (21) and (22) into this inequality, using \( \phi_{+1} (1 + \tau) = \phi \), dividing by \( \beta \) and recalling that \( i = \frac{1 - \beta + \tau}{\beta} \) it simplifies into

\[
i\phi_{+1} (p'_2 - p'_1) > \left( \beta'_2 + \frac{1 - \beta'_2}{2} \right) \sigma_2 [Q - \phi_{+1} P'_2],
\]

where the left-hand side is the marginal cost of holding \( p'_2 \) instead of \( p'_1 \) and the right-hand side is the marginal benefit. From (23) buyer 1's best-response correspondence is given by

\[
\theta'_1 = \begin{cases} 
0 & \text{if } \beta'_2 > \frac{2\phi_{+1} (p'_2 - p'_1)}{\sigma_2 (Q - \phi_{+1} P'_2)} - 1 \\
(0,1) & \text{if } \beta'_2 = \frac{2\phi_{+1} (p'_2 - p'_1)}{\sigma_2 (Q - \phi_{+1} P'_2)} - 1 \\
1 & \text{if } \beta'_2 < \frac{2\phi_{+1} (p'_2 - p'_1)}{\sigma_2 (Q - \phi_{+1} P'_2)} - 1 
\end{cases}
\]
A similar exercise shows that the best-response correspondence of buyer 2 is

\[
\theta_2' = \begin{cases} 
0 & \text{if } \theta_1' > \frac{2\theta_1' \left( \psi_2' - \phi_2' \right)}{\sigma_1(Q - \phi_1' \psi_2')} - 1 \\
(0, 1) & \text{if } \theta_1' = \frac{2\theta_1' \left( \psi_2' - \phi_2' \right)}{\sigma_1(Q - \phi_1' \psi_2')} - 1 \\
1 & \text{if } \theta_1' < \frac{2\theta_1' \left( \psi_2' - \phi_2' \right)}{\sigma_1(Q - \phi_1' \psi_2')} - 1
\end{cases}
\] (25)

Assuming \( \sigma_1 \in (0, 1) \) and \( \sigma_2 \in (0, 1) \), then both correspondences intersect at \((1, 0)\), \((0, 1)\) and \( \theta_1' \in (0, 1) \) and \( \theta_2' \in (0, 1) \). Clearly \( (\theta_1', \theta_2') = (1, 1) \) cannot be an equilibrium: if buyer 1 plays \( \theta_1' = 1 \) then buyer 2's best response is to play \( \theta_2' = 0 \) and vice versa. The equivalent argument holds for \( (\theta_1', \theta_2') = (0, 0) \). Since \( (\theta_1', \theta_2') = (1, 0) \) or \( (0, 1) \) correspond to coordinated strategies, it follows that if a symmetric equilibrium exists, then buyers randomize over money holdings, i.e. \( \theta_1' \in (0, 1) \) and \( \theta_2' \in (0, 1) \).

Similarly, if visiting seller 2, buyer 1's best response is given by

\[
\theta_1 = \begin{cases} 
0 & \text{if } \theta_2 < \frac{2\theta_2 \left( \psi_2' - \phi_1' \right)}{(1 - \sigma_2)(Q - \phi_2 \psi_1')} - 1 \\
(0, 1) & \text{if } \theta_2 = \frac{2\theta_2 \left( \psi_2' - \phi_1' \right)}{(1 - \sigma_2)(Q - \phi_2 \psi_1')} - 1 \\
1 & \text{if } \theta_2 > \frac{2\theta_2 \left( \psi_2' - \phi_1' \right)}{(1 - \sigma_2)(Q - \phi_2 \psi_1')} - 1
\end{cases}
\] (26)

and buyer 2's best response is given by

\[
\theta_2 = \begin{cases} 
0 & \text{if } \theta_1 > \frac{2\theta_1 \left( \psi_2' - \phi_2' \right)}{(1 - \sigma_1)(Q - \phi_2 \psi_1')} - 1 \\
(0, 1) & \text{if } \theta_1 = \frac{2\theta_1 \left( \psi_2' - \phi_2' \right)}{(1 - \sigma_1)(Q - \phi_2 \psi_1')} - 1 \\
1 & \text{if } \theta_1 < \frac{2\theta_1 \left( \psi_2' - \phi_2' \right)}{(1 - \sigma_1)(Q - \phi_2 \psi_1')} - 1
\end{cases}
\] (27)

respectively, also intersecting at \((1, 0)\), \((0, 1)\) and \( \theta_1 \in (0, 1) \) and \( \theta_2 \in (0, 1) \). Only the last one is a candidate for the symmetric subgame perfect equilibrium.

For buyer 1, visiting seller 1 can be done holding either \( p'_1 \) or \( p'_2 \). The corresponding expected payoff is given by

\[
\theta'_1 \left[ -\phi p'_1 + \beta V^{b_1}(p'_1) \right] + (1 - \theta'_1) \left[ -\phi p'_2 + \beta V^{b_1}(p'_2) \right].
\]

Recall from Lemma 1 that if a symmetric equilibrium exists then it implies \( \theta'_1 \in (0,1) \) and \( \theta'_2 \in (0,1) \). In this equilibrium buyer 1 randomizes between the two money holdings because the expected payoff to holding the low amount or the high amount of money while visiting seller 1 are equal. For buyer 1 this means that \( -\phi p'_1 + \beta V^{b_1}(p'_1) = -\phi p'_2 + \beta V^{b_1}(p'_2) \) so that (28) is simply equal to \( -\phi p'_1 + \beta V^{b_1}(p'_1) \). Similarly, buyer 1’s expected payment from visiting seller 2 is simply \( -\phi p'_1 + \beta V^{b_1}(p'_1) \). Therefore buyer 1 strictly prefers visiting seller 1 to visiting seller 2 if \( -\phi p'_1 + \beta V^{b_1}(p'_1) > -\phi p'_1 + \beta V^{b_1}(p'_1) \). Using the same simplification techniques used in the Proof of Lemma 1, this transforms into

\[
-i\phi_{+1} p'_1 + (1 - \sigma_2) [Q - \phi_{+1} p'_1] > -i\phi_{+1} p_1 + \sigma_2 [Q - \phi_{+1} p_1].
\]

Buyer 1’s best response correspondence is then given by

\[
\sigma_1 = \begin{cases} 
0 & \text{if } \sigma_2 > \frac{i\phi_{+1} (p_1 - p'_1) + Q - \phi_{+1} p'_1}{(Q - \phi_{+1} p'_1) + (Q - \phi_{+1} p_1)} \\
(0,1) & \text{if } \sigma_2 = \frac{i\phi_{+1} (p_1 - p'_1) + Q - \phi_{+1} p'_1}{(Q - \phi_{+1} p'_1) + (Q - \phi_{+1} p_1)} \\
1 & \text{if } \sigma_2 < \frac{i\phi_{+1} (p_1 - p'_1) + Q - \phi_{+1} p'_1}{(Q - \phi_{+1} p'_1) + (Q - \phi_{+1} p_1)}
\end{cases}
\]

Similarly, buyer 2’s best response correspondence is given by

\[
\sigma_2 = \begin{cases} 
0 & \text{if } \sigma_1 > \frac{i\phi_{+1} (p_1 - p'_1) + Q - \phi_{+1} p'_1}{(Q - \phi_{+1} p'_1) + (Q - \phi_{+1} p_1)} \\
(0,1) & \text{if } \sigma_1 = \frac{i\phi_{+1} (p_1 - p'_1) + Q - \phi_{+1} p'_1}{(Q - \phi_{+1} p'_1) + (Q - \phi_{+1} p_1)} \\
1 & \text{if } \sigma_1 < \frac{i\phi_{+1} (p_1 - p'_1) + Q - \phi_{+1} p'_1}{(Q - \phi_{+1} p'_1) + (Q - \phi_{+1} p_1)}
\end{cases}
\]
Both best-response correspondences intersect at \((1, 0), (0, 1)\) and at \(\sigma_1 = \sigma_2 = \sigma\) given in (12).

### A.3. Proof of Proposition 1

Using (13) to substitute \(p'_2\) into (17) and then (12) to substitute \(p'_1\) we obtain

\[
\Pi^1(\sigma, \theta'; p_1) = 2\sigma(1 - \sigma)K_1 + \sigma^2 \left[1 - (\theta')^2\right] \frac{Q\sigma(1 + \theta') + 2iK_1}{2i + \sigma(1 + \theta')}
\]

with

\[
K_1 = \frac{Q(1 - 2\sigma) + (i + \sigma)\phi_{+1}p_1}{1 + i - \sigma}.
\]

Similarly,

\[
\Pi^2(\sigma, \theta; p'_1) = 2\sigma(1 - \sigma)K_2 + (1 - \sigma)^2 \left(1 - \theta^2\right) \frac{Q(1 + \theta)(1 - \sigma) + 2iK_2}{2i + (1 + \theta)(1 - \sigma)}
\]

with

\[
K_2 = \frac{Q(2\sigma - 1) + (1 + i - \sigma)\phi_{+1}p'_1}{i + \sigma}.
\]

Given \((p_1, p_2)\) we define \(\sigma^*_1\) as the visit strategy by buyers that maximizes seller 1's profit, that is \(\sigma^*_1 = \arg \max_{\sigma \in [0,1]} \Pi^1(\sigma, \theta'; p_1)\). Given \((p_1, p_2)\) we define \(\theta^*_1\) as the monetary strategy by buyers that maximizes seller 1's profit, that is \(\theta^*_1 = \arg \max_{\theta \in [0,1]} \Pi^1(\sigma, \theta'; p_1)\). Similarly, given \((p'_1, p'_2)\) we denote \(\sigma^*_2 = \arg \max_{\sigma \in [0,1]} \Pi^2(\sigma, \theta; p'_1)\) and \(\theta^*_2 = \arg \max_{\theta \in [0,1]} \Pi^2(\sigma, \theta; p'_1)\) for seller 2. Seller 1's best-response correspondence is then given by

\[
\begin{align*}
\theta'_1 &= \arg \max_{\theta' \in [0,1]} \Pi^1(\sigma, \theta'; p_1) \\
\sigma^*_1 &= \arg \max_{\sigma \in [0,1]} \Pi^1(\sigma, \theta'; p_1)
\end{align*}
\]
Similarly, seller 2’s best-response correspondence is given by

\[
\begin{aligned}
\sigma_2^* &= \arg\max_{\sigma \in [0,1]} \Pi^2(\sigma, \theta; p_1') \\
\alpha_2^* &= \arg\max_{\sigma \in [0,1]} \Pi^2(\sigma, \theta; p_1')
\end{aligned}
\]  

(37)

To show that there is only one fixed point in the symmetric equilibrium, take the first-order condition of \( \Pi^2 \) with respect to \( \theta \) and insert \( \sigma = 1/2 \). This enables to extract

\[
\phi p_1' = \phi p_2 = \frac{Q(1 + \theta)[2i - \theta(1 + 6i) - \theta^2]}{2i[1 + 2i + 8i + \theta^2]}
\]  

(38)

Then take the first order condition of \( \Pi^2 \) with respect to \( \sigma \), insert \( \phi p_1' \) from (38) and set \( \sigma = 1/2 \) to obtain one equation in one unknown, \( \theta \), parameterized by \( i \), of the form \( g(\theta; i) = 0 \). It is easy to show that the function \( g \) is strictly increasing and that \( g(0; i) < 0 \) and \( g(1; i) > 0 \) so that there exists a unique \( \theta \) that maximizes seller 2’s profit given \( i \), \( p_1' \) and \( p_2' \). Finally, inserting \( p_1' \) from (38) into (13) and setting \( \theta' = \theta \) and \( \sigma = 1/2 \) one obtains

\[
\phi p_2' = \phi p_2 = \frac{Q(1 - \theta^2)}{1 + \theta(2 + 8i + \theta^2)}
\]  

(39)

To check that \( i \in I = [i, \bar{i}] \) simply insert \( p_2' \) from (38) and \( p_1' \) from (38) into \( i = \frac{\sigma(Q - \phi_1' p_2')}{\phi_1'(Q^2 - p_1')} \) and \( \bar{i} = \frac{Q - \phi_1' p_1'}{\phi_1'(Q^2 - p_1')} \). This yields \( i = \frac{1}{1 + \theta} < i \) and \( \bar{i} = \frac{2i}{1 + \theta} > i \).

A.4. The large economy

Suppose there is an infinite yet countable number \( n \) of buyers and the same number of sellers. The value function for a buyer holding \( p_1 \) on the DSM is given by

\[
V^b(p_1) = \psi_p \left[ Q + W^b_{i+1}(p_1 - p_1) \right] + (1 - \psi_p) W^b_{i+1}(p_1),
\]  

(40)
where $\psi_p$ is the probability of a pairwise match for a buyer. Noting $\psi_m$ the probability of a multilateral match in which the buyer wins the good, the value function of a buyer holding $p_2$ in the DSM is given by

$$V^b(p_2) = \psi_p \left[ Q + W^b_{n+1}(p_2 - p_1) \right] + \psi_m \left[ Q + W^b_{n+1}(p_2 - p_2) \right] + (1 - \psi_p - \psi_m) W^b_{n+1}(p_2).$$

In the symmetric equilibrium, since buyers use identical visit strategies, the matching technology corresponds to a standard urn-ball process. The probability for a buyer to face $n$ other buyers at a seller's shop is then given by $\Pr[Y = n] = \frac{n^n}{n!} e^{-\gamma}$ where $Y$ is the random variable equal to how many other buyers show up and $\gamma (= 1)$ is the buyer-seller ratio. The probability of a pairwise match for a buyer is then equal to the probability that no other buyer shows up, that is $\psi_p = \Pr[Y = 0] = e^{-\gamma}$. The value of $\psi_m$ is more demanding. To win the good in a multilateral match, the buyer must hold the high amount of money $p_2$ and be selected among all the buyers who also brought $p_2$. Denoting $k$ the number of buyers who also brought $p_2$ among the $n$ other buyers present at the seller's we have

$$\psi_m = \sum_{n \in \mathbb{N}^+} \Pr[Y = n] \left\{ \sum_{k=0}^n \frac{n^k}{k!} \theta^k (1 - \theta)^{n-k} \frac{1}{n - k + 1} \right\}.$$

To understand the terms inside the curly brackets, consider the case in which the buyer faces 2 other buyers so that $n = 2$. The buyer wins the good if he is the only one holding $p_2$ which happens with probability $\theta^2$, or if one of the two other buyers holds $p_2$, which happens with probability $\binom{2}{1} \theta (1 - \theta)$ in which case he wins the good with probability $\frac{1}{2}$, or if both of them hold $p_2$, which happens with probability $(1 - \theta)^2$ in which case he wins the good with probability $\frac{1}{3}$. This probability simplifies into

$$\frac{1 - \gamma (1 - \theta) e^{-\gamma} - e^{-\gamma(1-\theta)}}{\gamma (1 - \theta)}.$$
The value function for a buyer on the centralized market is given by

\[ W^b(m) = \phi(m + T) + \max_{\hat{m} \in \{p_1, p_2\}} \left\{ -\phi\hat{m} + \beta V^b(\hat{m}) \right\}. \tag{43} \]

Inserting (40) into (43), using the same simplification techniques as before and noting \( z_1 = \phi_+ p_1 \), a buyer’s net utility from holding \( p_1 \) is given by

\[ -iz_1 + e^{-\gamma}(Q - z_1). \tag{44} \]

Inserting (41) into (43) and noting \( z_2 = \phi_+ p_2 \), a buyer’s net utility from holding \( p_2 \) is given by

\[ -iz_2 + e^{-\gamma}(Q - z_1) + \frac{1 - \gamma(1 - \theta)e^{-\gamma} - e^{-\gamma(1 - \theta)}}{\gamma(1 - \theta)} (Q - z_2). \tag{45} \]

As for sellers, their value function on the DSM is given by

\[ V^s = \xi p W^s_{+1}(p_1) + \xi m W^s_{+2}(p_2) + (1 - \xi_p - \xi_m) W^s_{+1}(0) \tag{46} \]

where \( \xi_p \) is the probability of a pairwise match for a seller and \( \xi_m \) is the probability of a multilateral match in which at least one of the buyer holds \( p_2 \). Noting \( Z \) the random variable equal to how many buyers are present, we have

\[ \xi_p = \Pr[Z = 1] = \gamma e^{-\gamma} \tag{47} \]

and

\[ \xi_m = \sum_{n \geq 2} \Pr[Z = n] (1 - \theta^n). \tag{48} \]
This last probability simplifies into

\begin{equation}
1 - \gamma (1 - \theta) e^{-\gamma} - e^{-\gamma(1 - \theta)}.
\end{equation}

To compute the value function of a seller on the centralized market, we use a method developed by Montgomery (1991). The key to this method is to assume that sellers must provide buyers with a certain level of expected utility $U$, given by the market, which is later determined endogenously. Sellers are then thought of offering a combination of prices and a probability to trade that yields $U$ to buyers. Suppose a seller chooses a particular $(p_1, p_2)$ and buyers respond by visiting him with probability $\sigma$ and holding $p_1$ with probability $\theta$, implying probabilities to trade equal to $\psi_p(\gamma)$ and $\psi_m(\gamma)$. In a competitive economy no seller can beat the market by posting a different combination of prices and probabilities. Given that $\sigma = \sigma(\gamma)$ the seller's value function in the centralized market is then

\begin{equation}
W^s(m) = \phi m + \max_{p_1, p_2, \gamma, \theta} \beta V^s.
\end{equation}

Inserting (46) into (50) and simplifying, the seller's problem is

\begin{align}
\max_{z_1, z_2, \gamma, \theta} & \gamma e^{-\gamma} z_1 + \left[ 1 - \gamma (1 - \theta) e^{-\gamma} - e^{-\gamma(1 - \theta)} \right] z_2 \\
\text{s.t.} & -iz_1 + e^{-\gamma}(Q - z_1) = U, \\
\text{s.t.} & -iz_2 + e^{-\gamma}(Q - z_2) + \frac{1 - \gamma (1 - \theta) e^{-\gamma} - e^{-\gamma(1 - \theta)}}{\gamma (1 - \theta)}(Q - z_2) = U.
\end{align}

First use (52) to substitute $z_1$ into (53) and call it (53-bis). Then use (52) to substitute $z_1$ into (51), and use (53-bis) to substitute $z_2$ into (51). We are left with an unconstrained maximization problem in $\gamma$ and $\theta$, the solution to which is represented on Figure 2 in red.

\footnote{Other papers in the literature using this approach are Peters (2000), Peters and Severinov (1997), Burdett, Shi and Wright (2001).}
References


