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Partial state and unknown input estimation for time-delay systems

P.S. Teh \(^{a}\) & H. Trinh \(^{a}\)

\(^{a}\) School of Engineering, Deakin University, Geelong, VIC 3217, Australia

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Partial state and unknown input estimation for time-delay systems

P.S. Teh* and H. Trinh
School of Engineering, Deakin University, Geelong, VIC 3217, Australia
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This article considers the problem of estimating a partial set of the state vector and/or unknown input vector of linear systems driven by unknown inputs and time-varying delay in the state variables. Three types of reduced-order observers, namely, observers with delays, observers without internal delays and delay-free observers are proposed in this article. Existence conditions and design procedures are presented for the determination of parameters for each case of observers. Numerical examples are presented to illustrate the design procedures.

Keywords: time-delay systems; partial-state estimation; unknown-input estimation; reduced-order observers

1. Introduction

Estimation of the state and unknown inputs often arises in various engineering applications (Patton, Frank, and Clark 1987; Park and Stein 1988; Hou and Müller 1992; Corless and Tu 1998; Liao and Huang 1999; Boutayeb, Darouach, and Rafaralahy 2002; Xiong and Saif 2003; Ha and Trinh 2004; Zhang, Branicky, and Philips 2004). The problem of estimating the unknown inputs is motivated by situations where system inputs are often unmeasurable or inaccessible, or where the measurement of the system inputs is expensive to measure (Corless and Tu 1998). In such a situation, where the unknown inputs can be modelled as disturbances, uncertainties or generally represent the incipient failure of internal component of the system, their knowledge could be beneficial for robust control. The state and unknown input estimation can be of significant value for the purpose of fault detection and isolation (Patton et al. 1987). In particular, for chaotic systems, the problem of simultaneous estimation of the state and unknown inputs for a class of chaotic systems is becoming one of the emerging research areas (Liao and Huang 1999; Boutayeb et al. 2002) as one wishes to estimate not only the states for chaos synchronisation but also the information signal input for secure communication. It is also potentially useful in networked control systems with unknown input package delays and even losses (Zhang et al. 2004).

It is well known that the dynamic behaviour of many industrial processes contains inherent time delays (Malek-Zavarei and Jamshidi 1987; Niculescu 2001) due to the distributed nature of the systems. In processes with time delays, it is difficult to achieve satisfactory performance (Gorecki, Fuksa, Grabowski, and Korytowski 1989), especially when there are parametric uncertainties (Zhang et al. 2004) and time-varying delays (Wang, Wang, and Shi 2009a,b). They are often a source of instability and degradation in control performance, and thus the stability issue of systems with time-varying delays is therefore of theoretical and practical importance. Some recent improved results pertaining to the delay-dependent stability analysis for systems with time-varying delays have been reported (see, e.g. He, Wang, Lin, and Wu 2007a; He, Wang, Xie, and Lin 2007b; Lin, Wang, and Yang 2009). Estimation and control of time-delay systems with uncertainties have attracted extensive studies in the literature (see Richard 2003 for an overview). Recently, the problem of state and unknown input simultaneous estimation for time-delay systems have received increasing attention, and various related works have been reported in the literature. A new method for designing observers for linear time-delay systems with unknown inputs has been proposed in Fu, Duan, and Song (2004). Their proposed observer, however, is dependent on the derivatives of the output measurement. A design method which deals with functional estimation for time-delay systems with unknown inputs has been presented by Darouach (2007), where the proposed observer considers only the estimation of the state vector. Recent attention has been on the problem of simultaneous estimation of state and unknown inputs of time-delay systems where the unknown inputs are present in both the state and the measurement equations (Gao and Ding 2007; Trinh and Ha 2007).

*Corresponding author. Email: pste@deakin.edu.au

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Consider the following linear system with a time-varying delay to illustrate the main results for three different types of observers. Estimated low-order observers that can ensure asymptotic convergence of any given linear function of the state vector and/or unknown input vector of time-delay systems. This article considers the simultaneous estimation of a linear combination of the states and/or unknown inputs of linear systems driven by a time-varying delay and unknown inputs. In this article, the unknown inputs are present in both the state and the measurement equations. The contribution of this article is to develop an efficient design approach to generate low-order observers that can ensure asymptotic convergence of any given linear function of the state vector and/or unknown input vector. The order of the observer is also low and is the same as the number of linear functions to be estimated.

This article is organized as follows. Section 2 presents the problem statement. Section 3 presents the main results for three different types of observers. In Section 4, two numerical examples and an application to a wind tunnel model are given to illustrate the results. Finally, a conclusion is given in Section 5.

2. Problem Formulation

Consider the following linear system with a time-varying delay and unknown inputs:

\[
\dot{x}(t) = Ax(t) + A_dx(t - \tau(t)) + Bu(t) + Dd(t), \quad t > 0, \quad (1a)
\]

\[x(t) = \phi(t), \quad t \in [-\tau_u, 0], \quad (1b)\]

\[y(t) = Cx(t) + Wd(t), \quad (1c)\]

where \(x(t) \in \mathbb{R}^n\) is the state vector, \(u(t) \in \mathbb{R}^k\) is the control input vector, \(y(t) \in \mathbb{R}^r\) is the measurement output vector, \(d(t) \in \mathbb{R}^q\) is the unknown input vector and \(\phi(t)\) is a continuous initial function of the system. Matrices \(A \in \mathbb{R}^{n \times n}, A_d \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times k}, C \in \mathbb{R}^{r \times n}, D \in \mathbb{R}^{r \times q}\) and \(W \in \mathbb{R}^{r \times d}\) are known real constants. Here, we assume that \(\text{rank}(C) = r, r \geq q\) and \(\tau(t)\) is a known time-varying delay function satisfying

\[0 \leq \tau(t) \leq \tau_u, \quad \dot{\tau}(t) \leq \gamma < 1 \quad \forall t \geq 0, \quad (2)\]

where \(\tau_u\) is a positive scalar denoting the upper bound of the delay.

The system (1) can be expressed as the following augmented plant:

\[E\dot{\omega}(t) = \overline{A}\omega(t) + \overline{A}_d\dot{\omega}(t - \tau(t)) + B\overline{u}(t), \quad t > 0 \quad (3a)\]

\[\omega(t) = \overline{\phi}(t), \quad t \in [-\tau_u, 0] \quad (3b)\]

\[y(t) = \overline{C}\omega(t), \quad (3c)\]

where \(\omega(t) = \begin{bmatrix} x(t) \\ d(t) \end{bmatrix} \in \mathbb{R}^{(n+q)}, \overline{\phi}(t) = \begin{bmatrix} \phi(t) \\ 0_{1\times q} \end{bmatrix}, E = [I_n 0_{n \times q}], \overline{A} = [A A_d 0_{n \times q}], \overline{A}_d = [A_d 0_{n \times q}]\) and \(\overline{C} = [C W].\)

Remark 1: The augmented system (3) is in the form of a time-delay descriptor system. The design idea of transforming system (1) into the descriptor form (3) was inspired from the descriptor system approach proposed in Fridman and Shaked (2002). Therefore, designing an observer that simultaneously estimates both the state, \(x(t)\), and the unknown input, \(d(t)\), of system (1) is the same as designing a state observer for the state, \(\omega(t)\), of system (3). The work reported in Trinh and Ha (2007) dealt with the estimation of the entire state vector, \(\omega(t)\), of the descriptor time-delay system (3). Whereas in this article we consider the estimation of only a partial set of the state vector of (3). To our knowledge, so far, no reported work is available in the literature that addresses the design of linear functional observers for time-delay descriptor systems (3).

To estimate only a partial set of the state vector, \(\omega(t)\), let us define \(z(t) \in \mathbb{R}^p\) as follows:

\[z(t) = L\omega(t), \quad (4)\]

where \(L \in \mathbb{R}^{p \times (n+q)}\) is a given constant matrix. Without loss of generality, we assume that \(\text{rank}(L) = p\) and \(\text{rank}[L] = (r + p)\). Let us now propose the following observer structure for \(z(t)\)

\[\dot{\hat{z}}(t) = \hat{\xi}(t) + My(t), \quad (5a)\]

\[\dot{\hat{\xi}}(t) = Nx(t) + N_d\hat{\xi}(t - \tau(t)) + Gy(t) + G_d\hat{y}(t - \tau(t)) + Hu(t), \quad t > 0, \quad (5b)\]

\[\hat{\xi}(t) = \rho(t), \quad t \in [-\tau_u, 0], \quad (5c)\]

where \(\hat{z}(t)\) denotes the estimate of \(z(t)\), \(\hat{\xi}(t) \in \mathbb{R}^p\) is the observer state vector and \(\rho(t)\) is the continuous vector-valued initial function. Matrices \(N \in \mathbb{R}^{p \times p}, N_d \in \mathbb{R}^{p \times q}, G \in \mathbb{R}^{p \times k}, G_d \in \mathbb{R}^{p \times q}, H \in \mathbb{R}^{p \times k}\) and \(M \in \mathbb{R}^{p \times p}\) are to be determined such that \(\hat{z}(t) \rightarrow z(t)\) as \(t \rightarrow \infty\).

Remark 2: With the linear functional state vector as defined in (4), the proposed observer (5) can have great flexibility in estimating any linear combination of the state and/or the unknown input of the system (1).
For instance, if there is only the estimation of unknown input vector to be considered, then matrix $L$ in (4) can be described as $L = [L_1 \ 0_{p \times d}]$. On the other hand, if there is only a linear combination of the states to be estimated, i.e. $L_1 x(t)$, then matrix $L$ can be written as $L = [L_1 \ 0_{p \times d}]$. Also note that the order of the observer is low as it is the same as the number of functions to be estimated.

Remark 3: The knowledge of time-varying delay $\tau(t)$ in the system model and estimation equation is assumed to be known. This assumption is generally required in the literature on observer design for time-delay systems (Darouach 2001, 2007; Richard 2003; Gao and Ding 2007). Nevertheless, there are many applications utilising the knowledge of time-varying delay to modelling and control of concrete systems in various disciplines including biology, chemistry, economics, mechanics, physics, physiology, population dynamics, as well as in engineering sciences (Niculescu 2001). In Section 3.3, we will be able to remove this assumption completely by proposing delay-free observers (64) where the real-time knowledge of time-varying delay is neither required nor its rate of change is assumed to be bounded.

3. Main results

Let $X \in \mathbb{R}^{p \times n}$ and define error vectors $\varepsilon(t) \in \mathbb{R}^p$ and $\tilde{z}(t) \in \mathbb{R}^q$ as

$$
\varepsilon(t) = \xi(t) - XE \omega(t),
$$

$$
\tilde{z}(t) = \hat{z}(t) - z(t).
$$

Theorem 3.1: For the proposed observer (5), the estimate $\hat{z}(t)$ will converge asymptotically to $z(t)$ if the following conditions hold.

Condition (1): The error $\varepsilon(t)$ determined by the observer error system

$$
\dot{\varepsilon}(t) = N_{d} \varepsilon(t) + N_{a} \varepsilon(t - \tau(t)), \quad t > 0,
$$

$$
\varepsilon(t) = \rho(t) - XE \tilde{\omega}(t), \quad t \in [-\tau_u, 0],
$$

converges asymptotically to zero for all $\varepsilon(\theta)$, $\forall \theta \in [-\tau_u, 0]$.

Condition (2): There exists a matrix $X$ such that the following matrix equations hold:

$$
NXE + G \overline{C} - X \overline{A} = 0,
$$

$$
N_{d}XE + G_{d} \overline{C} - X \overline{A}_{d} = 0,
$$

$$
XE + M \overline{C} - L = 0,
$$

$$
H = XB.
$$

Proof: From (6a), the following error dynamics equation is obtained:

$$
\dot{\varepsilon}(t) = N \varepsilon(t) + N_{a} \varepsilon(t - \tau(t)) + (NXE + G \overline{C} - X \overline{A}) \omega(t)
$$

$$
+ (N_{d}XE + G_{d} \overline{C} - X \overline{A}_{d}) \omega(t - \tau(t))
$$

$$
+ (H - XB) \eta(t), \quad t > 0,
$$

$$
\varepsilon(t) = \rho(t) - XE \tilde{\omega}(t), \quad t \in [-\tau_u, 0].
$$

From (6b), the error vector $\varepsilon(t)$ can be expressed as

$$
\varepsilon(t) = \varepsilon(t) + (XE + M \overline{C} - L) \omega(t).
$$

From (9) and (10), it is clear that if all the conditions stated in Theorem 3.1 are satisfied, the error $\varepsilon(t) \to 0$, i.e. $\hat{z}(t) \to z(t)$, as $t \to \infty$. This completes the proof of Theorem 3.1.

3.1. General case

In the sequel, the design of reduced-order observers (5) to estimate a partial set of the state vector of the system (3) is reduced to finding matrices $N$, $N_{d}$, $X$, $G$, $G_{d}$, $M$ and $H$ such that all the two conditions in Theorem 3.1 are satisfied.

Let us first define $A_{u} \in \mathbb{R}^{(n+q) \times (n+q)}$ and $A_{ud} \in \mathbb{R}^{(n+q) \times (n+q)}$ as

$$
A_{u} = \begin{bmatrix} \overline{A} \\ 0_{q \times (n+q)} \end{bmatrix} \quad \text{and} \quad A_{ud} = \begin{bmatrix} \overline{A}_{d} \\ 0_{q \times (n+q)} \end{bmatrix}.
$$

Then $\overline{A}$ and $\overline{A}_{d}$ can be expressed as

$$
\overline{A} = EA_{u} \quad \text{and} \quad \overline{A}_{d} = EA_{ud}.
$$

By substituting (8c) and (12) into (8a) and (8b), we have

$$
NL = LA_{u} - [M \quad T] \Theta
$$

and

$$
N_{d}L = LA_{ud} - [M \quad T_{d}] \Theta_{d},
$$

where $T = (G - NM)$, $\Theta = \begin{bmatrix} CA_{u} \\ C \end{bmatrix}$, $T_{d} = (G_{d} - N_{d}M)$ and $\Theta_{d} = \begin{bmatrix} CA_{ud} \\ C \end{bmatrix}$.

Define the following full-row rank matrix:

$$
\begin{bmatrix} H_{1} & E_{1} \end{bmatrix} = \begin{bmatrix} L^{+} & (I - L^{+}L) \end{bmatrix},
$$

where $L^{+}$ is the Moore-Penrose inverse of $L$. Post-multiplying (13) and (14) by (15) gives

$$
N = LA_{u}H_{1} - [M \quad T] \Theta H_{1},
$$

$$
LA_{ud}E_{1} = \begin{bmatrix} M \quad T \end{bmatrix} \Theta E_{1}
$$
and

\[ N_d = LA_{ud}H_1 - \begin{bmatrix} M & T_d \end{bmatrix} \Theta_d H_1, \quad (17a) \]

\[ LA_{ud}E_1 = \begin{bmatrix} M & T_d \end{bmatrix} \Theta_d E_1. \quad (17b) \]

In (16) and (17), \( L, A_r, A_{nh}, \Theta, \Theta_d, H_1 \) and \( E_1 \) are known matrices. The unknown matrices are \( M, T \) and \( T_d \). Note that the knowledge of these unknown matrices is necessary for the determination of \( N_d \) and \( N_{\mu} \) in (16a) and (17a), respectively. Let us now augment (16b), (17b) and (8c) as follows:

\[
\begin{bmatrix}
M & T & T_d & \mathbf{X}
\end{bmatrix} \mathbf{\Omega} = \mathbf{\psi},
\]

where \( \mathbf{\Omega} \in \mathbb{R}^{(3r+n) \times 3(n+q)} \) and \( \mathbf{\psi} \in \mathbb{R}^{p \times 3(n+q)} \) are known matrices as defined by

\[
\mathbf{\Omega} = \begin{bmatrix}
\mathcal{C}A_u E_1 & \mathcal{C}A_{ud} E_1 & \mathcal{C} \\
\mathcal{C} E_1 & 0 & 0 \\
0 & \mathcal{C} E_1 & 0 \\
0 & 0 & \mathcal{E}_1 \\
L A_u & L A_{ud} & L \\
L & 0 & 0 \\
0 & L & 0
\end{bmatrix}, \quad (19a)
\]

\[
\mathbf{\psi} = \begin{bmatrix}
L A_u E_1 & L A_{ud} E_1 & L
\end{bmatrix}. \quad (19b)
\]

**Lemma 3.2:** There exist matrices \( M, T, T_d \) and \( \mathbf{X} \) such that (18) is satisfied if and only if

\[
\begin{bmatrix}
\mathcal{C}A_u & \mathcal{C}A_{ud} & \mathcal{C} \\
\mathcal{C} & 0 & 0 \\
0 & \mathcal{C} & 0 \\
0 & 0 & E \\
L A_u & L A_{ud} & L \\
L & 0 & 0 \\
0 & L & 0
\end{bmatrix} = \text{rank}
\begin{bmatrix}
\mathcal{C}A_u & \mathcal{C}A_{ud} & \mathcal{C} \\
\mathcal{C} & 0 & 0 \\
0 & \mathcal{C} & 0 \\
0 & 0 & E \\
L & 0 & 0 \\
0 & L & 0
\end{bmatrix}.
\]

**Proof:** Based on the general solution of linear matrix equations (Rao and Mitra 1971), the solution to Equation (18) exists if and only if

\[
\text{rank}\begin{bmatrix}
\mathbf{\Omega} \\
\mathbf{\psi}
\end{bmatrix} = \text{rank}(\mathbf{\Omega}). \quad (21)
\]

To prove Lemma 3.2, we will prove that (20) is equivalent to (21). For this let us define the following full-row rank matrix:

\[
\mathbf{X}_1 = \begin{bmatrix}
H_1 & E_1 & 0 & 0 & 0 \\
0 & 0 & H_1 & E_1 & 0 \\
0 & 0 & 0 & 0 & I_{(n+q)}
\end{bmatrix}. \quad (22)
\]

Using a well-known fact that \( \text{rank}(XY) = \text{rank}(X) \) where \( Y \) is a full-row rank matrix, the left-hand side of (20) can be expressed as

\[
\begin{bmatrix}
\mathcal{C}A_u & \mathcal{C}A_{ud} & \mathcal{C} \\
\mathcal{C} & 0 & 0 \\
0 & \mathcal{C} & 0 \\
0 & 0 & E \\
L A_u & L A_{ud} & L \\
L & 0 & 0 \\
0 & L & 0
\end{bmatrix} = \text{rank}
\begin{bmatrix}
\mathcal{C}A_u & \mathcal{C}A_{ud} & \mathcal{C} \\
\mathcal{C} & 0 & 0 \\
0 & \mathcal{C} & 0 \\
0 & 0 & E \\
L & 0 & 0 \\
0 & L & 0
\end{bmatrix} = 2p + \text{rank}(\mathbf{\Omega}).
\]

Similarly, the right-hand side of (20) can be expressed as

\[
\begin{bmatrix}
\mathcal{C}A_u & \mathcal{C}A_{ud} & \mathcal{C} \\
\mathcal{C} & 0 & 0 \\
0 & \mathcal{C} & 0 \\
0 & 0 & E \\
L & 0 & 0 \\
0 & L & 0
\end{bmatrix} = \text{rank}
\begin{bmatrix}
\mathcal{C}A_u & \mathcal{C}A_{ud} & \mathcal{C} \\
\mathcal{C} & 0 & 0 \\
0 & \mathcal{C} & 0 \\
0 & 0 & E \\
L & 0 & 0 \\
0 & L & 0
\end{bmatrix} = 2p + \text{rank}(\mathbf{\Omega}).
\]

From the above, it is clear that (20) is equivalent to (21). This completes the proof of Lemma 3.2. \( \square \)

**Remark 4:** The condition of Lemma 3.2 can be tested since it involves only checking for the rank of known matrices. Upon the satisfaction of (20), the unknown matrices \( M, T, T_d \) and \( \mathbf{X} \) can be derived. Then from (16a) and (17a), \( N \) and \( N_{\mu} \) are obtained and the asymptotic convergence of the error, \( \varepsilon(t) \), as defined in (7) can be established.

Now, upon the satisfaction of (20), a general solution to the matrix equation (18) always exists.
By using the generalized matrix inverse approach (Rao and Mitra 1971), we have

\[
\begin{bmatrix}
M & T & T_d & X
\end{bmatrix} = \psi \Omega^* + Z(I - \Omega \Omega^*),
\] (25)

where \(\Omega^*\) is the Moore–Penrose inverse of \(\Omega\), \(Z \in \mathbb{R}^{(2r+x) \times n}\) is an arbitrary matrix (which will be further utilised to establish asymptotic convergence of the observer error).

By substituting (25) into (16a) and (17a), we obtain

\[
N = \Delta - Z \Gamma,
\] (26)

\[
N_d = \Delta_d - Z \Gamma_d,
\] (27)

where \(\Delta, \Gamma, \Delta_d \text{ and } \Gamma_d\) are known matrices, defined as follows:

\[
\Delta = (LA_u H_1 - \psi \Omega^* \tilde{\Theta}), \quad \Gamma = (I - \Omega \Omega^*) \tilde{\Theta},
\]

\[
\tilde{\Theta} = \begin{bmatrix}
\mathcal{C} A_u H_1 \\
\mathcal{C} H_1 \\
0 \\
0
\end{bmatrix}.
\]

\[
\Delta_d = (LA_d H_1 - \psi \Omega^* \tilde{\Theta}_d), \quad \Gamma_d = (I - \Omega \Omega^*) \tilde{\Theta}_d
\]

\[
\tilde{\Theta}_d = \begin{bmatrix}
\mathcal{C} A_d H_1 \\
0 \\
\mathcal{C} H_1 \\
0
\end{bmatrix}.
\]

Remark 5: The observer design problem now rests with the determination of a matrix parameter \(Z\) such that Condition (1) of Theorem 3.1 is satisfied. That is, \(\dot{\varepsilon}(t) = N \varepsilon(t) + N_d \varepsilon(t - \tau(t))\) is asymptotically stable. From (26) and (27), the observer error dynamics can be expressed as \(\dot{\varepsilon}(t) = (\Delta - Z \Gamma) \varepsilon(t) + (\Delta_d - Z \Gamma_d) \varepsilon(t - \tau(t))\). In the following, we first establish stability of the matrix \((N + N_d)\) since this is necessary for ensuring asymptotic convergence of \(\varepsilon(t)\) for the case \(\tau(t) = 0\) and also in the subsequent derivation of a delay-dependent stability condition of (7).

From (26) and (27), \(\tilde{N}\) can be expressed as

\[
\tilde{N} = N + N_d = (\Delta + \Delta_d) - Z(\Gamma + \Gamma_d). \tag{28}
\]

Thus, \(\tilde{N}\) is stable if and only if the pair \((\Gamma + \Gamma_d, \Delta + \Delta_d)\) is detectable, i.e.

\[
\text{rank} \begin{bmatrix}
sL - (\Delta + \Delta_d) & -L A_u & -L A_d & -L
\end{bmatrix} = p, \quad \forall s \in \mathbb{C}, \text{Re}(s) \geq 0. \tag{29}
\]

Lemma 3.3: \(\tilde{N} = (N + N_d)\) is stable if and only if

\[
\begin{bmatrix}
\text{rank} \\
L - (A_u + A_d) & -LA_u & -LA_d & -L
\end{bmatrix} = 
\begin{bmatrix}
\mathcal{C} A_u & \mathcal{C} A_d & \mathcal{C} \\
\mathcal{C} & 0 & 0 & 0 \\
0 & \mathcal{C} & 0 & 0 \\
0 & 0 & E & 0 \\
0 & L & 0 & 0 \\
0 & L & 0 & 0
\end{bmatrix}, \quad \forall s \in \mathbb{C}, \text{Re}(s) \geq 0. \tag{30}
\]

Proof: Here, we show that the condition in Lemma 3.3 implies (29) and thus ensure that matrix \(\tilde{N}\) is stable. Let us define full-row rank matrices \(X_2\) and \(X_3\) and a full-column rank matrix \(X_d\) as

\[
X_2 = \begin{bmatrix}
I & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & -I & I & 0 \\
0 & 0 & 0 & I
\end{bmatrix}, \quad X_3 = \begin{bmatrix}
I & 0 \\
-\Omega^* \tilde{\Theta} & I
\end{bmatrix}
\]

and

\[
X_4 = \begin{bmatrix}
I & \psi \Omega^* \\
0 & (I - \Omega \Omega^*) \\
0 & \Omega \Omega^+
\end{bmatrix}, \tag{31}
\]

where

\[
\tilde{\Theta} = \begin{bmatrix}
\mathcal{C} A_u H_1 \\
\mathcal{C} H_1 \\
0
\end{bmatrix}.
\]
Using the full-row rank matrix $X_1$ as defined in (22), the left-hand side of (30) can be evaluated as

$$
\begin{bmatrix}
  sL - L(A_u + A_{ud}) & -LA_{ud} & -L \\
  \bar{C}(A_u + A_{ud}) & \bar{C}A_{ud} & \bar{C} \\
  \bar{C} & 0 & 0 \\
  0 & \bar{C} & 0 \\
  0 & 0 & E \\
  0 & L & 0
\end{bmatrix} = \text{rank}
\begin{bmatrix}
  sL - L(A_u + A_{ud}) & -LA_{ud} & -L \\
  \bar{C}(A_u + A_{ud}) & \bar{C}A_{ud} & \bar{C} \\
  \bar{C} & 0 & 0 \\
  0 & \bar{C} & 0 \\
  0 & 0 & E \\
  0 & L & 0
\end{bmatrix}
$$

$$X_1$$

$$\begin{bmatrix}
  sI_p - L(A_u + A_{ud})H_1 & -L(A_u + A_{ud})E_1 & -L \\
  \bar{C}(A_u + A_{ud})H_1 & \bar{C}(A_u + A_{ud})E_1 & \bar{C}A_{ud}E_1 & \bar{C} \\
  \bar{C}H_1 & \bar{C}E_1 & 0 & 0 \\
  0 & \bar{C}H_1 & \bar{C}E_1 & 0 \\
  0 & 0 & E
\end{bmatrix} = 0$$

$$X_2$$

$$\begin{bmatrix}
  sI_p - (\Delta + \Delta_d) & -\psi \\
  (\Gamma + \Gamma_d) & \Omega
\end{bmatrix} = \text{rank}(\Omega), \quad \forall s \in \mathbb{C}, \Re(s) \geq 0.
$$

Since the right-hand side of (30) is given in (24), it is then clear that the condition presented in (30) is equivalent to (29) and therefore provides the necessary and sufficient condition for matrix $\tilde{N}$ to be Hurwitz. This completes the proof of Lemma 3.3.

Now, we are in a position to establish asymptotic convergence of $\varepsilon(t)$ as defined in (7). Based on an Lyapunov functional approach, we will present a delay-dependent stability condition in terms of Linear Matrix Inequality (LMI).

**Theorem 3.4:** Upon the satisfaction of Lemmas 3.2 and 3.3, for given scalars $\tau_u > 0$ and $\gamma < 1$, there exists a matrix $Z$ such that the error $\varepsilon(t)$ of system (7) converges asymptotically to zero provided that there exist matrices $P = P^T > 0$ and $Y$; and positive scalars $\varsigma_1$ and $\varsigma_2$ such that the following LMI holds:

$$
\begin{bmatrix}
  M_{11} & \tau_u(P\Delta_d - Y\Gamma_d) & \tau_u(\Delta^T P - \Gamma^T Y^T) & \tau_u(\Delta^T P - \Gamma^T Y^T) \\
  \tau_u(\Delta^T P - \Gamma^T Y^T) & -\tau_u(\varsigma_1 + \varsigma_2)^{-1} P & 0 & 0 \\
  \tau_u(P\Delta - Y\Gamma) & 0 & -\tau_u\varsigma_1 P & 0 \\
  \tau_u(P\Delta_d - Y\Gamma_d) & 0 & 0 & -\tau_u\varsigma_2(1 - \gamma)^2 P
\end{bmatrix} < 0.
$$

Here, $M_{11} = (\Delta + \Delta_d)^T P + P(\Delta + \Delta_d) - Y(\Gamma + \Gamma_d) - (\Gamma + \Gamma_d)^T Y^T$ and the parameter matrix $Z$ is given by $Z = P^{-1} Y$.

**Proof:** The proof of Theorem 3.4 can be derived by following along the lines of Trinh and Ha (2007) and Kim (2001). Consider a Lyapunov functional candidate to be

$$
V(\varepsilon, t) = \varepsilon^T(t)P\varepsilon(t) + \frac{1}{\varsigma_1} \int_0^t \int_{t-\theta}^t \varepsilon^T(s)N^T P N \varepsilon(s) ds d\theta + \frac{1}{\varsigma_2(1 - \gamma)^2} \int_{t_{\nu}(t)}^{t_{\nu}(t)+\tau(t)} \varepsilon^T(s)N^T P N \varepsilon(s) ds d\theta,
$$

where $P = P^T > 0$, $\varsigma_1$ and $\varsigma_2$ are positive scalars.
By making use of the following relation:
\[
\varepsilon(t - \tau(t)) = \varepsilon(t) - \int_{t-\tau(t)}^{t} \dot{\varepsilon}(s) ds
\]
we can then rewrite (7a) as
\[
\dot{\varepsilon}(t) = (N + N_d)\varepsilon(t) - N_d \int_{t-\tau(t)}^{t} N_\varepsilon(s) ds
\]
\[
- N_d \int_{t-\tau(t)}^{t} N_d \dot{\varepsilon}(s - \tau(s)) ds,
\]
(35)

incorporating the relations of (40), (41) and (42), the time derivative of \( V(\varepsilon, t) \) in (34) can be obtained as
\[
\dot{V}(\varepsilon, t) \leq \varepsilon^T(t) \Pi \varepsilon(t),
\]
(43)
where
\[
\Pi = (N + N_d)^T P + P(N + N_d) + (\varepsilon_1 + \varepsilon_2) \tau_d P N_d P^{-1} N_d^T P
\]
\[
+ \frac{1}{\varepsilon_1} \tau_0 N_d^T P N_d + \frac{1}{\varepsilon_2} (1 - \gamma)^T \tau_0 N_d^T P N_d.
\]

Finally, by substituting (26) and (27) into (43) and letting \( Y = PZ \), (43) can be converted to the LMI (33) by applying Schur decomposition result (Boyd, El Ghaiou, Feron, and Balakrishnan 1994). If \( \dot{V}(\varepsilon, t) < 0 \), when \( \varepsilon(t) \neq 0 \), then \( \varepsilon(t) \to 0 \) as \( t \to \infty \), and thus Condition (1) of Theorem 3.1 is satisfied. This completes the proof of Theorem 3.4. □

Remark 6: Upon the satisfaction of the LMI (33), matrix gain \( Z \) can be derived from \( Z = P^{-1} Y \). Hence from (25)–(27), \( M, T, T_d, X, N \) and \( N_d \) can be obtained. Since \( T = (G - NM) \) and \( T_d = (G_d - N_dM) \), we can thus obtain matrices \( G \) and \( G_d \). Therefore, all the parameters of observer (5) are obtained. Also, for the case where the time delay is a constant, the results derived above are still applicable by making the value of \( \gamma \) in (33) to be 0.

Based on the above development, we can see that, subject to the satisfaction of Lemmas 3.2, 3.3 and Theorem 3.4, the design of a \( p \)-th-order observer (5) can be systematically carried out as presented in the following design procedure.

Algorithm 1: Observers with delays

1. Compute \( \Delta, \Delta_\varepsilon, \Gamma \) and \( \Gamma_\varepsilon \).
2. Solve the LMI (33) and obtain matrix \( Z = P^{-1} Y \). Use (26) and (27) to obtain matrices \( N \) and \( N_d \) respectively.
3. Use (25) to obtain \( M, T, T_d \) and \( X \).
4. Finally, obtain \( H = XB \), \( G = (T + NM) \) and \( G_d = (T_d + N_dM) \).

Remark 7: For the case where \( L \) is of dimension \( (n + q - r) \times (n + q) \) and \( \tau_c \) is non-singular, the knowledge of \( z(t) \) and \( y(t) \) can be used to obtain an \( (n + q - r) \)-order observer to estimate the state vector and unknown input vector of system (1), i.e. \( \hat{x}(t) = \left[ \hat{z}(t) \right]_{L} = \left[ \tau_c \right]^{-1} \left[ \hat{z}(t) \right]_{L} \). We can show that the two conditions of Lemmas 3.2 and 3.3 are reduced to the following two simplified conditions.

Corollary 3.5: When rank\( [\tau_c]_L = (n + q) \), the conditions defined in Lemmas 3.2 and 3.3 are reduced to the following simplified conditions:

\[
\text{rank}(W) = q, \text{ i.e. } W \text{ is a full column rank matrix,}
\]
(44)
Proof: When \( \text{rank} \begin{bmatrix} \mathcal{C} & \mathcal{C}A_u & \mathcal{C}A_{ud} & \mathcal{C} \\ \mathcal{C} & 0 & 0 & \mathcal{C} \\ 0 & \mathcal{C} & 0 & 0 \\ 0 & 0 & E & \mathcal{C} \\ \mathcal{C}A_u & \mathcal{C}A_{ud} & \mathcal{C} \\ LA_u & LA_{ud} & L \\ L & 0 & 0 & \mathcal{C} \\ 0 & L & 0 & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} \mathcal{C} & \mathcal{C}A_u & \mathcal{C}A_{ud} & \mathcal{C} \\ \mathcal{C} & 0 & 0 & \mathcal{C} \\ 0 & \mathcal{C} & 0 & 0 \\ 0 & 0 & E & \mathcal{C} \\ \mathcal{C}A_u & \mathcal{C}A_{ud} & \mathcal{C} \\ LA_u & LA_{ud} & L \\ L & 0 & 0 & \mathcal{C} \\ 0 & L & 0 & 0 \end{bmatrix} \), the left-hand side of (20) can be evaluated as

\[
2(n + q) + \text{rank} \begin{bmatrix} \mathcal{C} \\ LA_u & LA_{ud} & L \\ L & 0 & 0 \end{bmatrix} = (3n + 3q).
\] (46)

Similarly, the right-hand side of (20) can be evaluated as

\[
2(n + q) + \text{rank} \begin{bmatrix} \mathcal{C} \\ E \end{bmatrix} = (3n + 2q) + \text{rank}(W). \] (47)

As can be seen from (47) and (46), the condition in Lemma 3.2 is reduced to (44) where matrix \( W \) is of full-column rank.

The left-hand side of (30) can be reduced to

\[
\text{rank} \begin{bmatrix} (sL_n - A - A_d) - D^T \\ C \end{bmatrix} = (n + q), \quad \forall s \in \mathbb{C}, \text{Re}(s) \geq 0.
\] (45)

\[
= \text{rank} \begin{bmatrix} K_1 & K_2 \\ 0 & 0 \end{bmatrix} + (n + q) + \text{rank} \begin{bmatrix} (sI_n + A) - D^T \\ C \end{bmatrix} + \text{rank}(W), \quad \forall s \in \mathbb{C}, \text{Re}(s) \geq 0,
\] (48)

where \( \begin{bmatrix} K_1 & K_2 \end{bmatrix} = \left[ \begin{smallmatrix} \mathcal{C} \\ \mathcal{C} \\ \mathcal{C} \end{smallmatrix} \right]^{-1} \). From the simplified equations as shown in (47) and (48), one can see that the condition of Lemma 3.3 is reduced to (45). This completes the proof of Corollary 3.5.

### 3.2. Observers without internal delays

In this section we consider the case where the observer (5) is independent of internal delay, i.e. \( N_d = 0 \). For this case, the observer (5) is of a simpler structure, where

\[
\dot{z}(t) = \dot{x}(t) + My(t),
\] (49a)

\[
\dot{\xi}(t) = N\dot{x}(t) + G\dot{y}(t) + G_d\dot{y}(t - \tau(t)) + Hu(t), \quad t > 0,
\] (49b)

\[
\dot{\xi}(t) = \rho(t), \quad t \in [-\tau_w, 0].
\] (49c)

Accordingly, the conditions for asymptotic convergence of Theorem 3.1 are now reduced to the following corollary.

**Corollary 3.6:** For the proposed observer (49), the estimate \( \hat{z}(t) \) will converge asymptotically to \( z(t) \) if there exists a matrix \( X \) such that the following matrix equations hold:

\[
N \text{ is Hurwitz},
\] (50a)

\[
NXE + G\mathcal{C} - X\mathcal{A} = 0,
\] (50b)

\[
G_d\mathcal{C} - X\mathcal{A}_d = 0,
\] (50c)

\[
XE + MC - L = 0,
\] (50d)

\[
H = XB.
\] (50e)

From above, it is clear that the task of designing a linear functional observer (49) amounts to solving for the unknown matrices \( X, N, G, G_d \) and \( M \) such
that (50a)–(50d) hold. Note that matrix $H$ can be obtained from (50e) once matrix $X$ is found.

Here, by following the same lines as in the previous section, (50b)–(50d) can now be expressed as

$$
\begin{bmatrix}
M & T & G_d & X
\end{bmatrix} \Omega_1 = \psi_1,
$$

(51)

$$
N = LA_u H_1 - \begin{bmatrix} M & T \end{bmatrix} \Theta H_1,
$$

(52)

where $\Omega_1 \in \mathbb{R}^{(3n+\alpha) \times 3(n+\eta)}$ and $\psi_1 \in \mathbb{R}^{(3n+\eta) \times 3}$ are known matrices and are defined by

$$
\Omega_1 = \begin{bmatrix}
CA_u E_1 & 0 & 0 \\
C E_1 & 0 & 0 \\
0 & C & 0 \\
0 & -A_d & E
\end{bmatrix},
$$

(53a)

$$
\psi_1 = \begin{bmatrix}
LA_u E_1 & 0 & L
\end{bmatrix}.
$$

(53b)

From (51), we can derive an existence condition for the unknown matrix $[M \ T \ G_d \ X]$. This is now stated in the following corollary.

**Corollary 3.7:** There exist matrices $M$, $T$, $G_d$ and $X$ such that (51) is satisfied if and only if

$$
\begin{bmatrix}
\bar{C}A_u & 0 & \bar{C} \\
\bar{C} & 0 & 0 \\
0 & \bar{C} & 0 \\
0 & -A_d & E \\
LA_u & 0 & L \\
L & 0 & 0
\end{bmatrix}
= \text{rank}
\begin{bmatrix}
\bar{C}A_u & 0 & \bar{C} \\
\bar{C} & 0 & 0 \\
0 & \bar{C} & 0 \\
0 & -A_d & E \\
LA_u & 0 & L \\
L & 0 & 0
\end{bmatrix}.
$$

(54)

**Proof:** From the general solution of linear matrix equations (Rao and Mitra 1971), there exists a solution to Equation (51) if and only if

$$
\text{rank}
\begin{bmatrix}
\Omega_1 \\
\psi_1
\end{bmatrix}
= \text{rank}(\Omega_1).
$$

(55)

The left-hand side of (54) can be expressed as

$$
\begin{bmatrix}
\bar{C}A_u & 0 & \bar{C} \\
\bar{C} & 0 & 0 \\
0 & \bar{C} & 0 \\
0 & -A_d & E \\
LA_u & 0 & L \\
L & 0 & 0
\end{bmatrix}
= \text{rank}
\begin{bmatrix}
H_1 & E_1 & 0 & 0 \\
0 & 0 & I_{(n+\eta)} & 0 \\
0 & 0 & 0 & I_{(n+\eta)}
\end{bmatrix}.
$$

(56)

Similarly, the right-hand side of (54) can be expressed as

$$
\begin{bmatrix}
\bar{C}A_u & 0 & \bar{C} \\
\bar{C} & 0 & 0 \\
0 & \bar{C} & 0 \\
0 & -A_d & E \\
LA_u & 0 & L \\
L & 0 & 0
\end{bmatrix}
= \text{rank}
\begin{bmatrix}
H_1 & E_1 & 0 & 0 \\
0 & 0 & I_{(n+\eta)} & 0 \\
0 & 0 & 0 & I_{(n+\eta)}
\end{bmatrix}.
$$

(57)

From the above, it is clear that (54) implies (55). This completes the proof of Corollary 3.7. □

Accordingly, the general solution to (51) gives

$$
\begin{bmatrix} M & T & G_d & X \end{bmatrix} = \psi_1 \Omega_1^\top + Z_1 (I - \Omega_1 \Omega_1^\top). 
$$

(58)

Substituting $[M \ T]$ (as derived from (58)) into (52), $N$ can be expressed as

$$
N = \Delta_1 - Z_1 \Gamma_1,
$$

(59)

where

$$
\Delta_1 = LA_u H_1 - \psi_1 \Omega_1^\top \bar{\Theta}, \quad \Gamma_1 = (I - \Omega_1 \Omega_1^\top) \bar{\Theta}
$$

and

$$
\bar{\Theta} = \begin{bmatrix}
\bar{C}A_u H_1 \\
\bar{C}H_1 \\
0 \\
0
\end{bmatrix}.
$$

(60)

From the above, $\Delta_1$ and $\Gamma_1$ are known matrices and there exists a stable matrix $N$ if and only if the pair $\Gamma_1, \Delta_1$ is detectable. Detectability implies that the poles of the observer are stable but they cannot be arbitrarily assigned. When $(\Gamma_1, \Delta_1)$ is observable, then we can always find a $Z_1$ to place all the poles of $N$ at any location.

It can be seen from the above development that the design of an observer (49) amounts to just checking the
The rank condition of Corollary 3.7 and the detectability of the pair \((\Gamma_1, \Delta_1)\). Checking these two conditions can be carried out since they only involve known matrices. Given that the two conditions are satisfied, the design of a \(p\)-th order observer (49) can be systematically carried out as presented in the following design procedure.

**Algorithm 2:** Observers without internal delays

1. Obtain \(\Delta_1\) and \(\Gamma_1\) from (60).
2. Determine \(Z_1\) such that \(\mathbf{N}\) defined in (59) is stable.
3. Use (58) to obtain \(M, T, G_d\) and \(X\).
4. Finally, obtain \(H = XB\) and \(\mathbf{G} = (T + NM)\).

**Remark 8:** By following the same lines of derivation as in Corollary 3.5, we can show that for the case when \(\text{rank} \left[ \begin{array}{c} 1 \\ \mathbf{L} \end{array} \right] = (n + q)\), the condition of Corollary 3.7 is reduced to the following simplified condition.

**Corollary 3.8:** When \(\text{rank} \left[ \begin{array}{c} 1 \\ \mathbf{L} \end{array} \right] = (n + q)\), the condition defined in Corollary 3.7 is equivalent to the following simplified condition:

\[
\text{rank} \left[ \begin{array}{cc} C & W \\ A_d & 0 \end{array} \right] + q = \text{rank} \left[ \begin{array}{ccc} C & W & 0 \\ CA_d & 0 & W \end{array} \right]. \tag{61}
\]

**Proof:** When \(\text{rank} \left[ \begin{array}{c} 1 \\ \mathbf{L} \end{array} \right] = (n + q)\), the left-hand side of (54) can be evaluated as

\[
\text{rank} \left[ \begin{array}{ccc} C & W & 0 \\ A_d & 0 & W \end{array} \right] = \text{rank} \left[ \begin{array}{ccc} 0 & C & 0 \\ 0 & -A_d & E \end{array} \right] = (n + q) + \text{rank} \left[ \begin{array}{c} \mathbf{C} \\ A_d \end{array} \right]. \tag{62}
\]

Similarly, the right-hand side of (54) can be evaluated as

\[
\text{rank} \left[ \begin{array}{ccc} C & W & 0 \\ A_d & 0 & W \end{array} \right] = \text{rank} \left[ \begin{array}{ccc} 0 & C & 0 \\ 0 & -A_d & E \end{array} \right] = (n + q) + \text{rank} \left[ \begin{array}{c} \mathbf{C} \\ A_d \end{array} \right]. \tag{63}
\]

As can be seen from (62) and (63), the condition of Corollary 3.7 is simplified to (61). This completes the proof of Corollary 3.8.

### 3.3. Delay-free observers

In this section, we present the case where the observer (5) is independent of delay, i.e. \(N_d = 0\) and \(G_d = 0\). The design of functional observer for the system (3) is to design a delay-free observer

\[
\dot{\mathbf{z}}(t) = \mathbf{X} \dot{\mathbf{y}}(t) + M_y(t), \tag{64a}
\]

\[
\mathbf{y}(t) = N\mathbf{z}(t) + G_y(t) + H(t), \tag{64b}
\]

where \(N, G, M\) and \(H\) are constant matrices of appropriate dimensions to be determined. There is no time-delay in (64) and therefore there is no restriction imposed on the rate of change of the time-varying delay nor its exact real-time knowledge and upper bound are required. The proposed observer (64) is completely delay-free, low-order and therefore it is easy to implement and cost effective.

In this case, the conditions of Theorem 3.1 are reduced to the following.

**Corollary 3.9:** The state \(\dot{\mathbf{z}}(t)\) is an asymptotic estimate of \(\dot{\mathbf{z}}(t)\) for the proposed observer (64) if the following conditions hold:

\[
N \text{ is Hurwitz,} \tag{65a}
\]

\[
NXG + GNC - XA = 0, \tag{65b}
\]

\[
X A_d = 0, \tag{65c}
\]

\[
XE + MC - L = 0, \tag{65d}
\]

\[
H = XB. \tag{65e}
\]

Therefore to design a delay-free observer (64), all we need is to determine matrices \(N, G, M\) and \(X\) such that equations (65a)–(65d) are satisfied. Note that matrix \(H\) can be obtained from (65e) once matrix \(X\) is found.
Following the same lines of derivation as in Section 3.1, the coupled equations (65b)–(65d) can be described as

\[ \begin{bmatrix} M & T & X \end{bmatrix} \Omega_2 = \psi_2, \] (66)

\[ N = LA_u H_1 - \begin{bmatrix} M & T \end{bmatrix} \Theta H_1, \] (67)

where \( \Omega_2 \in \mathbb{R}^{(2r+n)\times 3(n+q)} \) and \( \psi_2 \in \mathbb{R}^{3\times (n+q)} \) are given by

\[ \Omega_2 = \begin{bmatrix} \overline{C} A_u E_1 & 0 & \overline{C} \\ \overline{C} E_1 & 0 & 0 \\ 0 & \overline{A_d} & E \end{bmatrix}, \]

\[ \psi_2 = \begin{bmatrix} LA_u E_1 & 0 & L \end{bmatrix}. \]

In the sequel, an existence condition for the solvability of the unknown matrix \([M T X]\) is provided in the following corollary.

**Corollary 3.10:** There exist matrices \( M, T \) and \( X \) such that (66) is satisfied if and only if the following condition holds:

\[ \text{rank} \begin{bmatrix} \overline{C} A_u & 0 & \overline{C} \\ 0 & \overline{C} & 0 \\ 0 & \overline{A_d} & E \end{bmatrix} = \text{rank} \begin{bmatrix} \overline{C} A_u & 0 & \overline{C} \\ \overline{C} & 0 & 0 \\ 0 & \overline{A_d} & E \end{bmatrix} \] (68)

The proof of Corollary 3.10 can be derived by following along the same lines as in the proof of Lemma 3.2 and Corollary 3.7, and thus is omitted here. Upon the satisfaction of the condition in (68), a general solution to (66) yields

\[ \begin{bmatrix} M & T & X \end{bmatrix} = \psi_2 \Omega_2^+ + Z_2 (I - \Omega_2 \Omega_2^+), \] (69)

and matrix \( N \) can be expressed as

\[ N = \Delta_2 - Z_2 \Gamma_2, \] (70)

where

\[ \Delta_2 = LA_u H_1 - \psi_2 \Omega_2^+ \tilde{\Theta}_2, \quad \Gamma_2 = (I - \Omega_2 \Omega_2^+) \tilde{\Theta}_2 \]

and

\[ \tilde{\Theta}_2 = \begin{bmatrix} \overline{C} A_u H_1 \\ \overline{CH}_1 \\ 0 \end{bmatrix}. \] (71)

It is clear that from (70) and (71), \( \Delta_2 \) and \( \Gamma_2 \) are known matrices and the observer gain matrix \( Z_2 \) can be obtained from any pole placement method. Matrix \( N \) is Hurwitz if and only if the pair \((\Gamma_2, \Delta_2)\) is detectable.

The task of designing a \( p \)-th order delay-free observer can be performed accordingly as summarised in the following design procedure provided that the rank condition of Corollary 3.10 and the detectability of the pair \((\Gamma_2, \Delta_2)\) are satisfied.

**Algorithm 3:** Delay-free observers

1. Obtain \( \Delta_2 \) and \( \Gamma_2 \) from (57).
2. Determine \( Z_2 \) such that \( N \) as defined in (70) is stable.
3. Use (69) to obtain the unknown matrices \( M, T \) and \( X \).
4. Obtain \( H = XB \) and \( G = (T + NM) \).

4. Numerical examples

In this section, three examples are presented in order to illustrate the proposed three types of functional observers. Examples 1 and 2 are academic examples and Example 3 is an application to a wind tunnel model.

**Example 1:** Design of functional observers with delays

To illustrate the design approach of Algorithm 1, let us present the design of a linear functional observer with delays described in (5), where

\[ A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad A_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad W = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad t_\alpha = 0.2 \text{ s}, \quad \dot{t}(t) = 0. \]

Clearly, the above system can be constructed into an augmented plant as in the form of a time-delay descriptor model (3). First, we check the existence conditions of three different types of observers. It can be found that the existence conditions for observers without internal delays and delay-free observers are not satisfied. On the other hand, the conditions for designing observers with delays are satisfied. Thus in this case, only observers with delays exist. Let us use the result of Algorithm 1 to design a linear functional observer with delays to estimate only the unknown input \( d(t) \). Therefore, matrix \( L \) is selected to be \( L = [0 \ 0 \ 1] \). Here, we have found that the conditions of Corollary 3.5 are satisfied and the design of a first-order \((p = 1)\) observer can now be carried out as follows.

**Step 1:** We obtain \( \Delta = -4, \Delta_d = 1, \Gamma = [0.25 \ 0.25 \ 0 \ -0.25 \ -0.25 \ 0] \) and \( \Gamma_d = [-0.111 \ -0.7441 \ 0.6747 \ 0.2220 \ 0.0139 \ -0.0833 \ -0.2498] \).

The delay of the state variable is assumed to be constant \((i.e. \dot{t} = 0)\) with \( \tau = 0.2 \text{ s} \). Hence, the delay-dependent criterion is then applied for \( \tau_\alpha = \tau \) and \( \gamma = 0 \).

**Step 2:** From the LMI (33), we obtain \( P = 1.3884 \) and \( Z = [-85.092 \ -680.8717 \ 0 \ 388.4753 \ 170.1840 \ 0 \ 388.4753 \ 170.1840] \).
\(-140.6892 -142.5565 -191.457\), then we deduce matrices \(N = -1.7123\) and \(N_2 = 1\).

**Step 3:** Based on the parameters given in the above steps and using (25), we obtain matrices \(M = [1 -2.2877] , \ T = [2 -2.2877], \ T_a = [-1 2.2877] \) and \(X = [2.2877 -1] \).

**Step 4:** \(H = 2.2877, \ G = [0.2877 1.6296] \) and \(G_d = [-0.0659 0.216] \).

Accordingly, the design of a first-order linear functional observer is completed. A simulation study was carried out and the results are exhibited in the following figures: Figure A1 shows the simulated responses of \(d(t)\) and \(\hat{d}(t)\) while Figure A2 shows that the estimation error \((d(t) - \hat{d}(t))\) converges to zero.

**Example 2:** Design of functional observers without internal delays

This example is given to illustrate the design method of Algorithm 2. Let us consider the following time-delay system:

\[
\dot{x}(t) = \begin{bmatrix}
-10 & 1 & 0 & 0 \\
-48.6 & -1.26 & 48.6 & 0 \\
0 & 0 & -2.2 & 1 \\
1.95 & 0 & -19.5 & -6 \\
\end{bmatrix} x(t) \\
+ \begin{bmatrix}
2 & -1 & 0 & 0 \\
-1 & 3 & 0 & 0 \\
1 & -1 & 0 & 0 \\
-2 & 1 & 0 & 0 \\
\end{bmatrix} x(t - \tau(t)) \\
+ \begin{bmatrix}
1 \\
0 \\
2 \\
0.5 \\
\end{bmatrix} u(t) + \begin{bmatrix}
0 \\
1 \\
0 \\
0 \\
\end{bmatrix} d(t) \\
y(t) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{bmatrix} x(t) + \begin{bmatrix}
0.5 \\
1 \\
\end{bmatrix} d(t),
\]

where \(\tau(t)\) can be any unknown time-varying delay satisfying \(0 \leq \tau(t) \leq \tau_u\) and \(\dot{\tau}(t) \leq \gamma < 1\).

The above plant can be augmented into the descriptor form of (3). The objective of this numerical example is to design an asymptotic linear functional observer without internal delays to estimate simultaneously the states \(x_3(t)\), \(x_4(t)\) and the unknown input, \(d(t)\). Hence the functional state vector can be described as

\[
z(t) = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} z(t).
\]

Here, \(\text{rank}(\tau) = (n + q) = 5\) and it is easy to check that the condition of Corollary 3.8 holds.

Accordingly, a third-order functional observer for \(z(t)\) can be carried out based on Algorithm 2 as follows.

**Step 1:** Matrices \(\Delta_1\) and \(\Gamma_1\) are obtained as

\[
\Delta_1 = \begin{bmatrix}
-12.28 & 1 & 3.712 \\
-19.5 & -6 & -0.975 \\
0 & 0 & -8 \\
\end{bmatrix}
\]

and \(\Gamma_1 = 0\).

**Step 2:** Thus, we have a stable matrix \(N = \Delta_1\) with matrix \(Z_1 = 0\).

**Step 3:** Using (58), the following matrices are obtained:

\[
M = \begin{bmatrix}
0.4 & -0.2 \\
2 & 0 \\
\end{bmatrix} , \quad T = \begin{bmatrix}
-5.72 & -0.652 \\
20 & -2 \\
\end{bmatrix}
\]

\[
G_d = \begin{bmatrix}
-2 & 1 \\
-4 & 2 \\
\end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix}
-0.4 & 0.2 & 1 \\
0 & 0 & 0 \\
-2 & 0 & 0 \\
\end{bmatrix}
\]

**Step 4:**

\[
H = \begin{bmatrix}
1.6 \\
0.5 \\
-2 \\
\end{bmatrix}
\]

and

\[
G = \begin{bmatrix}
-3.208 & 1.804 \\
-7.8 & 3.9 \\
4 & -2 \\
\end{bmatrix}
\]

Subsequently, a third-order observer without internal delay in the form of (49) is achieved and the effectiveness of the design approach can be verified by the following simulation results. Figures A3–A5 show the simulated responses of \(z(t)\) and \(\dot{z}(t)\). From these figures, it is clear that \(\dot{x}_3(t) \rightarrow x_3(t)\), \(\dot{x}_4(t) \rightarrow x_4(t)\), and \(\dot{d}(t) \rightarrow d(t)\). It can be seen that all the estimates for the states \(x_3(t)\), \(x_4(t)\) and the unknown input \(d(t)\) settle quickly to the actual signals. Thus it is clear from this study that the approach of designing a functional observer without internal delay, as presented in this article, is able to asymptotically estimate any given vector of the state and unknown input simultaneously.

**Example 3:** Design of delay-free observers

The approaches developed in Section 3 can be applied to estimate simultaneously a linear combination of the state and unknown input of a wind tunnel model. Let us consider the liquid nitrogen wind tunnel at NASA Langley Research Center in Hampton, VA (Armstrong and Tripp 1981). A simplified mathematical model of the Mach number dynamic response to guide vane changes in Manitius (1984) is a system of
three-state equations with a delay in one-state variable. This model has also been considered in Germani, Manes, and Pepe (2000), where an approximation approach is used to design a LQG control, in the presence of Gaussian noise. In steady-state operating conditions (fan speed, liquid nitrogen injection rate and gaseous-nitrogen vent rate), the dynamic response of the Mach number in a guide vane angle actuator is given in the following system involving some noises as described in Germani et al. (2000):

\[
\dot{x}(t) = \begin{bmatrix}
-a & 0 & 0 \\
0 & -\bar{\omega}^2 & -2\xi \bar{\omega} \\
0 & ka & 0
\end{bmatrix} x(t) \\
+ \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} x(t - \tau(t)) \\
+ \begin{bmatrix}
0 \\
0 \\
\bar{\omega}^2
\end{bmatrix} u(t) + Dd(t),
\]

\[
y(t) = \begin{bmatrix}
0 & 1 & 0
\end{bmatrix} x(t) + Wd(t),
\]

where \(x_1\) is the Mach number, \(x_2\) is the guide vane angle (actuator position), \(x_3 = \dot{x}_2\) (actuator rate) and the disturbance is a resistant torque on the input motor. Here, the disturbance can be treated as unknown input \(d(t)\). The parameters are given as follows: \((1/a) = 1.964\), \(k = -0.0117\), \(\xi = 0.8\), \(\bar{\omega} = 6.0\), \(D = [0 0 10]^T\) and \(W = 1.0\). Only the guide vane angle is measured, neither the measurement of the Mach number nor of the guide vane angle derivative is measured. The delay represents the transportation time between the guide vanes of the fan and the test section of the tunnel. The model can be constructed in an augmented plant in the form of (3). For illustrative purpose, let us now design a linear functional observer to estimate only the actuator rate and the resistant torque:

\[
z(t) = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \omega(t).
\]

It is easy to check that the rank condition of Corollary 3.10 and the detectability of the pair \((\Gamma_2, \Delta_2)\) are satisfied. Based on the design Algorithm 3, a second-order delay-free observer exists with the parameters computed as following.

**Step 1:** Matrices \(\Delta_2\) and \(\Gamma_2\) are obtained as

\[
\Delta_2 = \begin{bmatrix}
-9.6 & 46 \\
-1 & 0
\end{bmatrix}
\]

and \(\Gamma_2 = 0_{5 \times 2}\).

**Step 2:** Matrix \(N = \Delta_2\) is obtained with \(Z_2 = 0\).

**Step 3:** Using (69), the following matrices are obtained:

\[
M = \begin{bmatrix}
0 \\
1
\end{bmatrix}, \quad T = \begin{bmatrix}
-36 \\
0
\end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix}
0 & 0 & 1 \\
0 & -1 & 0
\end{bmatrix}.
\]

**Step 4:** Accordingly, we obtain matrices \(H = [36 10]\) and \(G = [10 0]\).

A simulation study was carried out to examine the performance of the proposed observer design approach. The simulation results are given in Figures A6 and A7 with the initial conditions \(x(0) = [0.2 0.1 0.6 0.35]^T\) and \(\xi(0) = [0.2 0]^T\) and simulated with time-varying delay subject to (2). Figures A6 and A7 clearly show that the observer converges to the true state of the system in less than 1 s. From the results obtained by applying the proposed methodology on a wind tunnel model, one can expect that there is a great applicability of the functional observer design as presented in this article to many real-life systems.

5. Conclusion

In this article, a design approach of low-order observers to estimate any given linear function of the state vector and unknown input vector simultaneously for linear time-delay systems with unknown inputs has been presented. Three types of reduced-order observers; namely, observers with delays, observers without internal delays and delay-free observers are proposed in the article. The order of the observer is low and is the same as the number of linear functions to be estimated. Conditions have been given for the existence of reduced-order observers. Design procedures for the determination of parameters for each case of observers have been provided. The proposed design procedures have been illustrated through the numerical examples and simulation results.

Notes on contributors

**P.S. Teh** received her BE (first-class Honours) degree from Deakin University, Australia in 2005. She worked as a Product Engineer with Spansion LLC, Malaysia, for 1 year. She is currently a PhD student in the School of Engineering, Deakin University, Australia. Her research interests include robust control and estimation, fault diagnosis and fault-tolerant control.

**H. Trinh** received his BE (Hons.), MEngSc and PhD degrees from the University of Melbourne, Australia, in 1990, 1992
and 1996, respectively. He is currently an Associate Professor and HDR Coordinator in the School of Engineering, Deakin University, Australia. His current research activities include robust control and estimation, fault diagnosis and fault-tolerant control, robotics and application of control theory to industrial systems.

References


Appendix A: Simulation results

Figure A1. Responses of $d(t)$ (solid line) and $\hat{d}(t)$ (dashed line) in Example 1.

Figure A2. Estimation error $d(t) - \hat{d}(t)$ in Example 1.

Figure A3. Responses of $x_3(t)$ (solid line) and $\hat{x}_3(t)$ (dashed line) in Example 2.

Figure A4. Responses of $x_4(t)$ (solid line) and $\hat{x}_4(t)$ (dashed line) in Example 2.
Figure A5. Responses of $d(t)$ (solid line) and $\hat{d}(t)$ (dashed line) in Example 2.

Figure A6. Responses of $x_3(t)$ (solid line) and $\hat{x}_3(t)$ (dashed line) in Example 3.

Figure A7. Responses of $d(t)$ (solid line) and $\hat{d}(t)$ (dashed line) in Example 3.