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Introducing linear functions: an alternative statistical approach

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\textbf{Abstract} The introduction of linear functions is the turning point where many students decide if mathematics is useful or not. This means the role of parameters and variables in linear functions could be considered to be ‘threshold concepts’. There is recognition that linear functions can be taught in context through the exploration of linear modelling examples, but this has its limitations. Currently, statistical data is easily attainable, and graphics or computer algebra system (CAS) calculators are common in many classrooms. The use of this technology provides ease of access to different representations of linear functions as well as the ability to fit a least-squares line for real-life data. This means these calculators could support a possible alternative approach to the introduction of linear functions. This study compares the results of an end-of-topic test for two classes of Australian middle secondary students at a regional school to determine if such an alternative approach is feasible. In this study, test questions were grouped by concept and subjected to concept by concept analysis of the means of test results of the two classes. This analysis revealed that the students following the alternative approach demonstrated greater competence with non-standard questions.

\textbf{Keywords} Linear functions \textendash{} Secondary school \textendash{} Statistical data \textendash{} Multiple representations \textendash{} Technology

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Introduction

Researchers have described five desirable strands of mathematical action for students: 
conceptual understanding, procedural fluency, strategic competence, adaptive reasoning 
and productive disposition (Kilpatrick et al. 2001). Procedural fluency is the ability 
to carry out calculations efficiently and have factual knowledge; strategic competence 
is the ability to devise, represent and solve mathematical problems; adaptive reasoning 
involves the capacity for logical thought; and productive disposition is present when 
students recognise that mathematics is something worthwhile that they could sensibly 
use (Kilpatrick et al. 2001). One field of mathematics where the five strands of 
mathematical action have great significance is the area of relations and functions, 
usually beginning with linear functions in the middle years of secondary school.

In the Australian Curriculum (Australian and Assessment and Reporting Authority 
2012), the scope and sequence for linear and non-linear relationships outline its 
development across the year levels. In year 7, students are introduced to the 
Cartesian plane and coordinate notation. They also investigate ‘interpret and analyse 
graphs for authentic data’. In years 8 and 9, students are expected to graph linear 
relationships on the Cartesian plane and, in year 9, find the gradient of a line segment 
(interval) on the Cartesian plane as well as sketch linear graphs using the coordinates of 
two points. For data representation and interpretation, data sets are compared in year 9, 
and scatterplots and the relationships between two numerical variables are considered 
in year 10. For the real numbers strand, year 9 students are expected to be able to solve 
problems involving direct proportion and explore the relationship between graphs and 
equations corresponding to simple rate problems. In year 9 science, students are 
expected to be able to ‘construct and use a range of representations, including graphs, 
keys and models to represent and analyse patterns or relationships, including using 
digital technologies as appropriate (ACSS129). [They] analyse patterns and trends in 
data, including describing relationships between variables and identifying inconsistencies’ (ACSS169) (Australian and Assessment and Reporting Authority 2012). So, a 
treatment in a mathematics class of linear functions which emphasises the connection 
to real-life data may facilitate and support transfer of the concepts related to linear 
functions into science and other subject areas.

The learning of linear functions¹ has been problematic in the past (Leinhardt et al. 
1990), so several approaches have been trialled. For example, linear functions can be 
taught in context through the exploration of linear modelling examples, but Bardini and 
Stacey (2006) noted that students continued to struggle with linear functions despite 
using this approach. In an era where statistical data is easily attainable and there is a 
need for students to be statistically literate (Craig 2009), the use of scatterplots and 
equations for least-squares lines of real-life data has the potential to enable students to 
develop concepts related to linear functions in a useful context. Gattuso (2006) asserted 
‘statistics is an ideal field to provide a meaningful context for the learning of many 
mathematical concepts’ and suggested that scatterplots could be used to explore linear 
relations, where the line of best fit can be drawn and the symbolic representation can be 
found from ‘two points or the y-intercept and the slope’ (p. 4).

¹ More correctly termed affine functions.
This study is an exploration of the efficacy of introducing linear functions through the use of scatterplots and least-squares lines. Specifically, the research questions are as follows:

- What is the effect on student outcomes for the topic of linear functions by adopting an alternative approach based on linear models of statistical data compared with the traditional textbook approach?
- Are there particular concepts within this topic where this alternative approach is more or less successful than the traditional textbook approach in improving student outcomes?
- Does this alternative approach hinder development of fluency with procedures usually associated with this topic, such as calculating gradient or graphing linear functions?

To assist in answering these questions, an alternative unit for introducing linear functions was produced using four linear models, with data sourced from the Australian Bureau of Statistics (ABS), Census in Schools (Australian Bureau of Statistics 2013). This paper reports on the results of an end-of-topic assessment of two year 9 mathematics classes where one class was taught using a textbook approach (Greenwood et al. 2011), and the other class was introduced to linear functions using linear models of statistical data. At the end of the topic, both classes were assessed using the same assessment tool and questions were analysed to ascertain the differences and similarities in students’ understandings of linear functions concepts.

Background

Hiebert and Carpenter (1992) stressed the importance of the teacher’s role in facilitating the attachment of new knowledge to existing cognitive structures. They extended Piaget’s notion of schema by emphasising the connections between schemata and claimed that the degree of understanding is related to the strength and number of the connections between internal representations and that new concepts are best internalised when they can be attached to existing cognitive structures. This forms the foundation on which new learning is built. New learning, disconnected from an existing schema, may be soon forgotten, so it is important that the design of instructional material takes into account students’ current understanding and existing schemata. Explicit connections help students to link current learning to established networks of schemata (Hiebert and Carpenter 1992). Consistent with Hiebert and Carpenter’s (1992) advice, Nathan and Kim’s (2007) study of younger students in the early stages of learning algebra showed that the greatest gains in students’ understanding were achieved when students were supported by technology to make links from their invented strategies and representations to more formal representations and methods.

Research into students’ learning has seen the introduction of the term ‘threshold concepts’ (Meyer and Land 2005), which are potentially troublesome concepts which are difficult to understand and teach and block the way to more advanced study. However, once acquired, they are concepts that are difficult to forget, and they enable the student to recognise the interrelatedness of otherwise abstract concepts (Mitchelmore and White 2004), such as statistics and algebra. An understanding of
functions as both a process and an object is described as a threshold concept in Pettersson and Scheja’s (2012) study involving prospective mathematics teachers and so may also be the case for the year 9 students in this study who are usually introduced to functions through consideration of linear functions (Australian and Assessment and Reporting Authority 2012).

A student demonstrates understanding when they have the ability to do a range of things within a topic such as generalising, applying, and, in the case of linear functions, working strategically with the multiple representations of linear functions: numeric, graphic and symbolic (Perkins and Blythe 1994). In recent years, the introduction of technology into the mathematics classroom has facilitated students’ access to these multiple representations of linear functions (Roschelle et al. 2003). Kendal and Stacey (2003) recommended that it was better to initially focus on these representations individually rather than introduce them all at once and introduce them over time. This suggests that there is value in exploring in more depth how students’ understanding of these multiple representations can be developed in the year 9 classroom through connecting numeric statistical data with graphical representations and finally symbolic representations.

A modelling approach supported by technology, with its ready access to multiple representations, has the potential to assist students to make connections between the concepts related to linear functions. Galbraith et al. (2005) reported on several approaches to teaching linear functions supported by technology. They found ‘[t] he use of a real-world interface to bring a virtual world into the classroom for analysis impacted on the teachers’ conceptualisation of the task of teaching linear functions’ (p. 9). Later, Stillman et al. (2007) used a linear modelling task ‘Barbie Bungee’ to formulate a framework for tracing transitions in modelling tasks. Bardini et al. (2004) analysed pre- and posttest results of a group of mathematically able year 8 students who had used a modelling approach with extensive use of graphics calculators. They found that this approach did indeed appear to have significant impact on the students’ understanding of linear functions. However, Bardini and Stacey’s (2006) study into students’ understanding of the parameters of $m$ and $c$ in $y=mx+c$, the symbolic representation of a linear function, used a common real-life example of hiring a tradesman. They noticed confusion about considering cost as a function of time and recommended using other contexts.

Scatterplots have been used to demonstrate linear relationships. O’Keefe (1997) claimed that a scatterplot of arm span against height could assist students to make connections between algebra and statistics. This example is particularly relevant to this study as the scatterplot shown in his article displays a linear correlation. O’Keefe asserted that ‘a primary purpose of the scatterplot is to enable us to look for general trends in data’ (p. 20) and suggested that a scatterplot could also be used to determine whether a connection, or correlation, exists between real-life variables. Similarly, Asp et al. (1995) employed the same example as O’Keefe in their book of classroom exercises exploring functions. Geiger et al. (2010) reported on a study where students were expected to create a scatterplot and an equation for a least-squares line from some real-life data related to carbon dioxide production in the Darling River. They found that unexpected outputs from the technology ‘provoked the rethinking of their original assumptions … and led to an adjustment to their approach to solving these problems’ (p. 65). This is consistent with Goos et al.’s (2013) observation that some students were
surprised when the scatterplot of height against walking speed was not linear. They asserted that the ‘use of personal data encouraged positive dispositions towards involvement in and completion of the task’ (p. 595).

Technology enables a student to move easily from using numerical techniques for solving a problem to a statistical analysis and/or graphical techniques and/or algebraic solution (Pierce 2005). Pierce also suggested that there are three important features which impacted on students: initially modelling contextual problems, following a functional approach and using a graphics calculator. The effects of this approach were explored through the analysis of students’ responses to pre- and posttest items, and although the use of graphical calculators enabled students to solve modelling problems, there was limited influence on the test results (Pierce 2005). In Victoria, computer algebra system (CAS) technology has been used in senior assessments for more than a decade (Garner and Leigh-Lancaster 2003). The use of this technology may afford younger students the opportunity to develop a better understanding of symbolic representations when they first meet functions enabling even more flexibility when they study calculus in later years (Herbert and Pierce 2011).

It appears that a modelling approach supported by technology is worth further investigation with a more detailed consideration of what could be considered to be genuine, relevant real-life context. Bivariate data sources, such as the Australian Bureau of Statistics, could be considered as options for authentic modelling examples as there is evidence that using real-life data can engage students’ interest in statistics (Chick and Pierce 2012). Hence, it was decided to trial a different approach to the introduction of linear functions, starting with examples of least-squares lines for real-life, Census at Schools (Australian Bureau of Statistics 2013) bivariate data. While it is expected that real-life data can engage students in this area of mathematics, it needs to be ascertained whether this approach results in a better understanding of the concept of variables (Fly and Adams 2012; Bardi et al. 2005) and use of parameters for the gradient and intercepts in a way that can be translated across to the more abstract ideas of functions and graphs on the Cartesian plane.

One way to do this is to compare student outcomes using this alternative approach with the more usual textbook approach. In Australian schools, textbooks are typically used to guide instruction (Vincent and Stacey 2008). Vincent and Stacey reported that Australian schools placed a great emphasis on the use of textbooks in classroom practice, and they found that in nine different textbooks, there was too much emphasis on repetitive questions of low-level complexity. They also discovered that a high proportion of problems were of low procedural complexity, with considerable repetition, and an absence of deductive and inductive reasoning. This means that students might develop a disjointed understanding of concepts (Stacey 2010), which does not augur well for students’ mathematical understanding, as they learn to expect that there must be a rule for every possible situation, and thus, classroom mathematics is seen as just a series of meaningless practices that are not necessarily meant to make sense (Goos et al. 1997).

The study reported in this paper seeks to explore the effectiveness of an alternative approach based on statistical data with less reliance on the textbook (Greenwood et al. 2011) at the school where this study takes place, in common with many Victorian schools. Consistent with Vincent and Stacey’s findings, in the chapter ‘Linear Relations’ of this textbook, all sections generally consist of many single or multiple
computation procedures usually with numerical or numeric expression answers. Some questions relate to the contextual features within linear graphs including intercepts, points, gradients, etc. The cognitive requirements of the text were largely procedural practice, with some conceptual understanding in the latter sections, but very little authentic problem solving and no reference to the use of technology.

This discussion of the literature has considered some approaches to alleviating the difficulties experienced by students studying linear functions in mathematics. The repetitive low-level complexity of problems in current textbooks does little to ease these difficulties and students tend to develop a disjointed, instrumental understanding (Stacey 2010). Students develop the ability to generalise, evaluate the adequacy and connect ideas when they are working with other students and faced with problems that they find interesting and could be considered to be real-life situations (Pierce et al. 2004). In the area of linear functions, however, there are still cognitive difficulties experienced by students when variables are used to represent actual or potential numbers and with the concept of generality in real-life modelling problems. The use of bivariate data examples offers problem-solving options where they can initially learn about the relationship between paired coordinate values and then develop generality through the process of fitting least-squares lines and may be similar to the modelling that students experience in other curriculum areas such as science.

Methods

This study investigated the effectiveness of an alternative approach to introducing linear functions by using linear models of real-life data. This study employs an exploratory data analysis approach (Tukey 1977) in order to investigate the data to reveal aspects of the approach which appear to warrant further examination. The statistical procedures are used to provide some advice regarding this potentially new approach, rather than being regarded as strong empirical confirmation. Examination of the data in this way could lead to the formation of hypotheses that could guide further large-scale research.

Participants

The participants in this study were a group of year 9 students from a Victorian rural independent school. There are seven mathematics classes in this year level: one class has a small group of students who are considered to be advanced in mathematics, another small class of weaker students and then five classes are considered to be mixed-ability classes who follow the same core curriculum and common assessment. The school aims to keep the composition of the five mixed-ability classes roughly equivalent with respect to ability as verified by numerous assessments on previous topics in mathematics.

For this topic, two of the five mixed-ability classes followed traditional approach and three followed the alternative approach, depending on the teacher’s preference. The participants for this research were two classes of 22 students and their teachers, taken from this middle group of mixed-ability classes. These classes were selected because their teachers were willing to be involved in the research, and they ensured that the
researcher had ethics permission to be involved in the research from all the students and their parents. One class followed the traditional textbook approach and the other followed the alternative approach. A pretest was not conducted because the classes were roughly equivalent in the range of ability of the students due to the school’s policy of streaming, and it was thought that the use of a pretest would compromise the investigation outcomes where the introduction of the topic was the key difference between the classes.

The two teachers taught the class they usually have for mathematics, which meant they had an established working relationship with the students in their classes, thus minimising the perhaps negative effect on classroom behaviour of an unfamiliar teacher. Both were experienced teachers with a similar teaching style and had a similar history of teaching year 9 mathematics at this school. Both classes were taught the linear functions topic in the final few weeks of the second semester after all the students had completed the following topics: Number and Financial Arithmetic, Pythagoras Theorem and the Trigonometric Ratios, Linear Equations and Formulas and Statistics and Handling Data. The topic Linear Equations and Formulas was taught following the textbook approach: algebraic expressions, simplifying algebraic expressions, expanding algebraic expressions, substituting values into formulae and solving linear equations. This approach emphasised symbolic manipulation of linear equations with no reference to the solution having any connection to the graphical representation. As part of the statistics topic, all the students in both classes were involved in the Australian Bureau Statistics’ Census at Schools’ (Australian Bureau of Statistics 2013) initiative and had participated in the 2013 survey. They had already used the random sampler examples and analysed this data using the statistics application of the CAS.

The school has a history of introducing the CAS calculator from the start of year 9, so all students in this study would have had some experience of this technology prior to beginning the topic of linear functions, but all were familiar with the use of the ‘Main’ section and the ‘Statistics’ applications of this technology used in the statistics topic. The teacher of the alternative approach used the technology to facilitate the handling of the messy ABS real-life data. The teacher of the traditional approach emphasised by-hand procedures with minimal employment of technology and took care to remain consistent with the approach as set out in the textbook. The teachers’ adherence to their respective approaches was confirmed by the researcher in discussions throughout the duration of the topic.

Traditional approach

In the traditional approach, the students followed the textbook approach and were taught the topic in the order and style suggested by the textbook. In the first lesson, the students revised the Cartesian plane, producing tables of values and using them to plot linear graphs. In the second lesson, the students were asked to sketch linear graphs by finding the intercepts. In the third lesson, the students looked at horizontal and vertical lines. Next, the students were taught about the gradient of a linear graph, how to find it from the graph or from two given points and direct proportion. The students then revisited the gradient-intercept form for a linear relationship, which they interpreted and then used to find the relationship for linear graphs. Finally, they applied these skills to
linear modelling problems. The class then diverted from the textbook to complete an exercise which looked at scatterplots and lines of best fit, so this could be included in assessments. It was only at this point that the students in this class used the graphing and statistical functions on their calculators.

**Alternative approach**

The alternative approach consisted of four introductory lessons utilising the Census at School data, followed by some sections from the textbook. The four data sets used were selected to give bivariate data with a strong correlation that would be appropriate for fitting linear least-squares lines. For example, data set 1: The dominant hand and non-dominant hand reaction times of 16 students (Fig. 1). The coefficient of determination was 94.9%. The equation of the least-squares line was \( \text{non-dominant times} = 0.141 + 0.632(\text{dominant times}) \).

In the alternative approach, the main objective of the first lesson was to familiarise the students with the idea of a scatterplot as a visual way of seeing the relationship between paired data. The students were expected to be able to draw up a set of axes for the range of data values, plot paired data values to get a scatterplot, answer questions about the scatterplot and be able to describe informally the scatterplot in terms of strength (strong, moderate or weak correlation) and direction (positive or negative correlation). In the second lesson, the students were asked to add a line of best fit to a scatterplot drawn by eye, to answer questions using the line of best fit and to interpret the meaning of the \( y \)-intercept and gradient for their lines. In the third lesson, the students used the statistics application of the CAS to find the equation of the least-squares line, rewrite the equation using the variables appropriate to the context and add the least-squares line to their scatterplot. Finally, in the fourth lesson, the students were asked to interpret the meaning of the gradient and intercept in the equation in the context of the statistical data. This introduction of four lessons was followed by six lessons using the textbook exercises, beginning with the exercise on gradients. The students then returned to the gradient-intercept form for a linear relationship and revisited the meaning of the gradient and \( y \)-intercept referring to the ABS data they

![Fig. 1 Scatterplot for data set 1](image-url)
had previously experienced in the introduction. They also used gradient and $y$-intercept to find the relationship for linear graphs and compared this with the equations for least-squares lines generated from the ABS data. Next, they were taught how to sketch a linear graph by finding the intercepts. Finally, they worked through the exercises on direct proportion and linear modelling. Technology was used when appropriate, and frequent reminders about the work with the ABS data assisted students to make connections between the concepts and develop an awareness of the usefulness of the mathematics to real-life problems. Table 1 shows a comparison of the sequence of lessons taken by the two classes with exercises referring to those in G.

**Assessment**

At this school, there is a tradition of pen and paper technology-free testing, and the same test was administered to all students in the five year 9 mixed-ability classes, including the two classes involved in this research. The assessment was similar to the type of assessment used in the past at this school that included both short answer skill questions and application style questions. The questions were taken from the teacher resource for the textbook with an extra question on scatterplots. Each question could also be directly related back to a course descriptor of the Australian Curriculum (Australian and Assessment and Reporting Authority 2012) and similar in style to the sample questions provided by ACARA. For the class following the traditional approach, the results were comparable with those from previous years. It consisted of

<table>
<thead>
<tr>
<th>Lessons</th>
<th>Traditional approach</th>
<th>Alternative approach</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 and 2</td>
<td>Exercise A: revision of the Cartesian plane, tables of values, introduction of the linear relations equations into the form $y=mx+c$</td>
<td>Plotting scatterplots and looking at the relationships between the two variables. Fitting a line of best fit by eye and informal interpretation of the gradient, $y$-intercept</td>
</tr>
<tr>
<td>3</td>
<td>Exercise B: sketching linear graphs using the intercepts</td>
<td>Finding the least-squares line on the calculator and adding this line to the scatterplot. Reading off the linear graph</td>
</tr>
<tr>
<td>4</td>
<td>Exercise C: graphing and interpreting horizontal and vertical lines</td>
<td>Interpreting of coefficients of the least-squares line. Introduction of the idea of finding the gradient of two points</td>
</tr>
<tr>
<td>5 and 6</td>
<td>Exercise D/E: finding the gradient of lines and direct proportion</td>
<td>Exercise D/F: finding the gradient of lines and direct proportion. The gradient-intercept form of a linear relation equation</td>
</tr>
<tr>
<td>7</td>
<td>Exercise F: the gradient and intercept form of linear relation equation</td>
<td>Exercise G: finding the equation of a line</td>
</tr>
<tr>
<td>8</td>
<td>Exercise G: finding the equation of the line</td>
<td>Exercise B: sketching linear graphs using the intercepts</td>
</tr>
<tr>
<td>9 and 10</td>
<td>Exercise K: linear modelling and a worksheet on lines of best fit. Calculator would be used</td>
<td>Exercise E/K: direct proportion and linear modelling</td>
</tr>
</tbody>
</table>
12 questions: 9 short answer skill questions and 3 extended response application questions. The maximum marks for the assessment was 35.

The results for this test of the two different classes of students were compared to determine efficacy of the alternative approach. The results of the assessment were considered as a whole where the similarities were noted and comparisons made between the boxplot results for each of the classes. Exploratory t tests were used to examine the test questions more closely to decide whether there was any difference between the results for the two groups. Two tailed tests were used because there was no assumption that one approach would be better than the other. Questions were grouped according to the concepts addressed in the students’ study of linear functions, and results were compared to explore whether there was any difference in the student responses to specific types of questions.

Results

The distribution of data for the test results of the traditional class, shown in Fig. 2, was negatively skewed with three students doing particularly badly with results less than 10. The highest result for the class was 25. The mean result for the traditional class was a score of 16.18. This mean score is less than 50%, indicating that these students did not perform fluently the procedures tested in this assessment. The test results of the alternative class were distributed more evenly and with higher results. The lowest result was 10, and the highest result was 31. The mean result for this class was 19.36 which was still quite low, and approximately one third had results less than 50%. The lack of results less than 10 seemed to imply that all students in this class were able to attempt at the majority of the questions.

The range for the results for the class following the traditional approach was wider than the range for the results for the alternative class (see Fig. 2) because of the very low values in the traditional class; however, the middle 50% of the traditional class’ results were quite consistent with an interquartile range of only 8 marks compared with the larger interquartile range of 13 for the alternative class.

![Boxplot Comparison of Results](image-url)

Fig. 2 Boxplot comparison of the test results

[Springer]
Questions testing similar concepts were grouped often according to Australian Curriculum code and exploratory $t$ tests conducted comparing the mean scores for the traditional class with the mean scores for the alternative class for each grouping. This analysis showed some differences in the students’ correct responses.

**Plotting linear relationships**

Figures 3 and 4 show two questions grouped by AC code ACMNA193: *Plot linear relationships on the Cartesian plane with and without the use of digital technologies.* When the marks for the two questions were added, the mean mark for the traditional class for these questions was 1.591 and the mean mark for the alternative class was 1.818 with $p=0.382$. This means that when considering whether or not the students grasp the concept that the equation describes the relationship between $x$ and $y$ in the Cartesian plane, there appears to be no difference between the classes in the students’ ability to produce a table of values and plot a graph from this table and their ability to recognise to apply this concept to points that are not in a table.

**Finding the gradient of a line segment**

Figures 5 and 6 show two questions grouped by AC code ACMNA294: *Find the gradient of a line segment (interval) on the Cartesian plane using a range of strategies, including graphing software.* These questions tested students’ ability to find the gradient of a line segment, regardless of whether the students were given a graph or simply given the coordinates of two points. This was one of the key course descriptors for year 9. Students from both classes attempted to solve the first problem by reading from the graph, and the errors were similar, generally reading the points incorrectly. The second question assumed the recall and use of the gradient formula, although there was space to sketch a graph and some students attempted to do this. When the marks for the two questions were added, the mean mark for the traditional class was 3.091 and the mean mark for the alternative class was 3.136 with $p=0.923$. There was no noticeable difference between the groups.

**Linear graphs and their symbolic representation**

Figures 7 and 8 show two questions grouped as they gauged whether the students, without context, were able to recall the meaning of ‘$m$’ and ‘$c$’ and recognise or place the values that give the gradient and $y$-intercept in a linear relationship. This concept was emphasised with the use of the calculator to find the least-squares line in the alternative approach, but there was, however, no apparent difference between the classes for these questions since when the marks for the two questions were added, the mean mark for the traditional class was 2.409 and the mean result for the alternative
Which of the following points lie on the line $y = -2x + 3$?

A $(-3, 9)$  B $(1, -2)$  C $(0, 3)$  D $(2, -1)$  E $(-1, 5)$

**Fig. 4** Question 8 (1 mark)

class was 2.364, with $p$ value 0.916. Identifying, in the symbolic representation of linear function, the coefficient of the $x$ variable as the gradient and the constant term as the $y$-intercept does not necessarily equate to an understanding of the relevance of these values in a real-life application. Both classes of students were able to do this quite easily. A common error in the traditional class was to read the first number, which is the number 18, as the gradient because the equations in the text were generally written in the $y = mx + c$ form as opposed to the equally appropriate, $y = c + mx$. Whereas, this error was not as common for the alternative class, where students were encouraged to write the equation for least-squares lines for the last two sets of data, which had negative correlation, with the constant first.

**Vertical and horizontal lines**

Figure 9 shows a question relating to the concept of gradient of vertical and horizontal lines, the mean mark for the traditional class was 1.455 and the mean mark for the alternative class was 0.727. The $p$ value was 0.010, so this suggests there was a difference between the results of the two classes for the question testing the students’ ability to recognise the equation of horizontal and vertical lines, indicating the traditional class was better able to recognise these equations as horizontal or vertical lines. Horizontal and vertical lines were not really discussed at length when the alternative class was looking at the scatterplots, as the data used did not have this type of linear relationship. There were, however, situations where students were expected to read points from the least-squares line and this involved drawing a horizontal or vertical line, but apparently, no connection was made between the idea of drawing a line to read data values off a graph given and explanatory variables and the equation of a vertical line.

**Fig. 5** Question 2 (4 marks)
Find the gradient of the following line joining the two points, (−4, 1) and (4, −11)

Fig. 6 Question 3 - parts a and b (2 marks)

**Sketching linear functions by finding two points**

In comparing the results of the two questions shown in Figs. 10 and 11, grouped by AC code ACMNA215: *Sketch linear graphs using the coordinates of two points*, it was important to consider each separately as it was expected that there may be a difference between the two classes as the first question was written in the gradient-intercept form and the second written in a more general form. For question 5, the mean mark for the traditional class was 0.636 and the mean for the alternative class was 1.091. The $p$ value was 0.048, so this suggests that there was a difference between the classes in the students’ ability to sketch the linear graph when it was written in the gradient-intercept form, possibly reflecting a better understanding of the application of the gradient concept. It appears from this result that the students who used the alternative approach had a better understanding of concepts of the $y$-intercept and gradient or alternatively were able to plot two points, the $y$-intercept and one another point using the rule. Although most students who got full marks for this question did plot the point $(5, -1)$, marks were awarded for any second point. The examples using bivariate data used approximate decimal values rather than fractions for the coefficients, and a few in the alternative class did plot the point $(1, 1.4)$ which was expected as a possible answer.

For question 6, the mean mark for the traditional class was 1.591 and the mean mark for the alternative class was 1.409 with a $p$ value of 0.640, suggesting that there is no significant difference between the means for the classes for this question. This was one procedure that was drilled as an exercise for the traditional class. It was anticipated that these students would do considerably better in this question than the alternative class, but this was not the case.

**Direct proportion**

Figure 12 shows a question addressing AC code ACMNA208: *Solve problems involving direct proportion. Explore the relationship between graphs and equations corresponding to simple rate problems*. Solving problems involving direct proportion was also one of the key course descriptors for year 9, and this question involved recognising direct proportion examples either by the description of a real-life scenario or by recognising that the graph was linear passing through the origin. For this question, the mean mark for the traditional class was 1.545 and the mean mark for the alternative class was 3.000. The $p$ value was 0.038, so this suggests there appears to be a difference between the classes in the students’ ability to recognise and convert a worded direct proportion question into a graphical representation. In the alternative introduction, one of
Write the rule of the straight line whose gradient = $\frac{5}{6}$ and y-intercept = -8.

Fig. 8  Question: 7 (2 marks)

the examples, height against belly button height, was a situation where the assumption made by the students was that the line of best fit drawn by most students went through the origin. This did lead to quite a lengthy discussion about the appropriateness of the assumption, and this may have contributed to the students’ understanding of direct proportion. This was a technology-free assessment, and some students struggled with the multiplication and division by 12 and were also not able to plot the graph, so it was difficult to ascertain whether students would have understood the graphing concept if they had the appropriate numeracy skills.

Scatterplots

Figure 13 shows a question addressing AC code ACMSP251: Use scatterplots to investigate and comment on relationships between two numerical variables. The mean mark for the traditional class was 2.818 and the mean mark for the alternative class was 3.045. The $p$ value of 0.798 does not indicate any difference between the classes in their ability to interpret a linear least-squares line on a scatterplot. Very few of the students were able to find the equation of the line of best fit although they were able to find the gradient. Initially, it was expected that the alternative class would find this question easier, and although the results were slightly higher, there was no significant difference in their ability to tackle this type of question. Although this was written as a scatterplot question, there was no context; the skills involved were plotting two points, reading points off a line and finding the equation of line with $x$ and $y$ as variables given two points.

Linear modelling

Figure 14 shows a question requiring modelling with a linear function for its solution. The mean mark for the traditional class was 1.045 and the mean mark for the alternative class was 2.772. The $p$ value of 0.027 implies there was a difference between the
classes in the students’ ability to apply the linear relationship skills to modelling questions with the alternative class achieving greater success with this question. This was the last question of the test, and there were eight students from the traditional class who did not attempt this question, compared with only two students from the alternative class. It was unclear whether this was because they were actually unable to attempt the question because they did not have the skills or whether it was because they did not understand the final question or perhaps they ran out of time. Also, quite a few only wrote answers for a (i) and b (i), and this could also have been related more to limited numeracy skills rather than a lack of understanding of the linear modelling.

Summary of results

Table 2 shows a summary of the analysis of the questions in the test. The alternative class did appear to have a better understanding of gradients and intercepts demonstrating the ability to interpret these parameters to sketch graphs and apply the knowledge to linear modelling examples. The traditional class, however, were significantly more able to recognise the equations for vertical or horizontal lines, which was not demonstrated by the alternative class. Both classes had similar results for the skill questions: finding and plotting points, finding gradients, sketching a line using the intercepts and finding the equation of a line.

From the results of this assessment, it seems that the students in the class who trialled the alternative approach were not disadvantaged as they seem to have grasped the basic skills in a similar way to the other class. This is perhaps not surprising since they did use the textbook after the four initial introductory lessons. However, they did seem to be more able to apply these skills in a linear modelling context. There were none of the very low marks seen in the traditional class which suggests that there were no students in the alternative class who were totally unable to engage or participate in this topic. It appears that there is merit in continuing to develop this alternative approach for use with all students, but more specifically with students who may be put off by the more algebraic and abstract textbook approach.

Discussion

TIMMS research showed that textbooks were commonly used in secondary schools to guide mathematics instruction and further research. Vincent and Stacey (2008) found there were too many repetitive questions of low-level complexity within these texts.
The exchange rate for Australian Dollar in to South Africa Rand is $1(Aus) = 12 Rand. This can be calculated using the $R = 12A$ where $R$ is the number of Rand and $A$ is the number of Australian Dollars.

a) On the axes show the graph for this linear relationship with $R$ on the vertical axes and $A$ on the horizontal axes. Label at least two points clearly.

b) Complete the following

(i) $80$ is equivalent to ______________________________

(ii) $1440$ Rand is equivalent to $\$ \underline{\hspace{2cm}}$ (Australian)

c) Explain why this is an example of direct proportion

Fig. 12 Question 10 (5 marks)

Consider the variables $x$ and $y$

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>7</td>
<td>6</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>3</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

The line of best fit for this data is displayed on the graph. The line passes through the points $(3,5)$ and $(8,2)$

a The line passes through the points $(3,5)$ and $(8,2)$. Mark these two points with a cross.

b Using the line of best fit to estimate:

i $y$ when $x = 6.5$ __________________________________________

ii $x$ when $y = 5.5$ __________________________________________

c Find the equation of the line of best fit

Fig. 13 Question 11 (5 marks)
Trent is looking to hire a carpenter to build a desk for his office. He is trying to choose between two carpenters. Both charge a fixed price for the materials required to build the desk (wood, nails, etc.) and also an hourly labour charge for the time spent constructing the desk.

a) The first carpenter’s total cost, $C$, is given by $C = 30n + 200$, where $n$ is the number of labour hours.
   
   i. State the fixed price for materials, and also the hourly labour fee the carpenter charges.
   
   ii. How much will Trent pay for the desk if the carpenter requires 6.5 hours to build?
   
   iii. If the final cost of the desk is $320, how many hours did the carpenter require to build it?

b) The second carpenter charges $100 for materials and a labour fee of $50 per hour.

   i. Write a rule for the total cost, $C$, for the second carpenter if the desk requires $n$ labour hours to be built.
   
   ii. If both carpenters would require 5 hours to complete the desk, which is the cheaper option?

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Fig. 14  Question 12 (5 marks)

Just as Ely and Adams (2012) found, these traditional approaches to the teaching of linear functions, similar to those followed by one class in this study, led to some confusion about the idea that the parameters $m$ and $c$ can be used to represent the gradient and intercept for graphs and linear modelling questions. The analysis of the assessment questions in this study seems to confirm that these concepts were also not understood by students in the traditional class, where the mean results for the questions were 0.636 out of 2 for the question on sketching the graph using the gradient and $y$-intercept, 1.545 out of 3 for the direct proportion question and 1.045 out of 5 for the linear modelling question.

In the alternative approach, students began their study of linear functions by exploring scatterplots of Census at School data (Australian Bureau of Statistics 2013), fitting ‘lines of best fit’ by eye and predicting using interpolation from the line of best fit, using technology to find the best ‘line of best fit’, drawing the ‘line of best fit’ on the scatterplot and interpreting the gradient and intercept with reference to the given variables. The ABS data needed to be selected so that there was high correlation between the variables and therefore an obvious linear trend for each of the examples. Consistent with the research of Bardini et al. (2004), the students in the alternative class did appear to develop a better understanding of the gradient and intercept concepts when the ideas were introduced in context using modern technology to support the development of the concepts through access to multiple representations. This was most evident in the application question in the assessment where the alternative class appeared to perform significantly better than the traditional class.

In their research, Chick and Pierce (2012) suggested that using real-life data can engage students’ interest. Similarly, this study showed that there were fewer students in the alternative class who achieved very low results in the assessment, compared to
several students with very low results in the traditional class. Perhaps, this indicates a higher level of engagement by the students in the alternative class. The last question, however, was not attempted by some of the traditional class, and it was unclear as to whether this was due to a lack of understanding or because they ran out of time. Either way, it was an indication that the students in the alternative class had a better understanding of the topic or were more efficient when completing the assessment which does imply that they were comfortable with the topic. Consistent with the Australian Curriculum’s suggestion that there is a need for the today’s students to be numerate and statistically literate, the use of personal data such as reaction times, height and time taken to get to school appears to be an effective way to introduce linear functions.

The ability to sketch horizontal and vertical lines given the equation was the only skill where the traditional class appeared to have a better understanding than the alternative class. This is not a particularly difficult concept, and a slight modification to the sequence to ensure sketching of horizontal and vertical lines or finding suitable real-life data resulting in vertical or horizontal lines would alleviate this discrepancy.

Throughout the lessons for the alternative class, there was extensive use of technology to ensure that the students continued to work with the different representations. The students were ultimately expected to complete a technology-free assessment at the end of the topic, but this appears not to have been an impediment to these students. The question-by-question analysis suggests that these students were better able to interpret gradients and \( y \)-intercepts, and this was not at the expense of other skills.

**Conclusion**

There is sufficient evidence from this study to suggest that the introduction of linear functions through the use of scatterplots and least-squares lines of relevant data may be an appropriate alternative approach to the commonly used textbook approach. The use of technology provides ease of access to multiple representations and so allows students to investigate and consolidate the symbolic representations of linear functions. When students study scatterplots and equations for least-squares lines, they recognise the need to use a sensible domain as they choose starting and finishing points to their graphs, develop an understanding of variables and are introduced to relevant mathematical language that could enable them to make connections and hence improve their relational understanding (Skemp 1976). This alternative approach was relevant to the students; they were familiar with the data; and there was a recognition that least-squares lines are genuinely used for problem solving in a real-life context.

The current Australian and Assessment and Reporting Authority (2012) scope and sequence recognises that interpretation of authentic graphs are within the ability of year 7 students but emphasises linear graphs within the abstract approach to linear functions on the Cartesian plane. Plotting bivariate data scatterplots is a natural application of the coordinate plane. Fitting a least-squares line to statistical data with the use of technology is an easily accessible application of linear graphs. This would also connect linear graphs with the statistical analysis of numerical data sets as well as offering a skill that can be used to process and analyse data in science and other curriculum areas.
<table>
<thead>
<tr>
<th>Test qu. no.</th>
<th>Question type</th>
<th>Mean result: traditional class</th>
<th>Mean result: alternative class</th>
<th>p value</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 and 8</td>
<td>Plot linear relationships</td>
<td>1.591</td>
<td>1.818</td>
<td>0.382</td>
<td>No difference</td>
</tr>
<tr>
<td>2 and 3</td>
<td>Finding the gradient of a line segment</td>
<td>3.091</td>
<td>3.136</td>
<td>0.923</td>
<td>No difference</td>
</tr>
<tr>
<td>4 and 7</td>
<td>Linear graphs and their symbolic representation</td>
<td>2.409</td>
<td>2.364</td>
<td>0.916</td>
<td>No difference</td>
</tr>
<tr>
<td>9</td>
<td>Vertical and horizontal lines</td>
<td>1.455</td>
<td>0.727</td>
<td>0.010</td>
<td>The traditional class did better</td>
</tr>
<tr>
<td>5</td>
<td>Sketching linear functions by finding two points from the symbolic representation</td>
<td>0.636</td>
<td>1.091</td>
<td>0.048</td>
<td>The alternative class did better</td>
</tr>
<tr>
<td>6</td>
<td>Sketching linear functions by finding two points from Intercepts</td>
<td>1.591</td>
<td>1.409</td>
<td>0.640</td>
<td>No difference</td>
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<tr>
<td>10</td>
<td>Direct proportion</td>
<td>1.545</td>
<td>3.000</td>
<td>0.038</td>
<td>The alternative class did better</td>
</tr>
<tr>
<td>11</td>
<td>Scatterplots</td>
<td>2.818</td>
<td>3.045</td>
<td>0.798</td>
<td>No difference</td>
</tr>
<tr>
<td>12</td>
<td>Linear modelling</td>
<td>1.045</td>
<td>2.772</td>
<td>0.027</td>
<td>The alternative class did better</td>
</tr>
</tbody>
</table>

Although there were differences in the students’ ability to apply linear function skills for particular questions, this was a relatively small sample, one school and only one assessment. To be truly confident in the efficacy of the alternative approach, it would need to be trialled on a much larger scale, possibly as part of a longitudinal study using a range of different assessments. This study was unable to show if either class had understood these concepts well enough to be able to recall the information and apply these skills at a later date. The scope of this study did not really allow for a longitudinal investigation of the students’ progress to explore the effect of using scatterplots and least-squares lines to introduce and develop linear function skills as they continued with mathematics, either in the study of functions or in the study of statistics, as well as other subjects. Another area that could also have been analysed or measured is whether the students were more engaged by this process compared with the traditional textbook approach. These are all potential areas for further research.

This study highlights the importance of sequencing of topics so that each one builds on the one before (Hiebert and Carpenter 1992). This is not usually the case where chapters of the textbook are usually treated as separate disconnected topics. In particular, this can be seen in the textbook used at this school, where the topic of Linear Equations and Formulas is presented before and in isolation from Linear Relations. This study demonstrates the manner in which a smooth transition from a study of statistics may be blended into the introduction to linear functions, contributing to connections between schemata. In addition, the emphasis on real-life data related to students’ previous experience builds stronger links with prior knowledge, in this case their involvement in the collection of data for the ABS’ Census at School. This
connection to previous data collection emphasises that each dot in the scatterplot represents the data from one student and that there may be relationships between the data which is not necessarily causal. This awareness regarding relationships between variables may better inform students as they move into higher levels of school, and their understanding of linear functions may also transferable to other subjects such as science and other curriculum areas where data is often generated or collected. This study demonstrates the power of introducing topics strongly connected to the previous topic, in this case just the first four lessons.

References


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Introduction to linear functions


