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Strong Fuzzy Subsethood Measures and Strong Equalities Via Implication Functions

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In this work we present the definition of strong fuzzy subsethood measure as a unifying concept for the different notions of fuzzy subsethood that can be found in the literature. We analyze the relations of our new concept with the definitions by Kitainik ([20]), Young ([26]) and Sinha and Dougherty ([23]) and we prove that the most relevant properties of the latter are preserved. We show also several construction methods.

Keywords: Fuzzy subsethood measure; strong fuzzy subsethood measure; fuzzy sets; fuzzy equality; implication function; aggregation function

1 INTRODUCTION

Traditionally, fuzzy set inclusion is defined, following the ideas of Zadeh [27], as follows: For two fuzzy sets A and B defined over the referential set X, A ⊆ B if and only if μ_A(x) ≤ μ_B(x) for every x ∈ X.

Many researchers pointed out that this definition is too rigid [5, 20, 23, 24, 26], since, given two fuzzy sets A and B over the same referential, either A is a subset of B (that is, every membership value of an element of the referential...
to the set $A$ is less than or equal the membership of the same element to the set $B$) or it is not a subset at all, even if there exists only one single element in the referential whose membership to $A$ is greater than its membership to $B$. Trying to solve this problem, in 1980 Bandler and Kohout [2] suggested the following definition: Given two fuzzy sets $A$ and $B$, the degree to which $A$ is a subset of $B$ is given by $\inf_{x \in X} j(\mu_A(x), \mu_B(x))$, where $j : [0, 1]^2 \rightarrow [0, 1]$ is a mapping such that $j(0, 0) = j(0, 1) = j(1, 1) = 1$ and $j(1, 0) = 0$. This definition induced (see [13]) to consider mappings $\sigma : F(X) \times F(X) \rightarrow [0, 1]$ to represent how much a fuzzy set $A$ is contained in another fuzzy set $B$. Here $F(X)$ denotes the class of fuzzy sets defined over the universe $X$.

Several different proposals for the axioms to be fulfilled by such mappings have been presented in the literature [8, 10, 13, 20, 23, 26], even considering some extensions [4]. They provide different approaches to the same idea, differing essentially on how the idea of maximal inclusion must be understood, i.e., which condition should be linked to the maximal possible value of the mapping $\sigma$. Note that this concept of inclusion is related to the equality concept, since equality between two fuzzy sets $A$ and $B$ can be defined as simultaneous maximal inclusion of $A$ in $B$ and of $B$ in $A$. Moreover, the concept of inclusion can also be related to the concept of union of fuzzy sets.

Our main goal in this work is to unify the different approaches to the concept of subsethood in a single definition that catches the most important features common to all the previous definitions. We intend to provide a unified framework that covers most of the already existing approaches and would be of interest for future possible theoretical developments as well as for applications. To do so we consider the definition of subsethood in the strong sense proposed by Dubois et al. [12] as a reference.

In this way, we introduce the concept of strong fuzzy subsethood measure, for which a minimal set of axioms is provided. These axioms are common to both Young’s [26] and Sinha and Dougherty’s [23] definitions, so our definition generalizes these two ones. Moreover, strong fuzzy subsethood measures can be built in terms of aggregation functions and implication functions, and this fact allows us to define strong equality indexes [7] by means of strong fuzzy subsethood measures.

The structure of this paper is as follows. We start recalling some basic concepts in Section 2. Section 3 is devoted to recall some of the definitions of subsethood in the literature. In Section 4 we present the concept of strong fuzzy subsethood measure. In Section 5 we introduce several construction methods in terms of aggregation functions and implications. Section 6 studies the relation of our concept with that of strong inclusion introduced by Dubois et al. Next, we relate our concepts with strong equality indexes and we show how the later can be obtained from the former. We finish with some concluding remarks.
2 PRELIMINARIES

A fuzzy set $A$ over a referential set $X$ is given as $\{(x, \mu_A(x)) \mid x \in X\}$, where $\mu_A : X \rightarrow [0, 1]$ is a mapping. We only deal in this work with finite referential sets, and, if not otherwise stated, we will assume that $\text{card}(X) = n$. We denote by $F(X)$ the class of all fuzzy sets defined over the referential set $X$. For $A, B \in F(X)$ we define the union and intersection of $A$ and $B$ as

$$A \cup B = \{(x, \max(\mu_A(x), \mu_B(x))) \mid x \in X\}$$

and

$$A \cap B = \{(x, \min(\mu_A(x), \mu_B(x))) \mid x \in X\},$$

respectively. Moreover, by abuse of notation, for $a, b \in [0, 1]$ we will also denote $a \lor b = \max(a, b)$ and $a \land b = \min(a, b)$.

Given two fuzzy sets $A, B \in F(X)$ $A$ is said to be included in $B$ in the sense of Zadeh if and only if $\mu_A(x) \leq \mu_B(x)$ for every $x \in X$. In this case, we write $A \leq B$.

Given a mapping $s : X \rightarrow X$, if $A \in F(X)$, we denote by $s(A)$ the set $\{(x, \mu_{sA}(x) = \mu_A(s(x))) \mid x \in X\}$. If $k \in [0, 1]$, by abuse of notation we also denote by $k$ the fuzzy set $A$ such that $\mu_A(x) = k$ for every $x \in X$.

A (strong) negation is a decreasing and involutive mapping $c : [0, 1] \rightarrow [0, 1]$ such that $c(0) = 1$ and $c(1) = 0$. For every strong negation $c$ there exists a single point $e \in [0, 1]$ such that $c(e) = e$. This point is called equilibrium point of the negation $c$. Given a strong negation $c$ and $A \in F(X)$, the complementary of $A$, denoted by $A_c$, is the fuzzy set

$$A_c = \{(x, \mu_{A_c}(x) = c(\mu_A(x))) \mid x \in X\}.$$

For us, an ($n$-ary) aggregation function [9] is a mapping $M : [0, 1]^n \rightarrow [0, 1]$ such that

(A1) $M(x_1, \ldots, x_n) = 0$ if and only if $x_1 = \cdots = x_n = 0$;

(A2) $M(x_1, \ldots, x_n) = 1$ if and only if $x_1 = \cdots = x_n = 1$;

(A3) For every $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in [0, 1]^n$ such that $x_i \leq y_i$ for every $i = 1, \ldots, n$ it holds that $M(x_1, \ldots, x_n) \leq M(y_1, \ldots, y_n)$.

Observe that (A3) refers to the monotonicity of $M$. If this monotonicity is strict, then we will say that $M$ satisfies axiom (A3S). Note that this definition is more restrictive than the usual one given for aggregation functions [19].
On the other hand, sometimes we will also demand symmetry, that is, that $M$ fulfills the axiom

\[ M(x_1, \ldots, x_n) = M(x_{\pi(1)}, \ldots, x_{\pi(n)}) \] for any $\pi$, where $\pi$ denotes a permutation, $\pi: \{1, \ldots, n\} \to \{1, \ldots, n\}$.

Suppose $X = \{x_1, \ldots, x_n\}$. Given a fuzzy set $A$ and an aggregation function $M$, we denote $M(A) = M(\mu_\sigma(x_1), \ldots, \mu_\sigma(x_n))$.

Another important concept in this work is that of implication. An implication function (in the sense of Fodor and Roubens, [14,16], see also Baczynski and Jayaram [1]) is a mapping $I: [0, 1]^2 \to [0, 1]$ such that, for every $x, y, z, t \in [0, 1]$:

\begin{align*}
(I1) & \text{ If } x \leq z \text{ then } I(x, y) \geq I(z, y); \\
(I2) & \text{ If } y \leq t \text{ then } I(x, y) \leq I(x, t); \\
(I3) & I(0, x) = 1; \\
(I4) & I(x, 1) = 1; \\
(I5) & I(1, 0) = 0.
\end{align*}

Other properties that may be demanded to implication functions are the following (see [1]):

\begin{align*}
(I6) & I(1, x) = x; \\
(I7) & I(x, I(y, z)) = I(y, I(x, z)); \\
(I8) & I(x, y) = 1 \text{ if and only if } x \leq y; \\
(I9) & I(x, 0) = c(x) \text{ is a strong negation}; \\
(I10) & I(x, y) \geq y; \\
(I11) & I(x, x) = 1; \\
(I12) & I(x, y) = I(c(y), c(x)) \text{ for a given strong negation } c; \\
(I13) & I \text{ is continuous}.
\end{align*}

Relations between these properties have been exhaustively studied, for instance, in [6] and [22].

3 DIFFERENT AXIOMATIZATIONS OF THE CONCEPT OF FUZZY SUBSETHOOD MEASURES

In this section we review some different axiomatizations that have been proposed for fuzzy subsethood measures. We deal with a fixed strong negation $c$ and we start with Kitainik’s axiomatization.
Definition 1 [20]. A fuzzy subsethood measure in the sense of Kitainik is a mapping $\sigma_K : F(X) \times F(X) \rightarrow [0, 1]$ such that, for every $A, B, C \in F(X)$

(K1) $\sigma_K(A, B) = \sigma_K(B_c, A_c)$;
(K2) $\sigma_K(A, B \land C) = \land\{\sigma_K(A, B), \sigma_K(A, C)\}$;
(K3) For every one-to-one mapping $s : X \rightarrow X$, it holds that $\sigma_K(A, B) = \sigma_K(s(A), s(B))$;
(K4) $\sigma_K$ restricted to crisp sets coincides with the usual set inclusion.

Regarding this axiomatization, Fodor and Yager proved the following result.

Theorem 1 [17]. A mapping $\sigma_K : F(X) \times F(X) \rightarrow [0, 1]$ satisfies (K1)-(K4) if and only if there exists an implication function $I$ satisfying (I12) and such that, for every $A, B \in F(X)$

$$\sigma_K(A, B) = \min_{i=1}^{n} I(\mu_A(x_i), \mu_B(x_i)).$$

Remark 1. For a fixed $\sigma_K$ characterized in Theorem 1, and a fixed level $\alpha \in [0, 1]$, we can introduce the $\alpha$-subsethood relation $R_\alpha$ by

$$(A, B) \in R_\alpha \text{ if and only if } \sigma_K(A, B) \geq \alpha.$$  

Then the smallest fuzzy set $C \in F(X)$ such that $(A, C) \in R_\alpha$ and $(B, C) \in R_\alpha$ can be seen as the $\alpha$-union of the fuzzy sets $A$ and $B$. For example, consider $I(a, b) = \min(1, 1 - a + b)$ (Łukasiewicz implication). Then, for any fixed $A, B \in F(X)$,

$$\mu_C(x) = \inf\{z \in [0, 1] \mid I(\mu_A(x), z) \geq \alpha \text{ and } I(\mu_B(x), z) \geq \alpha\} = \max(0, \mu_A \lor \mu_B(x) + \alpha - 1) = T_L(\alpha, \mu_A \lor \mu_B(x)),$$

where $T_L$ is the Łukasiewicz $t$-norm, see [21]. Obviously, for $\alpha = 1$ the $\alpha$-union is just the standard Zadeh’s union of fuzzy sets.

Later on, Sinha and Dougherty proposed the following axiomatization.

Definition 2 [23]. A fuzzy subsethood measure in the sense of Sinha and Dougherty is a mapping $\sigma_{SD} : F(X) \times F(X) \rightarrow [0, 1]$ such that, for every $A, B, C \in F(X)$

(SD1) $\sigma_{SD}(A, B) = 1$ if and only if $A \leq B$;
(SD2) $\sigma_{SD}(A, B) = 0$ if and only if there exists $x \in X$ such that $\mu_A(x) = 1$ and $\mu_B(x) = 0$.
Apart from this axioms, Sinha and Dougherty also proposed the following three optional axioms:

(SD10) \( \sigma_{SD}(A, B) + \sigma_{SD}(A_c, B_c) \geq 1 \);

(SD11) if \( A \) is a refinement of \( B \), that is, if \( \mu_A(x_i) \leq \mu_B(x_i) \) when \( \mu_B(x_i) \leq e \) and \( \mu_A(x_i) \geq \mu_B(x_i) \) when \( \mu_B(x_i) \geq e \) with \( c(e) = e \), then

\[
\sigma_{SD}(A \cup A_c, A \wedge A_c) \leq \sigma_{SD}(B \cup B_c, B \wedge B_c);
\]

(SD12) if \( \frac{1}{n} \sum_{i=1}^{n} \gamma(1 - \mu_A(x_i), \mu_B(x_i)) = 1 \), then \( \sigma_{SD}(A, B) \geq \frac{1}{2} \).

Burillo et al. proved in [3] that Axiom (SD9) is a consequence of Axiom (SD3). Moreover, Kitainik in [20] also settled that for a subsumption measure fulfilling (K1)-(K4) axioms (SD3) and (SD4) also hold. So eventually Sinha and Dougherty’s and Kitainik’s axiomatizations are equivalent, except for axioms (SD1) and (SD2), since Kitainik only impose them to crisp sets. In particular, the expression in Theorem 1 does not fulfill Axiom (SD2).

In an attempt to overcome the weaknesses of both Kitainik’s and Sinha and Dougherty’s proposals, Young ([26]) proposed the following axiomatization of what she called weak fuzzy subsethood measure.

**Definition 3 [26].** A fuzzy subsumption measure in the sense of Young is a mapping \( \sigma_Y : F(X) \times F(X) \rightarrow [0, 1] \) such that, for every \( A, B, C \in F(X) \):

\[
(Y1) \quad \sigma_Y(A, B) = 1 \text{ if and only if } A \subseteq B,
\]

\[
(Y2) \quad \text{if } e \subseteq A, \text{ then } \sigma_Y(A, A_c) = 0 \text{ if and only if } A = 1 \text{ (Here, } e \text{ is the equilibrium point of the negation } c \text{ used to build the complementary } A_c),
\]

\[
(Y3) \quad \text{if } A \subseteq B \subseteq C \text{ then } \sigma_Y(C, A) \leq \sigma_Y(B, A) \text{; and if } A \subseteq B \text{ then } \sigma_Y(C, A) \leq \sigma_Y(C, B).
\]

However, Young realized that there are many examples in the literature for which (Y1) does not hold. For this reason, she introduced the following definition.
Definition 4 [26]. A weak fuzzy subethood measure in the sense of Young is a mapping $\sigma_{WY}: F(X) \times F(X) \rightarrow [0, 1]$ which, for every $A, B, C \in F(X)$, it satisfies (Y2) and (Y3) in Definition 3 and such that there exist $A, B \in F(X)$ with $A \subseteq B$ but $\sigma_{WY}(A, B) < 1$.

Finally, in order to provide appropriate boundary conditions, Fan and Xie [13] propose the following definition of a weak fuzzy *-subsethood measure.

Definition 5 [13]. A weak fuzzy *-subsethood measure is a mapping $\sigma_*: F(X) \times F(X) \rightarrow [0, 1]$ such that, for every $A, B, C \in F(X)$:

(*1) $\sigma_*(0, 0) = \sigma_*(0, 1) = \sigma_*(1, 1) = 1$.
(*2) $\sigma_*(1, 0) = 0$.
(*3) if $A \subseteq B \subseteq C$, then $\sigma_*(C, A) \leq \sigma_*(B, A)$ and $\sigma_*(C, A) \leq \sigma_*(C, B)$.

4 STRONG FUZZY SUBETHOOD MEASURES ON X

A view of inclusion stronger than the ones considered in the definitions of the previous section is to consider that $A$ is included in $B$ if all the elements in $A$ are prototypes of $B$; i.e., the mapping $\sigma_S: F(X) \times F(X) \rightarrow [0, 1]$ should verify

$$A \subseteq_S B \text{ if and only if } A \cup B = 1,$$

that is, $\mu_A(x_i) = 0$ or $\mu_B(x_i) = 1$ for all $x_i \in X$ (cf. [11]).

In [12] it is proposed that when the considered fuzzy subethood measure is the strong one, the axioms that must be satisfied by the fuzzy subethood measure are (SD2), (SD3), (SD4) and Axiom (SD1) should be replaced by the following condition:

$$\sigma(A, B) = 1 \text{ if and only if } A \subseteq_S B. \quad (1)$$

This proposal has suggested us to introduce and study of fuzzy subethood measures with Axiom (SD1) replaced by condition (1).

Definition 6. A mapping $\sigma_S: F(X) \times F(X) \rightarrow [0, 1]$ is a strong fuzzy subethood measure on $X$, if:

(SF1) $\sigma_S(A, B) = 1$ if and only if $\mu_A(x_i) = 0$ or $\mu_B(x_i) = 1$ for all $i \in \{1, \ldots, n\}$, that is, $A \subseteq_S B$.
(SF2) \( \sigma(S, A, B) = 0 \) if and only if \( A = 1 \) and \( B = 0 \);
(SF3) if \( A \leq B \), then \( \sigma(S, A, C) \geq \sigma(S, B, C) \) and \( \sigma(S, C, A) \leq \sigma(S, C, B) \).

Regarding axiom (SF3) in this definition, note that we have the following result, whose proof is obvious.

**Proposition 1.** Let \( A, B \in F(X) \) such that \( A \subseteq B \). Then \( A \leq B \).

**Example 1.** The following are examples of strong fuzzy subsethood measures in the sense of Definition 6. Note that they are built by means of complementation using the standard negation \( c(x) = 1 - x \).

1. \[ \frac{1}{n} \sum_{i=1}^{n} (1 - \mu_A(x_i), \mu_B(x_i)). \] This is known as weak inclusion [11].
2. \[ \frac{1}{n} \sum_{i=1}^{n} (1 - \mu_A(x_i) + \mu_A(x_i) \cdot \mu_B(x_i)). \]
3. \[ \frac{1}{n} \sum_{i=1}^{n} (1 - \mu_A(x_i) + \mu_A(x_i) \cdot \mu_B(x_i)). \]
4. \[ \frac{1}{n} \sum_{i=1}^{n} (\vee(1 - \mu_A(x_i), \mu_B(x_i)))^\lambda \], with \( 1 < \lambda < \infty \).

**Theorem 2.**

(i) Every strong fuzzy subsethood measure is a weak fuzzy subsethood measure in the sense of Young;
(ii) Every strong fuzzy subsethood measure is a weak fuzzy subsethood measure in the sense of Fan and Xie.

**Proof.**

(i) Since \( \sigma(S, A, B) = 1 \) if and only if \( A \subseteq B \), by taking \( A \leq B \) with \( A \) non empty and \( B \) a fuzzy set which is not normal we see that \( \sigma(S, A, B) < 1 \). On the other hand, if (SF2) holds, it follows that if \( A \geq B \), \( \sigma(S, A, B) = 0 \) if and only if \( A = 1 \). Finally, if \( A \leq B \leq C \), then we have that, on one hand, \( \sigma(S, C, A) \leq \sigma(S, B, A) \) and, on the second hand, \( \sigma(S, C, A) \leq \sigma(S, C, B) \), from (SF3).

(ii) If \( A = B = 0 \), we have that \( A \subseteq B \) so \( \sigma(S, A, B) = 1 \). An analogous calculations completes the proof for (1) and (2). Finally, the proof for (3) is very similar to that for (Y3) above.

Note that, for example, the following expression

\[ \sigma(F, A, B) = \frac{\vee_{i=1}^{n} (1 - \mu_A(x_i)), \mu_B(x_i))}{\vee_{i=1}^{n} (\mu_A(x_i), 1 - \mu_A(x_i), \mu_B(x_i), 1 - \mu_B(x_i))}. \]
is a weak fuzzy subsethood measure in the sense of Fan and Xie ([13]), and
is not a strong fuzzy subsethood measure in the sense of the definition above,
because it does not fulfill (SF3).

It is easy to see in the definition above that

1. If \( A = 0 \), then \( \sigma_s(A, B) = 1 \);
2. If \( B = 1 \), then \( \sigma_s(A, B) = 1 \);
3. Every strong fuzzy subsethood measure satisfies Axioms (SD3) and
   (SD4) of Sinha and Dougherty.

**Corollary 1.** Let \( \sigma_s \) be a strong fuzzy subsethood measure. The following
items hold:

(i) \( \sigma_s \) satisfies Axioms (SD9) and (SD11);
(ii) \( \sigma_s \) satisfies the inequalities:

\[
\begin{align*}
\sigma_s(A \lor B, C) & \leq \land \{\sigma_s(A, C), \sigma_s(B, C)\}, \\
\sigma_s(A, B \land C) & \leq \land \{\sigma_s(A, B), \sigma_s(A, C)\}, \\
\sigma_s(A \land B, C) & \geq \lor \{\sigma_s(A, C), \sigma_s(B, C)\}.
\end{align*}
\]

**Proof.** It is straight from the monotonicity condition (SF3).

From this corollary we see that strong fuzzy subsethood measures do not
fulfill Axiom (SD7), but they fulfill the inequality:

\[
\sigma_s(A \lor B, C) \leq \land \{\sigma_s(A, C), \sigma_s(B, C)\}.
\]

They do not fulfill Axiom (SD8), either. However they do fulfill the follow-
ing inequality:

\[
\sigma_s(A, B \land C) \leq \land \{\sigma_s(A, B), \sigma_s(A, C)\}.
\]

**Proposition 2.** Let \( \sigma_{s1}, \ldots, \sigma_{sn} \) be n strong subsethood measures on \( X \) and
let \( M : [0, 1]^n \rightarrow [0, 1] \) be an aggregation function, i.e., such that (A1), (A2)
and (A3) hold. Then

\[
\sigma_s(A, B) = M^n_{i=1} \sigma_{si}(A, B)
\]

is a strong fuzzy subsethood measure on \( X \).

**Proof.**

(SF1) Since \( M \) satisfies (A2) we have \( \sigma_s(A, B) = 1 \) if and only if
\( \sigma_{si}(A, B) = 1 \) if and only if \( A \subseteq S B \).
(SF2) Since $M$ satisfies (A1) we immediately have $\sigma_S(A, B) = 0$ if and only if $\sigma_S(A, B) = 0$ if and only if $A = 1$ and $B = 0$.

(SF3) If $A \leq B$, then bearing in mind that $M$ satisfies (A3) and $\sigma_{S_i}(C, A) \leq \sigma_{S_i}(C, B)$ for all $i \in \{1, \ldots, n\}$, we have

$$\sigma_S(C, A) = M_{i=1}^n \sigma_{S_i}(C, A) \leq M_{i=1}^n \sigma_{S_i}(C, B) = \sigma_S(C, B).$$

On the other hand, $\sigma_{S_i}(A, C) \geq \sigma_{S_i}(B, C)$. Therefore

$$\sigma_S(A, C) = M_{i=1}^n \sigma_{S_i}(A, C) \geq M_{i=1}^n \sigma_{S_i}(B, C) = \sigma_S(B, C).$$

Since $A \subseteq_S B$ implies that $A \leq B$, the result follows.

**Example 2.** Let $\sigma_{S_1}, \ldots, \sigma_{S_n}$ be $n$ strong fuzzy subsethood measures on $X$ and let

$$M(x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^n x_i.$$

Then

$$\sigma_S(A, B) = \prod_{i=1}^n \sigma_{S_i}(A, B)$$

is a strong fuzzy subsethood measure.

**Example 3.** Let $\sigma_{S_1}, \ldots, \sigma_{S_n}$ be $n$ strong fuzzy subsethood measures on $X$ and let

$$M(x_1, \ldots, x_n) = \frac{n \sum_{i=1}^n x_i}{n \sum_{i=1}^n x_i + \sum_{i=1}^n 1 - x_i}.$$

The following expression is a strong fuzzy subsethood measure on $X$.

$$\sigma_S(A, B) = \frac{n \sum_{i=1}^n \sigma_{S_i}(A, B)}{n \sum_{i=1}^n \sigma_{S_i}(A, B) + \sum_{i=1}^n \sigma_{S_i}(A, B)}.$$

**Proposition 3.** Let $\sigma_{S_1}, \ldots, \sigma_{S_n}$ be $n$ strong fuzzy subsethood measures on $X$ and let $M : [0, 1]^n \rightarrow [0, 1]$ be such that it satisfies (A2), (A3) and the property $M(x_1, \ldots, x_n) = 0$ if and only if $\min(x_1, \ldots, x_n) = 0$. Then, $\sigma_S(A, B) = M_{i=1}^n \sigma_{S_i}(A, B)$ is a strong fuzzy subsethood measure on $X$. 
Proof.

(SF1) Similar to the one for (SF1) in Proposition 2.

(SF2) If $\sigma_S(A, B) = 0$, then $M_{i=1}^{n} \sigma_{S_i}(A, B) = 0$. Therefore there exists at least one $\sigma_{S_i}(A, B) = 0$, so $A = 1$ and $B = 0$.

On the other hand, if $A = 1$ and $B = 0$, then $\sigma_{S_i}(A, B) = 0$, so $\sigma_S(A, B) = M_{i=1}^{n} \sigma_{S_i}(A, B) = 0$.

(SF3) Similar to the one done in (SF3) in Proposition 2.

Example 4. Let $\sigma_{S_1}, \ldots, \sigma_{S_n}$ be $n$ strong fuzzy subsethood measures on $X$. Then

$$
\sigma_S(A, B) = \bigwedge_{i=1}^{n} \sigma_{S_i}(A, B),
$$

$$
\sigma_S(A, B) = \bigvee_{i=1}^{n} \sigma_{S_i}(A, B)
$$

are strong fuzzy subsethood measures.

Proposition 4. Let $\sigma_S^\mu$ be a strong fuzzy subsethood measure on $X$. Then the following items hold:

(i) $\sigma_S(A, B) = \land(\sigma_S^\mu(A, B), \sigma_S^\mu(B_c, A_c))$ is a strong fuzzy subsethood measure on $X$;

(ii) $\sigma_S(A, B) = \sigma_S^\mu(A, B) \cdot \sigma_S^\mu(B_c, A_c)$ is a strong fuzzy subsethood measure on $X$.

Proof.

(i) (SF1) $\sigma_S(A, B) = 1$ if and only if $\sigma_S^\mu(A, B) = \sigma_S^\mu(B_c, A_c) = 1$ if and only if $A \subseteq S B$.

(SF2) If $\sigma_S(A, B) = 0$, then two things can happen.

i. $\sigma_S^\mu(A, B) = 0$, then $A = 1$ and $B = 0$.

ii. $\sigma_S^\mu(B_c, A_c) = 0$, therefore $B_c = 1$ and $A_c = 0$; that is, $A = 1$ and $B = 0$.

On the other hand, if $A = 1$ and $B = 0$, then $\sigma_S^\mu(A, B) = 0$, therefore $\sigma_S(A, B) = 0$.

(SF3) If $A \subseteq B$, then $A_c \geq B_c$. We know that

$$
\sigma_S^\mu(A, C) \geq \sigma_S^\mu(B, C), \quad \sigma_S^\mu(C, A_c) \geq \sigma_S^\mu(C, B_c),
$$
so we arrive at the inequality

\[ \sigma_S(A, C) = \wedge(\sigma'_S(A, C), \sigma'_S(C_c, A_c)) \geq \wedge(\sigma'_S(B, C), \sigma'_S(C_c, B_c)) = \sigma_S(B, C). \]

We also know

\[ \sigma'_S(C, A) \leq \sigma'_S(C, B), \quad \sigma'_S(A_c, C_c) \leq \sigma'_S(B_c, C_c). \]

So again we arrive at

\[ \sigma_S(C, A) = \wedge(\sigma'_S(C, A), \sigma'_S(A_c, C_c)) \leq \wedge(\sigma'_S(B, C), \sigma'_S(B_c, C_c)) = \sigma_S(C, B). \]

(ii) Similar to the previous one.

5 CONSTRUCTION OF STRONG FUZZY SUBSETHOOD MEASURES ON \( X \)

In this section we present two methods of constructing strong fuzzy subsethood measures on \( X \). It is important to indicate that in reality the second one is a particular case of the first one.

5.1 First construction method

**Proposition 5.** Let \( c \) be a strong negation such that \( c(e) = e \), \( M : [0, 1]^n \rightarrow [0, 1] \) be an aggregation function which satisfies (A1) and (A3S) and let the functions \( g, h : [0, 1]^2 \rightarrow [0, 1] \) be such that

1. \( g(x, y) \leq h(x, y) \) for all \( x, y \in [0, 1] \);
2. \( g(x, y) = h(x, y) \) if and only if \( x = 0 \) or \( y = 1 \);
3. \( g(x, y) = 0 \) if and only if \( x = 1 \) and \( y = 0 \);
4. If \( x \leq y \), then
   - \( g(z, x) \leq g(z, y) \),
   - \( g(y, z) \leq g(x, z) \),
   - \( h(z, y) \leq h(z, x) \),
   - \( h(x, z) \leq h(y, z) \).
In these conditions \( \sigma_S : F(X) \times F(X) \to [0, 1] \) given by

\[
\sigma_S(A, B) = \frac{M_{i=1}^n(g(\mu_A(x_i), \mu_B(x_i)))}{M_{i=1}^n(h(\mu_A(x_i), \mu_B(x_i)))}
\]

is a strong fuzzy subsethood measure on \( X \).

Proof. Under the hypothesis of this proposition it can never happen that 
\( h(\mu_A(x_i), \mu_B(x_i)) = 0 \), since if this was true, then by (1) we see that 
\( g(\mu_A(x_i), \mu_B(x_i)) = 0 \), and therefore, by (2), \( \mu_A(x_i) \leq \mu_B(x_i) \) holds and by 
(3) we have \( \mu_A(x_i) = 1 \) and \( \mu_B(x_i) = 0 \). That is, a contradiction.

(SF1) If it holds that \( \sigma_S(A, B) = 1 \), then it follows that

\[
M_{i=1}^n(g(\mu_A(x_i), \mu_B(x_i))) = M_{i=1}^n(h(\mu_A(x_i), \mu_B(x_i))).
\]

By hypothesis we know that \( g(x, y) \leq h(x, y) \). Since \( M \) satisfies \( (A3S) \) we have that if there exists an \( x \) such that 
\( g(\mu_A(x_i), \mu_B(x_i)) < h(\mu_A(x_i), \mu_B(x_i)) \), then

\[
M_{i=1}^n(g(\mu_A(x_i), \mu_B(x_i))) < M_{i=1}^n(h(\mu_A(x_i), \mu_B(x_i))).
\]

So, for all \( i \in \{1, \ldots, n\} \) we have that the identity \( g(\mu_A(x_i), \mu_B(x_i)) = h(\mu_A(x_i), \mu_B(x_i)) \) holds, and by (2) we have \( \mu_A(x_i) = 0 \) or 
\( \mu_B(x_i) = 1 \), for all \( i \in \{1, \ldots, n\} \). On the other hand, if \( \mu_A(x_i) = 0 \) or \( \mu_B(x_i) = 1 \) for all \( i \in \{1, \ldots, n\} \), then by (2) we know that 
\( g(\mu_A(x_i), \mu_B(x_i)) = h(\mu_A(x_i), \mu_B(x_i)) \), therefore \( \sigma_S(A, B) = 1 \).

(SF2) and (SF3) follow from a straight calculation taking into account the 
monotonicity of \( M \) ((A3)) and the properties demanded to \( g \) and \( h \).

Example 5. Let the functions

\[
g(x, y) = \frac{1-x+y}{4} \quad \text{if } x=0 \text{ or } y=1
\]

\[
h(x, y) = \begin{cases} 
\frac{1}{3}, & \text{if } x=0 \text{ or } y=1 \\
\frac{1}{2}, & \text{otherwise}
\end{cases}
\]

It is easy to see that these functions satisfy the conditions (1)-(4) of the proposition above. If we take \( M(x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^n x_i \), which satisfies \( (A1) \) and
(A3S), then

$$
\sigma_S(A, B) = \frac{1}{n} \sum_{i=1}^{n} \frac{1 - 0.5 (\mu_A(x_i) + \mu_B(x_i))}{4} \mu_A(x_i) \mu_B(x_i)
$$

if \( \mu_A(x_i) = 0 \) or \( \mu_B(x_i) = 1 \)

otherwise

is a strong fuzzy subsethood measure.

**Corollary 2.** In the conditions of Proposition 5, the following items hold.

(i) If \( M \) satisfies A4, then \( \sigma_S(A, B) = \sigma_S(s(A), s(B)) \) for all \( A, B \in F(X) \) and for any one-to-one mapping \( s : X \rightarrow X \).

(ii) If \( g(x, y) = g(c(y), c(x)) \) and \( h(x, y) = h(c(y), c(x)) \) for all \( x, y \in [0, 1] \), then \( \sigma_S(A, B) = \sigma_S(B, A) \).

(iii) If \( g(x, y) + g(x, c(y)) \geq 1 \) for all \( x, y \in [0, 1] \) and

$$
M_{i=1}^{n} (x_1, \ldots, x_\alpha) + M_{i=1}^{n} (1 - x_1, \ldots, 1 - x_\alpha) \geq 1,
$$

then \( \sigma_S(A, B) + \sigma_S(A, B_c) \geq 1 \).

**Proof.** It follows from a straight calculation.

**Example 6.** Let \( g(x, y) = \vee (1 - x, y), h(x, y) = 1 \) and \( M(x_1, \ldots, x_\alpha) = \frac{1}{n} \sum_{i=1}^{n} x_i \). Then

$$
\sigma_S(A, B) = \frac{1}{n} \sum_{i=1}^{n} \vee (1 - \mu_A(x_i), \mu_B(x_i))
$$

is a strong fuzzy subsethood measure that satisfies Axioms (SD5), (SD6), (SD10) and (SD12).

**5.2 Second construction method**

In this section we present constructions of strong fuzzy subsethood measures using aggregation and implication functions. These constructions are a particular case of those of the previous section, it is enough to take \( h(x, y) = 1 \) in Proposition 5

**Proposition 6.** Let \( c \) be a strong negation and let \( \sigma_S : F(X) \times F(X) \rightarrow [0, 1] \), given by:

$$
\sigma_S(A, B) = M_{i=1}^{n} (I(\mu_A(x_i), \mu_B(x_i))),
$$
for all \( A, B \in F(X) \), where \( M : [0, 1]^n \to [0, 1] \) is a function that satisfies (A1), (A2), (A3) and \( I \) is a function of \([0, 1]^2\) in \([0, 1]\) that satisfies (I1), (I2) and

\[
I(x, y) = 0 \text{ if and only if } x = 1 \text{ and } y = 0,
\]

\[
I(x, y) = 1 \text{ if and only if } x = 0 \text{ or } y = 1.
\]  

Then \( \sigma_S \) is a strong fuzzy subsethood measure on \( X \).

Proof.

(SF1)  If

\[
\sigma_S(A, B) = 1
\]

\[
= M(I(\mu_A(x_1), \mu_B(x_1)), \ldots, I(\mu_A(x_n), \mu_B(x_n))),
\]

then, since \( M \) satisfies (A2), we have that \( I(\mu_A(x_i), \mu_B(x_i)) = 1 \) for all \( i \in \{1, \ldots, n\} \), so \( \mu_A(x_i) = 0 \text{ or } \mu_B(x_i) = 1 \). If \( \mu_A(x_i) = 0 \) or \( \mu_B(x_i) = 1 \) for all \( i \in \{1, \ldots, n\} \), then \( I(\mu_A(x_i), \mu_B(x_i)) = 1 \). Since \( M \) satisfies (A2) then

\[
1 = M(I(\mu_A(x_1), \mu_B(x_1)), \ldots, I(\mu_A(x_n), \mu_B(x_n))) = \sigma_S(A, B).
\]

(SF2)  If \( \sigma_S(A, B) = 0 \), then

\[
0 = M(I(\mu_A(x_1), \mu_B(x_1)), \ldots, I(\mu_A(x_n), \mu_B(x_n))),
\]

since \( M \) satisfies (A1) we have \( I(\mu_A(x_i), \mu_B(x_i)) = 0 \) for all \( i \in \{1, \ldots, n\} \), then \( \mu_A(x_i) = 1 \text{ and } \mu_B(x_i) = 0 \) for all \( i \in \{1, \ldots, n\} \). If \( \mu_A(x_i) = 1 \) for all \( i \in \{1, \ldots, n\} \) and \( \mu_B(x_i) = 0 \), then we see that \( I(\mu_A(x_i), \mu_B(x_i)) = 0 \) bearing in mind that \( M \) satisfies (A1) we have

\[
0 = M(I(\mu_A(x_1), \mu_B(x_1)), \ldots, I(\mu_A(x_n), \mu_B(x_n))) = \sigma_S(A, B).
\]

(SF3)  If \( A \subseteq B \), then \( I(\mu_A(x_i), \mu_C(x_i)) \geq I(\mu_B(x_i), \mu_C(x_i)) \) for all \( i \in \{1, \ldots, n\} \), bearing in mind that \( M \) satisfies (A3) we have

\[
\sigma_S(A, C) = M(I(\mu_A(x_1), \mu_C(x_1)), \ldots, I(\mu_A(x_n), \mu_C(x_n)))
\]

\[
\geq M(I(\mu_B(x_1), \mu_C(x_1)), \ldots, I(\mu_B(x_n), \mu_C(x_n)))
\]

\[
= \sigma_S(B, C).
\]
If \( A \leq B \), then \( I(\mu_C(x_i), \mu_A(x_i)) \leq I(\mu_C(x_i), \mu_A(x_i)) \) for all \( i \in \{1, \ldots, n\} \), therefore

\[
\sigma_S(C, A) = M(I(\mu_C(x_1), \mu_A(x_1)), \ldots, I(\mu_C(x_n), \mu_A(x_n))) \\
\leq M(I(\mu_C(x_1), \mu_B(x_1)), \ldots, I(\mu_C(x_n), \mu_B(x_n))) \\
= \sigma_S(C, B).
\]

In the following corollary we study the conditions under which strong fuzzy subsethoods fulfill Axioms (SD5), (SD6), (SD10) and (SD12). The other axioms have been studied in Corollary 1.

**Corollary 3.** In the same conditions as in Proposition 6, the following items hold:

(i) If \( M \) satisfies \( A_4 \), then \( \sigma_S \) satisfies \( \sigma_S(A, B) = \sigma_S(s(A), s(B)) \) for all \( A, B \in F(X) \) and for every one-to-one mapping \( s: X \rightarrow X \).

(ii) If \( I \) satisfies (112), then \( \sigma_S \) satisfies Axiom (SD6).

(iii) If \( M(x_1, \ldots, x_n) + M(c(x_1), \ldots, c(x_n)) \geq 1 \) and \( I \) is an \( S \)-implication, then \( \sigma_S \) satisfies Axiom (SD10).

(iv) If \( M \) is idempotent and \( I \) satisfies (110) and (112), then \( \sigma_S \) satisfies Axiom (SD12).

**Proof.** It follows from a straight calculation.

5.3 Characterization of the strong fuzzy subsethood measures on \( X \) with \( M \) fixed

**Theorem 3.** Let \( c \) be a strong negation and let \( \sigma_S: F(X) \times F(X) \rightarrow [0, 1] \) be given by

\[
\sigma_S(A, B) = M^n_{i=1}(I(\mu_A(x_i), \mu_B(x_i)))
\]

for all \( A, B \in F(X) \), with \( M: [0, 1]^n \rightarrow [0, 1] \) a function that satisfies (A1), (A2), (A3) and is idempotent and \( I \) a function from \([0, 1]^2 \) to \([0, 1] \). Then \( \sigma_S \) is a strong fuzzy subsethood measure on \( X \) that satisfies Axiom (SD6) and the property \( \sigma_S(1, A) = M(A) \) if and only if \( I \) satisfies (II), (I6), (II2) and (I(x, y) = 1 if and only if \( x = 0 \) or \( y = 1 \)).

**Proof.** (Sufficiency) By hypothesis \( I \) satisfies (I6) and (II2), so \( I \) also satisfies (I9) [6]. Moreover, since \( I \) satisfies (II1) and (II2), it also satisfies (I2). On the other hand, as \( I \) satisfies (II2), (I6) and (I9), then \( I \) satisfies the property
$I(x, y) = 0$ if and only if $x = 1$ and $y = 0$. In these conditions, by Proposition 6 we have that $\sigma_S$ is a strong fuzzy subsethood measure on $X$.

The rest of the demonstration a straight verification.

**Corollary 4.** In the same conditions as in the theorem above, $\sigma_S$ satisfies Axiom (SD12).

Since we know [6] that if $I : [0, 1] \rightarrow [0, 1]$ satisfies strict (I1), strict (I2), (I7), (I9) and (I13), then $I$ satisfies (2), in this section we are going to see that from strict strong fuzzy subsethood measures we can construct fuzzy entropies. We begin characterizing the strict strong fuzzy subsethood measures on $X$.

**Theorem 4.** Let $c$ be a strong negation and let $\sigma_S : F(X) \times F(X) \rightarrow [0, 1]$ be given by

$$\sigma_S(A, B) = M^n_{i=1}(I(\mu_A(x_i)), \mu_B(x_i))$$

for all $A, B \in F(X)$, where $M : [0, 1]^n \rightarrow [0, 1]$ is a function that satisfies (A1), (A2), (A3S) and is idempotent and $I$ is a function of $[0, 1]^2$ in $[0, 1]$ that satisfies (I7) and (I13). In these conditions the following items are equivalent:

(i) $\sigma_S$ is a strict strong fuzzy subsethood measure on $X$ that satisfies Axiom (SD6) and $\sigma_S(1, A) = M(A)$;
(ii) $I$ satisfies strict (I1), strict (I2), (I6) and (I12);
(iii) There exists an automorphism $\varphi$ of the unit interval such that

$$\sigma_S(A, B) = M^n_{i=1} \cdot c \cdot \varphi^{-1}(\varphi(\mu_A(x_i)) \cdot \varphi(c(\mu_B(x_i))))$$

**Proof.** (iii) $\Rightarrow$ (i) The only point which is not trivial is (SF2). If $\sigma_S(A, B) = 0$, then

$$M^n_{i=1} \cdot c \cdot \varphi^{-1}(\varphi(\mu_A(x_i)) \cdot \varphi(c(\mu_B(x_i)))) = 0.$$ 

Since $M$ satisfies (A1) we have

$$\varphi^{-1}(\varphi(\mu_A(x_i)) \cdot \varphi(c(\mu_B(x_i)))) = 1.$$
so \(\varphi(\mu_A(x_i)) \cdot \varphi(c(\mu_B(x_i))) = 1\), and therefore \(\mu_A(x_i) = 1\) and \(\mu_B(x_i) = 0\) for all \(i \in \{1, \ldots, n\}\). The sufficiency is straight. The rest of the demonstration is straight.

It is important to note that if in the conditions of the theorem above, if \(c(x) = \varphi^{-1}(1 - \varphi(x))\) for all \(x \in [0, 1]\), then

\[
\sigma_S(A, B) = M_n \varphi^{-1} 1 - \varphi(\mu_A(x_i)) + \varphi(\mu_A(x_i)) \cdot \varphi(\mu_B(x_i))
\]

**Corollary 5.** The following items hold.

(i) Let \(c\) be a strong negation, \(f: [0, 1]^2 \to [0, 1]\) a continuous strictly increasing function and \(\varphi\) an automorphism of the unit interval. Then

\[
\sigma_S(A, B) = f^{-1} \frac{1}{n} \sum_{i=1}^{n} f(\varphi^{-1}(\varphi(\mu_A(x_i)) \cdot \varphi(c(\mu_B(x_i)))))
\]

is a continuous, strict, strong fuzzy subsethood measure on \(X\) that satisfies Axioms (SD6) and (SD12).

(ii) Let \(c\) be the standard negation, \(f: [0, 1]^2 \to [0, 1]\) a continuous, strictly increasing and convex function and \(\varphi\) an automorphism of the unit interval. Then

\[
\sigma_S(A, B) = f^{-1} \frac{1}{n} \sum_{i=1}^{n} f(1 - \varphi^{-1}(\varphi(\mu_A(x_i)) \cdot \varphi(1 - \mu_B(x_i))))
\]

is a continuous, strict, strong fuzzy subsethood measure on \(X\) and satisfies Axioms (SD6), (SD10) and (SD12).

(iii) Let \(c\) be the standard negation and let \(\varphi\) be an automorphism of the unit interval such that \(\varphi(x) + \varphi(y) = 1\) if and only if \(x + y = 1\). Then

\[
\sigma_S(A, B) = \varphi^{-1} \frac{1}{n} \sum_{i=1}^{n} 1 - \varphi(\mu_A(x_i))(1 - \varphi(\mu_B(x_i)))
\]

is a continuous, strict strong fuzzy subsethood measure on \(X\) that satisfies Axioms (SD6), (SD10) and (SD12).

**Proof.** It is straight.
5.4 Characterization of the strong fuzzy subsethood measures on $X$ with $I$ fixed

**Theorem 5.** Let $c$ be a strong negation and let $\sigma_S: F(X) \times F(X) \to [0, 1]$ be given by

$$\sigma_S(A, B) = M_{i=1}^{n}(I(\mu_A(x_i), \mu_B(x_i)))$$

for all $A, B \in F(X)$, where $M: [0, 1]^n \to [0, 1]$ and $I$ is a function from $[0, 1]^2$ to $[0, 1]$ which satisfies (I1), (I2), (I6) and (2). In these conditions $\sigma_S$ is a strong fuzzy subsethood measure on $X$ such that $\sigma_S(1, A) = M(A)$ if and only if $M$ satisfies (A1), (A2), (A3).

**Proof.** It is straight taking into account Proposition 6.

**Example 7.** If we take $I(x, y) = \vee(1 - x, y)$ and $M(x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i$, then we have:

$$\sigma_S(A, B) = \frac{1}{n} \sum_{i=1}^{n} \vee(1 - \mu_A(x_i), \mu_B(x_i)).$$

If we take $I(x, y) = 1 - x + x \cdot y$ and $M(x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i$ we have:

$$\sigma_S(A, B) = \frac{1}{n} \sum_{i=1}^{n} 1 - \mu_A(x_i) + \mu_A(x_i) \cdot \mu_B(x_i).$$

6 CHARACTERIZATION OF THE STRONG INCLUSION OF DUBOIS AND PRADE

In this section we present the characterization of the expression of Dubois and Prade:

$$\sigma_D(A, B) = \frac{1}{n} \sum_{i=1}^{n} \vee(1 - \mu_A(x_i), \mu_B(x_i)),$$

from our constructions. It is necessary to point out that in 1980 Dubois and Prade called this expression weak inclusion.
Theorem 6. Let \( c \) be a strong negation and let \( \sigma_S : F(X) \times F(X) \to [0, 1] \) be given by

\[
\sigma_S(A, B) = M_{\text{idem}}^{\mu}(I(\mu_A(x_i), \mu_B(x_i))),
\]

for all \( A, B \in F(X) \), where \( M : [0, 1]^n \to [0, 1] \) satisfies (A1), (A2), (A3) and is idempotent and \( I \) is a function from \([0, 1]^2\) to \([0, 1]\). In these conditions \( \sigma_S \) is a strong fuzzy subsethood measure on \( X \) such that it satisfies Axiom 6, \( \sigma_S(1, A) = M(A) \) and if \( A = k \), then \( \sigma_S(k, c(k)) = c(k) \).

If and only if

\( I \) satisfies (I1), (I6), (I12), \( I(x, c(x)) = c(x) \) for all \( x \in [0, 1] \) and \( I(x, y) = 1 \) if and only if \( x = 0 \) or \( y = 1 \).

Proof. We know by Theorem 3 that if \( M : [0, 1]^n \to [0, 1] \) is a function that satisfies (A1), (A2), (A3) and is idempotent and \( I \) is a function from \([0, 1]^2\) to \([0, 1]\), then \( \sigma_S \) is a strong fuzzy subsethood measure on \( X \) that satisfies Axiom 6 and the property \( \sigma_S(1, A) = M(A) \) if and only if \( I \) satisfies (I1), (I6), (I12) and \( I(x, y) = 1 \) if and only if \( x = 0 \) or \( y = 1 \). So we only have to prove the condition \( \sigma_S(k, c(k)) = c(k) \).

(Necessity) If \( A = k \in [0, 1] \), then since \( M \) is idempotent we have \( I(k, c(k)) = M(I(k, c(k)), \ldots, I(k, c(k))) = \sigma_S(k, c(k)) = c(k) \). Therefore for all \( x \in [0, 1] \) we have \( I(x, c(x)) = c(x) \).

(Sufficiency) If \( I(x, c(x)) = c(x) \) for all \( x \in [0, 1] \) we have that if we take \( k \in [0, 1] \) and \( A = k \), then since \( M \) is idempotent, \( \sigma_S(k, c(k)) = M(I(k, c(k)), \ldots, I(k, c(k))) = M(c(k), \ldots, c(k)) = c(k) \).

Looking at the conditions that we demand from the functions \( I \) in the theorem above and bearing in mind the well-known relations between the different properties that can be demanded to implication operators ([6]):

\( I \) satisfies (I1) and (I12), therefore it satisfies (I2). Since it satisfies (I6) and (I12) it also satisfies (I9). Since it verifies (I9) and (I12) it satisfies (I3), since it satisfies (I12) and (I3) it also satisfies (I4). Lastly, since it satisfies (I6) it satisfies (I5). Therefore \( I \) is an implication in the sense of Fodor and Roubens, that satisfies in addition (I6) and (I12).

In these conditions, if we impose on \( I \) to satisfy (I7) we know by [16, 24] that \( I \) is an S-implication with an appropriate t-conorm and a strong negation \( c \). Since besides, we demand \( I \) to satisfy the condition \( I(x, c(x)) = c(x) \) for all \( x \in [0, 1] \), by Theorem 1.6 we have that the only S-implication that satisfies this property is \( I(x, y) = \vee(c(x), y) \). All these considerations bring us to the following corollary.
Corollary 6. Let $\sigma_S: F(X) \times F(X) \rightarrow [0, 1]$ be given by

$$\sigma_S(A, B) = M^n_{i=1}(I(\mu_A(x_i), \mu_B(x_i))),$$

for all $A, B \in F(X)$, where $M: [0, 1]^n \rightarrow [0, 1]$ is a function that satisfies (A1), (A2), (A3) and is idempotent and $I$ is a function of $[0, 1]^2$ in $[0, 1]$ that satisfies (I7). In these conditions, $\sigma_S$ is a strong fuzzy subsethood measure on $X$ which satisfies Axiom (SD6), $\sigma_S(1, A) = M(A)$ and the property (if $A = k$, then $\sigma_S(k, c(k)) = c(k))$ if and only if $I(x, y) = \vee(1 - x, y)$.

Proof. We only need to bear in mind the theorem above and the fact that $I(x, y) = \vee(1 - x, y)$ satisfies (I1), (I6), (I12), $I(x, c(x)) = c(x)$ for all $x \in [0, 1]$ and $I(x, y) = 1$ if and only if $x = 0$ or $y = 1$.

Remark 2.

1. If we take $M(x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i$ (obviously it satisfies (A1), (A2), (A3) and is idempotent), the expression that we obtain in the conditions of the corollary above is:

$$\sigma_S(A, B) = \frac{1}{n} \sum_{i=1}^{n} I(\mu_A(x_i), \mu_B(x_i)).$$

If the negation considered is the standard, this expression is the one presented by Dubois and Prade in 1980 [11] with the name of weak inclusion.

2. If in our constructions we use any R-implication as function $I$, it results that the $\sigma_S$ constructed from $M$ and $I$ by Theorem 3 or Proposition 6, does not fulfill the condition (SF1); that is, it does not fulfill

$$\sigma_S(A, B) = 1 \text{ if and only if } A \subseteq_S B.$$ 

Therefore, it is not a strong fuzzy subsethood measure.

3. If $I$ is a QL-implication, we know that in general these implications do not satisfy the property (I1), therefore nothing can be said about the fulfillment of the condition (SF3) when we use our constructions. Fodor studies in [15], the conditions under which the QL-implications satisfy (I1). We leave for a near future the analysis of the way in which these conditions influence our strong fuzzy subsethood measures.
7 STRONG EQUALITY INDEXES

The concept of strong equality index can be defined as follows [7].

**Definition 7.** A mapping $\Psi_{SE} : \mathcal{F}X \times \mathcal{F}(X) \rightarrow [0, 1]$ is a strong equality index if:

(SE1) $\Psi_{SE}(A, B) = 1$ if and only if $A = B$ with $A, B$ crisp sets;
(SE2) $\Psi_{SE}(A, B) = 0$ if and only if $A$ and $B$ are complementary crisp sets;
(SE3) $\Psi_{SE}(A, B) = \Psi_{SE}(B, A)$ for every $A, B \in \mathcal{F}(X)$.

In [7] the following result is proved.

**Corollary 7.** Let $M : [0, 1]^n \rightarrow [0, 1]$ be a mapping that satisfies (A1) and (A2). Let $I : [0, 1]^2 \rightarrow [0, 1]$ be such that (2) hold. Then, the mapping $\Psi_{SE} : \mathcal{F}X \times \mathcal{F}(X) \rightarrow [0, 1]$ given by

$$\Psi_{SE}(A, B) = M^n_{i=1}(\land(I(\mu_A(x_i), \mu_B(x_i)), I(\mu_B(x_i), \mu_A(x_i))))$$

is a strong equality index.

Note that

$$\land(I(\mu_A(x_i), \mu_B(x_i)), I(\mu_B(x_i), \mu_A(x_i))) = E(\mu_A(x_i), \mu_B(x_i))$$

where $E$ is the bi-implication related to $I$, see [18]. So if we take into account this corollary and we bear in mind Theorem 5 we have immediately the following result.

**Theorem 7.** In the same conditions of Theorem 5, if $M$ is such that

$$M^n_{i=1}(\land(x_i, y_i)) = \land(M^n_{i=1}x_i, M^n_{i=1}y_i),$$

then it follows that the mapping $\Psi_{SE} : \mathcal{F}X \times \mathcal{F}(X) \rightarrow [0, 1]$ given by

$$\Psi_{SE}(A, B) = \land(\sigma_5(A, B), \sigma_5(B, A))$$

is a strong equality index.

Note that $M$ satisfies the constraints of the previous theorem if and only if $M(x_1, \ldots, x_n) = \min(f_1(x_1), \ldots, f_n(x_n))$, where the functions
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$f_1, \ldots, f_n : [0, 1] \to [0, 1]$ are increasing and satisfy, for each $i \in \{1, \ldots, n\}$, that $f_i(x) = 0$ if and only if $x = 0$, and $f_i(x) = 1$ if and only if $x = 1$.

8 CONCLUDING REMARKS

In this work we have presented the concept of strong fuzzy subsethood measure taking into account the ideas of Dubois et al. for the inclusion between fuzzy sets. This definition allows to defined strong equality between two fuzzy sets, $A$ and $B$ as follows:

$$(A =_S B) \text{ if } (A \subseteq S B \text{ and } B \subseteq S A).$$

We have also linked this concept to that of strong equality index by means of the use of appropriate aggregation functions and implication functions.

In the future we hope to connect this notion of strong subsethood with that of overlap function and overlap index. Moreover, we think this concept could find wide applicability in image processing to compare two or more images or to detect objects in a given picture.

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