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A New Algorithm to Design Minimal Multi-Functional Observers for Linear Systems

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Abstract

Designing minimum possible order (minimal) observers for Multi-Input Multi-Output (MIMO) linear systems have always been an interesting subject. In this paper, a new methodology to design minimal multi-functional observers for Linear Time-Invariant (LTI) systems is proposed. The approach is applicable, and it also helps in regulating the convergence rate of the observed functions. It is assumed that the system is functional observable or functional detectable, which is less conservative than assuming the observability or detectability of the system. To satisfy the minimality of the observer, a recursive algorithm is provided that increases the order of the observer by appending the minimum required auxiliary functions to the desired functions that are going to be estimated. The algorithm increases the number of functions such that the necessary and sufficient conditions for the existence of a functional observer are satisfied. Moreover, a new methodology to solve the observer design interconnected equations is elaborated. Our new algorithm has advantages with regard to the other available methods in designing minimal order functional observers. Specifically, it is compared with the most common schemes, which are transformation based. Using numerical examples it is shown that under special circumstances, the conventional methods have some drawbacks. The problem partly lies in the lack of sufficient numerical degrees of freedom proposed by the conventional methods. It is shown that our proposed algorithm can resolve this issue. A recursive

algorithm is also proposed to summarize the observer design procedure. Several numerical examples and simulation results illustrate the efficacy, superiority and different aspects of the theoretical findings.
Keywords: Functional observer, functional observability and detectability, generalized inverse, constrained Sylvester equations

1 Introduction

A functional observer is a general form of a Luenberger observer that can be used for estimation of one or multiple functions of the states of a system. This subject has recently been the focus of much research ([1, 2, 3, 4], etc.). It is cost effective to use functional observers whenever the number of the states of a system are considerably larger than the number of the outputs and we do not need to observe all of them. For example, in fault detection and isolation of a large scale electromechanical system, mostly the states related to the current, velocity, and acceleration are desired.

Recently, one of the important research directions in this field has been devoted to finding the least possible order for a functional observer [5, 3, 6, 7, 8], and designing an observer with that order. In [3, 6] the concept of *functional observability* is introduced and the necessary and sufficient conditions for functional observability or functional detectability of a system are derived. Fernando and Trinh [9], propose a scheme for designing functional observers for linear systems if the necessary and sufficient conditions of Darouach [1] are not satisfied, and even the system is not observable/detectable, but it is functional observable/detectable. Although the proposed method might lead to the design of a reduced order observer, the resulted observer is not necessarily *minimal* (of minimum order).

In the present paper, to address the minimality requirement of the functional observer, the definitions of functional observability are revisited. As one of our contributions, we elaborate an algorithm to increase the order of the functional in a way that the necessary and sufficient conditions of the existence of a functional observer are satisfied. This is accomplished by appending the minimum required number of auxiliary functions to be observed.

As the second contribution, we derive a new methodology to obtain the observer parameters. One important problem that is addressed in most of the significant previous works in this field, is solving a number of coupled matrix equations called interconnected *generalized* (or constrained) *Sylvester* equations [10]. This set of equations might have infinite number of solutions and each solution is a set of observer parameters for the system that should satisfy the observer equations. However, to the best of our knowledge, there are three state of the art methodologies to solve the constrained Sylvester equations in the literature: 1) transformation based approach [11, 1, 2, 5], 2) parametric approach [10, 8], and 3) direct approach [7].

The first method, which has been the most popular one, is based on a number of matrix transformations that break the unknown matrices into smaller sub-matrices and increase the number of observer equations [1, 2]. This approach can also be classified to three different schemes that are more illustrated in Section 2.1. The transformation-based approaches have been applied to single and multi functional observers, as well as unknown-input functional observers ([12, 13, 14, 15, 16]). It can be stated that most of the recent contributions in this field of research have used one of the schemes of this approach in their design procedures. This is because the methodologies in this class can be simply converted to numerical algorithms, and can also be applied to general MIMO linear systems as well as a class of nonlinear systems. However, it is shown in this paper that the schemes of this class have different numerical properties and might have some numerical issues in special situations. Motivated by this observation, we have tried to partly fill this gap by presenting a *new* transformation-based design algorithm that improves performance of the observer. The contributions in this regard are more explained in Section 3.

Hence the main novelty of the present paper lies in the development of a new methodology to design minimal multi-functional observers for linear systems via solving the constrained Sylvester equations. Our observations show that the proposed algorithm can be more reliable and effective than the other available methods. The necessary and sufficient conditions are derived for the existence of an asymptotic functional observer for the system using the novel design approach. The equivalence between the obtained new conditions and the renown conditions proposed by Darouach [1] are also verified. Numerical examples and simulation results show the effectiveness of our design algorithms as well as some of the issues related to the other conventional design schemes.

Notations: throughout the paper, the following simplified notations are used. The expression $[A_1; A_2]$ is equivalent to $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$; rows$(X)$ indicates the number of rows of the matrix X; I_k and $I_{k\times l}$, $k, l \in \mathbb{N}$ respectively denote the $k \times k$ and $k \times l$ identity matrices, and $\mathbf{a}_{k\times l}$ denotes a $k \times l$ matrix with elements all equal to number "a". Furthermore, $X^\dagger$ represents the Moore-Penrose pseudo-inverse or the generalized inverse of the matrix X, and $X^\perp$ denotes the right orthogonal matrix of X, in which $XX^\perp = \mathbf{0}$. In addition, denote $\mathcal{R}(M)$ and $\mathcal{N}(M)$ as the row space and the null space of the matrix M, respectively. In addition, $[[\mathbb{S}]]$ denotes a matrix of row bases vectors for the subspace $\mathbb{S}$. Furthermore, let us define $H_2 := [[\mathcal{R}([C; CA; L])]]$, $H_3 := \left[\left[\mathcal{R}\left(\begin{bmatrix} H_2 \\ LA \end{bmatrix}\right)\right]\right]$, $a := rank(H_2)$, $b := rank(H_3)$.

2 Problem Statement and Preliminaries

Our problem is to propose a new algorithm to design a minimal functional observer for an LTI system described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \\ z(t) &= Lx(t) \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $L \in \mathbb{R}^{l\times n}$ are constant and known matrices. The main aim of the observer is to reconstruct a functional $z_0(t) = L_0x(t)$ with $z_0 \in \mathbb{R}^{l_0}$, where $l_0 \leq l$. The matrix $L = [L_0; R_1; R_2]$ is defined in a way that a stable functional observer can be designed for the system. If $l > l_0$, auxiliary functions have been added to the vector $z_0(\cdot)$ to form the new vector $z(\cdot)$ in an effort to design the minimum order functional observer *for the new system*. Apparently, if any auxiliary functions are added to the system's nominal vector z_0, the designed observer could be of *minimal* order instead of *minimum* one. One of our goals is thus to find R_1 and R_2 with the minimum possible number of rows.

The following assumptions are considered in the paper,

(A) The matrices C and L_0 are of full row rank. Moreover, $rank([C; L_0]) = p + l_0$.

(B) The number of functions to be observed are not larger than the difference between the number of states and outputs of the system ($l \leq n-p$).

(C) The triple $\Sigma = (A, C, L_0)$ is functional observable or at least functional detectable.

Assumptions (A) and (B) do not fail the generality of the paper. For example, if assumption (B) is not satisfied, the least possible order for the functional observer is $n-p$, which is already solved using the reduced order Luenberger observer [17, 18]. Furthermore, if assumption (C) does not hold, then it is not possible to design a functional observer for the triplet Σ [19, 3]. On the other hand, if assumption (C) is satisfied, then the rows of R_1 and R_2 can be determined (this is later described in Algorithm 1).

The following theorem shows a simple method for checking the functional observability/detectability of the system Σ,

Theorem 2.1 ([3, 6])**.** *The system (A, C, L_0) is functional detectable if and only if*

$$rank \begin{bmatrix} sI_n - A \\ C \\ L_0 \end{bmatrix} = rank \begin{bmatrix} sI_n - A \\ C \end{bmatrix}, \quad \forall s \in \mathbb{C}^+ \tag{2}$$

Moreover, it is functional observable if and only if (2) holds for all $s \in \mathbb{C}$. Equivalently, Σ is functional observable if and only if

$$rank \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \\ L_0 \\ L_0A \\ \vdots \\ L_0A^{n-1} \end{bmatrix} = rank \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{3}$$

Remark 1. The condition (2) can be readily testified by MATLAB using the *invariant zeros* of the *Rosenbrock's* matrix. If the matrices of both sides of (2) have similar invariant zeros, then their ranks are equal.

The following (*l'th* order) functional observer structure, which is the minimal order observer for the system Σ is given as below,

$$\begin{aligned} \dot{\omega}(t) &= F\omega(t) + Gu(t) + Hy(t) \\ \hat{z}(t) &= \omega(t) + Vy(t) \end{aligned} \tag{4}$$

with $F \in \mathbb{R}^{l\times l}$, $G \in \mathbb{R}^{l\times m}$, $H \in \mathbb{R}^{l\times p}$ and $V \in \mathbb{R}^{l\times p}$ as the design parameters to be defined. Let us define the following error signal, $e(t) := \hat{z}(t) - z(t)$.

Then from [1] the necessary and sufficient conditions of the existence of a stable functional observer (4) for the system (1) are as follows,

Condition I:

$$rank \begin{bmatrix} LA \\ CA \\ C \\ L \end{bmatrix} = rank \begin{bmatrix} CA \\ C \\ L \end{bmatrix} \tag{5}$$

Condition II:

$$rank \begin{bmatrix} sL - LA \\ CA \\ C \end{bmatrix} = rank \begin{bmatrix} CA \\ C \\ L \end{bmatrix} \quad \forall s \in \mathbb{C}^{+} \tag{6}$$

In fact, the matrices R_1 and R_2 in L are added to satisfy the above-mentioned Conditions I and II, respectively. Note that Condition I is equivalent to $\mathcal{R}(H_2) = \mathcal{R}(H_3)$, or $a = b$. If Conditions I and II are fulfilled, then the following necessary and sufficient conditions for asymptotic stability of the observer (4), in the sense that $\lim_{t\to\infty} e(t) = \mathbf{0}$ can be satisfied by appropriately choosing the observer parameters [1],

(a) The observer matrix F is Hurwitz.

(b) There exists a matrix $T \in \mathbb{R}^{l\times n}$, such that the set of equations

$$FT - TA + HC = \mathbf{0} \tag{7a}$$

$$G = TB \tag{7b}$$

$$L - T - VC = \mathbf{0} \tag{7c}$$

are satisfied using appropriate design matrices F, H, G and V.

2.1 Conventional Observer Design Approaches

Although the set of matrix equations (7) might have infinite number of solutions if they are solvable, finding a suitable solution for them is difficult. The reason for this lies in the difficulties arising from the interconnected Sylvester equations (7a) and (7c). The Sylvester equation (7a) confers that matrix $\bar{H} := TA - FT \in \mathbb{R}^{l\times n}$ must lie in the row space spanned by matrix

C. It means that although $\bar{H}$ can be of rank l, it must be at most of rank p, i.e the rank of matrix C. However, l might be larger than p and the main challenge in solving (7a) is to find matrices T and F such that $\bar{H}$ lies in the row space of matrix C, and F be Hurwitz. Moreover T must also satisfy (7c), which is thus tangled to (7a) that increases the complexity of the problem.

From the main three approaches proposed to solve this problem and briefly mentioned in the introduction, the transformation-based approach [11, 8, 2] is likely to be more effective and easier to design and apply. That is why this method has been the most commonly used one in solving (7). By the way, this approach can also be classified into three different methods, two of them are briefly studied here, and the third one, which is the most recent method, will be briefly described in Remark 9.

First, a square matrix $\bar{C} \in \mathbb{R}^{n\times n}$, which is always of full rank (due to Assumption (A)) is defined as

$$\bar{C} := \begin{bmatrix} C^{\dagger} & C^{\perp} \end{bmatrix} \tag{8}$$

where $C^{\dagger}$ and $C^{\perp}$ are the generalized inverse and right-orthogonal matrices of C, respectively. Next, $T_1 \in \mathbb{R}^{l\times p}, T_2 \in \mathbb{R}^{l\times(n-p)}, A_{11} \in \mathbb{R}^{p\times p}, A_{12} \in \mathbb{R}^{p\times(n-p)}, A_{21} \in \mathbb{R}^{(n-p)\times p}, A_{22} \in \mathbb{R}^{(n-p)\times(n-p)}, L_1 \in \mathbb{R}^{l\times p}$, and $L_2 \in \mathbb{R}^{l\times(n-p)}$ are defined as below,

$$\begin{bmatrix} T_1 & T_2 \end{bmatrix} := T\bar{C}$$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} := \bar{C}^{-1}A\bar{C}$$

$$\begin{bmatrix} L_1 & L_2 \end{bmatrix} := L\bar{C}$$

Finally, the interconnected Sylvester equations (7a) and (7c) are post-multiplied by $\bar{C}$ that results

$$V = L_1 - T_1 \tag{9}$$

$$T_2 = L_2 \tag{10}$$

and,

$$H = T_1A_{11} + T_2A_{21} - FT_1 \tag{11}$$

$$FT_2 - T_1A_{12} - T_2A_{22} = \mathbf{0} \tag{12}$$

Now from (10) and (12) we have

$$\left[\begin{array}{cc} F & -T_1 \end{array}\right]\Omega_1 = L_2A_{22} \tag{13}$$

where $\Omega_1 := [L_2; A_{12}] \in \mathbb{R}^{(l+p)\times(n-p)}$. According to [20, 1], the relation (13) has a solution for F and T_1 if and only if

$$L_2A_{22}(I_{n-p} - \Omega_1^{\dagger}\Omega_1) = \mathbf{0} \tag{14}$$

where $\Omega_1^{\dagger}$ is the generalized inverse of Ω_1. This is equivalent to,

$$rank\left(\left[\begin{array}{c} \Omega_1 \\ L_2A_{22} \end{array}\right]\right) = rank\,(\Omega_1) \tag{15}$$

It can be shown using the same methodology as in [1, 21, 2], that (15) is equivalent to Condition I described in (5). Hence, if Condition I is satisfied, then from (13)

$$\left[\begin{array}{cc} F & -T_1 \end{array}\right] = U_1 + Z_1U_2 \tag{16}$$

where $U_1 := L_2A_{22}\Omega_1^{\dagger} \in \mathbb{R}^{l\times(p+l)}$, $U_2 = I_{p+l} - \Omega_1\Omega_1^{\dagger} \in \mathbb{R}^{(p+l)\times(p+l)}$, and $Z_1 \in \mathbb{R}^{l\times(p+l)}$ is an arbitrary design matrix, which is used to obtain F and T_1 later. By appropriately partitioning U_1 and U_2 as $U_1 = \left[\begin{array}{cc} U_{11} & U_{12} \end{array}\right]$, and $U_2 = \left[\begin{array}{cc} U_{21} & U_{22} \end{array}\right]$ the following are obtained from (16),

$$T_1 = -U_{12} - Z_1U_{22} \tag{17}$$

$$F = U_{11} + Z_1U_{21} \tag{18}$$

To define Z_1 in (18) in a way that the matrix F becomes Hurwitz, it is necessary and sufficient that the pair (U_{11}, U_{21}) to be observable or at least detectable. It can be shown by some algebraic manipulations as described in [1, 21, 2] that this is equivalent to Condition II proposed in (6). If Condition II is satisfied, the matrix Z_1 can be computed in a way that the matrix F is Hurwitz with the most possible predefined eigenvalues. The eigenvalues are defined to achieve the required observer performance. Accordingly, T_1, H, V, and G are obtained from (17), (11), (9), and (7b), respectively.

Besides, Darouach in [1] propose another transformation based method to solve the interconnected equations (7). The transformation matrix $\bar{L} = \left[\begin{array}{cc} L^{\dagger}, & I_n - L^{\dagger}L \end{array}\right]$ is used in this methodology. Post-multiplying (7a) by $\bar{L}$ gives,

$$F = TAL^\dagger - KCL^\dagger \tag{19}$$

$$T\bar{A} = K\bar{C} \tag{20}$$

where $\bar{A} = A(I_n - L^\dagger L)$, $\bar{C} = C(I_n - L^\dagger L)$, and $K = H - FV$. Substituting T from (7c) into (20), gives

$$\begin{bmatrix} V & K \end{bmatrix} \bar{\Sigma} = L\bar{A} \tag{21}$$

where $\bar{\Sigma} = \begin{bmatrix} C\bar{A} \\ \bar{C} \end{bmatrix}$. If Condition I is satisfied, then rank $\begin{bmatrix} L\bar{A} \\ \bar{\Sigma} \end{bmatrix} = \text{rank}\bar{\Sigma}$. Consequently

$$\begin{bmatrix} V & K \end{bmatrix} = L\bar{A}\bar{\Sigma}^\dagger + Z_2(I_{2p} - \bar{\Sigma}\bar{\Sigma}^\dagger) \tag{22}$$

Further, after some algebraic manipulations it is obtained that

$$F = N_1 - Z_2 N_2 \tag{23}$$

where $N_1 = LAL^\dagger - L\bar{A}\Sigma^\dagger \begin{bmatrix} CAL^\dagger \\ CL^\dagger \end{bmatrix}$, $N_2 = (I_{2p} - \Sigma\Sigma^\dagger) \begin{bmatrix} CAL^\dagger \\ CL^\dagger \end{bmatrix}$, and $Z_2 \in \mathbb{R}^{l \times 2p}$ is the observer design matrix with similar role to Z_1 in the previous scheme. It is shown in [1] that the detectability of the pair (N_1, N_2) is equivalent to Condition II, which is the necessary and sufficient condition for the existence of a matrix Z_2 such that F in (23) becomes Hurwitz. By finding a suitable matrix Z_2 through a pole-placement approach, the matrix F and the other observer parameters are found accordingly.

Remark 2. Although the conventional approaches described above solve the same problem, they use different approaches, and have distinct numerical properties, and give different results for the same observer features. The observer equations usually have infinite number of solutions even for a specific convergence rate. Each method finds a set of these solutions. However, there are also some sources of numerical error generation, such as multiple pseudo-inverses, inverses, and null space calculations that are performed during the design procedure. In addition, the pole-placement technique that is employed in the final step is a numerical method that can induce some errors. Hence, lacking enough degrees of freedom provided by the design parameter, can result in insufficient performance of the observer. By the way, both of these approaches suffer from some numerical issues in special situations. This is illustrated via some numerical examples in Section 4. To exemplify, the first approach usually faces with numerical issues, whenever the matrix C is not in the canonical form $C = [I_p, \mathbf{0}]$ (see Examples 2 and 4 in Section 4). The second conventional method on the other hand can confront with numerical deficiencies whenever $\bar{\Sigma}$ in (21) is close to a singular configuration (see Example 3 in Section 4). This phenomena have instigated a desire to increase the numerical flexibility of the observer design parameter (matrices Z_1 and Z_2 in the first and second conventional algorithms, respectively).

3 Main Results

First, a recursive algorithm to obtain the minimal matrix L is proposed using the concept of functional observability [3]. Then, the new functional observer design approach is illustrated, and compared to the conventional schemes. Finally, a recursive algorithm for designing a functional observer using the new methodology is presented.

Algorithm 1 (a procedure for finding the minimal matrix L):

1. Testify whether condition (2) of Theorem 2.1 is satisfied. If it is satisfied, then set $i = 0$, $L_i = L_0$, and go to the next step. Otherwise, there is no functional observer for the system and the algorithm stops.

2. Check for Condition I. If it is satisfied set $L_\beta = L_0$, and go to Step 6. Otherwise go to the next step. L_β is the (modified) matrix L that satisfies Conditon I.

3. Define $H_2^i := [[\mathcal{R}([C; CA; L_i])]]$, $H_3^i := [[\mathcal{R}([H_2^i; L_iA])]]$, $a_i = rank(H_2^i)$, and $b_i = rank(H_3^i)$.

4. Calculate $\Pi_i := [[\mathcal{R}(\Theta_i)^\perp]]\Phi_i H_3^i$, where Φ_i and Θ_i are calculated as follows,
$$\begin{bmatrix} \Phi_i & \Psi_i \end{bmatrix} := \left[\left[\mathcal{N}\left(\begin{bmatrix} H_3^j A \\ H_3^j \end{bmatrix}\right)\right]\right] \tag{24}$$
and
$$\begin{bmatrix} \Theta_i & \Gamma_i \end{bmatrix} := \left[\left[\mathcal{N}\left(\begin{bmatrix} \Phi_i H_3^i \\ H_2^i \end{bmatrix}\right)\right]\right] \tag{25}$$
Next, for $j = \{1, \cdots, \text{rows}(\Pi_i)\}$ define $L_i^j = [L_i^{j-1}; q_j]$, and $L_i^1 = [L_i; q_1]$, where q_j is the $j'th$ row of Π_i. If L_i^j satisfies Condition I, then select $L_\beta = L_i^j$ and go to Step 6. Otherwise proceed to the next step.

5. Construct $\Lambda_i := \left[\left[\mathcal{R}(H_3^i) \cap \mathcal{R}\left(\begin{bmatrix} H_2^i \\ \Pi_i \end{bmatrix}\right)^{\perp}\right]\right]$. Next define $L_{i+1} = [L_i; \Pi_i; \Lambda_i]$, set $i = i+1$, and go back to Step 3.

6. Examine Condition II. If it is satisfied, choose $L = L_\beta$ and the algorithm stops. Otherwise, calculate $\begin{bmatrix} \Xi_1 & \Delta_1 \end{bmatrix} := \left[\left[\mathcal{N}\left(\begin{bmatrix} CA^2 \\ H_2^\beta \end{bmatrix}\right)\right]\right]$, in which the number of columns of Ξ_1 is equal to the number of the rows of C. Moreover, define $\Pi = \mathcal{R}(\Xi_1 CA)$. Now, for $j = \{1, \cdots, \text{rows}(\Pi)\}$, let $L_\beta^j = [L_\beta^{j-1}; q_j]$, and $L_\beta^1 = [L_\beta; q_1]$, where q_j is the $j'th$ row of the matrix Π. If L_β^j satisfies Condition II , then set $L = L_\beta^j$, and the algorithm is terminated.

As an illustration of Algorithm 1, it is worthwhile to mention that Steps 4 and 5 are given to find the minimum number of rows to append to L_0 such that Condition I is satisfied. Besides, (24) and (25) simply indicate $\Phi_i H_3^j A + \Psi_i H_3^j = \mathbf{0}$ and $\Theta_i \Phi_i H_3^i + \Gamma_i H_2^i = \mathbf{0}$, respectively. To satisfy Condition I, it is required that each appended row to L_0 increase the right-hand side of (5), while keeping its left-hand side unaltered. Hence, each selected row must be in the product space spanned by H_3^i and Φ_i, while orthogonal to Θ_i. This is exactly what Π_i defined in Step 4 demonstrates. In other words, each row selected from Π_i in Step 4 increases a_i by one, while does not alter b_i. On the other hand, each row selected from Λ_i in Step 5, increases both a_i and b_i by one [3].

Analogously, Step 6 of the algorithm is aimed at finding the minimum number of rows to be appended to L_β to satisfy Condition II, whereas Condition I is not violated. It can be shown that each of these row vectors (named q for example), should lie in the row space of CA [3], and to simultaneously satisfy Condition I, qA must be in the range space of H_2^β. This results in the last step of the algorithm.

Remark 3. The concept of functional observability works as a certificate for the existence of a functional observer for the system (1). In fact in Step 1 of the algorithm, it is assured that there exists a functional observer for the system. Then it is looked for the minimum possible number of rows to be appended to L, such that Conditions I and II are satisfied.

Remark 4. In Step 6 of the algorithm, Condition II can be checked for all variable s in the complex plane. Then asymptotic stability of the observer

(4) can be assured with arbitrary convergence rate. Hence, this step is flexible and the minimum required order of the observer can be increased for increasing the performance of the observer.

Remark 5. Algorithm 1 is independent of the method chosen to solve the interconnected equations (7). After finding a suitable matrix L that satisfies Conditions I and II, any effective method can be used to solve (7), including the conventional methods summarized in Section 2.1.

Remark 6. Although Algorithm 1 might look algebraically complicated at the first glance, it can be simply implemented in any matrix programming software like MATLAB.

3.1 New Observer Design Approach

If Conditions I and II are satisfied, the design parameters F, H, V, G and T can always be found such that equations (7) are satisfied. Unlike the conventional approaches explained in the previous section that start with equation (7a), our design procedure starts with equation (7c), which is rearranged as,

$$\begin{bmatrix} T & V \end{bmatrix} \begin{bmatrix} I_n \\ C \end{bmatrix} = L \tag{26}$$

Denote $M := [I_n; C] \in \mathbb{R}^{(n+p)\times n}$. Since $n + p > n$, and the matrix M is of full column rank, from (26) we have

$$\begin{bmatrix} T & V \end{bmatrix} = LM^{\dagger} + Z(I_{n+p} - MM^{\dagger}) \tag{27}$$

where $M^{\dagger}$ is the Moore-Penrose pseudo-inverse of M, and $Z \in \mathbb{R}^{l\times(n+p)}$ is an arbitrary matrix of appropriate dimension, which is our *first* design parameter. In fact, it plays the same role as the matrix T in what follows. In addition, let us partition the matrices $M^{\dagger}$, and $I_{n+p} - MM^{\dagger}$ as

$$\begin{bmatrix} M_1 & M_2 \end{bmatrix} := M^{\dagger}$$

and,

$$\begin{bmatrix} \mu_1 & \mu_2 \end{bmatrix} := I_{n+p} - MM^{\dagger}$$

where $M_1, M_2, \mu_1,$ and μ_2 are $n \times n$, $n \times p, (n + p) \times n$, and $(n + p) \times p$ constant matrices, respectively. It gives,

$$T = LM_1 + Z\mu_1 \tag{28}$$

$$V = LM_2 + Z\mu_2 \tag{29}$$

If the left hand side of (7a) is post-multiplied by $\bar{C}$ defined as in (8), the following are obtained,

$$H = -FTC^{\dagger} + TAC^{\dagger} \tag{30}$$

and,

$$FTC^{\perp} = TAC^{\perp} \tag{31}$$

Substituting T from (28) into (31) yields,

$$FLM_1C^{\perp} + FZ\mu_1C^{\perp} = LM_1AC^{\perp} + Z\mu_1AC^{\perp} \tag{32}$$

It can be shown by some algebraic manipulations that the matrix μ_1 is orthogonal to $C^{\perp}$, i.e. $\mu_1C^{\perp} = \mathbf{0}$. This is proved in Appendix (A). Hence, (32) is modified as,

$$FLM_1C^{\perp} = LM_1AC^{\perp} + Z\mu_1AC^{\perp} \tag{33}$$

Now, to find suitable matrices F and Z from (33) such that the Sylvester equation is satisfied and F is Hurwitz, (33) is reformulated as below,

$$\begin{bmatrix} F & -Z \end{bmatrix} \begin{bmatrix} LM_1C^{\perp} \\ \mu_1AC^{\perp} \end{bmatrix} = LM_1AC^{\perp} \tag{34}$$

Denoting $\Omega := \begin{bmatrix} LM_1C^{\perp} \\ \mu_1AC^{\perp} \end{bmatrix} \in \mathbb{R}^{(n+p+l)\times(n-p)}$ and $\Phi := LM_1AC^{\perp} \in \mathbb{R}^{l\times(n-p)}$, (34) becomes

$$\begin{bmatrix} F & -Z \end{bmatrix} \Omega = \Phi \tag{35}$$

It is well-known that (35) has a solution if and only if the following important rank condition is satisfied [20, 2].

Condition III:

$$rank\left(\begin{bmatrix} \Omega \\ \Phi \end{bmatrix}\right) = rank(\Omega) \tag{36}$$

Hence, if Condition III is satisfied, the matrices F, and Z can be obtained from,

$$\begin{bmatrix} F & -Z \end{bmatrix} = \Phi\Omega^{\dagger} + \tilde{Z}(I_{n+p+l} - \Omega\Omega^{\dagger}) \tag{37}$$

where $\tilde{Z} \in \mathbb{R}^{l\times(n+p+l)}$ is an arbitrary matrix that is to be defined in the following. Let us denote $N_1 := \Phi\Omega^{\dagger} \in \mathbb{R}^{l\times(n+p+l)}$, and $N_2 := (I_{n+p+l} - \Omega\Omega^{\dagger}) \in \mathbb{R}^{(n+p+l)\times(n+p+l)}$. Moreover, partition N_1 and N_2 as $N_1 = \begin{bmatrix} N_{11} & N_{12} \end{bmatrix}$, and $N_2 = \begin{bmatrix} N_{21} & N_{22} \end{bmatrix}$, where N_{11}, N_{12}, N_{21}, and N_{22} are of dimensions $l \times l$, $l \times (n+p)$, $(n+p+l)\times l$, and $(n+p+l)\times(n+p)$, respectively. As a result we have,

$$F = N_{11} + \tilde{Z}N_{21} \tag{38}$$

and

$$Z = -N_{21} - \tilde{Z}N_{22} \tag{39}$$

If the pair (N_{11}, N_{21}) is observable or detectable, the matrix $\tilde{Z}$ can be defined such that the observer gain F is Hurwitz. This satisfies the condition (a) that was mentioned in Section 2. Hence the second necessary condition for the existence of an asymptotic observer with structure (4) for the system is the following rank condition.

Condition IV:

$$rank\left(\begin{bmatrix} sI_l - N_{11} \\ N_{21} \end{bmatrix}\right) = l \quad \forall s \in \mathbb{C}^{+} \tag{40}$$

Apparently, if (40) is satisfied for all complex values of $s \in \mathbb{C}$, then the pair (N_{11}, N_{21}) is observable and arbitrary eigenvalues can be assigned for F. Otherwise, only the observable poles can be placed in arbitrary positions of the complex plane.

Remark 7. If the matrix L that is the augmented version of L_0, R_1, and R_2 is defined by employing Algorithm 1, then Conditions III and IV are automatically satisfied. Hence, it will be sufficient to use the conventional rank conditions (5) and (6) instead of the obtained new ones (conditions (36) and (40)). To show this, it is sufficient to display the equivalence between (36) and (5), as well as the equivalence between (40) and (6).

As a conclusion of the above descriptions, the following theorem and lemmas are proposed.

Theorem 3.1. *Conditions III and IV together are the necessary and sufficient conditions for the existence of an asymptotic functional observer with structure (4) for the system (1).*

Lemma 3.2. *The rank condition (36) is equivalent to Condition I described in (5).*

Proof. This lemma is verified in Appendix (B). □

Showing the equivalence between (40) and (6) is straightforward.

Lemma 3.3. *The rank condition (40) is equivalent to Condition II described in (6).*

Proof. The lemma is proved using contradiction. Suppose that these conditions are not equivalent. Since Conditions I and II are necessary and sufficient for the existence of an asymptotic functional observer ([1]), and Conditions I and III are equivalent (Lemma 3.2), it can be concluded that Conditions III and IV are *not* necessary and sufficient for the existence of an asymptotic functional observer. This is a contradiction according to Theorem 3.1, and the result follows. □

To summarize the aforementioned design methodology, a recursive algorithm is given as below that covers both of the functional observability and functional detectability scenarios of the triplet Σ.

Algorithm 2:

1. Set $L = L_0$
2. Compute the matrices $M, M_1, M_2, \mu_1, \mu_2$, $\bar{C}, \Omega$, and Φ from their corresponding definitions.
3. Check if Condition III is satisfied. If so, then go to the next step. Otherwise append a row vector to the matrix L as articulated in Algorithm 1 and go back to Step 2.
4. Testify Condition IV. If it holds, then continue to the next step. Otherwise append a row vector to the matrix L, selected according to Algorithm 1. Then go back to Step 2.
5. Examine the observability or the detectability of the pair (N_{11}, N_{21}). If it is observable, then go to the next step. If it is detectable and the triplet Σ is functional detectable, or if it is not essential to assign all of the poles of the dynamic observer, then go to Step 7. Otherwise, if the pair (N_{11}, N_{21}) is detectable, but the triplet Σ is functional observable, and in addition we need to assign all the observer's eigenvalues, then increase the order of the observer by one. To do this, append another row vector to the latest updated matrix L as instructed in Algorithm 1, and go back to Step 2.
6. Solve for $\tilde{Z}$ in (38) by employing a pole-placement algorithm such that the matrix F is Hurwitz with pre-defined eigenvalues that are selected by the designer.
7. Solve for $\tilde{Z}$ in (38) using a pole-placement approach such that the observable eigenvalues of N_{11} are assigned with the desired values.
8. Compute the remaining observer parameters Z, T, V, H, and G from (39), (28), (29), (30), and (7b), respectively.

Remark 8. To compare the design Algorithm 2 and the conventional ones summarised in Section (2.1), it is noted that each of these schemes use a different approach to solve the observer problem. It is clear that by using the new design method (Algorithm 2) the dimension of the observer design parameter is increased (that is $l \times (n+p+l)$ instead of $l \times (p+l)$ and $l \times 2p$ in the first and second conventional schemes, respectively). In addition, in

(35) the matrix Ω has the size of $(n + p + l) \times (n - p)$. In other words it cannot have more columns than rows. Additionally, let us call the situations resulting to numerical issues that some of them summarised in Remark 2 as *near-singular* configurations. These configurations are created partly due to the size of the free matrix parameters that should satisfy the resulted observer equations. By increasing the number of the free elements in these matrices, more accurate mathematical operations that are approximate in nature would follow, due to obtaining numerically better-behaved matrices. As a result, by employing the new methodology, such near-singular configurations are less likely to appear, because the increased size of the observer parameter (as our free parameter) helps the mathematical operations to be more reliable. This helpful point is more explained and exemplified in Section 4.

Remark 9. Recently Fernando and Trinh [5] have also proposed a methodology to design a minimal multi-functional observer for LTI systems only if the system Σ is functional observable and even Conditions I and II are not satisfied. To the best of our knowledge, this is the only published paper in the Literature that address this specific problem. The proposed approach designs a minimum possible order functional observer for the system (1) using the reduced order Luenberger observer design technique, and system decomposition based on some matrix transformations proposed in [22]. Although it solves a similar problem as ours, but the methodology of [5] involve complex algebraic and numeric process. Moreover, it does not cover the case when the system Σ is functional detectable but not functional observable. In addition, it is not clear that if it is possible to extend the developed method to unknown-input functional observer design as well as the other classes of dynamical systems such as linear-time-varying, nonlinear, and time-delay systems.

4 Numerical Examples

The following input signal was arbitrarily selected in all of the upcoming examples, which is more illustrated in Fig. 1. Since observer design is conceptually independent of the controller design, controllability and stability of the studied systems are not of our concern in the present paper.

$$u(t) = 2 + 10e^{-0.4t}cos(2t), \quad t \geq 0 \tag{41}$$

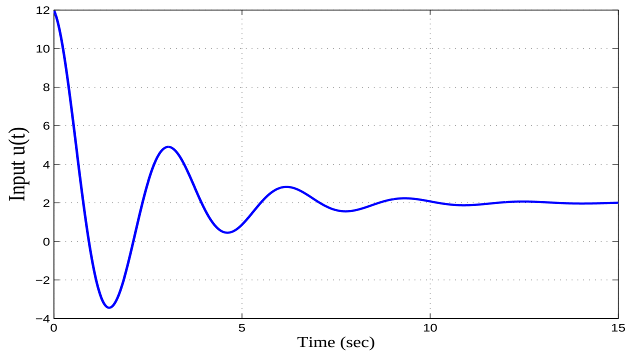

Figure 1: The arbitrary input of the system u(t)

Example 1:

In this example, the capabilities of Algorithm 1 are shown using a numerical example. Consider matrices A, C, L_0 as follows,

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & -3 & 0 & -4 & -1 & 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, & \mathbf{0}_{2\times 8} \end{bmatrix}, \quad L_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

By checking condition (3) of Theorem 2.1, it can be shown that the system $\Sigma = (A, C, L_0)$ is functional observable. However, Condition I is not satisfied for $L = L_0$, because $a = rank(H_2) = 5$ but $b = rank(H_3) = 7$. Hence, Algorithm 1 should be used to find two auxiliary rows to be appended to L_0. The result was obtained as below,

$$L_\beta = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & 1 & 0 & 0 \\ 0.0948 & -0.0948 & -0.1897 & -0.0222 & 0.0948 & 0.0948 & -0.0636 & 0.1908 & -0.0625 & 0 \\ 0.1030 & -0.1030 & -0.2061 & -0.0554 & 0.1030 & 0.1030 & 0.0637 & -0.1911 & -0.3335 & 0 \end{bmatrix}$$

Examining Condition II, it is found that L_β does *not* satisfy this condition. Hence using Step 6 of Algorithm 1, an additional row vector $q = \begin{bmatrix} 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ was found, such that $L = [L_\beta; q]$ satisfies both Conditions I and II. That is a functional observer of order five is the minimum possible order observer for this system, which is still less than the order of a reduced order Luenberger observer that is $n - p = 8$ for this case.

Example 2:

Consider the following system,

$$A = \begin{bmatrix} 1 & 0 & 0 & -6 & 0 & 1 & 0 & 2 & -2 \\ 0 & -1 & 29 & 136 & -64 & -64 & -123 & -91 & 112 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & -2 & 3 & 2 & 0 & 1 & -2 \\ -3 & 1 & 0 & 1 & 0 & -1 & 1 & 1 & 0 \\ -1 & 0 & 1 & 3 & 0 & -1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 4 & -5 & 1 & -1 \\ 1 & 1 & 1 & 0 & 1 & -2 & -3 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & -1 & 2 & 4 & -3 \end{bmatrix}$$

$L_0 = \begin{bmatrix} 1 & 4 & 1 & 21 & -16 & -16 & -9 & -15 & 20 \end{bmatrix}$, $B = [1; -3; -5; 5; -2; 0; -1; 3; 8]$, and $C = \begin{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, & \mathbf{0}_{2\times 7} \end{bmatrix}$. The triple (A, C, L_0) is functional observable that can be simply checked via Theorem 2. However, L_0 does not satisfy Condition II. Using Algorithm 1 it was obtained that appending $q = \begin{bmatrix} 2 & 0 & 0 & -12 & 0 & 2 & 0 & 4 & -4 \end{bmatrix}$, which lies in the eigen-space of CA, to L_0, resolves this issue. Hence, $L = [L_0; q]$ satisfies both Conditions I and II. Now, the three methodologies explained in the paper are used to design a functional observer for the system (A, C, L). Since Σ is *functional observable* it is possible to assign arbitrary eigenvalues to the observer dynamics. Hence, the pair $(-3, -6)$ was selected as the arbitrary eigenvalues for the observer. By performing Algorithm 2, the following observer parameters were obtained: $\tilde{Z} = \begin{bmatrix} \begin{bmatrix} 0 & -14.0833 & 28.1667 \\ 0 & -11 & 22 \end{bmatrix}, & \mathbf{0}_{2\times 8}, & \begin{bmatrix} -14.0833 & 0 \\ -11 & 0 \end{bmatrix} \end{bmatrix}$, $F =$

$\begin{bmatrix} 1 & -14 \\ 2 & -10 \end{bmatrix}$, $T = \begin{bmatrix} -28 & 1 & 1 & 21 & -16 & -16 & -9 & -15 & 20 \\ -22 & 0 & 0 & -12 & 0 & 2 & 0 & 4 & -4 \end{bmatrix}$, $V = \begin{bmatrix} 14.5 & 3 \\ 12 & 0 \end{bmatrix}$, $H = \begin{bmatrix} -118 & 8 \\ -102 & -14 \end{bmatrix}$, and $G = \begin{bmatrix} 225 \\ -102 \end{bmatrix}$.

The obtained parameters satisfy the constrained Sylvester equations (7). On the other hand, using the first conventional design approach (equations (9)-(18)), the following observer parameters were obtained: $Z_1 = \begin{bmatrix} 0 & -14 & 14 & 0 \\ 0 & -10.5 & 10.5 & 0 \end{bmatrix}$, $T = \begin{bmatrix} -14 & 1 & 1 & 21 & -16 & -16 & -9 & -15 & 20 \\ -11 & 0 & 0 & -12 & 0 & 2 & 0 & 4 & -4 \end{bmatrix}$, $F = \begin{bmatrix} 1 & -14 \\ 2 & -10 \end{bmatrix}$, $V = \begin{bmatrix} 14.5 & 3 \\ 12 & 0 \end{bmatrix}$, $H = \begin{bmatrix} -118 & 8 \\ -102 & -14 \end{bmatrix}$, and $G = \begin{bmatrix} 239 \\ -91 \end{bmatrix}$. However, surprisingly in this case equations (7a) and (7c) are *not* satisfied. That is $E_1 := TA - FT - HC = \begin{bmatrix} 154 & 0 & 0 & -84 & 0 & 14 & 0 & 28 & -28 \\ 93 & 0 & 0 & -66 & 0 & 11 & 0 & 22 & -22 \end{bmatrix}$, and $E_2 := L - PT - VC = \begin{bmatrix} -14 & \mathbf{0}_{2\times 8} \\ -11 & \end{bmatrix}$. This example shows that it is possible that Conditions I and II are satisfied, while the first conventional method does not work properly due to numerical deficiencies. On the other hand, the second conventional algorithm (equations (19)-(23)) was also tested to solve equations (7) that resulted in the following observer parameters: $Z_2 = \begin{bmatrix} 21.5 & 0 & 0 & 0 \\ 14 & 0 & 0 & 0 \end{bmatrix}$, $T = \begin{bmatrix} -42 & 1 & 1 & 21 & -16 & -16 & -9 & -15 & 20 \\ -26 & 0 & 0 & -12 & 0 & 2 & 0 & 4 & -4 \end{bmatrix}$, $V = \begin{bmatrix} 21.5 & 3 \\ 14 & 0 \end{bmatrix}$, $F = \begin{bmatrix} 1 & -21 \\ 2 & -12 \end{bmatrix}$, $H = \begin{bmatrix} -118 & 8 \\ -102 & -14 \end{bmatrix}$, and $G = \begin{bmatrix} 211 \\ -106 \end{bmatrix}$. In this case, the observer parameters satisfy equations (7), and as a result the algorithm works properly.

Simulation results conducted in the Simulink environment using the input signal (41), with initial conditions $x_0 = \mathbf{2}_{9\times 1}$ and $z_0 = \mathbf{2}_{2\times 1}$ are reported in Fig. 2. It shows that the designed observers using the new algorithm and the second conventional method are asymptotically stable, while the observer obtained from the first conventional algorithm is not asymptotically stable. Moreover, the observer obtained from our new algorithm have better performance characteristics compared to the observer obtained via the second conventional scheme.

Example 3:

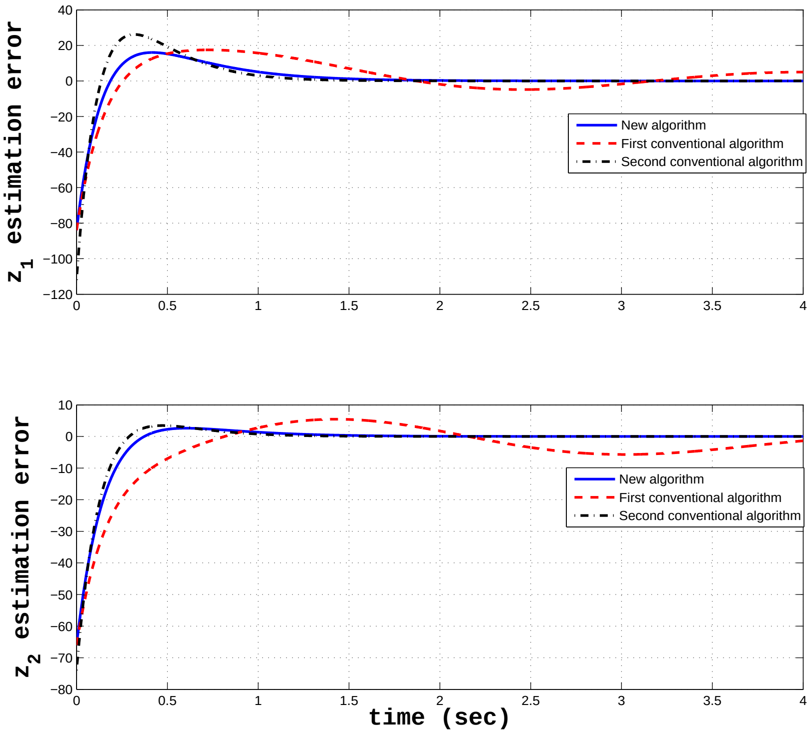

Figure 2: The estimation errors of the functions of example 2 using different design algorithms

In this example, a similar matrix A as Example 1 is considered, but the other system parameters are as follows,

$$B = [1; -3; -5; 5; -2; 3; 5; 8; 2; 1], \; C = C_1 = \begin{bmatrix} I_2 & \mathbf{0}_{2\times 8} \end{bmatrix}$$

$$L_0 = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 \\ -0.1767 & -0.0256 & -0.2836 & -0.2836 & -0.2886 & -0.2886 & -0.3662 & -0.3662 & 1 & 0 \end{bmatrix}$$

It can be readily tested that the pair (A, C) is *not observable*, but the triple (A, C, L_0) is functional observable. Nevertheless, it can be shown that Condition I is satisfied with $L = L_0$, but Condition II is not satisfied. Hence, Algorithm 1 was used to add an auxiliary row to L_0 to satisfy Condition II.

The auxiliary row was obtained as $q = \begin{bmatrix} 1 & 3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. As a result, $L = [L_0; q]$ satisfies both Conditions I and II.

Now, the results of applying the new and conventional algorithms to solve the observer problem are reported. By applying Algorithm 2, the following observer parameters were obtained for assigning the eigenvalues of the matrix F to arbitrary values $(-3, -6, -7, -8)$:

$$\tilde{Z} = 10^3 \left[\mathbf{0}_{4\times3}, \begin{bmatrix} -2.9692 & 2.9692 & 2.9692 \\ -1.2379 & 1.2379 & 1.2379 \\ -0.3192 & 0.3192 & 0.3192 \\ -0.0315 & 0.0315 & 0.0315 \end{bmatrix}, \mathbf{0}_{4\times8}, \begin{bmatrix} -2.9692 & -2.9692 \\ -1.2379 & -1.2379 \\ -0.3192 & -0.3192 \\ -0.0315 & -0.0315 \end{bmatrix} \right],$$

$$T = \begin{bmatrix} -2969 & -2969 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ -1238 & -1238 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 \\ -319.3 & -319.3 & -0.3 & -0.3 & -0.3 & -0.3 & -0.4 & -0.4 & 1 & 0 \\ -31 & -31 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, F =$$

$$\begin{bmatrix} 3 & 0 & 0 & -2970 \\ 0.1 & 1.6 & 1.9 & -1237.6 \\ -0.6 & 0 & 0.9 & -319 \\ 0.4 & -0.5 & -0.4 & -29.5 \end{bmatrix}, V = \begin{bmatrix} 2969 & 2969 \\ 1238 & 1238 \\ 319.1 & 319.3 \\ 32 & 34 \end{bmatrix}, H = 10^4 \begin{bmatrix} -8.6132 & -9.207 \\ -3.6708 & -3.9184 \\ -1.188 & -1.2519 \\ -0.0584 & -0.0646 \end{bmatrix},$$

and $G = 10^3 \begin{bmatrix} 5.954 \\ 2.488 \\ 0.6355 \\ 0.062 \end{bmatrix}$. The obtained parameters satisfy the equations (7), which guarantees the asymptotic stability of the observer.

Similarly, using the first conventional design scheme in obtaining the unknown observer parameters, yielded in the same results as the previous algorithm except for $Z_1 = 10^3 \left[\mathbf{0}_{4\times3}, \begin{bmatrix} -2.9693 & 2.9693 & 2.9693 \\ -1.2379 & 1.2379 & 1.2379 \\ -0.3192 & 0.3192 & 0.3192 \\ -0.0305 & 0.0305 & 0.0305 \end{bmatrix} \right]$. Likewise, the constrained Sylvester equations are fulfilled in this case.

However, unlike the previous algorithms, the second conventional algorithm is not successful in satisfying the Sylvester equation (7a). The following results were obtained from this methodology: $Z_2 = 10^4 \begin{bmatrix} -7.3704 & -1.2073 & -4.5181 & 3.157 \\ -1.2334 & -0.4398 & -0.6454 & 0.1751 \\ 0.1446 & 0.0405 & 0.0808 & -0.037 \\ -0.2118 & -0.0027 & -0.1447 & 0.1382 \end{bmatrix}$

$$T = \begin{bmatrix} -605.9603 & 376.4509 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -20.8024 & 160.3568 & 0 & 0 & -1 & -1 & 1 & 1 & 0 \\ 6.0902 & -14.3731 & -0.2836 & -0.2836 & -0.2886 & -0.2886 & -0.3662 & -0.3662 & 1 \\ -29.8792 & -3.9329 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$F = \begin{bmatrix} -12.3376 & 3.2237 & 83.0857 & -25.6485 \\ -2.7117 & 2.1675 & 17.2046 & 86.8105 \\ -0.3233 & -0.1119 & -0.8202 & -5.6845 \\ -0.0284 & -0.4295 & 1.8177 & -13.0097 \end{bmatrix}, V = \begin{bmatrix} 605.9603 & -376.4509 \\ 20.8024 & -160.3568 \\ -6.2669 & 14.3475 \\ 30.8792 & 6.9329 \end{bmatrix},$$

$H = \begin{bmatrix} -8720.7 & 5144.8 \\ 974.7 & 1520.7 \\ -368.8 & 87.5 \\ -440.8 & 10.9 \end{bmatrix}$, and $G = \begin{bmatrix} -1719.3 \\ -489.9 \\ 46.2 \\ -18.1 \end{bmatrix}$. Substituting the above parameters in (7a) yields:

$$E_1 = \begin{bmatrix} -566.6306 & 223.0139 & -542.4112 & 440 & 42.5398 & 42.5398 & 42.5398 & 42.5398 & -67.7481 & 0 \\ -104.4881 & 41.1243 & -100.022 & 81.1371 & 7.8445 & 7.8445 & 7.8445 & 7.8445 & -12.4929 & 0 \\ 11.8028 & -4.6453 & 11.2983 & -9.1651 & -0.8861 & -0.8861 & -0.8861 & -0.8861 & 1.4112 & 0 \\ -14.9652 & 5.89 & -14.3255 & 11.6208 & 1.1235 & 1.1235 & 1.1235 & 1.1235 & -1.7893 & 0 \end{bmatrix}$$

This example shows that in some cases the second conventional approach does not work successfully, even when Conditions I and II are satisfied. To magnify these findings, simulation results obtained from the Simulink environment are illustrated in Figs. 3 and 4. As it is shown in Fig. 3 the performance of our new methodology and the first conventional method are similar, which was expected due to their similarly obtained observer parameters. However, the observer working based on the parameters obtained from the second conventional algorithm is not convergent that is clear from Fig. 4 that reports 0.1 seconds of the simulation time.

Example 4:

In this example, the following system matrices are considered,

$$A = \begin{bmatrix} -5 & -2 & 5 & 0 & 1 \\ 2 & -6 & 1 & 0 & -3 \\ 0 & -2 & -8 & 0 & 0 \\ -6 & 5 & 7 & -5 & -5 \\ -2 & 0 & -4 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -3 \\ -5 \\ 5 \\ -2 \end{bmatrix}, C = \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, & \mathbf{0}_{2\times 3} \end{bmatrix}, L_0 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Here the system Σ is *not* functional observable, but it is functional detectable. This can be shown via condition (2) of Theorem 2.1. Considering $L = L_0$, Condition I is satisfied, but Condition II is practically not satisfied. This is because the system is *nearly undetectable*. To resolve this issue,

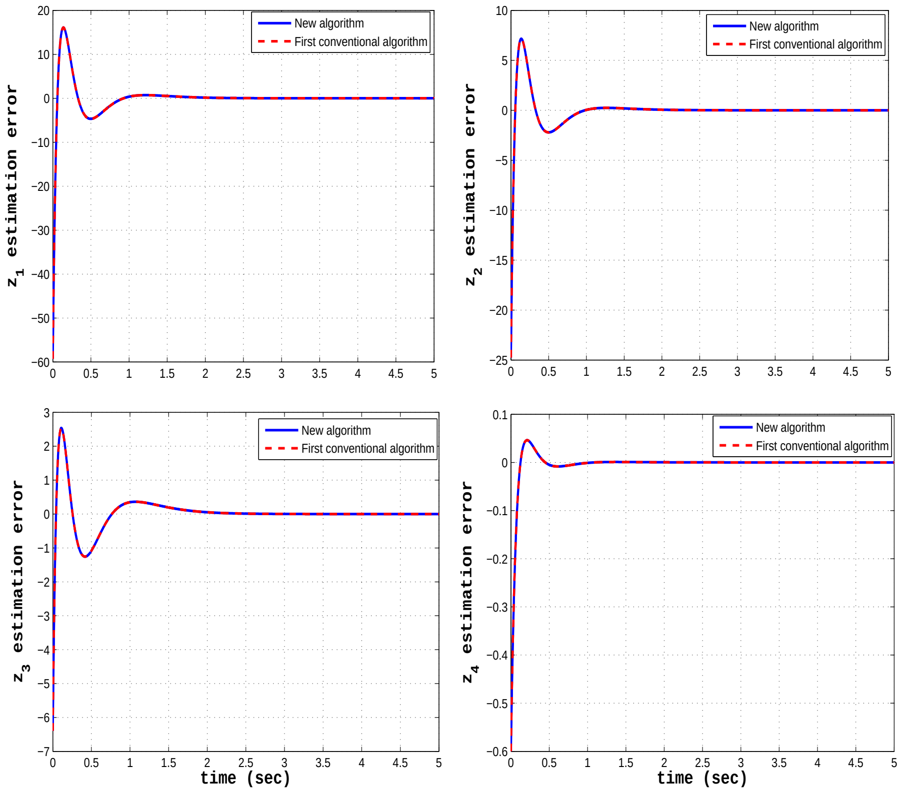

Figure 3: The estimation errors of the functions of example 3 obtained from using different design algorithms

Algorithm 1 was used to increase the order of the observer. As a result, a row vector $q = \begin{bmatrix} -3 & -8 & 6 & 0 & -2 \end{bmatrix}$ was obtained as an auxiliary row to be appended to L_0. Hence, $L = [L_0; q]$ satisfies both Conditions I and II, or analogously Conditions III and IV. Calculating the invariant zeros of the matrix pencil $S = \begin{bmatrix} sI_n - A & \mathbf{0}_{n\times d} \\ \begin{bmatrix} C \\ L \end{bmatrix} & \mathbf{0}_{(p+l)\times d} \end{bmatrix}$, with d as an arbitrary integer, it is obtained that the invariant zero of the Rosenbrock's matrix of the system is equal to "-5". Hence, $\lambda = -5$ is the undetectable eigenvalue of the system and should be included in the desired observer's eigenvalues list.

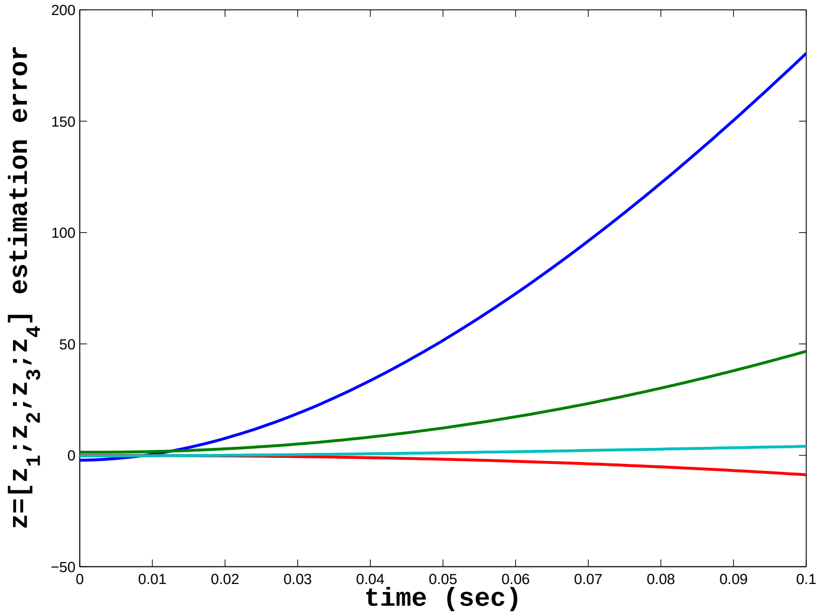

Figure 4: The estimation errors of the functions of example 3 obtained from using the second conventional method

Applying Algorithm 2 for obtaining the observer parameters to assign the observer's pole locations to $\{-10.2648, -5, -9\}$, where $\{-10.2648, -9\}$ were selected arbitrarily, the following parameters were obtained:

$$\tilde{Z} = \left[\begin{bmatrix} -8.6776 & 0 & 0.8397 & 0.7873 & -0.2974 \\ 0.06 & 0 & -1.0066 & 0.9953 & 1.0028 \\ 1.2371 & 0 & -5.2751 & 5.0432 & 5.1978 \end{bmatrix} \ \mathbf{0}_3 \ \begin{bmatrix} 1.0847 & -0.7873 \\ -0.0075 & -0.9953 \\ -0.1546 & -5.0432 \end{bmatrix} \right],$$

$T = \begin{bmatrix} -0.1875 & -0.0625 & 1 & 0 & 0 \\ -1 & -2 & 0 & 1 & 0 \\ -2.7648 & -7.7648 & 6 & 0 & -2 \end{bmatrix}$, $F = \begin{bmatrix} -9 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -10.2648 \end{bmatrix}$, $V = \begin{bmatrix} -0.125 & 0.1875 \\ 1 & 1 \\ 0 & -0.2352 \end{bmatrix}$, $H = \begin{bmatrix} -0.9375 & -0.875 \\ 19 & -10 \\ -13.5 & -26.0857 \end{bmatrix}$, and $G = \begin{bmatrix} -5 \\ 10 \\ -5.4704 \end{bmatrix}$.

Apparently, these parameters satisfy the equations (7). Likewise, using the second conventional algorithm resulted in the same observer parameters satisfying (7), except for the matrix Z_2 that was obtained as $Z_2 =$

$$\begin{bmatrix} -0.148 & 0.1875 & -0.074 & 0.9619 \\ 1.046 & 1 & 0.523 & -6.7989 \\ 4.1034 & -0.2352 & 2.0517 & -26.6724 \end{bmatrix}$$. On the contrary, applying the first conventional scheme yields in the observer parameters that does not satisfy conditions (7a) and (7c):

$$Z_1 = \begin{bmatrix} -8.7429 & 0 & 0.7259 & -1.0929 & 0.9134 \\ 0.0302 & 0 & -0.5028 & 0.0038 & 0.4972 \\ 1.2854 & 0 & -6.6353 & 0.1607 & 6.3943 \end{bmatrix}, T = \begin{bmatrix} 0.125 & -0.1875 & 1 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 \\ -5 & -2.7648 & 6 & 0 & -2 \end{bmatrix},$$

$G = \begin{bmatrix} -4.3125 \\ 7 \\ -22.7056 \end{bmatrix}$, and the other parameters were obtained as the previous algorithms. Substituting the obtained parameters in (7a) and (7c) gives $E_1 =$

$$\begin{bmatrix} 1 & -1 & 1.4375 & 0 & 0.6875 \\ 2 & -1 & 1 & 0 & -3 \\ -1.7679 & 25.7944 & -6.176 & 0 & -17.2352 \end{bmatrix}, \text{ and } E_2 = \begin{bmatrix} \begin{bmatrix} -0.3125 & 0.125 \\ 0 & -1 \\ 2.2352 & -5 \end{bmatrix} & \mathbf{0}_3 \end{bmatrix},$$

respectively.

Simulation results are illustrated in Fig. 5. As this figure shows, the observers obtained from the new algorithm and the second conventional scheme have the same tracking performance, which was expected due to their similar observer parameters. However, the observer designed via the first conventional algorithm is not convergent.

5 Conclusions

The problem of designing minimal multi-functional observers for LTI systems has been addressed. A new design algorithm has been proposed that allows finding a new and more reliable way to solve the observer equations. Our observer scheme has only assumed the functional observability/detectability of the system, which is necessary and sufficient for the existence of an asymptotic functional observer for the system. In the new observer design scheme, more numerical degrees of freedom are obtained for the design parameter that results in better performance and simplicity with higher reliability with regard to the other available approaches. The necessary and sufficient conditions have been obtained for the existence of an asymptotic observer when using the new methodology, and the equivalence of these conditions to renown Conditions I and II has been verified. An appropriate recursive algorithm for designing a functional observer has been given to summarize the design

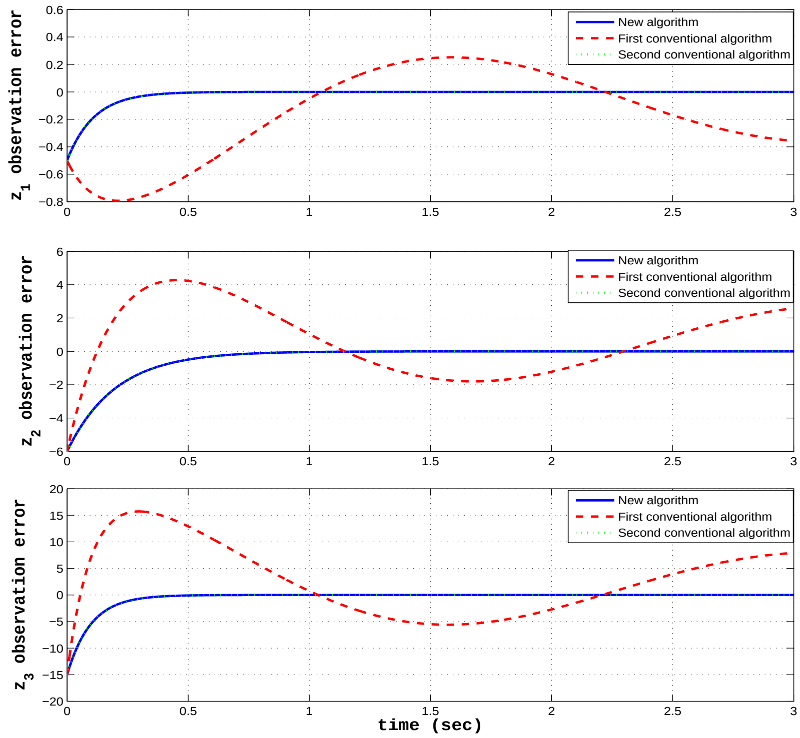

Figure 5: The estimation errors of the functions of example 4 obtained from using different design algorithms

procedure. Several numerical examples and simulation results clarified the usefulness and superiority of the proposed design method, as well as some drawbacks of the conventional observer design schemes.

Future work will benchmark all of the available methodologies for solving the functional observer equations by applying them on a large-scale electromechanical system. Finding the exact numerical conditions under which each methodology fails, needs an extensive research that might result in better design techniques. In addition, the extension of the proposed methodology to unknown-input functional observers is another ongoing study.

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A Proof of the statement: $\mu_1 C^{\perp} = \mathbf{0}$

To show this, we show that μ_1 lies in the range space of the matrix C. Recalling the computation of the matrix μ_1 we have,

$$\begin{bmatrix} \mu_1 & \mu_2 \end{bmatrix} = \left(I_{n+p} - \begin{bmatrix} I_n \\ C \end{bmatrix} \begin{bmatrix} I_n \\ C \end{bmatrix}^{\dagger} \right) \tag{42}$$

There are a variety of approaches in computing of the generalized inverse of a matrix ([23, 20]). One well-known method in calculating the pseudo-inverse of a matrix $X \in \mathbb{R}^{k \times s},\ k > s,\ k, s \in \mathbb{N}$, is the following relation,

$$X^{\dagger} = \left(X^T X \right)^{-1} X^T \tag{43}$$

Using (43), we have $X^{\dagger} X = I_l$ if and only if X is of full column rank. However, noting that $k > s$, the multiplication $X X^{\dagger}$ is not the identity matrix. Taking (43) into account, (42) is reformulated as,

$$\begin{bmatrix} \mu_1, & \mu_2 \end{bmatrix} = I_{n+p} - \begin{bmatrix} I_n \\ C \end{bmatrix} \left(I_n + C^T C \right)^{-1} \begin{bmatrix} I_n & C^T \end{bmatrix} \tag{44}$$

that is equivalent to,

$$\begin{bmatrix} \mu_1, & \mu_2 \end{bmatrix} = I_{n+p} - \begin{bmatrix} \left(I_n + C^T C \right)^{-1} & \left(I_n + C^T C \right)^{-1} C^T \\ C \left(I_n + C^T C \right)^{-1} & C \left(I_n + C^T C \right)^{-1} C^T \end{bmatrix} \tag{45}$$

Hence, the matrix μ_1 is given as below,

$$\mu_1 = \begin{bmatrix} I_n - \left(I_n + C^T C \right)^{-1} \\ -C \left(I_n + C^T C \right)^{-1} \end{bmatrix} \tag{46}$$

Now, noting that

$$\left(I_n + C^T C \right)^{-1} = I_n - \left(I_n + C^T C \right)^{-1} C^T C \tag{47}$$

equation (46) can be written as

$$\mu_1 = \begin{bmatrix} \left(I_n + C^T C \right)^{-1} C^T C \\ C - C \left(I_n + C^T C \right)^{-1} C^T C \end{bmatrix} \tag{48}$$

Accordingly, it is clear that since $C C^{\perp} = \mathbf{0}$, the multiplicative term $\mu_1 C^{\perp}$ is zero.

B Proof of Lemma 3.2

To prove Lemma 3.2, first it is noticed that

$$rank\left(\begin{bmatrix} \Omega \\ \Phi \end{bmatrix}\right) = rank\left(\begin{bmatrix} LM_1C^{\perp} \\ \mu_1 AC^{\perp} \\ LM_1AC^{\perp} \end{bmatrix}\right) \tag{49}$$

According to its definition and (47), M_1 can be written as,

$$\begin{aligned} M_1 &= (I_n + C^T C)^{-1} \\ &= I_n - (I_n + C^T C)^{-1} C^T C \end{aligned} \tag{50}$$

Accordingly,

$$\begin{aligned} LM_1C^{\perp} &= L(I_n - (I_n + C^T C)^{-1} C^T C)C^{\perp} \\ &= LC^{\perp} \end{aligned} \tag{51}$$

In addition, since the matrix $(I_n + C^T C)^{-1}$ is full rank, we have

$$\begin{aligned} rank\left[LM_1AC^{\perp}\right] &= rank\left[L(I_n + C^T C)^{-1} AC^{\perp}\right] \\ &= rank\left[LAC^{\perp}\right] \end{aligned} \tag{52}$$

Moreover, according to the definition of μ_1 in (46) and the useful relation (47), the following can be obtained,

$$\begin{aligned} rank\left[\mu_1 AC^{\perp}\right] &= rank\begin{bmatrix} C^T CAC^{\perp} \\ -CAC^{\perp} \end{bmatrix} \\ &= rank\left[CAC^{\perp}\right] \end{aligned} \tag{53}$$

As a result, the rank condition (49) can be written as

$$rank\left(\begin{bmatrix} \Omega \\ \Phi \end{bmatrix}\right) = rank\left(\begin{bmatrix} LC^{\perp} \\ CAC^{\perp} \\ LAC^{\perp} \end{bmatrix}\right) \tag{54}$$

In a similar way, the right hand side of (36) is equivalent to

$$rank\left[\Omega\right] = rank\left(\begin{bmatrix} LC^{\perp} \\ CAC^{\perp} \end{bmatrix}\right) \tag{55}$$

On the other side, the left hand side of (5) is manipulated as

$$
\begin{aligned}
rank\left(\begin{bmatrix} L \\ C \\ CA \\ LA \end{bmatrix}\right) &= rank\left(\begin{bmatrix} LA \\ CA \\ C \\ L \end{bmatrix}\begin{bmatrix} C^{\dagger} & C^{\perp} \end{bmatrix}\right) \\
&= rank\left(\begin{bmatrix} LAC^{\dagger} & LAC^{\perp} \\ CAC^{\dagger} & CAC^{\perp} \\ I_p & \mathbf{0}_p \\ LC^{\dagger} & LC^{\perp} \end{bmatrix}\right) \\
&= p + rank\left(\begin{bmatrix} LC^{\perp} \\ CAC^{\perp} \\ LAC^{\perp} \end{bmatrix}\right)
\end{aligned} \tag{56}
$$

Likewise, the right hand side of (5) is obtained as

$$
rank\left(\begin{bmatrix} L \\ C \\ CA \end{bmatrix}\right) = p + rank\left(\begin{bmatrix} LC^{\perp} \\ CAC^{\perp} \end{bmatrix}\right) \tag{57}
$$

Hence, from (54), (55), (56), and (57), it is clear that the condition (36) can always be concluded from (5), and the proof is complete.