Functional State Observers Design for Interconnected
Time-Delay Systems

by

Wei Yin Leong
B.Eng (Hons)

Submitted in fulfilment of the requirements for the degree of
Doctor of Philosophy

Deakin University
February, 2016
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<th>Full Form</th>
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<tbody>
<tr>
<td>I/O</td>
<td>Input/Output</td>
</tr>
<tr>
<td>FDI</td>
<td>Fault Detection and Isolation</td>
</tr>
<tr>
<td>LKF</td>
<td>Lyapunov–Krasovskii Functional</td>
</tr>
<tr>
<td>LMI</td>
<td>Linear Matrix Inequality</td>
</tr>
<tr>
<td>UIO</td>
<td>Unknown Input Observer</td>
</tr>
<tr>
<td>DDO</td>
<td>Disturbance Decoupled Observer</td>
</tr>
<tr>
<td>UIFO</td>
<td>Unknown Input Functional Observer</td>
</tr>
<tr>
<td>DDFO</td>
<td>Disturbance Decoupled Functional Observer</td>
</tr>
<tr>
<td>NUIO</td>
<td>Nonlinear Unknown Input Observer</td>
</tr>
<tr>
<td>NUIFO</td>
<td>Nonlinear Unknown Input Functional Observer</td>
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</table>
## List of Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$A$</td>
<td>the system state matrix</td>
</tr>
<tr>
<td>$B$</td>
<td>the system input matrix</td>
</tr>
<tr>
<td>$C$</td>
<td>the system output matrix</td>
</tr>
<tr>
<td>$D$</td>
<td>the system unknown input matrix</td>
</tr>
<tr>
<td>$F$</td>
<td>the linear functional matrix</td>
</tr>
<tr>
<td>$A_{ii}$</td>
<td>the state matrix of an $i$th subsystem</td>
</tr>
<tr>
<td>$A_{dii}$</td>
<td>the delayed state matrix of an $i$th subsystem</td>
</tr>
<tr>
<td>$A_{ij}$</td>
<td>the interconnection matrix of an $i$th subsystem</td>
</tr>
<tr>
<td>$B_i$</td>
<td>the input matrix of an $i$th subsystem</td>
</tr>
<tr>
<td>$C_i$</td>
<td>the output matrix of an $i$th subsystem</td>
</tr>
<tr>
<td>$F_i$</td>
<td>the linear functional matrix of an $i$th observer</td>
</tr>
<tr>
<td>$A(\tau)$</td>
<td>the system state polynomial matrix as a function of $\tau$ operator</td>
</tr>
<tr>
<td>$B(\tau)$</td>
<td>the system input polynomial matrix as a function of $\tau$ operator</td>
</tr>
<tr>
<td>$C(\tau)$</td>
<td>the system output polynomial matrix as a function of $\tau$ operator</td>
</tr>
<tr>
<td>$\dot{x}(t)$</td>
<td>the rate of change of $x(t)$</td>
</tr>
<tr>
<td>$x(t)$</td>
<td>the system state vector</td>
</tr>
<tr>
<td>$u(t)$</td>
<td>the system input vector</td>
</tr>
</tbody>
</table>
\[ y(t) \]  the system output vector
\[ z(t) \]  the linear functional vector
\[ \dot{x}_i(t) \]  the rate of change of \( x_i(t) \)
\[ x_i(t) \]  the local state vector of an \( i \)th subsystem
\[ x_j(t) \]  the remote state vector of an \( i \)th subsystem
\[ u_i(t) \]  the local input vector of an \( i \)th subsystem
\[ y_i(t) \]  the local output vector of an \( i \)th subsystem
\[ z_i(t) \]  the linear functional vector of an \( i \)th observer
\[ \phi_i(t) \]  the initial function of the \( i \)th subsystem
\[ n_i \]  the dimension of the state of an \( i \)th subsystem
\[ n_j \]  the dimension of the state of an \( j \)th subsystem
\[ m_i \]  the number of inputs of an \( i \)th subsystem
\[ p_i \]  the number of outputs of an \( i \)th subsystem
\[ q_i \]  the number of functions to be estimated for an \( i \)th subsystem
\[ t \]  time
\[ \tau_i \]  the constant time delay of an \( i \)th subsystem
\[ \tau(t) \]  the time-varying delay function
\[ \tau_{ii}(t) \]  the time-varying delay of an \( i \)th subsystem
\[ \tau_{ji}(t) \]  the time-varying delay of an \( i \)th subsystem
\[ \tau_{ji}(t) \]  the time-varying delay associated with an \( j \)th remote subsystem
\[ \tau_{max} \]  the highest amplitude of \( \tau_{ii}(t) \) and \( \tau_{ji}(t) \)
\[ \tau_{li} \]  the lower-bound delay
\( \tau_{ii}^u \)  the upper-bound delay
\( \hat{x}(t) \)  the estimate of \( x(t) \)
\( \hat{z}(t) \)  the estimate of \( z(t) \)
\( \hat{x}_i(t) \)  the estimate of \( x_i(t) \) of an \( i \)th subsystem
\( \hat{z}_i(t) \)  the estimate of \( z_i(t) \) of an \( i \)th subsystem
\( \varepsilon_i(t) \)  the estimation error vector of an \( i \)th subsystem
\( I_n \)  an \( n \times n \) identity matrix
\( \alpha_{i}/2 \)  the observer’s exponential convergence rate
Abstract

The purpose of this thesis is to provide a comprehensive treatment on the subject of functional state estimation for interconnected time-delay systems. The thesis commences with an exhaustive study of the design of low-order functional observers for interconnected systems, with particular attention paid to both the interconnection effects and relaxation of the observers’ existence conditions. As they are an inherent part of an interconnected system, time-delay effects are considered simultaneously. In the process, different observer structures are proposed along with the provision of simple constructive procedures, in which progressive improvements and refinements are made, in terms of relaxing the observers’ restrictive existence conditions. Additionally, observer schemes are tailored to cater for the needs and challenges of interconnected systems, which are subject to different types of time delay. In this regard, efforts are made to treat the time delay challenges in the specific interconnected system, ranging from a single constant time delay through to multiple heterogenous time-varying delays. Advancement is achieved in regulating the observer’s error dynamics, from merely guaranteeing an asymptotic convergence to enabling a prescribable exponential rate of convergence. Lastly, the efficacy and practicality of these design methods are verified and illustrated through numerical examples and simulation results.
Chapter 1

Introduction

1.1 Background

The problem of state estimation/observation has received significant attention over the past few decades and many solutions to the problem have been proposed by various researchers covering diverse disciplines [120, 29, 127, 30, 17, 64, 65, 2, 10, 15]. The states of a system which carry the vital information for the derivation and execution of control strategies are commonly obtainable from its outputs, measurable with the use of sensors [32]. These physical sensors are not without shortcomings of their own, as they are susceptible to measurement errors and noise [43, 32]. In fact, it is well recognised that the errors introduced by the measurement instruments may be much greater than that of the ones produced by a state observer [85], making state observers a preferable option when it comes to achieving satisfactory performance and greater reliability of a control system. Furthermore, sensors are normally able to only sense a limited number of output quantities of interest, and are highly prone to destruction if operated under harsh or extreme environments for an extended period of time. In addition, it is also relatively costly to deploy a
large quantity of sensors into a system, especially if this involves a large-scale system. When all these considerations are taken into account, there is no doubt that the application of observers opens up an attractive avenue in replacing sensors for the estimation of unmeasurable state variables, which is the motivation behind this research.

In this modern era of proliferating complexity and dimensionality, the emergence of large-scale systems has become an indispensable part of our lives. To cater for the expansion of our societal needs, there is an ongoing development of strongly interconnected, highly interactive, increasingly complex systems as the technology advances. The notion of large-scale systems can often be associated with a heterogeneous collection of simpler interacting systems commonly referred to as interconnected subsystems [18, 99, 9]. Large-scale systems consisting of systems of interconnections are common these days and are prevalent in many diverse fields and disciplines, ranging from engineering to biological systems. As a result, the uphill task of controlling and regulating these complex systems has become more challenging than ever before. The challenge to the research community is due to the fact that large-scale systems are inherently complex and are most commonly accompanied by high levels of uncertainties and therefore remain a major stumbling block to the theory and practice of control systems. With the increasing numbers of inputs and outputs combined with the frequent interactions and communications amongst the subsystems in the multidimensional system as a result of the proliferation of the overall system's complexity, the task of measuring all the numerous output quantities becomes out of the question. As such, the reconstruction of state information
using state observation techniques proves to be essential in this regard.

State observation for interconnected systems can be performed in two fashions: centralised or decentralised estimation schemes. In centralised observer design topology, the outputs of all the subsystems are channeled into a single processing facility or the centralised observer, and eventually the reconstructed state estimates are transmitted to the corresponding controller. In the case of a highly complex interconnected system, this method proves to be futile as the volume of incoming information that a centralised observer has to handle far exceeds its processing capabilities. Moreover, the need for real-time high volume information transfer of output information from all the corresponding subsystems to a single location, which is in this case the centralised observer, puts constraints onto the communication network as the bandwidth of the communication channel is finite. In order to estimate the states of all the subsystems with a centralised topology, a substantially high-order observer is required. Such a complex implementation of state observation technique puts forward challenges and inflates the construction cost of the observer as the required hardware will be of considerable scale.

While the decentralisation technique for controllers and observers design renders the benefit of having to deal with lesser system inputs or outputs (I/Os), nevertheless, it is not without challenges of its own. One of the most critical factors in selecting the use of decentralised observation schemes is governed by the strength of the interconnection couplings between the many subsystems. In order to address the problem of state observation for interconnected systems with weak couplings, a number of appealing methods [59, 119]
suggesting the treatment of subsystem interconnection effects as unknown inputs have been proposed in an attempt to deal with the subsystems interactions. For example, in [113], the design strategy of a decentralised observer together with the observer’s existence conditions were presented and proved through a simple direct approach. Despite the fact that a simpler observer structure can be formulated based on such principles, nonetheless, it is well known that such a scheme suffers from an unnecessarily restrictive existence condition [59]. In addition, when it comes to strongly-coupled interconnected systems, such an approach might not be viable. Taking into account these considerations, the methods reported in this thesis consider a quasi-decentralised approach that is able to handle systems with strongly-coupled subsystems [79, 80, 76, 77].

For a large-scale composite system made up of various subsystems spreading across multiple locations, the state information exchange between these subsystems is not instantaneous considering the limited bandwidth of the communication channels. In fact, one of the issues pertaining to the design of state estimators to reconstruct the states of the inter-coupled subsystems has been driven by the effects of time delays which cannot be overlooked. Time delays have become part of the subsystems’ interconnection data exchange due to the geographical proximity between these subsystems if real-world physical systems as well as suboptimal communication networks are taken into consideration. As such, state estimation problems have to take into account the inter-subsystem data transfer latency. On that note, the results presented in this thesis propose observer structures which take care of the realistic issue of
time delay.

Armed with the well-known observer theories and the well-developed innovative computational approaches in the field of matrix algebra amongst other mathematical theorems, this research endeavours to exploit the many techniques available to tackle the state observation challenges brought about by these large-scale systems. Whilst taking advantage of the readily available techniques, new approaches and methods will be established on the basis of the previous novel discoveries, which attempt to simplify the design procedures and also to extend the utility of existing methods in the field of complex large-scale systems. Due to the difficulties involved in dealing with large-scale interconnected systems which are very complicated and time-consuming, the low-order functional estimator solutions reported in the literature are vastly insufficient and, if they do exist, they are usually limited to a particular class of system. In view of the need for observer-based solutions to problems occurring in large-scale systems and the lack of those available currently, this research attempts to provide some systematic, straightforward and effective estimator design methods specifically for large-scale interconnected time-delay systems.

1.2 Main Contributions of the Thesis

The main contributions of this thesis are set out as follows:

- The development of low-order partially-decentralised functional observers that are capable of reconstructing the state functionals of a class of interconnected systems subject to constant interconnection time delays;
• Proposal and derivation of functional observer schemes which contribute towards the relaxation and improvement of the observers’ existence conditions for interconnected time-delay systems. This is achieved with the construction of functional observers that accept the transmission of additional delayed output information from other remote subsystems;

• Extension of state estimation problems to interconnected systems in the presence of time-varying delays in both the interconnections and internal state vectors;

• The development of distributed observer schemes that place no specific requirement for information exchange amongst the observers, and the operation of these observers is independent of one another. The observer scheme is able to handle systems with very strong interconnections, in contrast to those found currently in the literature;

• The application of coordinate transformation method into dealing with interconnected time-delay problems, which enables the construction of the existing well-known observer schemes for interconnected time-delay systems.

• The establishment and resolution of interval time-varying delay problems and stability conditions of the observers’ error dynamics by the application of linear matrix inequality (LMI) formulation to achieve a prescribed degree of stability.
1.3 List of Publications


1.4 Thesis Outline

The contents of this thesis pertaining to the development of low-order functional observers for interconnected time-delay systems, is outlined in the following chapters:

Chapter 2 provides a survey on the theory and motivation of state observation together with an overview on the development of the functional observers. These studies provide an important basis for the establishment of new results in the subsequent chapters of this thesis.

Chapter 3 [79, 80] proposes two low-order functional observer schemes for a class of interconnected linear systems that is subject to time delays in the interconnections. Unlike observers which consider only the ideal non-delayed output information transfer, the proposed observer is capable of dealing with delayed output information from geographically distant subsystems. It is demonstrated that by accepting measurement data from other subsystems, the conditions under which an observer exists can be made less conservative. Existence conditions, systematic and straightforward procedures for the synthesis of the observers, are given along with numerical examples illustrating the effectiveness and simplicity of the design algorithms.

Chapter 4 [76] presents a partially-distributed functional observer scheme for a class of interconnected linear systems with a very strong non-instantaneous subsystem interaction and with time delays in the local
states and in the transmission of output information from the remote subsystems. The procedure for computing and solving the observer parameters is presented through a numerical example of a dual interconnected system. The proposed observer structure is tailored for a time-delay system, with possible application and extension to systems possessing a higher number of interconnected subsystems. The proposed scheme has been found to be effective, and simulation results prove the feasibility of this approach. The robustness assessment of the designed observer against the deviation in the system matrices provides suggestions for possible future research endeavours.

**Chapter 5** [77] provides a comprehensive treatment on the problem of distributed functional state observation of a large-scale interconnected system with the presence of time delays in the interconnections and the local state vectors. The resulting observer scheme is suitable for strongly coupled subsystems with multiple time-varying delays, and is shown to give superior results for systems with very strong interconnections which only require the satisfaction of some mild existence conditions. A set of existence conditions are derived along with a computationally simple observer constructive procedure. With the use of Lyapunov–Krasovskii functional method, the time-varying delay stability conditions are established and, upon the solution of linear matrix inequalities (LMI), the observer parameter that guarantees the exponential stability of the observer error dynamics can be obtained. All the developed results are
tested and simulated with a numerical example of a three-area interconnected system to demonstrate the feasibility and effectiveness of the proposed method.

Chapter 6 [78] considers the design of state observers for interconnected time-delay systems using a coordinate transformation method. Through such a transformation, the system that has interconnection and state delays is metamorphosed into a new system that injects time-delay information into its input and output terms, before reintroducing them back into the latter system, effectively coupling the delay terms into the I/O injection terms and eliminating the delay values from the state variables. Next, full-order and reduced-order observers are designed based on the transformed system. Finally, the observed states of the transformed system that correspond to the original system is used to deduce the estimates of the original system. A numerical example is provided of an interconnected time-delay system.

Chapter 7 summarises the key contributions presented in this thesis, draws the conclusions, and proposes some suggestions for potential further research related to this thesis.
Chapter 2

Literature Survey

2.1 Theory, Motivation And Importance of State Observers

Observers, which have made possible the rapid development and implementation of control law to its present state of refinement, have assumed an increasing important role in control theory, in industrial manufacturing plant [29, 125], in chemical and biochemical processes [30, 120, 47] and in various branches of engineering and scientific research [17, 64, 27, 65, 70, 144, 31]. An observer is itself a dynamic system being represented in the state-space form just like the physical system whose internal state variables are being estimated by the observer [43]. Its main function is to provide an approximation of the system state vector, especially those that are not directly available from on-line measurement. For example, an observer can be employed to produce the state estimate \( \hat{x}(t) \) of the state vector \( x(t) \) of the system of interest [47]. In order for an observer to be built around the target system, the physical system has to be mathematically modelled into a state-space representation in which the
inputs, outputs and state variables of a system are to be related by a set of first-order differential equations as depicted in Figure 2.1.

\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t), \quad (2.1) \\
y(t) &=Cx(t), \quad (2.2)
\end{align*}

where \(x(t)\) is an \(n \times 1\) state vector, \(u(t)\) is an \(m \times 1\) input vector and \(y(t)\) is an \(p \times 1\) output vector. Matrices \(A \in \mathbb{R}^{n \times n}\), \(B \in \mathbb{R}^{n \times m}\) and \(C \in \mathbb{R}^{p \times n}\) are known constant matrices. The past behaviour of the system is governed by the state vector \(x(t)\) while the first-order ordinary differential equation (2.1) summarises future behaviour of the system [83]. Constant matrices \(A\), \(B\) and \(C\) dictate the properties of the system [83].

In control system engineering, particularly in the design of controller or...
regulator implementing a specific control law, there is a common assumption that all the internal state variables of the system under control are available or accessible [83]. Unfortunately, in practical situations, this is not always the case. Instead, in most cases, only a very limited number of measurable state variables can be obtained from the system output [32]. It is, therefore, necessary to have a mechanism which is able to provide the missing unmeasurable state variables to the controller for the generation of control signal. Or, alternatively, a completely new controller scheme can be formulated to account for the nonavailability of the entire state vector. It is apparent that the former approach of providing an approximate of the system state vector is vastly simpler as compared to devising a totally new controller. An observer, also commonly referred to as an estimator, provides a practical and elegant solution to this problem. In order to exploit the benefits that an observer can offer, it is important to understand the basic principle of operation as follows.

Figure 2.2: Block diagram of an observer acting as an auxiliary system
The concept behind state observation is that of creating an auxiliary dynamic system adjacent to the actual system or plant to be controlled or monitored, as illustrated in Figure 2.2. Under such a model, the measurable outputs along with the input signals acting on the original system will be used to drive an observer or estimator. The observer will in turn go through a state reconstruction process and produce approximation of the system states. In the case of a state feedback control system, the reconstruction of unknown state variables is essential for the proper operation of its controller, be it a state feedback controller or any other control strategy. A typical application of an observer in a feedback control system is demonstrated in the block diagram of Figure 2.3.

![Block diagram of an observer providing reconstructed state variables to the controller](image)

Figure 2.3: Block diagram of an observer providing reconstructed state variables to the controller

In the case of a closed-loop system, where a feedback controller is applied to regulate the system, there is a common assumption that the entire system
state vector is required to form the feedback controller. However, in practical situations, not all of the elements of the system state vector are available through direct measurement [112]. This could be due partly to the unavailability of the states via direct measurement or that it is impractical to do so because of cost constraints. In that respect, an observer seeks to reconstruct the missing state variables and supply that to the controller which utilises these system states information to implement the linear state feedback control law. It is not always possible or desirable to measure every variable that we intend to control, and the observer provides a means for inferring the missing information.

2.2 Survey of the Development of Functional State Observer

This section seeks to critically investigate the history of the advent and development of the state observation or estimation theory in the past few decades up to the present time. It is almost impossible to give a complete account of each and every single work of all contributions due to the vast number of new concepts, methods and results that have been generated to date. Nevertheless, a broad summary of the key achievements and contributions to the field of state observation theory is provided.

The concept of state observation was first introduced by Rudolph E. Kalman in the Sixties with the introduction of the well-known Kalman filter [67]. It is essentially an optimal linear estimator consisting of a set of mathematical
equations implementing a predictor-corrector algorithm that is capable of inferring the missing information from indirect measurement in a noisy system [137, 51]. The Kalman filter has been utilised extensively in the state estimation of navigation systems for aircraft, ships and even trajectory estimation of the Apollo space program.

Subsequently, the minimal state observer [83] was proposed by Luenberger in 1964 as a substitute for the Kalman filter applicable in optimal feedback control schemes when the signal to noise ratios are high. The significant achievement of this so-called Luenberger observer is the reduced observer dimension which can be clearly distinguishable from the full-order Kalman filter, thereby simplifying the observer dynamic order from \( n \) to \((n-m)\) for an \( n \)-th order system with \( m \) outputs [100]. The minimal observer design method however, has a restrictive assumption that the state matrix and observer matrix must not share the common eigenvalues. Yuksel and Bongiorno [143] and Newmann [95] later removed this restriction by extending the work of Luenberger. Luenberger was also the first to show a single linear functional observer of order \((v-1)\) where \( v \) is known as the observability index, (see [84]), that is capable of reconstructing any single linear combination of the state vector. \( v \) is defined as the smallest integer for which the following holds [111, 131]

\[
\text{rank} \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{v-1} \end{bmatrix} = n. \tag{2.3}
\]
The procedure for designing a linear functional observer having a lower order than that of Luenberger as well as possessing the advantage that plant and observer can have common poles as opposed to Luenberger observer was introduced by Fortmann and Williamson [42]. In their work, observers construction methods were proposed for both single-output and multi-output plants together with the provision of necessary and sufficient conditions for the existence of the observers. It was proven that single-output design procedure can be applied to multi-output systems by first reducing the system to an output-coupled set of single-output systems using a canonical form. Additional to this is the approach based on the simple sequential design method proposed by Murdoch [92, 93] which makes possible the generation of multiple linear state functionals. While not requiring the reduction of the multi-output system to a number of single-output systems as is the case of [42], the design approach may not produce an observer of minimal order.

Most multi-functional observer construction techniques proposed by authors such as [42], [92] and [91] were meant only for special cases until methods for more generic cases were introduced [33]. The treatment intended for a more general case were subsequently contributed by [110], who related this problem with those of the partial realization, as well as [71] who applied the geometric approach. Later, [33] introduced a reduced-order multi-functional observer. Such an observer relies on the unified output generated by appending the functionals to the system outputs which were then used to establish the Luenberger canonical form. The need for the preliminary computation of a system canonical form before an observer can be designed was eventually
eliminated by [53], who presented a simple single functional observer design procedure which also allowed the arbitrary assignment of eigenvalues. However, no extension of the design procedure to multi-functional observers have been reported.

Along with these developments, efforts on further reducing the order of multi-functional observer were attempted by a number of authors. One rather practical approach presented by [134], utilised the Jordan form of the observer state matrix which had essentially given rise to the possibility of significantly reducing the order to less than that of $m(v-1)$ where $m$ is the number of functionals and $v$ is the observability index. Shortly after this, an even lower order observer was achieved by the same author [135], who improved the previous [134] design procedure, resulting in a further reduction of observer order equivalent to the addition of all the descending order observability indices minus the required number of functionals. For high dimensional systems consisting of greater numbers of outputs than inputs, [4] devised a simple implementation approach that only required the solution of a set of consistent linear algebraic equations. Although this method rendered a lower order observer realization than that of [135] with the consideration of the ratio of the number of independent output measurements to the number of independent inputs, nevertheless, the approach only provided sufficient conditions for the existence of such an observer whilst the necessary conditions had been omitted. Both necessary and sufficient conditions for the existence and stability of a $p$-th order linear functional observer which coincided with the number of functionals were subsequently obtained by [22]. Furthermore, [22] also provided design procedures
for both continuous and discrete time systems together with the corresponding necessary and sufficient conditions for the existence for each scenario. In addition, [132] also proposed methods of designing both the reduced-order scalar and multiple functional observers based on a parametric approach.

Yet another interesting development of linear functional observers has been for the system subject to unknown inputs or disturbances, commonly known as unknown input functional observer (UIFO) or disturbance decoupled functional observer (DDFO) [60]. A huge amount of attention has been devoted to this class of system in the literature, be it full-order, minimal-order or reduced-order observer design (see, for example [62, 142, 13, 52, 90, 35, 102, 86]). More recently, necessary and sufficient conditions for the existence and design of an UIFO was given by [37] which was less restrictive as compared to the one of [62] as no satisfaction of the observer matching condition is required.

2.3 Survey of Functional Observers for Time-Delay Systems

Time delay can commonly exist in many parts of a system and is inherent in various engineering systems [109, 146]. In fact, time delay is said to be a source of instability and consequently renders any control or state estimation tasks difficult [97, 109, 12]. For a particular system with internal state delay, state observers, whether full-order, reduced-order or functional observers, had been proposed by a number of authors (see for example,
[81, 61, 103, 12, 114, 14, 25, 45, 41, 94] and the references therein). In addition, part of the necessary and sufficient conditions were also being derived by a few researchers [133, 24]. However, despite the intensive development and various works presented for the design of linear functional observers for systems influenced by internal state delays, when large-scale interconnected systems comprising many subsystems come into perspective, the consideration of the impact of time-delay would not merely be confined to the delayed internal state variables. Rather, delays occurring in the information that are transferred to a particular subsystem from its interrelated subsystems must be taken into serious consideration, instead of just within the single subsystem itself. Therefore, a more holistic approach must be devised to tackle the delay problems occurring in the process of multiple information transfer among the subsystems and this transmission could be the measurement information transmitted from other subsystems or the subsystems' interaction information received from the interconnections.

Unfortunately, a recent search in the literature suggests that the problem of designing an observer that takes into account the delayed output measurements from other subsystems has not been adequately addressed and hence no satisfactory solution has been found for this problem despite the limitations it presents to the area of control and automation. Thus, there is a pressing need for a solution to this issue considering its impact on most physical interconnected systems, which are expanding every year in terms of system complexity.
In this regard, the subsequent chapters of the thesis investigate the possibility of designing low-order linear functional observers that take into consideration the effects of time delays to the overall interconnected system. First, a constant delay will be assumed to simplify the development process and, more importantly, to assess the feasibility of the approach. In most real-life scenarios, time-delay would not always remain constant, rather they change over time. As such, a more pragmatic assumption of delay type will be used in the subsequent chapters, which in this case is a time-varying delay. The cases mentioned above have assumed that the value of delay is within a known range of values or functions. The final stage of the research would look into the practical problem of designing functional observers for interconnected systems in the presence of multiple time-varying delays, namely the delays occurring in the interconnections, within the subsystem itself, and during the transmission of output information from the remote subsystems.
Chapter 3

Design of Distributed Functional Observers for Interconnected Time-Delay Systems

This chapter deals with the practical aspects of reduced-order distributed functional state observers design for interconnected linear systems subject to time delays in the interconnections [79, 80]. Contrary to some estimation strategies which only take the ideal instantaneous output information into account, the proposed schemes incorporate output information that is inevitably encountered with time delays in the course of its transmission from the distanced subsystems. It is proved that such estimators possess less restrictive existence conditions with the acceptance of measurement data from other interrelated subsystems. Upon the satisfaction of the established existence conditions, it will be demonstrated through simple design procedures and simulation results that feasible observers can be realised for the given numerical systems.
3.1 Introduction

The challenging problem of decentralised state estimation and control has attracted enormous interest with extensive studies being conducted in the past decades (see, for example [119, 59, 113]). Classical state estimation approach generally involves the use of high-order centralised observers requiring a large amount of measurement data to be transmitted, processed and stored in the central processing station. In such circumstances, a much longer execution and communication time is needed as large volume of state information streamlined into the single central estimator from multiple subsystems.

From the standpoint of interconnected systems, several authors [59, 119] reported some interesting findings that the subsystems interconnection effects can be accounted for by treating them as disturbances or unknown inputs to the corresponding local subsystem. For example, in [113], the design strategy of a decentralised observer together with the observers existence conditions were presented and proved through simple direct approach. Despite the fact that simpler observer structure can be formulated based on such principal, nonetheless, it is well-known that such scheme suffers from an unnecessarily restrictive existence condition [59].

Delay occurs in any practical system and it affects the reliability, stability and performance of observers and controllers. It was shown in [82] that delay occurring in the real-time information exchange of a power system has inevitable impacts on the performance of its controller. Moreover, the interesting topic of state estimation with the presence of time delays in the output measurement and/or system equation has garnered serious attention and much
literature has been devoted to this problem [24, 48, 44, 45, 121, 138, 46, 96, 74]. In the case of [24], functional state reconstruction for systems with delays in state variables is achieved by incorporating internal delays into the observer structure. Furthermore, state estimation by means of a delay-free estimator reported in [133] allows for a simple observer structure which minimises computational efforts.

In the context of a large-scale composite system made up of various subsystems spreading across multiple locations, the state information exchange between these subsystems are not instantaneous considering the limited bandwidth of the communication channels. As such, state estimation problem has to take into consideration the inter-subsystem communication delay and data transfer latency.

The capability of estimating only the desired functional subset of system states have been made possible with the advent of functional observers [131]. Moreover, it has been demonstrated that low-dimensional state reconstruction schemes can be achieved through the deployment of functional observers [130, 131, 40, 139] to asymptotically estimate only the desired linear functions of the state vector. Such functional observers offer the potential of providing the missing state information so as to complement the state feedback control strategy implemented in, for instance, an output tracking controller [55]. This is particularly attractive for the cost-effective implementation of a state-feedback controller governing mechanical systems [89] which often requires the approximation of unmeasurable state variables from readily available ones. To the best of the author’s knowledge, the formulation of linear functional
estimator with output delays from the viewpoint of complex large-scale inter-
connected systems has not been reported. In fact, only a very limited amount
of work [122, 48] addressing the problem of state estimation through delayed
output and yet these solutions consider systems with no interconnected inter-
actions.

In anticipation of the aforementioned circumstances, this chapter seeks to
offer distributed functional observers design techniques for a class of inter-
connected systems while not neglecting the delays inherent in these physical
systems. The fundamental concept of our approach is that by having mea-
surement information transferred from other subsystems, the highly restrictive
existence conditions as in the case of a totally decentralised observer [113] ap-
plying unknown input observer (UIO) design scheme can be relaxed. However,
in practical situations, the output information transmitted from other remote
subsystem would experience a certain amount of time delay before it reaches
a local observer and the delay is non-negligible. Assuming that the delay is
known, an observer structure which accepts output information subject to a
specific unit of constant delay to account for the non-availability of instantan-
eous output information will be proposed in this chapter. It is worth noting
that even though this observer scheme requires the transmission of measure-
ment data from other subsystem, however, no exchange of information among
the local observers are required.

This chapter is organised as follows. The description of the class of sys-
tem of interest is stated in Section 3.2. In Section 3.3, the main results are
reported together with the associated derivations and proofs. The design procedure as well as a numerical example are presented in Sections 3.4.1 and 3.4.2, respectively. Finally, the conclusion is drawn in Section 3.6.

3.2 Problem Formulation

The class of interconnected linear time-invariant dynamical systems we are interested in is governed by

\[
\begin{align*}
\dot{x}_i(t) &= A_{ii}x_i(t) + B_iu_i(t) + \sum_{j=1, j\neq i}^{N} A_{ij}x_j(t - \tau_j) \\
y_i(t) &= C_ix_i(t) \\
z_i(t) &= F_ix_i(t); \quad i = 1, 2, \ldots, N,
\end{align*}
\]

where \(x_i(t) \in \mathbb{R}^{n_i}, u_i(t) \in \mathbb{R}^{m_i} \) and \(y_i(t) \in \mathbb{R}^{p_i} \) are the state, control input and the output measurement for a particular subsystem, respectively. The state information coming from its coupled/related subsystems is denoted by \(x_j(t) \in \mathbb{R}^{n_j} \). The subscript \(i = 1, 2, \ldots, N \) denotes the subsystem index in which \(N \) represents the total number of subsystems the composite system is composed of. \(z_i(t) \in \mathbb{R}^{q_i} \) is the linear functional to be estimated. Matrices \(A_{ii} \in \mathbb{R}^{n_i \times n_i}, A_{ij} \in \mathbb{R}^{n_i \times n_j}, B_i \in \mathbb{R}^{n_i \times m_i}, C_i \in \mathbb{R}^{p_i \times n_i} \) and \(F_i \in \mathbb{R}^{q_i \times n_i} \) are real known matrices. Without loss of generality, it is assumed that rank \(C_i = p_i\) and rank \(F_i = q_i\). Information arriving at a particular local subsystem from different neighbouring subsystems may experience different magnitude of time delay and this interconnection time delay is designated by the term \(\tau_j > 0\) with \(j \) representing the interconnected subsystem index where the information is originated from. The assumption is made such that the interconnection time
delays are constant but can be of different values.

For the simplicity of notational representation, the following two-area \((N = 2)\) linear time-invariant dynamical system will be used for the subsequent explanation.

\[
\begin{align*}
\dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t - \tau_2) + B_1u_1(t) \quad (3.4) \\
y_1(t) &= C_1x_1(t) \\
z_1(t) &= F_1x_1(t) \\
\dot{x}_2(t) &= A_{22}x_2(t) + A_{21}x_1(t - \tau_1) + B_2u_2(t) \quad (3.7) \\
y_2(t) &= C_2x_2(t) \\
z_2(t) &= F_2x_2(t) \quad (3.9)
\end{align*}
\]

where \(x_1(t) \in \mathbb{R}^{n_1}, u_1(t) \in \mathbb{R}^{m_1}\) and \(y_1(t) \in \mathbb{R}^{p_1}\) are the state, input and the output measurement for subsystem 1. Vectors \(x_2(t) \in \mathbb{R}^{n_2}, u_2(t) \in \mathbb{R}^{m_2}\) and \(y_2(t) \in \mathbb{R}^{p_2}\) are the state, input and the output measurement for subsystem 2, respectively. \(z_1(t) \in \mathbb{R}^{q_1}\) and \(z_2(t) \in \mathbb{R}^{q_2}\) are the linear functions to be estimated. Matrices \(A_{11} \in \mathbb{R}^{n_1 \times n_1}, A_{12} \in \mathbb{R}^{n_1 \times n_2}, A_{21} \in \mathbb{R}^{n_2 \times n_1}, A_{22} \in \mathbb{R}^{n_2 \times n_2},\)
\(B_1 \in \mathbb{R}^{n_1 \times m_1}, B_2 \in \mathbb{R}^{n_2 \times m_2}, C_1 \in \mathbb{R}^{p_1 \times n_1}, C_2 \in \mathbb{R}^{p_2 \times n_2}, F_1 \in \mathbb{R}^{q_1 \times n_1}\) and \(F_2 \in \mathbb{R}^{q_2 \times n_2}\) are real known matrices. Without loss of generality, it is assumed that rank \(C_i = p_i\) and rank \(F_i = q_i\) \((i = 1, 2)\).

### 3.3 Main Results

This chapter proposes two different observer structures, one of which is an improvement on the other. The first observer structure makes use of the delayed
output information generated locally, as well as the delayed input information obtained from the remote subsystem. Conversely, the second observer structure simplifies the observer design processes by eliminating the need for both the delayed local output and the delayed remote inputs to be transmitted to the local observer. This section begins with the explanation of the first observer structure with the second structure to be discussed in the following sections.

3.4 Observer Structure 1 (with additional delayed input and output information)

Derived on the basis of Luenberger observer [85], the structure of the proposed observer for subsystem 1 is shown in Figure 5.1 and can be mathematically expressed as

\[
\dot{z}_1(t) = w_1(t) + G_{11}y_1(t) + G_{12}y_2(t - \tau_2), \quad (3.10)
\]

\[
\dot{w}_1(t) = N_1w_1(t) + J_{11}y_1(t) + M_{11}y_1(t - \tau_1 - \tau_2) + J_{12}y_2(t - \tau_2) + H_1u_1(t) + Q_1u_2(t - \tau_2), \quad (3.11)
\]

where \( w_1(t) \in \mathbb{R}^{q_1} \) is the observer state vector and \( \dot{z}_1(t) \in \mathbb{R}^{q_1 \times 1} \) denotes the estimate of \( z_1(t) \). \( G_{11} \in \mathbb{R}^{q_1 \times p_1}, G_{12} \in \mathbb{R}^{q_1 \times p_2}, N_1 \in \mathbb{R}^{q_1 \times q_1}, J_{11} \in \mathbb{R}^{q_1 \times p_1}, M_{11} \in \mathbb{R}^{q_1 \times p_1}, J_{12} \in \mathbb{R}^{q_1 \times p_2}, Q_1 \in \mathbb{R}^{q_1 \times m_2} \) and \( H_1 \in \mathbb{R}^{q_1 \times m_1} \) are constant matrices to be computed based on the design procedure in the subsequent sections. \( \tau_1 > 0 \) is a constant interconnection time delay from subsystem 1.
$\tau_2 > 0$ is a constant output measurement delay and interconnection time delay from subsystem 2.

Figure 3.1: Block diagram of the proposed distributed functional observers
Taking into account the interconnected nature of the subsystems, a more holistic approach is employed in determining the functional state estimation error of a local area observer. As such, the estimation error dynamic equation is formulated with the consideration that the remote subsystem output information $y_2(t - \tau_2)$ has also been incorporated into the observer, apart from the locally obtainable ones. Next, the goal would be to ensure zero convergence of the estimation errors of the proposed local functional observer scheme in (3.10) and (3.11) for the class of subsystem described by (3.4)-(3.6). To accomplish this, sufficient conditions for the asymptotic convergence of the functional estimates are derived and stated in the following theorem.

**Theorem 3.4.1.** To ensure the asymptotic convergence of estimate $\hat{z}_1(t)$ to $z_1(t)$ for any initial condition $w_1(0)$ and any $u_1(t)$ and $u_2(t)$, the following conditions must hold.

\[
N_1 \text{ is Hurwitz,} \quad (3.12)
\]

\[
J_{11} C_1 + L_1 A_{11} - N_1 L_1 = 0, \quad (3.13)
\]

\[
M_{11} C_1 + G_{12} C_2 A_{21} = 0, \quad (3.14)
\]

\[
J_{12} C_2 + L_1 A_{12} + G_{12} C_2 A_{22} - N_1 G_{12} C_2 = 0, \quad (3.15)
\]

\[
H_1 + L_1 B_1 = 0, \quad (3.16)
\]

\[
Q_1 + G_{12} C_2 B_2 = 0. \quad (3.17)
\]

**Proof.** Define error vector $\varepsilon_1(t) \in \mathbb{R}^{n_1}$ as the error between $z_1(t)$ and $\hat{z}_1(t)$ as

\[
\varepsilon_1(t) = \hat{z}_1(t) - z_1(t) \quad (3.18)
\]

\[
w_1(t) = \varepsilon_1(t) - L_1 x_1(t) - G_{12} C_2 x_2(t - \tau_2),
\]

where $L_1 = (G_{11} C_1 - F_1) \in \mathbb{R}^{n_1 \times n_1}$. 
The time derivative of (3.18) or the error dynamics equation would be

\[
\dot{\varepsilon}_1(t) = N_1\varepsilon_1(t) + \{J_{11}C_1 + L_1A_{11} - N_1L_1\}x_1(t) \\
+ \{M_{11}C_1 + G_{12}C_2A_{21}\}x_1(t - \tau_1 - \tau_2) \\
+ \{J_{12}C_2 + L_1A_{12} + G_{12}C_2A_{22} - N_1G_{12}C_2\}x_2(t - \tau_2) \\
+ \{H_1 + L_1B_1\}u_1(t) \\
+ \{Q_1 + G_{12}C_2B_2\}u_2(t - \tau_2).
\]  

(3.19)

Accordingly, if conditions (3.12)-(3.17) of Theorem 3.4.1 are satisfied, the error converges to zero. This completes the proof of Theorem 3.4.1. \(\square\)

The ultimate goal of this observer construction rests on the solution of the unknown matrices, namely \(G_{11}, G_{12}, N_1, J_{11}, M_{11}, J_{12}, H_1\) and \(Q_1\), satisfying conditions (3.12)-(3.17). The non-trivial solutions for the unknown matrices is governed by the following theorem.

**Theorem 3.4.2.** There exists a stable \(q_1\) order local observer of the form (3.10)-(3.11) to estimate the unmeasurable state variables of subsystem 1 given by (3.4)-(3.6) if and only if

\[
\text{rank} \begin{bmatrix} \Psi_1 \\ \Phi_1 \end{bmatrix} = \text{rank}(\Psi_1),
\]  

(3.20)
where

\[
\Psi_1 = \begin{bmatrix}
F_{1(2)} & 0 & 0 \\
A_{11(b)} & 0 & C_1 A_{12} \\
0 & C_1 & 0 \\
0 & C_2 A_{21} & C_2 A_{22} \\
0 & 0 & C_2
\end{bmatrix} \in \mathbb{R}^{(q_1+2p_1+2p_2) \times (2n_1-p_1+n_2)}, \quad (3.21)
\]

\[
\Phi_1 = \begin{bmatrix}
F_{1(1)} A_{11(b)} + F_{1(2)} A_{11(d)} \\
0 \\
F_1 A_{12}
\end{bmatrix}^T \in \mathbb{R}^{q_1 \times (2n_1-p_1+n_2)}. \quad (3.22)
\]

**Proof.** Define a non-singular matrix \( P_1 \in \mathbb{R}^{n_1 \times n_1} \) as

\[
P_1 = \begin{bmatrix} C_1^+ & C_1^\perp \end{bmatrix}, \quad (3.23)
\]

where \( C_1^+ \in \mathbb{R}^{n_1 \times p_1} \) denotes the Moore-Penrose pseudoinverse of \( C_1 \) and \( C_1^\perp \in \mathbb{R}^{n_1 \times (n_1-p_1)} \) denotes an orthogonal basis for the null-space of \( C_1 \).

Therefore, we obtain

\[
C_1 P_1 = \begin{bmatrix} I_{p_1} & 0 \end{bmatrix}, \quad (3.24)
\]

\[
P_1^{-1} A_{11} P_1 = \begin{bmatrix} A_{11(a)} & A_{11(b)} \\
A_{11(c)} & A_{11(d)} \end{bmatrix}, \quad (3.25)
\]

\[
F_1 P_1 = \begin{bmatrix} F_{1(1)} & F_{1(2)} \end{bmatrix}, \quad (3.26)
\]

where the dimensions of the sub-matrices are \( A_{11(a)} \in \mathbb{R}^{p_1 \times p_1} \), \( A_{11(b)} \in \mathbb{R}^{p_1 \times (n_1-p_1)} \), \( A_{11(c)} \in \mathbb{R}^{(n_1-p_1) \times p_1} \), \( A_{11(d)} \in \mathbb{R}^{(n_1-p_1) \times (n_1-p_1)} \), \( F_{1(1)} \in \mathbb{R}^{q_1 \times p_1} \) and \( F_{1(2)} \in \mathbb{R}^{q_1 \times (n_1-p_1)} \). The notation \( I_{p_1} \in \mathbb{R}^{p_1 \times p_1} \) denotes a \( p_1 \) by \( p_1 \) identity matrix.

In respect to (3.24)-(3.26), it follows that equation (3.13) can be decomposed into the following sub-equations by post-multiplying that by \( P_1 \), thus
giving

\[ J_{11} = F_{1(1)}A_{11(a)} + F_{1(2)}A_{11(c)} + N_1 G_{11} - G_{11}A_{11(a)} - N_1 F_{1(1)}, \]  

(3.27)

and

\[ N_1 F_{1(2)} - F_{1(1)}A_{11(b)} - F_{1(2)}A_{11(d)} + G_{11} A_{11(b)} = 0. \]  

(3.28)

Furthermore, (3.15) can be represented as

\[ V_1 C_2 + G_{11} C_1 A_{12} - F_1 A_{12} + G_{12} C_2 A_{22} = 0, \]  

(3.29)

where \( V_1 = (J_{12} - N_1 G_{12}) \in \mathbb{R}^{q_1 \times p_2}. \)

Now, augmenting (3.28), (3.14) and (3.29) produces

\[ \Omega_1 \Psi_1 = \Phi_1, \]  

(3.30)

where

\[ \Omega_1 = \begin{bmatrix} N_1 & G_{11} & M_{11} & G_{12} & V_1 \end{bmatrix} \in \mathbb{R}^{q_1 \times (q_1+2p_1+2p_2)}. \]  

(3.31)

In a similar manner as [108], solution exists for \( \Omega_1 \) in (3.30) if the following holds

\[ \text{rank} \begin{bmatrix} \Psi_1 \\ \Phi_1 \end{bmatrix} = \text{rank} (\Psi_1). \]

As all the unknown matrices of interest reside in the vector \( \Omega_1 \) in (3.30), therefore if \( \Omega_1 \) is solved, all the unknown matrices could be computed. According to [108], the solution for (3.30) can be expressed as

\[ \Omega_1 = \Phi_1 \Psi_1^+ + Z_1 \left( I_{(q_1+2p_1+2p_2)} - \Psi_1 \Psi_1^+ \right), \]  

(3.32)
where \( Z_1 \in \mathbb{R}^{q_1 \times (q_1+2p_1+2p_2)} \) is an arbitrary matrix, \( \Psi_1^+ \in \mathbb{R}^{(2n_1-p_1+n_2)\times(q_1+2p_1+2p_2)} \) is the pseudo-inverse of \( \Psi_1 \).

Hence, the individual unknown matrices represented in standalone form would be

\[
\begin{align*}
N_1 &= N_{1(a)} + Z_1 N_{1(b)}, & (3.33) \\
G_{11} &= G_{11(a)} + Z_1 G_{11(b)}, & (3.34) \\
M_{11} &= M_{11(a)} + Z_1 M_{11(b)}, & (3.35) \\
G_{12} &= G_{12(a)} + Z_1 G_{12(b)}, & (3.36) \\
V_1 &= V_{1(a)} + Z_1 V_{1(b)}, & (3.37)
\end{align*}
\]

with \( N_{1(a)}, N_{1(b)}, G_{11(a)}, G_{11(b)}, M_{11(a)}, M_{11(b)}, G_{12(a)}, G_{12(b)}, V_{1(a)} \) and \( V_{1(b)} \) having appropriate dimensions being expressed as

\[
\begin{align*}
N_{1(a)} &= \Phi_1 \Psi_1^+ e_1, & (3.38) \\
N_{1(b)} &= (I_{q_1+2p_1+2p_2} - \Psi_1 \Psi_1^+) e_1, & (3.39) \\
G_{11(a)} &= \Phi_1 \Psi_1^+ e_2, & (3.40) \\
G_{11(b)} &= (I_{q_1+2p_1+2p_2} - \Psi_1 \Psi_1^+) e_2, & (3.41) \\
M_{11(a)} &= \Phi_1 \Psi_1^+ e_3, & (3.42) \\
M_{11(b)} &= (I_{q_1+2p_1+2p_2} - \Psi_1 \Psi_1^+) e_3, & (3.43) \\
G_{12(a)} &= \Phi_1 \Psi_1^+ e_4, & (3.44) \\
G_{12(b)} &= (I_{q_1+2p_1+2p_2} - \Psi_1 \Psi_1^+) e_4, & (3.45) \\
V_{1(a)} &= \Phi_1 \Psi_1^+ e_5, & (3.46) \\
V_{1(b)} &= (I_{q_1+2p_1+2p_2} - \Psi_1 \Psi_1^+) e_5, & (3.47)
\end{align*}
\]

in which \( e_1 \in \mathbb{R}^{(q_1+2p_1+2p_2)\times q_1}, e_2 \in \mathbb{R}^{(q_1+2p_1+2p_2)\times p_1}, e_3 \in \mathbb{R}^{(q_1+2p_1+2p_2)\times p_1}, \).
\( e_4 \in \mathbb{R}^{(q_1+2p_1+2p_2)\times p_2} \) and \( e_5 \in \mathbb{R}^{(q_1+2p_1+2p_2)\times p_2} \) are respectively given by

\[
e_1 = \begin{bmatrix} I_{q_1} & 0_{q_1,p_1} \\ 0_{p_1,q_1} & I_{p_1} \\ 0_{q_1,0} & 0_{p_1,0} \\ 0_{p_2,0} & 0_{p_2,0} \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0_{q_1,0} & I_{p_1} \\ 0_{p_1,0} & 0_{p_1,0} \\ 0_{q_1,0} & 0_{p_1,0} \\ 0_{p_2,0} & 0_{p_2,0} \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0_{q_1,0} & I_{p_1} \\ 0_{p_1,0} & 0_{p_1,0} \\ 0_{q_1,0} & 0_{p_1,0} \\ 0_{p_2,0} & 0_{p_2,0} \end{bmatrix},
\]

where \( I_\phi \) denotes an identity matrix of dimension \( \phi \) by \( \phi \) whereas \( 0_{\sigma,\rho} \) denotes a zero matrix of dimension \( \sigma \) by \( \rho \).

\( N_1 \) is Hurwitz if pair \((N_1(a),N_1(b))\) is detectable. Detectability of the pair \((N_1(a),N_1(b))\) is explicitly given by [26]

\[
\text{rank} \begin{bmatrix} sI_{q_1} - N_1(a) \\ N_1(b) \end{bmatrix} = q_1, \quad \forall s \in \mathbb{C}, \quad \text{Re}(s) \geq 0.
\]

(3.49)

If \((N_1(a),N_1(b))\) is observable, \(Z_1\) can be found by placing all the poles of \(N_1\) at pre-specified locations. Matrices \(N_1, G_{11}, M_{11}, G_{12}\) and \(V_1\) can then be found by substituting \(Z_1\) into (3.33), (3.34), (3.35), (3.36) and (3.37), respectively.

### 3.4.1 Design Procedure

**Algorithm 3.4.1. Distributed Functional Observer**

1. Compute \(P_1\) from (3.23) and submatrices \(A_{11(a)}, A_{11(b)}, A_{11(c)}, A_{11(d)}, F_1(1)\) and \(F_1(2)\) from (3.25) and (3.26).
2. Compute $\Psi_1$ and $\Phi_1$ from (3.21) and (3.22), respectively. From (3.38)-(3.47), obtain $N_{1(a)}$, $N_{1(b)}$, $G_{11(a)}$, $G_{11(b)}$, $M_{11(a)}$, $M_{11(b)}$, $G_{12(a)}$, $G_{12(b)}$, $V_{1(a)}$ and $V_{1(b)}$. Matrices $e_1$ to $e_5$ are defined in (3.48).

3. Next, check conditions (3.20) and (3.49). If not satisfied, stop, as no observer exists. Otherwise, go to the next step.

4. Determine $Z_1$ such that $N_1$ defined in (3.33) is stable. This can be done by arbitrarily picking the poles of $N_1$ to be at the left-half s-plane and compute $Z_1$ using pole-placement method if the pair is observable. $N_1$, $G_{11}$, $M_{11}$, $G_{12}$ and $V_1$ can subsequently be obtained from (3.33)-(3.37).

5. Next, compute $J_{11}$ from (3.27). Obtain $J_{12}$ from $J_{12} = V_1 + N_1 G_{12}$. $H_1$ and $Q_1$ are solved for according to (3.16) and (3.17), respectively.

3.4.2 Numerical Example

In order to demonstrate the efficacy of the proposed design method, a numerical example will be utilised. For brevity, the computation of only a single observer for one subsystem is presented even though the same technique has been applied to design the local observer for subsystem two. Let us consider a 7th-order system with 4 outputs and matrices $A_{11}$, $A_{12}$, $A_{21}$, $A_{22}$, $B_1$, $B_2$, $C_1$, $C_2$, $F_1$ and $F_2$ are as given below.
$A_{11} = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & -9 & 1 & -1 & 1 & 0 & 0 \\ 0 & 3 & -4 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & -5 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & -2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 4 \\ -1 \\ -2 \\ -5 \\ -8 \\ -1 \end{bmatrix}$

$A_{12} = \begin{bmatrix} -2 & 0 & 3 & -7 & 5 & -1 & 6 \\ -8 & -3 & -4 & -8 & 3 & -3 & 1 \\ -1 & -5 & 3 & 4 & 2 & 3 & -2 \\ 6 & 6 & 2 & 4 & -4 & 2 & 0 \\ 3 & -1 & 5 & 4 & -2 & -3 & -1 \\ -6 & -2 & 0 & 0 & 0 & 4 & -3 \\ 7 & 2 & -2 & -1 & 3 & -1 & 5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ -2 \\ -2 \\ -9 \\ -18 \\ 4 \\ 6 \end{bmatrix}$

$A_{21} = \begin{bmatrix} 2 & -2 & -4 & -2 & -2 & 0 & -4 \\ -2 & 2 & 4 & 2 & 2 & 0 & 4 \\ 1 & -1 & -2 & -1 & -1 & 0 & -2 \\ 0 & 4 & 3 & 3 & 0 & -1 & 4 \\ -3 & -1 & 3 & 0 & 3 & 1 & 2 \\ -1 & 1 & 2 & 1 & 1 & 0 & 2 \\ -1 & 1 & 2 & 1 & 1 & 0 & 2 \end{bmatrix}$

$A_{22} = \begin{bmatrix} -21 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -18 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$
\[ C_1 = C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} , \]

\[ F_1 = F_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} . \]

**Step 1:** \( P_1 \) is found to be \( P_1 = I_{7 \times 7} \). Submatrices \( A_{11(a)}, A_{11(b)}, A_{11(c)}, A_{11(d)}, F_{1(1)} \) and \( F_{1(2)} \), respectively are computed as

\[ A_{11(a)} = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 2 & -9 & 1 & -1 \\ 0 & 3 & -4 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} , \]

\[ A_{11(b)} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} , \]

\[ A_{11(c)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} , \]

\[ A_{11(d)} = \begin{bmatrix} -5 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & -2 \end{bmatrix} , \]

\[ F_{1(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } F_{1(2)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \]

**Step 2:** \( \Psi_1 \) and \( \Phi_1 \) are computed from (3.21) and (3.22). Matrices \( N_{1(a)}, N_{1(b)}, G_{11(a)}, G_{11(b)}, M_{11(a)}, M_{11(b)}, G_{12(a)}, G_{12(b)}, V_1(a) \) and \( V_1(b) \), respectively
are computed as

\[
N_{1(a)} = \begin{bmatrix}
-1.42 & 0.34 \\
-1.05 & -2.89
\end{bmatrix}, \quad N_{1(b)} = 0_{18 \times 2},
\]

\[
G_{11(a)} = \begin{bmatrix}
-0.12 & -0.76 & 0.76 & -0.34 \\
0.75 & 0.84 & 0.16 & 0.89
\end{bmatrix}, \quad G_{11(b)} = 0_{18 \times 4},
\]

\[
M_{11(a)} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad M_{11(b)} = 0_{18 \times 4},
\]

\[
G_{12(a)} = \begin{bmatrix}
0.33 & 0.23 & -0.19 & 0 \\
-0.45 & -0.06 & 0.79 & 0
\end{bmatrix}.
\]

**Step 3:** \( \text{rank} \begin{bmatrix}
\Psi_1 \\
\Phi_1
\end{bmatrix} = 16 \) and \( \text{rank}(\Psi_1) = 16 \), which means (3.20) is satisfied. (3.49) is satisfied as well since both

\[
q_1 = 2 \quad \text{and} \quad \text{rank} \begin{bmatrix}
sI_{r_1} - N_{1(a)} \\
N_{1(b)}
\end{bmatrix} = 2.
\]

**Step 4:** Since pair \( (N_{1(a)}, N_{1(b)}) \) are detectable, thus \( Z_1 \) is arbitrarily chosen as \( Z_1 = 0_{2 \times 18} \). The matrices obtained are

\[
N_1 = \begin{bmatrix}
-1.42 & 0.34 \\
-1.05 & -2.89
\end{bmatrix},
\]

\[
G_{11} = \begin{bmatrix}
-0.12 & -0.76 & 0.76 & -0.34 \\
0.75 & 0.84 & 0.16 & 0.89
\end{bmatrix},
\]

\[
M_{11} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
G_{12} = \begin{bmatrix}
0.33 & 0.24 & -0.19 & 0 \\
-0.45 & -0.06 & 0.79 & 0
\end{bmatrix},
\]

\[
V_1 = \begin{bmatrix}
-2.53 & 4.39 & -5.25 & -8.6 \\
0.47 & -0.78 & 0.78 & 6.7
\end{bmatrix}.
\]
Step 5: Matrices $J_{11}$, $J_{12}$, $H_1$ and $Q_1$ are computed to be

$$J_{11} = \begin{bmatrix} 2.82 & -7.76 & 2.78 & -0.87 \\ -2.97 & 6.45 & -1.46 & 0.55 \end{bmatrix},$$

$$J_{12} = \begin{bmatrix} -3.16 & 4.03 & -4.71 & -8.6 \\ 1.43 & -0.85 & -1.29 & 6.73 \end{bmatrix},$$

$$H_1 = \begin{bmatrix} -4.76 \\ -3.16 \end{bmatrix} \text{ and } Q_1 = \begin{bmatrix} -0.25 \\ 1.90 \end{bmatrix}.$$

This completes the observer design.

3.4.3 Simulation Results

The simulation for the proposed functional observers was performed and the responses of the estimated states relative to the corresponding actual states were depicted in Figures 3.3 - 3.6. Figure 3.2 shows the control input applied to the system. It is evident from these curves that, the reconstructed functional states track the actual functional states at different delay scenarios. The global asymptotic convergence of the observed functional states towards the true functional states has been demonstrated and confirmed. Thus, the designed observer having the form of (3.10)-(3.11) is proven to be working satisfactorily.
Figure 3.2: Input signal.
Figure 3.3: Actual, $z_1$ vs estimate, $\hat{z}_1$ for $\tau_1 = 0.01$, $\tau_2 = 0.05$. 
Figure 3.4: Actual, $z_1$ vs estimate, $\hat{z}_1$ for $\tau_1 = 0.1$, $\tau_2 = 0.5$. 
Figure 3.5: Actual, $z_2$ vs estimate, $\hat{z}_2$ for $\tau_1 = 0.01$, $\tau_2 = 0.05$. 
Figure 3.6: Actual, $z_2$ vs estimated, $\hat{z}_2$ for $\tau_1 = 0.1$, $\tau_2 = 0.5$. 
3.5 Observer Structure 2 (without additional delayed input and output information)

The observer structure given in Section 3.4 is effective when both the delayed remote input \( u_2(t - \tau_2) \) and the delayed local output \( y_1(t - \tau_1 - \tau_2) \) information are readily available to the observer. However, in the situation where it is not possible to obtain the knowledge of the local and remote time-delay information, such an observer structure will be unpractical for use. Therefore, this section provides an alternative structure that would perform the functional state estimation task without needing the aforementioned delayed information. As a result, a much simpler observer structure is formulated.

The compact form of the proposed observer structure for the composite system represented by (3.1)-(3.2) is given as

\[
\begin{align*}
\dot{\hat{z}}_i(t) & = w_i(t) + G_{ii}y_i(t), \\
\dot{w}_i(t) & = N_iw_i(t) + J_{ii}y_i(t) + \sum_{j=1, j\neq i}^{N} J_{ij}y_j(t - \tau_j) \\
& \quad + H_iu_i(t); \quad i = 1, 2, \ldots, N,
\end{align*}
\]

where \( w_i(t) \in \mathbb{R}^n \) is the observer state vector and \( \hat{z}_i(t) \in \mathbb{R}^{q_i \times 1} \) denotes the estimate of \( z_i(t) \). \( G_{ii} \in \mathbb{R}^{q_i \times p_i} \), \( N_i \in \mathbb{R}^{q_i \times q_i} \), \( J_{ii} \in \mathbb{R}^{q_i \times p_i} \), \( J_{ij} \in \mathbb{R}^{q_i \times p_j} \) and \( H_i \in \mathbb{R}^{q_i \times m_i} \) are the constant matrices to be computed following the design procedures in the subsequent sections. Considering the conjecture that it takes the same amount of time for information to travel from an adjacent subsystem to a particular subsystem as well as to its locally based observer, therefore, \( \tau_j \) also symbolises the output information transmission latency from the adjacent subsystem.
Figure 3.7: Block diagram of the two-area functional observers
It is worth noting that in constructing such an observer scheme, besides not requiring the transmission of remote delayed input information, it has also eliminated the need of possessing the knowledge of the local and remote time-delay values.

To facilitate understanding, the two-area \((N = 2)\) interconnected system (3.4)-(3.9) will be presented, even though the observer structure in question can be readily applied to systems with larger number of subsystems. A block diagram of such a system together with its associated local observers is illustrated in Figure 5.1. In order to avoid confusion, the explanation of our approach will be given from the perspective of subsystem 1 and its local observer.

In order to guarantee zero convergence of the estimation errors of the proposed local functional observer scheme in (3.50) and (3.51) for the class of subsystem described by (3.4)-(3.6), sufficient conditions for the asymptotic convergence of the functional estimates are derived and stated in the following theorem.

**Theorem 3.5.1.** The asymptotic convergence of \(\hat{z}_1(t)\) to \(z_1(t)\) is guaranteed for any initial condition of \(w_1(0)\) and any values for \(u_1(t)\) and \(u_2(t)\) if the following conditions are satisfied.

\[
N_1 \text{ is Hurwitz} \\
J_{11}C_1 + L_1A_{11} - N_1L_1 = 0 \\
J_{12}C_2 + L_1A_{12} = 0 \\
H_1 + L_1B_1 = 0.
\]

**Proof.** Define \(\varepsilon_1(t) \in \mathbb{R}^{n_1}\) as the error between \(z_1(t)\) and \(\hat{z}_1(t)\) as

\[
\varepsilon_1(t) = \hat{z}_1(t) - z_1(t).
\]
Substitute (3.50) and (3.3) into (3.56) (with \(i = 1\) for subsystem 1). Differentiating the resulted equation with respect to \(t\), and substituting (3.51) and (3.1) yields the error dynamics equation of

\[
\dot{\varepsilon}_1(t) = N_1 \varepsilon_1(t) + \{J_{11} C_1 + L_1 A_{11} - N_1 L_1\} x_1(t) \\
+ \{J_{12} C_2 + L_1 A_{12}\} x_2(t - \tau_2) \\
+ \{H_1 + L_1 B_1\} u_1(t),
\]

(3.57)

where \(L_1 = (G_{11} C_1 - F_1) \in \mathbb{R}^{q_1 \times n_1}\).

Hence, the error dynamics equation will be reduced to \(\dot{\varepsilon}_1(t) = N_1 \varepsilon_1(t)\) if conditions (3.53)-(3.55) of Theorem 3.5.1 are met, resulting in \(\varepsilon_1(t) \to 0\) regardless of any initial conditions and inputs. The proof for Theorem 3.5.1 is therefore completed.

\(\square\)

In a sense, it becomes obvious that this observer synthesis problem boils down to the solution of the unknown matrices, namely \(G_{11}, N_1, J_{11}, J_{12}\) and \(H_1\), satisfying conditions (3.52)-(3.55). If these matrices are indeed solvable, it follows that the proposed observer structure is valid and vice versa. The following theorem provides the prerequisite to obtain the non trivial solution for these unknown matrices.

**Theorem 3.5.2.** There exists a stable \(q_1\) order local observer of the form (3.50)-(3.51) to estimate the unmeasurable linear state functionals of subsystem 1 given by (3.1)-(3.3) provided that

\[
\text{rank} \left[ \begin{array}{c} \Psi_1 \\ \Phi_1 \end{array} \right] = \text{rank} (\Psi_1),
\]

(3.58)
where
\[
\Psi_1 = \begin{bmatrix} C_2 & 0 \\ C_1A_{12} & A_{11(b)} \\ 0 & F_{1(2)} \end{bmatrix} \in \mathbb{R}^{(p_2+p_1+q_1) \times (n_2+n_1-p_1)},
\]
(3.59)
\[
\Phi_1 = \begin{bmatrix} F_1A_{12} \\ F_{1(1)}A_{11(b)} + F_{1(2)}A_{11(d)} \end{bmatrix}^T \in \mathbb{R}^{q_1 \times (n_2+n_1-p_1)}.
\]
(3.60)

**Proof.** First, we define an invertible regular matrix \( \beta_1 \in \mathbb{R}^{n_1 \times n_1} \) as
\[
\beta_1 = \begin{bmatrix} C_1^+ & C_1^\perp \end{bmatrix},
\]
(3.61)
where \( C_1^\perp \in \mathbb{R}^{n_1 \times (n_1-p_1)} \) denotes an orthogonal basis for the null-space of \( C_1 \) and \( C_1^+ \in \mathbb{R}^{n_1 \times p_1} \) refers to the generalised inverse of \( C_1 \).

Next, we can subdivide the matrices \( C_1, A_{11} \) and \( F_1 \) to be as follows, in which
\[
C_1\beta_1 = \begin{bmatrix} I_{p_1} & 0 \end{bmatrix},
\]
(3.62)
\[
\beta_1^{-1}A_{11}\beta_1 = \begin{bmatrix} A_{11(a)} & A_{11(b)} \\ A_{11(c)} & A_{11(d)} \end{bmatrix},
\]
(3.63)
\[
F_1\beta_1 = \begin{bmatrix} F_{1(1)} & F_{1(2)} \end{bmatrix},
\]
(3.64)
where the notation \( I_{p_1} \in \mathbb{R}^{p_1 \times p_1} \) denotes a \( p_1 \) by \( p_1 \) identity matrix. The dimensions of the subdivided sub-matrices are \( A_{11(a)} \in \mathbb{R}^{p_1 \times p_1}, A_{11(b)} \in \mathbb{R}^{p_1 \times (n_1-p_1)}, A_{11(c)} \in \mathbb{R}^{(n_1-p_1) \times p_1}, A_{11(d)} \in \mathbb{R}^{(n_1-p_1) \times (n_1-p_1)}, F_{1(1)} \in \mathbb{R}^{q_1 \times p_1} \) and \( F_{1(2)} \in \mathbb{R}^{q_1 \times (n_1-p_1)} \).

In view of the relationship of (3.62)-(3.64), it follows that we can decompose equation (3.53) into the following sub-equations by post-multiplying that by
$\beta_1$, thus yielding

$$J_{11} = F_{1(1)}A_{11(a)} + F_{1(2)}A_{11(c)} + N_1G_{11} - G_{11}A_{11(a)} - N_1F_{1(1)},$$  \hspace{1cm} (3.65)$$

and

$$N_1F_{1(2)} - F_{1(1)}A_{11(b)} - F_{1(2)}A_{11(d)} + G_{11}A_{11(b)} = 0. \hspace{1cm} (3.66)$$

Now, rearranging and augmenting (3.66) and (3.54) together produces

$$\Omega_1\Psi_1 = \Phi_1, \hspace{1cm} (3.67)$$

where

$$\Omega_1 = \left[ \begin{array}{ccc} J_{12} & G_{11} & N_1 \end{array} \right] \in \mathbb{R}^{q_1 \times (p_2+p_1+q_1)}. \hspace{1cm} (3.68)$$

As suggested in [108], solution exists for $\Omega_1$ in (3.67) if the following equality holds

$$\text{rank} \left[ \begin{array}{c} \Psi_1 \\ \Phi_1 \end{array} \right] = \text{rank} (\Psi_1).$$

The vector $\Omega_1$ being part of (3.67) contains all the unknown matrices that we intend to decipher. It is logical to say all these unknown matrices will be uncovered as soon as $\Omega_1$ is solved. According to [108], the solution for (3.67) can be expressed as

$$\Omega_1 = \Phi_1\Psi_1^+ + Z_1 \left( I_{(p_2+p_1+q_1)} - \Psi_1\Psi_1^+ \right), \hspace{1cm} (3.69)$$

where $\Psi_1^+ \in \mathbb{R}^{(n_2+n_1-p_1) \times (p_2+p_1+q_1)}$ is the pseudo-inverse of $\Psi_1$ whereas $Z_1 \in \mathbb{R}^{q_1 \times (p_2+p_1+q_1)}$ is an arbitrary matrix.
Hence, in their standalone form, the individual unknown matrices could be expressed as

\[ J_{12} = J_{12(a)} + Z_1 J_{12(b)}, \quad (3.70) \]
\[ G_{11} = G_{11(a)} + Z_1 G_{11(b)}, \quad (3.71) \]
\[ N_1 = N_{1(a)} + Z_1 N_{1(b)}, \quad (3.72) \]

with \( J_{12(a)}, J_{12(b)}, G_{11(a)}, G_{11(b)}, N_{1(a)} \) and \( N_{1(b)} \) having appropriate dimensions being given by

\[ J_{12(a)} = \Phi_1 \Psi_1^+ e_1, \quad (3.73) \]
\[ J_{12(b)} = (I_{p_2+p_1+q_1} - \Psi_1 \Psi_1^+) e_1, \quad (3.74) \]
\[ G_{11(a)} = \Phi_1 \Psi_1^+ e_2, \quad (3.75) \]
\[ G_{11(b)} = (I_{p_2+p_1+q_1} - \Psi_1 \Psi_1^+) e_2, \quad (3.76) \]
\[ N_{1(a)} = \Phi_1 \Psi_1^+ e_3, \quad (3.77) \]
\[ N_{1(b)} = (I_{p_2+p_1+q_1} - \Psi_1 \Psi_1^+) e_3, \quad (3.78) \]

in which \( e_1 \in \mathbb{R}^{(p_2+p_1+q_1) \times p_2}, e_2 \in \mathbb{R}^{(p_2+p_1+q_1) \times p_1} \) and \( e_3 \in \mathbb{R}^{(p_2+p_1+q_1) \times q_1} \) are respectively specified by

\[ e_1 = \begin{bmatrix} I_{p_2} \\ 0_{p_1,p_2} \\ 0_{q_1,p_2} \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0_{p_2,p_1} \\ I_{p_1} \\ 0_{q_1,p_1} \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0_{p_2,q_1} \\ 0_{p_1,q_1} \\ I_{q_1} \end{bmatrix}, \quad (3.79) \]

where \( I_\mu \) denotes an identity matrix of dimension \( \mu \) by \( \mu \) whereas \( 0_{\rho,\sigma} \) denotes a zero matrix of dimension \( \rho \) by \( \sigma \).

If pair \((N_{1(a)}, N_{1(b)})\) is detectable, \( N_1 \) is said to be Hurwitz. Detectability
of the pair \( (N_{1(a)}, N_{1(b)}) \) is explicitly given by [26]

\[
\text{rank} \begin{bmatrix} sI_{q_1} - N_{1(a)} \\ N_{1(b)} \end{bmatrix} = q_1, \forall s \in \mathbb{C}, \ Re(s) \geq 0. \tag{3.80}
\]

Alternatively, if \( (N_{1(a)}, N_{1(b)}) \) is observable, \( Z_1 \) can be computed by arbitrarily placing all the eigenvalues of \( N_1 \) at pre-specified locations in the left-half \( s \)-plane using pole-placement method. Matrices \( J_{12}, G_{11} \) and \( N_1 \) can then be computed with the substitution of \( Z_1 \) into (3.70), (3.71) and (3.72), respectively.

### 3.5.1 Design Procedure

This section sets out the procedure for the establishment of a local area observer for subsystem 1.

**Algorithm 3.5.1. Distributed Functional Observer**

1. Find submatrices \( A_{11(a)}, A_{11(b)}, A_{11(c)}, A_{11(d)}, F_{1(1)} \) and \( F_{1(2)} \) according to (3.63) and (3.64) after obtaining \( \beta_1 \) from (3.61).

2. From (3.59) and (3.60), get \( \Psi_1 \) and \( \Phi_1 \). Next, obtain vectors \( e_1 \) to \( e_3 \) from (3.79) and then compute \( J_{12(a)}, J_{12(b)}, G_{11(a)}, G_{11(b)}, N_{1(a)} \) and \( N_{1(b)} \) from (3.73)-(3.78).

3. Determine the equality of equation (3.58) of Theorem 3.5.2 and the detectability of the pair \( (N_{1(a)}, N_{1(b)}) \) as defined in (3.80). If either condition is false, stop, as no observer exists. Otherwise, proceed to the next step.

4. Compute \( Z_1 \) such that \( N_1 \) defined in (3.72) is stable. \( J_{12}, G_{11} \) and \( N_1 \) can then be acquired from (3.70)-(3.72).

5. Finally, solve \( J_{11} \) and \( H_1 \) from (3.65) and (3.55), respectively.
3.5.2 Numerical Example

For the purpose of verifying the proposed observer design method, a numerical example has been carefully chosen to demonstrate the feasibility of the approach. Applying the step-by-step design algorithm stated in the previous section, two local observers corresponding to the two interconnected subsystems have been mathematically realised with the observers matrices successfully computed accordingly. A 14th-order composite system composed of two 7th-order subsystems with each subsystem having 3 outputs will be considered and the associated systems matrices $A_{11}$, $A_{12}$, $A_{21}$, $A_{22}$, $B_1$, $B_2$, $C_1$ and $C_2$ are given below. In this case, we are only interested in observing 3 state functionals denoted by $F_1$ and $F_2$ and thus reduced third-order observers will be formulated.

$$A_{11} = \begin{bmatrix}
-3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -5 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & -7 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & -1 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & -9 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & -2
\end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 4 \\ -1 \\ -1 \end{bmatrix},$$

$$A_{12} = \begin{bmatrix}
-4 & 2 & -3 & -6 & 4 & 4 & -3 \\
10 & -2 & 0 & 0 & -4 & -4 & 0 \\
2 & -8 & 5 & 3 & 5 & 5 & 6 \\
10 & 2 & 2 & 2 & -6 & -6 & 0 \\
0 & 2 & -9 & -8 & -2 & -2 & -7 \\
-4 & 0 & 0 & -2 & 4 & 4 & 0 \\
-2 & 0 & -5 & -2 & -2 & -2 & -3
\end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix},$$
$A_{21} = \begin{bmatrix}
1 & -1 & 9 & 7 & -1 & -4 & 7 \\
-3 & 0 & 1 & 3 & 5 & 0 & 2 \\
-2 & 1 & -8 & -3 & 3 & 2 & -4 \\
3 & -2 & 4 & 4 & 0 & -2 & 4 \\
-5 & 2 & -4 & -5 & 3 & 4 & -5 \\
4 & -1 & -2 & 4 & -2 & -4 & 3 \\
-3 & 1 & -7 & 1 & 5 & 0 & -1
\end{bmatrix}$, \\

$A_{22} = \begin{bmatrix}
-21 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -12 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -5 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & -18 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & -3 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & -2 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}$, \\

$C_1 = C_2 = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}$, \\

$F_1 = F_2 = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$. 
Step 1: Since $\beta_1 = I_{7 \times 7}$, submatrices $A_{11(a)}, A_{11(b)}, A_{11(c)}, A_{11(d)}, F_{1(1)}$ and $F_{1(2)}$ are found to be

$$A_{11(a)} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -5 \end{bmatrix},$$

$$A_{11(b)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

$$A_{11(c)} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$A_{11(d)} = \begin{bmatrix} -7 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -9 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix},$$

$$F_{1(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } F_{1(2)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Step 2: $\Psi_1$ and $\Phi_1$ are obtained from (3.59) and (3.60). Matrices $J_{12(a)}, J_{12(b)}, G_{11(a)}, G_{11(b)}, N_{1(a)}$ and $N_{1(b)}$, respectively are computed as

$$J_{12(a)} = \begin{bmatrix} -1.33 & -4 & -2.67 \\ 0.11 & 0.33 & 0.22 \\ -1.22 & -3.67 & -2.44 \end{bmatrix}, J_{12(b)} = 0_{9 \times 3},$$

$$G_{11(a)} = \begin{bmatrix} 1 & 0.67 & -0.67 \\ 0.44 & -0.28 & 0.22 \\ 0.11 & 0.06 & -0.44 \end{bmatrix}, G_{11(b)} = 0_{9 \times 3},$$

$$N_{1(a)} = \begin{bmatrix} -0.33 & 0.67 & -1 \\ -0.22 & -9.22 & 0 \\ 0.44 & 0.44 & -2 \end{bmatrix}, N_{1(b)} = 0_{9 \times 3}.$$
Step 3: Condition (3.58) and (3.80) are valid because

$$
\text{rank} \begin{bmatrix} \Psi_1 \\ \Phi_1 \end{bmatrix} = \text{rank}(\Psi_1) = 9, \text{ and }
$$

$$
\text{rank} \begin{bmatrix} sI_{r_1} - N_{1(a)} \\ N_{1(b)} \end{bmatrix} = 3 \text{ and } q_1 = 3.
$$

Step 4: Since pair \((N_{1(a)}, N_{1(b)})\) is detectable, thus \(Z_1\) is arbitrarily chosen as \(Z_1 = 0_{3 \times 10}\). The matrices obtained are

\[
J_{12} = \begin{bmatrix}
-1.3 & -4 & -2.7 \\
0.1 & 0.3 & 0.2 \\
-1.2 & -3.7 & -2.4
\end{bmatrix},
\]

\[
G_{11} = \begin{bmatrix}
1 & 0.7 & -0.7 \\
0.4 & -0.3 & 0.2 \\
0.1 & 0.1 & -0.4
\end{bmatrix}, N_1 = \begin{bmatrix}
-0.3 & 0.7 & -1 \\
-0.2 & -9.2 & 0 \\
0.4 & 0.4 & -2
\end{bmatrix}.
\]

Step 5: The solutions for matrices \(J_{11}\) and \(H_1\) are

\[
J_{11} = \begin{bmatrix}
3.85 & 4.87 & -2.52 \\
-2.99 & 1.19 & -0.79 \\
1.75 & 0.51 & -1.53
\end{bmatrix} \text{ and } H_1 = \begin{bmatrix}
-9.33 \\
-7.11 \\
-1.78
\end{bmatrix}.
\]

The observer design is thus completed.

In a similar manner to the way local observer for subsystem 1 is constructed, the local observer 2 matrices have been determined with reference to the same design procedure found in Algorithm 3.5.1. Hence, the observer structure for subsystem 2 is conveniently expressed as

\[
\dot{z}_2(t) = w_2(t) + G_{22}y_2(t), \quad (3.81)
\]

\[
\dot{w}_2(t) = N_2w_2(t) + J_{22}y_2(t) + J_{21}y_1(t - \tau_1) + H_2u_2(t) \quad (3.82)
\]
whereby the matrices \( G_{22}, N_2, J_{22}, J_{21} \) and \( H_2 \) are respectively found to be

\[
G_{22} = \begin{bmatrix}
  -1.4 & 0.8 & -0.8 \\
  2.2  & -1.4 &  2.4 \\
  1    &  0   &  2
\end{bmatrix},
\ N_2 = \begin{bmatrix}
  -2.2  & 0.8  & -1  \\
  -2.4  &-4.4  &  0  \\
  -2    &-2    & -1
\end{bmatrix},
\ J_{22} = \begin{bmatrix}
  -24.56 & 6.72 & -2.32 \\
  39.88  &-11.56&  3.36 \\
  19.4   & 1.2  &  4.8 
\end{bmatrix},
\ J_{21} = \begin{bmatrix}
  -2.8 & 1.4 & 1.4 \\
  2.4  &-1.2 &-1.2 \\
  0    &  0  &  0
\end{bmatrix},
\ H_2 = \begin{bmatrix}
  -16.6 \\
  3.8  \\
   9
\end{bmatrix}.
\]

### 3.5.3 Simulation Results

The developed observer scheme for the above-mentioned numerical system has been analysed and the performance has been evaluated by means of simulation. In order to display the effect time delays have on the observed variables, different constant interconnection or output time delays have been selected. As illustrated in Figure 5.6 and Figure 5.4 which correspond to the actual-vs-estimated state vectors of subsystem 1, all the functional state estimates are tracking the actual state variables despite having different \( \tau_1 \) values. The same thing happens to the state functionals of subsystem 2 and its functional estimates generated by the local observer as depicted in Figures 5.10 and 5.8.

It is therefore evident that the reconstructed functional states track the actual functional states under different interconnection delay scenarios. The global asymptotic convergence of the observed functional states towards the actual functional states has been demonstrated and ascertained. Thus, the designed observer having the form of (3.50)-(3.51) is proven to be working satisfactorily.
Figure 3.8: $z_1$ vs. $\hat{z}_1$ for $\tau_1 = 0.02, \tau_2 = 0.01$. 
Figure 3.9: $z_1$ vs. $\hat{z}_1$ for $\tau_1 = 0.03$, $\tau_2 = 0.01$. 
Figure 3.10: $z_2$ vs. $\hat{z}_2$ for $\tau_1 = 0.02$, $\tau_2 = 0.01$. 
Figure 3.11: $z_2$ vs. $\hat{z}_2$ for $\tau_1 = 0.03$, $\tau_2 = 0.01$. 
Figure 3.12: Estimation Error of $\hat{z}_1$. 
Figure 3.13: Estimation Error of $\hat{z}_2$. 
3.6 Conclusion

In this chapter the problem of linear functional observers design for interconnected systems, under the assumption that the interconnection time delays are constant, has been studied. The effects of the subsystems' interconnections in the presence of time delays, have been accounted for by utilising the delayed output measurements both locally and remotely. Two observer schemes have been proposed to deal with the class of interconnected time-delay system described by (3.1)-(3.2). This provides the flexibility of choosing the most appropriate observer structure under different scenarios, depending on the availability of the remote information. Numerical examples have been presented where the designed observers taking into account the effect of delayed output measurements were tested under different constant interconnection time delays. It has been shown through the simulation results, that the estimation errors converge to zero and the proposed technique is therefore proven to be effective.

In practical situations, not only are the entire interconnected systems be affected by the interconnection delays, in reality, within each individual subsystem, there also exists internal state delays. As a result, the task of devising an estimation scheme becomes undeniably more complex. Following this, the subsequent chapter deals with such challenges by introducing a new observer structure that is capable of accounting for both the subsystems' internal state delays and interconnection delays, by building on the results obtained in this chapter. The new scheme has also relaxed the existence conditions of the observers, rendering it capable of handling very strong interconnection effects.
like never before.
Chapter 4

A Practical Functional Observer Scheme for Interconnected Time-Delay Systems

This chapter proposes a partially-distributed functional observer scheme for a class of interconnected linear systems with a very strong non-instantaneous subsystem interaction and with time delays in the local states and in the transmission of output information from the remote subsystems [76]. A set of easily verifiable existence conditions is established and upon its satisfaction, simple distributed observers are designed using a straightforward design procedure. Simulation results of a numerical example are given to substantiate the feasibility of the approach.

4.1 Introduction

The rapid technological advancement and exponential growth in demand have played a major role in the ever-increasing size and complexity of physical systems. These large-scale systems are usually formed by multiple subsystems which are inter-related and interconnected. Even if they exist in a centralised
or integrated form, these high dimensional systems can be mathematically decomposed into a number of lower-order subsystems [36] and it is possible to design self-contained observers or controllers for each subsystem.

The subject of state estimation for large-scale interconnected systems [20, 9] has received considerable attention with a large body of literature [136, 119, 19, 129, 3, 141, 69, 5, 105] devoted to this topic. Notably, efforts have been concentrated on addressing the key observer design issues of inhibitive information structure imposed by the interconnection couplings; significant dimensional complexity of the observers; and various delays occurring in such large-scale systems.

In [123, 118, 7, 124, 129], information from other subsystems and (or) estimators have been exploited to counter the interconnection effects between the different subsystems, but this requires the exchange of estimates between the local observers which is not always possible and adds up to the data processing complication. To account for the subsystems’ interactions, [113, 59, 3], on the other hand, proposed observer structure that treats the subsystems’ interactions or interconnections as unknown inputs. However, these are only feasible for subsystems with weak couplings or are confined to certain interconnection patterns or convention. In [113], a totally-decentralised observer design technique leveraging from the well-established unknown input observer [13, 52, 62] approach was presented. Despite the advantage of possessing mutually independent local observers, without the requirement of information exchange among them, nonetheless, it is well-known that such a scheme suffers from unnecessarily restrictive existence conditions [59] and more so when
strongly-interconnected subsystems are involved.

It is worth mentioning that all the above-mentioned schemes are of either full-order or reduced-order structures, thus appear to be less attractive in state observation for large-scale systems, when only a particular set of state estimates is needed. The idea of functional observers [22, 24, 40, 111, 131, 39] offers an effective resolution to such a problem of high observer dimensionality.

To date, the problem of designing low-order distributed observers for interconnected time-delay systems, to the best of the author’s knowledge, has not yet received adequate attention. There are articles addressing the state observation problem of interconnected systems, but in-depth consideration has not been given to all the extensive range of delays present in the states and interconnections as well as delays encountered during the transmission of remote subsystems’ output measurements. It is therefore, the objective of this chapter to provide a solution approach to the afore-mentioned challenging issues of state reconstruction.

A brief outline of the chapter is as follows: Session 4.2 contains the discussion on the category of interconnected time-delay system under study. Following this the derivations of the sufficient conditions for the observer to exist; observer parameter matrices and their corresponding proofs are provided in Section 4.3. Subsequently, a step-by-step design algorithm for the partially-distributed observer is demonstrated with a numerical example and the computed observer matrices are shown. Finally, a conclusion is drawn in Section 4.5.
4.2 System Description

Consider the following class of interconnected time-delay systems

\[
\dot{x}_i(t) = A_{ii}x_i(t) + A_{di}x_i(t - \tau_{ii}) + \sum_{j=1, j \neq i}^{N} A_{ij}x_j(t - \tau_{ji}) + B_i u_i(t) \tag{4.1}
\]

\[
y_i(t) = C_i x_i(t) \tag{4.2}
\]

\[
z_i(t) = F_i x_i(t); \quad i = 1, 2, ..., N, \tag{4.3}
\]

where \(x_i(t) \in \mathbb{R}^{n_i}, \ u_i(t) \in \mathbb{R}^{m_i}, \) and \(y_i(t) \in \mathbb{R}^{p_i}\) are the state, input and the measured output for the \(i-th\) subsystem, respectively. The expression \(x_j(t) \in \mathbb{R}^{n_j}\) denotes the interconnection state information transmitted from other remote subsystems. The subscript \(i = 1, 2, ..., N\) denotes the subsystem index in which \(N\) representing the total number of subsystems the composite system is composed of. On the other hand, \(j\) is the index number for the remote subsystem that is linked to a particular \(i-th\) subsystem. The delayed internal state vector is given by the term \(x_i(t - \tau_{ii})\), with \(\tau_{ii} > 0\) indicating the time delay associated with the internal state vector for that specific local subsystem.

Matrices \(A_{ii} \in \mathbb{R}^{n_i \times n_i}, \ A_{di} \in \mathbb{R}^{n_i \times n_i}, \ A_{ij} \in \mathbb{R}^{n_i \times n_j}, \ B_i \in \mathbb{R}^{n_i \times m_i}, \ C_i \in \mathbb{R}^{p_i \times n_i}, \) and \(F_i \in \mathbb{R}^{q_i \times n_i}\) are real known system matrices. At a particular subsystem, information exchange from its distant interacting subsystems may encounter distinct latencies, depending on the distance between the transmitting and receiving ends; the magnitude of data traffics; and the network throughput or the bandwidth of the communication channels. Such an interconnection time delay is designated by the term \(\tau_{ji} > 0\) with \(j\) representing the identifier of the interconnected subsystem where the information is originated from. In this case, an assumption is made such that the interconnection time delays are
constant but can be of different values. The linear functions of the state vector to be reconstructed is represented by \( z_i(t) \in \mathbb{R}^{q_i} \). Without loss of generality, it is assumed that \( \text{rank } C_i = p_i \) and \( \text{rank } F_i = q_i \).

In this chapter, we propose a Darouach-type observer for interconnected systems with known delays, based on the general form of Darouach’s observer [22] and is built upon, but improved from the structure reported in [79]. This developed observer structure for multiple subsystems can be written compactly as

\[
\hat{z}_i(t) = w_i(t) + G_{ii}y_i(t), \quad (4.4)
\]

\[
\dot{w}_i(t) = N_iw_i(t) + J_{ii}y_i(t) + K_{ii}y_i(t - \tau_{ii}) + H_iu_i(t) + \sum_{j=1, j \neq i}^{N} J_{ij}y_j(t - \tau_{ji}); \quad i = 1, 2, ..., N, \quad (4.5)
\]

where \( \hat{z}_i(t) \in \mathbb{R}^{n_i} \) is the estimate of \( z_i(t) \). Vector \( w_i(t) \in \mathbb{R}^{n_i} \) constitutes the observer state variables and \( G_{ii} \in \mathbb{R}^{q_i \times p_i} \), \( N_i \in \mathbb{R}^{n_i \times q_i} \), \( J_{ii} \in \mathbb{R}^{n_i \times p_i} \), \( K_{ii} \in \mathbb{R}^{n_i \times p_i} \), \( J_{ij} \in \mathbb{R}^{n_i \times p_j} \) and \( H_i \in \mathbb{R}^{n_i \times m_i} \) are the unknown observer parameters in the form of constant matrices, that are to be solved in the process of designing the functional observers [22, 131]. Considering the conjecture that it takes the same amount of time for information to travel from an adjacent subsystem to a particular subsystem as well as to its locally based observer, therefore, \( \tau_{ji} \) also symbolises the output information transmission latency from the geographically distributed subsystem.

In order for this observer scheme to work, the time delay related to the internal state of the local subsystem (denoted as \( \tau_{ii} \)) is assumed to be known and constant. However, the knowledge of remote output time delay, defined
as $\tau_{ji}$, is optional. This is because the observer obtains the delayed remote output information $y_j(t - \tau_{ji})$ directly in its original form and that fulfils the input requirement of the term $\sum_{j=1, j\neq i}^{N} J_{ij} y_j(t - \tau_{ji})$ in the observer equation (5.13). Therefore, an assumption is made such that the internal state delay, that is, $\tau_{ii}$, for a local subsystem has to be known and constant whereas the remote output transport delays, namely, $\tau_{ji}$, are to be constant but can be unknown.

Despite being able to apply the observer scheme to systems with significant number of interacting subsystems, however, for simplicity purposes, a two-area ($N = 2$) interconnected system will be the subject of our discussion throughout this chapter. The corresponding state-space equations for a two-area interconnected system based on the generic form of (5.7) and (5.8) can be elaborated as

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{d11}x_1(t - \tau_{11}) + A_{12}x_2(t - \tau_{21}) + B_1u_1(t), \quad (4.6)$$
$$y_1(t) = C_1x_1(t), \quad (4.7)$$
$$\dot{x}_2(t) = A_{22}x_2(t) + A_{d22}x_2(t - \tau_{22}) + A_{21}x_1(t - \tau_{12}) + B_2u_2(t), \quad (4.8)$$
$$y_2(t) = C_2x_2(t). \quad (4.9)$$

For estimating the desired states of the two subsystems, two local observers are to be constructed for each subsystem. Based on equations (5.12) and (5.13),
the mathematical expressions of the observer structures are

\[
\begin{align*}
\dot{z}_1(t) &= w_1(t) + G_{11}y_1(t), \\
\dot{w}_1(t) &= N_1w_1(t) + J_{11}y_1(t) + K_{11}y_1(t - \tau_{11}) + J_{12}y_2(t - \tau_{21}) + H_1u_1(t), \\
\dot{z}_2(t) &= w_2(t) + G_{22}y_2(t), \\
\dot{w}_2(t) &= N_2w_2(t) + J_{22}y_2(t) + K_{22}y_2(t - \tau_{22}) + J_{21}y_1(t - \tau_{12}) + H_2u_2(t).
\end{align*}
\]

As our objective is to obtain an approximation of a particular linear functional state of the state vector for each subsystem, we define these linear functions for both subsystems as

\[
\begin{align*}
z_1(t) &= F_1x_1(t), \\
z_2(t) &= F_2x_2(t).
\end{align*}
\]

A block diagram of the system under consideration and its corresponding local observers is illustrated in Fig 5.1. For simplicity, the elucidation of our approach will be done from the standpoint of subsystem 1 and its local observer.

As depicted in Fig 5.1, the system of interest consists of two subsystems linked together via the interconnections. These interconnections form the communication path for the information exchange between these subsystems, which are normally located far apart from one another. As a result of that, there will be potential data transmission latencies and are denoted by both \(\tau_{12}\) and \(\tau_{21}\). Not only does time delay occur in the interconnections, but if one
Figure 4.1: Block diagram of two decoupled functional observers
concentrates on a single subsystem, say subsystem 1, one would realise that delay occurs within the internal state vector of the subsystem as well, and that corresponds to $\tau_{11}$ in equation (4.6).

For the purpose of estimating the linear functions of the state vector of such a subsystem, in this case subsystem 1, the functional observer 1 will require local input information, $u_1(t)$, local output information, $y_1(t)$, as well as the output information from the remote subsystem(s), $y_2(t - \tau_{21})$, which under practical scenario, experiences a certain amount of time delay during its transmission from a distant subsystem. In the block diagram, such a delay is represented by $\tau_{21}$. It is worth noting that while instantaneous local output is available, nevertheless the observer will still employ delayed local output, in which the delay, indicated as $\tau_{11}$, is equivalent to the internal state delay of subsystem 1. The reasoning behind this is to create a less conservative coupled equation which is capable of countering the effect of the internal state delay by having delayed local output information sent to the local observer. Such a coupled equation will be derived from the error dynamic equation and will be demonstrated later.

4.3 Main Results

In this section, an elaboration on the straightforward design techniques for observer having the form of (4.10)-(4.11) will be given. This encompasses on i) the formulation of observer’s estimation error equation; ii) the specification of conditions ensuring the asymptotic convergence of the state estimates based on the error equation; and iii) the computation of the unknown observer matrices
satisfying such defined conditions.

First, we establish the sufficient conditions for ensuring that the functional estimates \( \hat{z}_1(t) \), converge asymptotically towards the true functional state \( z_1(t) \). The following theorem states these delay-free conditions and the proof section describes how we arrived at such a theorem.

**Theorem 4.3.1.** The following delay-independent conditions will guarantee that the observer’s estimation error, \( \varepsilon_1(t) \), converges asymptotically to zero for any initial condition of \( w_1(0) \) and any value for \( u_1(t) \).

\[
\begin{align*}
N_1 & \text{ is Hurwitz} \quad (4.16) \\
J_{11}C_1 + [G_{11}C_1 - F_1]A_{11} - N_1[G_{11}C_1 - F_1] &= 0 \quad (4.17) \\
K_{11}C_1 + [G_{11}C_1 - F_1]A_{d11} &= 0 \quad (4.18) \\
J_{12}C_2 + [G_{11}C_1 - F_1]A_{12} &= 0 \quad (4.19) \\
H_1 + [G_{11}C_1 - F_1]B_1 &= 0. \quad (4.20)
\end{align*}
\]

**Proof.** Define \( \varepsilon_1(t) \in \mathbb{R}^{n_1} \) to be the error between \( z_1(t) \) and \( \hat{z}_1(t) \) as

\[
\varepsilon_1(t) = \hat{z}_1(t) - z_1(t). \quad (4.21)
\]

Substituting (4.14) and (4.10) into (4.21), we obtain

\[
\varepsilon_1(t) = w_1(t) + [G_{11}C_1 - F_1]x_1(t). \quad (4.22)
\]

Rearranging the above equation gives

\[
w_1(t) = \varepsilon_1(t) - [G_{11}C_1 - F_1]x_1(t). \quad (4.23)
\]

Taking the time derivative of (4.22) and subsequently substituting (4.11) and (4.6), followed by replacing (4.23) into the resultant equation, the error
If conditions (5.16)-(5.19) of Theorem 4.3.1 are satisfied, the error dynamics equation (5.24) will be reduced to \( \dot{\varepsilon}_1(t) = N_1 \varepsilon_1(t) \). Provided that \( N_1 \) is stable, or (5.15) is met, it follows that \( \varepsilon_1(t) \to 0 \) as \( t \to \infty \), regardless of any initial conditions and inputs. This concludes the proof for Theorem 4.3.1.

It can be seen that, due to the interconnected nature of the subsystems, the estimation error dynamic equation has been formulated with the consideration that the delayed output information \( y_2(t - \tau_{21}) \) of the remote subsystem has also been incorporated into the observer, apart from the locally obtainable ones.

Obviously, the goal of designing an observer has been narrowed down to solving the multiple coupled equations of (5.16)-(5.19) defined in Theorem 4.3.1 to obtain the unknown parameter matrices, namely \( G_{11}, N_1, J_{11}, K_{11}, J_{12} \) and \( H_1 \). The viability of constructing such an observer is very much dependent on the solvability of these unknown observer parameters, in that no observer of
such a structure could be formulated if such matrices have no known solutions. As such, it is vital to have a means of determining if this is possible beforehand, and Theorem 4.3.2 offers an instrument for the investigation of whether or not an observer can exist in an early stage before the observer construction even takes place. Such means will only make use of sub-matrices that can be easily decomposed from the available known system matrices and therefore no excessive computation is required.

To derive Theorem 4.3.2, first, the system matrices have to be decomposed into sub-matrices. In order to do so, we define an invertible regular matrix $T_1 \in \mathbb{R}^{n_1 \times n_1}$ as

$$T_1 = \begin{bmatrix} C_1^+ & C_1^\perp \end{bmatrix},$$

(4.25)

where $C_1^\perp \in \mathbb{R}^{n_1 \times (n_1 - p_1)}$ denotes an orthogonal basis for the null-space of $C_1$ and $C_1^+ \in \mathbb{R}^{n_1 \times p_1}$ refers to the generalised inverse of $C_1$.

Next, we can subdivide the matrices $C_1$, $A_{11}$ and $F_1$ to be as follows, in which

$$C_1 T_1 = \begin{bmatrix} I_{p_1} & 0 \end{bmatrix},$$

(4.26)

$$T_1^{-1} A_{11} T_1 = \begin{bmatrix} A_{11(a)} & A_{11(b)} \\ A_{11(c)} & A_{11(d)} \end{bmatrix},$$

(4.27)

$$F_1 T_1 = \begin{bmatrix} F_{1(1)} & F_{1(2)} \end{bmatrix},$$

(4.28)

where the notation $I_{p_1} \in \mathbb{R}^{p_1 \times p_1}$ denotes a $p_1$ by $p_1$ identity matrix. The dimensions of the subdivided sub-matrices are $A_{11(a)} \in \mathbb{R}^{p_1 \times p_1}$, $A_{11(b)} \in \mathbb{R}^{p_1 \times (n_1 - p_1)}$, $A_{11(c)} \in \mathbb{R}^{(n_1 - p_1) \times p_1}$, $A_{11(d)} \in \mathbb{R}^{(n_1 - p_1) \times (n_1 - p_1)}$, $F_{1(1)} \in \mathbb{R}^{p_1 \times p_1}$ and $F_{1(2)} \in \mathbb{R}^{q_1 \times (n_1 - p_1)}$. 
The usefulness of these subdivided sub-matrices will be evident once the reader has reached the Proof section of Theorem 4.3.2.

We now move our focus to solve the said unknown parameter matrices. The following theorem provides a condition for the solvability of $G_{11}$, $N_1$, $J_{11}$, $K_{11}$, $J_{12}$ and $H_1$, in such a way that the solutions conform with conditions stipulated in Theorem 4.3.1. The solvability of these observer matrices, in a way determines the possibility for the existence of the proposed observer scheme.

**Theorem 4.3.2.** There exists a stable $q_1$-th order local observer of the form (4.10)-(4.11) to estimate the unmeasurable linear state functionals of subsystem 1 given by (4.6)-(4.7) provided that

$$\text{rank} \begin{bmatrix} \Psi_1 \\ \Phi_1 \end{bmatrix} = \text{rank} (\Psi_1), \quad (4.29)$$

where

$$\Psi_1 = \begin{bmatrix} F_{1(2)} & 0 & 0 \\ A_{11(b)} & C_1A_{d11} & C_1A_{12} \\ 0 & C_1 & 0 \\ 0 & 0 & C_2 \end{bmatrix} \in \mathbb{R}^{(q_1+2p_1+p_2) \times (2n_1-p_1+n_2)}, \quad (4.30)$$

$$\Phi_1 = \begin{bmatrix} F_{1(1)}A_{11(b)} + F_{1(2)}A_{11(d)} \\ F_1A_{d11} \\ F_1A_{12} \end{bmatrix}^T \in \mathbb{R}^{q_1 \times (2n_1-p_1+n_2)}. \quad (4.31)$$

**Proof.** In view of the relationship of (4.26)-(4.28), it follows that we can decompose equation (5.16) into the following sub-equations by post-multiplying
that by $T_1$, thus yielding

$$J_{11} = F_{1(1)}A_{11(a)} + F_{1(2)}A_{11(c)} + N_1G_{11} - G_{11}A_{11(a)} - N_1F_{1(1)}, \quad (4.32)$$

and

$$N_1F_{1(2)} - F_{1(1)}A_{11(b)} - F_{1(2)}A_{11(d)} + G_{11}A_{11(b)} = 0. \quad (4.33)$$

Now, rearranging and augmenting (4.33), (5.17) and (5.18) together produces

$$\Omega_1 \Psi_1 = \Phi_1, \quad (4.34)$$

where

$$\Omega_1 = \begin{bmatrix} N_1 & G_{11} & K_{11} & J_{12} \end{bmatrix} \in \mathbb{R}^{q_1 \times (q_1 + 2p_1 + p_2)}. \quad (4.35)$$

As suggested in [108], solution exists for $\Omega_1$ in (4.34) if the following equality holds

$$\text{rank} \begin{bmatrix} \Psi_1 \\ \Phi_1 \end{bmatrix} = \text{rank} (\Psi_1). \quad (4.36)$$

The vector $\Omega_1$ being part of (4.34) contains all the unknown matrices that we intend to decipher. It is logical to say all these unknown matrices will be uncovered as soon as $\Omega_1$ is solved. According to [108], the solution for (4.34) can be expressed as

$$\Omega_1 = \Phi_1 \Psi_1^+ + Z_1 \left( I_{(q_1 + 2p_1 + p_2)} - \Psi_1 \Psi_1^+ \right), \quad (4.37)$$

where $\Psi_1^+ \in \mathbb{R}^{(2n_1 - p_1 + n_2) \times (q_1 + 2p_1 + p_2)}$ is the pseudo-inverse of $\Psi_1$ whereas $Z_1 \in \mathbb{R}^{q_1 \times (q_1 + 2p_1 + p_2)}$ is an arbitrary matrix.
Hence, in their standalone form, the individual unknown matrices could be expressed as

\[ N_1 = N_{1(a)} + Z_1 N_{1(b)}, \quad (4.38) \]
\[ G_{11} = G_{11(a)} + Z_1 G_{11(b)}, \quad (4.39) \]
\[ K_{11} = K_{11(a)} + Z_1 K_{11(b)}, \quad (4.40) \]
\[ J_{12} = J_{12(a)} + Z_1 J_{12(b)}, \quad (4.41) \]

with \( N_{1(a)}, N_{1(b)}, G_{11(a)}, G_{11(b)}, K_{11(a)}, K_{11(b)}, J_{12(a)} \) and \( J_{12(b)} \) having appropriate dimensions being given by

\[ N_{1(a)} = \Phi_1 \Psi_1^+ e_1, \quad N_{1(b)} = (I_{q_1+2p_1+p_2} - \Psi_1 \Psi_1^+) e_1, \quad (4.42) \]
\[ G_{11(a)} = \Phi_1 \Psi_1^+ e_2, \quad G_{11(b)} = (I_{q_1+2p_1+p_2} - \Psi_1 \Psi_1^+) e_2, \quad (4.43) \]
\[ K_{11(a)} = \Phi_1 \Psi_1^+ e_3, \quad K_{11(b)} = (I_{q_1+2p_1+p_2} - \Psi_1 \Psi_1^+) e_3, \quad (4.44) \]
\[ J_{12(a)} = \Phi_1 \Psi_1^+ e_4, \quad J_{12(b)} = (I_{q_1+2p_1+p_2} - \Psi_1 \Psi_1^+) e_4, \quad (4.45) \]

in which

\[ e_1 \in \mathbb{R}^{(q_1+2p_1+p_2) \times q_1}, \quad e_2 \in \mathbb{R}^{(q_1+2p_1+p_2) \times p_1}, \quad e_3 \in \mathbb{R}^{(q_1+2p_1+p_2) \times p_1} \] and

\[ e_4 \in \mathbb{R}^{(q_1+2p_1+p_2) \times p_2} \] are respectively specified by

\[ e_1 = \begin{bmatrix} I_{q_1} \\ 0_{p_1,q_1} \\ 0_{p_1,q_1} \\ 0_{p_2,q_1} \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0_{q_1,p_1} \\ I_{p_1} \\ 0_{p_1,p_1} \\ 0_{p_2,p_1} \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0_{q_1,p_1} \\ 0_{p_1,p_1} \\ I_{p_1} \\ 0_{p_2,p_1} \end{bmatrix}, \quad e_4 = \begin{bmatrix} 0_{q_1,p_2} \\ 0_{p_1,p_2} \\ 0_{p_1,p_2} \\ I_{p_2} \end{bmatrix}, \quad (4.46) \]

where \( I_\mu \) denotes an identity matrix of dimension \( \mu \) by \( \mu \) whereas \( 0_{\rho,\sigma} \) denotes a zero matrix of dimension \( \rho \) by \( \sigma \).

To satisfy condition (5.15) of Theorem 4.3.1, the matrix \( N_1 \) must be stable. Considering the breakdown form of \( N_1 \) in (4.38), it follows that if pair
The pair \((N_{1(a)}, N_{1(b)})\) is detectable, \(N_1\) is said to be a Hurwitz matrix or stable. Conversely, if the pair is not detectable or observable, an observer with zero estimation error is not achievable. The following statement provides the definition of detectability for \(N_1\).

Detectability of the pair \((N_{1(a)}, N_{1(b)})\) is explicitly given by [26]

\[
\text{rank} \begin{bmatrix} sI_{q_1} - N_{1(a)} \\ N_{1(b)} \end{bmatrix} = q_1, \quad \forall s \in \mathbb{C}, \quad \text{Re}(s) \geq 0. \tag{4.47}
\]

Alternatively, if \((N_{1(a)}, N_{1(b)})\) is observable, \(Z_1\) can be computed by arbitrarily placing all the eigenvalues of \(N_1\) at pre-specified locations in the left-half \(s\)-plane using pole-placement method. Matrices \(N_1, G_{11}, K_{11}\) and \(J_{12}\) can then be computed with the substitution of \(Z_1\) into (4.41), (4.39), (4.40) and (4.38), respectively.

4.3.1 Design Procedure

For the purpose of assisting the readers to systematically construct such observers, this section sets out, in a chronological order, a step-by-step procedure for the establishment of a local area observer for subsystem 1. The same procedure can then be reiterated for other local observers, say, local observer 2. \((i = 2)\)

Algorithm 4.3.1. Distributed Functional Observer \((i=1)\)

1. Based on (4.27) and (4.28), decompose matrices \(A_{11}\) and \(F_1\) into submatrices \(A_{11(a)}, A_{11(b)}, A_{11(c)}, A_{11(d)}, F_{1(1)}\) and \(F_{1(2)}\) after obtaining \(T_{1}\) from (4.25).
2. Determine $\Psi_1$ and $\Phi_1$ from (5.26) and (5.27) and subsequently evaluate the quality of (4.29) in Theorem 4.3.2. If the equality is invalid, increase the dimension of $F_1$ and begin again from Step 1. If the increment does not work, no observer is available. Otherwise, proceed to the next step.

3. Compute vectors $\epsilon_1$ to $\epsilon_4$ from (4.46) and hence matrices $N_{1(a)}, N_{1(b)}, G_{11(a)}, G_{11(b)}, K_{11(a)}, K_{11(b)}, J_{12(a)}$ and $J_{12(b)}$ can then be obtained according to (4.42)-(4.45).

4. Evaluate the detectability of the pair $(N_{1(a)},N_{1(b)})$ as defined in (4.47). If the condition is untrue, stop, as no observer exists.

5. Compute $Z_1$ such that $N_1$ defined in (4.38) is stable. $N_1, G_{11}, K_{11}$ and $J_{12}$ can then be acquired from (4.38)-(4.41).

6. Finally, solve $J_{11}$ and $H_1$ from (4.32) and (5.19), respectively. Thus, all the observer parameter matrices $G_{11}, N_1, J_{11}, J_{12}, H_1$ and $K_{11}$ are now obtained.

4.4 Numerical Example

A numerical example of a complex 14-th order composite system comprising of two strongly-coupled subsystems will be employed to demonstrate and validate the proposed observer design approach. Since this example involves the estimation of one state functional as given by the matrix $F_1$, the construction of a first-order observer using the aforementioned design algorithm, will be displayed. The system matrices $A_{11}, A_{12}, A_{d1}, A_{d2}, A_{21}, A_{22}, B_1, B_2, C_1$ and
$C_2$ are as follows.

$$A_{11} = \begin{bmatrix} -14.5 & 0 & 2.34 & 0 & 0 & 0 & 0 \\ 0 & -12.8 & 0 & 6.84 & 0 & -5.15 & 0 \\ 0 & 0 & -1.3 & 0 & 1.1 & 1.21 & 0 \\ 0 & -3.32 & 0 & -17.3 & 0 & 1.34 & 0 \\ 1.93 & 0 & 0 & 5.37 & -1.2 & 0 & -1.71 \\ 0 & 1.04 & 2.55 & 0 & 0 & -15.1 & 0 \\ 1.87 & 0 & 0 & 2.79 & 0 & -2.32 & -10.2 \end{bmatrix},$$

$$A_{12} = \begin{bmatrix} -10.3 & 3.22 & 7.18 & -2.23 & -1.1 & -4.3 & 3.22 \\ 6.38 & -4.3 & -8.1 & -4.72 & -2.7 & -2.8 & -8.4 \\ -11.3 & 0 & 5.02 & -10.5 & -2.4 & -5.6 & -6.2 \\ 2.19 & 2.42 & -2.1 & -3.08 & 3.37 & -5.3 & -3.1 \\ 5.32 & 5.71 & 1.52 & 10.1 & 4.03 & 3.31 & 13.2 \\ 3.21 & -3.4 & -4.2 & 2.82 & -3.7 & -1.9 & -3.3 \\ -2.16 & -5.1 & -3.7 & 1.22 & -7.3 & -6.2 & -8.1 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} 5.24 & 3.24 & 4.24 & 0 & -3.1 & -3.48 & 4.57 \\ 3.25 & 0 & -2.92 & 4.25 & 1.52 & 3.92 & 1.64 \\ -4.2 & -4.6 & -10.5 & 2.02 & 9.68 & 6.16 & -6.6 \\ 2.57 & -4.9 & -4.2 & 4.95 & -4.2 & 2.05 & 2.78 \\ 3.07 & 1.54 & 6.15 & -6.1 & 2.03 & -2.04 & 0 \\ 1.03 & 2.24 & 4.84 & -4.9 & -5.3 & -7.47 & 3.15 \\ 5.28 & 5.24 & 4.72 & 6.53 & -2.5 & 2.61 & 4.84 \end{bmatrix},$$

The execution of the step-by-step computational procedure is given along with the values of the computed matrices.

**Step 1:** Since $\beta_1 = I_{7 \times 7}$, submatrices $A_{11(a)}$, $A_{11(b)}$, $A_{11(c)}$, $A_{11(d)}$, $F_{1(1)}$ and $F_{1(2)}$ are found to be

$$A_{11(a)} = \begin{bmatrix} -14.5 & 0 & 2.34 & 0 & 0 \\ 0 & -12.8 & 0 & 6.84 & 0 \\ 0 & 0 & -1.3 & 0 & 1.1 \\ 0 & -3.32 & 0 & -17.3 & 0 \\ 1.93 & 0 & 0 & 5.37 & -1.2 \end{bmatrix}, \quad A_{11(b)} = \begin{bmatrix} 1.21 & 0 \\ 0 & -1.71 \end{bmatrix}$$
\[
A_{11(c)} = \begin{bmatrix}
0 & 1.04 & 2.55 & 0 & 0 \\
1.87 & 0 & 0 & 2.79 & 0
\end{bmatrix}, \quad A_{11(d)} = \begin{bmatrix}
-15.1 & 0 \\
-2.32 & -10.2
\end{bmatrix},
\]

\[
F_{1(1)} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0
\end{bmatrix} \quad \text{and} \quad F_{1(2)} = \begin{bmatrix}
0 & 1
\end{bmatrix}.
\]

**Step 2:** \(\Psi_1\) and \(\Phi_1\) are obtained from (5.26) and (5.27). The equality of (4.29) holds as

\[
\text{rank} \begin{bmatrix}
\Psi_1 \\
\Phi_1
\end{bmatrix} = \text{rank}(\Psi_1) = 16.
\]

**Step 3:** Matrices \(J_{12(a)}, J_{12(b)}, G_{11(a)}, G_{11(b)}, N_{1(a)}\) and \(N_{1(b)}\), respectively are computed as

\[
J_{12(a)} = \begin{bmatrix}
7.14 & -2.21 & -5.59 & 9.7 & -2.55
\end{bmatrix}, \quad J_{12(b)} = 0_{16 \times 5},
\]

\[
G_{11(a)} = \begin{bmatrix}
1.11 & 0.35 & -0.4 & -0.03 & -0.86
\end{bmatrix}, \quad G_{11(b)} = 0_{16 \times 5},
\]

\[
N_{1(a)} = \begin{bmatrix}
-11.67
\end{bmatrix}, \quad N_{1(b)} = 0_{16 \times 1}.
\]

**Step 4:** The pair \((N_{1(a)}, N_{1(b)})\) is detectable according to (4.47) and therefore an observer is available.

**Step 5:** Since \(N_{1(a)}\) is already stable, so \(Z_1\) can be arbitrarily chosen as \(Z_1 = 0_{1 \times 16}\). The matrices obtained are

\[
N_1 = \begin{bmatrix}
-11.67
\end{bmatrix}, \quad G_{11} = \begin{bmatrix}
1.11 & 0.35 & -0.4 & -0.03 & -0.86
\end{bmatrix},
\]

\[
K_{11} = \begin{bmatrix}
3.04 & -3.29 & 1.76 & -1.23 & -2.28
\end{bmatrix},
\]

\[
J_{12} = \begin{bmatrix}
7.14 & -2.21 & -5.59 & 9.7 & -2.55
\end{bmatrix}.
\]

**Step 6:** The solutions for matrices \(J_{11}\) and \(H_1\) are

\[
J_{11} = \begin{bmatrix}
6.68 & 0.3 & 1.57 & 4.86 & 9.43
\end{bmatrix} \quad \text{and} \quad H_1 = \begin{bmatrix}
-5.84
\end{bmatrix}.
\]
The observer design is thus completed.

In a similar manner to the way local observer for subsystem 1 is constructed, the local observer 2 matrices have been determined with reference to the same design procedure found in Algorithm 4.3.1. Hence, the observer structure for subsystem 2 is conveniently expressed as

$\dot{z}_2(t) = w_2(t) + G_{22}y_2(t), \quad (4.48)$

$\dot{w}_2(t) = N_2w_2(t) + J_{22}y_2(t) + K_{22}y_2(t - \tau_{22}) + J_{21}y_1(t - \tau_{12}) + H_2u_2(t) \quad (4.49)$

whereby the matrices $G_{22}$, $K_{22}$, $N_2$, $J_{22}$, $J_{21}$ and $H_2$ are respectively found to be

$G_{22} = \begin{bmatrix} -0.06 & -0.77 & -0.7 & 0.02 & 0.19 \\ -0.27 & 0.82 & -0.06 & 1.55 & 2.12 \end{bmatrix}$,

$J_{21} = \begin{bmatrix} 0.25 & -0.97 & -5.6 & 0.85 & 2.17 \\ -6.73 & 10.16 & 1.06 & 8.44 & -1.75 \end{bmatrix}$,

$J_{22} = \begin{bmatrix} -18.56 & 33.09 & -0.17 & -3.1 & -2.92 \\ -36.85 & 81.9 & 29.73 & 56.5 & 107.99 \end{bmatrix}$,

$K_{22} = \begin{bmatrix} -4.88 & -2.71 & 2.8 & -0.19 & 5.43 \\ -12.96 & -2.51 & 11.45 & -5.8 & 13.38 \end{bmatrix}$,

$N_2 = \begin{bmatrix} -29.25 & -11.25 \\ 15.83 & -0.75 \end{bmatrix}$,

$H_2 = \begin{bmatrix} 1.58 \\ 11.68 \end{bmatrix}$.

Since the pair $(N_2(a), N_2(b))$ is observable, the poles have been chosen to be $(-10; -20)$ to enable a faster convergence time.

The developed observer scheme for the above-mentioned numerical system
has been analysed and the performance has been evaluated by means of simulation. In order to display the effect of time delays on the observed variables, different constant interconnection or output time delays have been selected. As illustrated in Fig. 5.6, 5.7, 5.4 and 5.5 which correspond to the actual-vs-estimated state vectors of subsystem 1, all the functional state estimates are tracking the actual state variables despite having different $\tau_{11}$ and $\tau_{22}$ values. The same thing happens to the state functionals of subsystem 2 and its functional estimates generated by the local observer as depicted in Figs. 5.10, 5.11, 5.8 and 5.9.

![Figure 4.2: $z_1$ vs. $\hat{z}_1$ for $\tau_{12} = 2$, $\tau_{21} = 2$, $\tau_{11} = 3$, $\tau_{22} = 5$.](image)

It is therefore evident that the reconstructed functional states track the
Figure 4.3: Estimation Error of $\hat{z}_1$: $\tau_{12} = 2$, $\tau_{21} = 2$, $\tau_{11} = 3$, $\tau_{22} = 5$. 
Figure 4.4: $z_1$ vs. $\hat{z}_1$ for $\tau_{12} = 2$, $\tau_{21} = 2$, $\tau_{11} = 5$, $\tau_{22} = 3$. 
Figure 4.5: Estimation Error of $\hat{z}_1$: $\tau_{12} = 2$, $\tau_{21} = 2$, $\tau_{11} = 5$, $\tau_{22} = 3$. 
Figure 4.6: $z_2$ vs. $\hat{z}_2$ for $\tau_{12} = 3$, $\tau_{21} = 5$, $\tau_{11} = 2$, $\tau_{22} = 2$. 
Figure 4.7: Estimation Error of  ̂z_2:  \( \tau_{12} = 3, \tau_{21} = 5, \tau_{11} = 2, \tau_{22} = 2 \).
Figure 4.8: $z_2$ vs. $\hat{z}_2$ for $\tau_{12} = 5$, $\tau_{21} = 3$, $\tau_{11} = 2$, $\tau_{22} = 2$. 
Figure 4.9: Estimation Error of $\hat{z}_2$: $\tau_{12} = 5$, $\tau_{21} = 3$, $\tau_{11} = 2$, $\tau_{22} = 2$. 
actual functional states under different interconnection delay scenarios. The
global asymptotic convergence of the observed functional states towards the
actual functional states has been demonstrated and ascertained. Thus, the
designed observer having the form of (5.12)-(5.13) is proven to be working
satisfactorily.

Concerning robustness testing, the same observer with the previously given
parameters is used to estimate the functional states of the system when its
system matrices are deviated. The variation is reflected in the following system
matrices. Simulation results in Figs. 4.10 and 4.11 show that the existing
observer with unchanged parameters, is able to estimate the functional states
of the system having an altered set of matrices, depending on the magnitude
of variation in the system matrices. However, there is no guarantee as far as
zero convergence error is concerned.

$$A_{11} = \begin{bmatrix}
-14.4 & 0 & 2.31 & 0 & 0 & 0 & 0 \\
0 & -12.5 & 0 & 6.88 & 0 & -5.19 & 0 \\
0 & 0 & -1.5 & 0 & 1.2 & 1.25 & 0 \\
0 & -3.38 & 0 & -17.2 & 0 & 1.31 & 0 \\
1.95 & 0 & 0 & 5.32 & -1.3 & 0 & -1.75 \\
0 & 1.08 & 2.52 & 0 & 0 & -15.3 & 0 \\
1.82 & 0 & 0 & 2.71 & 0 & -2.37 & -10.4 \\
\end{bmatrix},$$
\[
A_{12} = \begin{bmatrix}
-10.2 & 3.21 & 7.12 & -2.29 & -1.2 & -4.2 & 3.28 \\
6.33 & -4.6 & -8.2 & -4.75 & -2.5 & -2.5 & -8.2 \\
-11.5 & 0 & 5.07 & -10.3 & -2.1 & -5.8 & -6.6 \\
5.35 & 5.72 & 1.57 & 10.3 & 4.08 & 3.39 & 13.3 \\
3.26 & -3.3 & -4.1 & 2.84 & -3.3 & -1.7 & -3.5 \\
-2.12 & -5.2 & -3.8 & 1.25 & -7.2 & -6.1 & -8.2 
\end{bmatrix},
\]

\[
A_{21} = \begin{bmatrix}
5.29 & 3.28 & 4.26 & 0 & -3.2 & -3.42 & 4.52 \\
3.21 & 0 & -2.99 & 4.21 & 1.58 & 3.97 & 1.62 \\
-4.3 & -4.4 & -10.4 & 2.08 & 9.62 & 6.12 & -6.5 \\
2.52 & -4.8 & -4.1 & 4.91 & -4.1 & 2.04 & 2.72 \\
3.02 & 1.51 & 6.11 & -6.2 & 2.08 & -2.06 & 0 \\
1.08 & 2.29 & 4.81 & -4.8 & -5.4 & -7.41 & 3.13 \\
5.22 & 5.25 & 4.78 & 6.57 & -2.6 & 2.67 & 4.82 
\end{bmatrix},
\]

\[
A_{22} = \begin{bmatrix}
0 & 20.2 & -6.99 & -18.6 & 10.1 & 31.5 & -19.4 \\
-27.4 & -7.12 & -19.3 & 2.11 & 9.08 & -28.2 & -1.38 \\
20.2 & 7.07 & 2.08 & 7.86 & 0 & 11.1 & 0 \\
37.1 & -35.4 & 3.21 & -13.4 & -41.3 & 25.2 & -13.5 \\
-9.28 & 12.2 & -7.36 & 6.23 & 10.5 & -12.5 & -11.8 \\
42.2 & 0 & -4.1 & 21.3 & 1.79 & 32.4 & -41.4 
\end{bmatrix},
\]

\[
A_{d11} = \begin{bmatrix}
-6.3 & 7.57 & 5.42 & 0 & 6.59 & 0 & -3.2 \\
5.13 & 1.11 & 5.66 & -1.3 & -4.4 & -1.3 & -1.4 \\
1.87 & 6.74 & 7.11 & -2.7 & -1.2 & 1.59 & -4.3 \\
0 & 4.58 & 5.08 & -1.5 & 0 & -4.4 & -5.5 \\
-1.4 & 3.03 & -1.7 & -2.5 & -1.6 & -3.6 & -5.2 \\
5.79 & -2.7 & -3.2 & -1.3 & -6.4 & -6.3 & -3.57 \\
-2.2 & 0 & 8.39 & 1.61 & 5.11 & 2.31 & 2.39 
\end{bmatrix},
\]
In the event of a situation in which the actual precise delay values are not available to the observer, approximations of delays would need to be used. To replicate such a situation, time delay values are slightly altered during the simulation of the observer so that it differs from the actual time delay values.
Figure 4.11: Estimation Error of \( \hat{z}_2 \) for altered system matrices. \( \tau_{12} = 5, \tau_{21} = 3, \tau_{11} = 2, \tau_{22} = 2 \).
in the system itself. In this case, the delay values of $\tau_{12} = 4.9$, $\tau_{21} = 3.1$, $\tau_{11} = 2.1$, $\tau_{22} = 1.9$ are substituted into the observer equations instead of the exact values of $\tau_{12} = 5$, $\tau_{21} = 3$, $\tau_{11} = 2$, $\tau_{22} = 2$. The simulated results are depicted in Figs. 4.12 and 4.13.

Figure 4.12: $z_1$ vs. $\hat{z}_1$ for altered observer delay. $\tau_{12} = 1.8$, $\tau_{21} = 1.8$, $\tau_{11} = 2.8$, $\tau_{22} = 4.8$. 
Figure 4.13: Estimation Error of $\hat{z}_1$ for altered observer delay. $\tau_{12} = 1.8$, $\tau_{21} = 1.8$, $\tau_{11} = 2.8$, $\tau_{22} = 4.8$.

4.5 Conclusion

In this chapter, the problem of designing Darouach-type linear functional observers for a class of interconnected systems with the presence of time delays has been studied. The procedure for computing and solving the observer parameters has been presented through a numerical example of a dual interconnected system. The proposed observer structure is tailored for a time-delay system, having the form of (4.6)-(4.9) with possible application and extension
to systems possessing a higher number of interconnected subsystems. The proposed scheme has been found to be effective, and simulation results prove the feasibility of the approach. The robustness assessment of the designed observer against the deviation in the system matrices provides suggestions for possible future research endeavours.

Considering the fact that the time delays encountered in interconnected systems are very diverse and are usually time-varying, the succeeding chapter provides a strategy for designing partially-decentralised functional observers, which addresses the problem of multiple time-varying delays. The chapter also deals with the stability issues accompanying these delays. Rather than achieving merely an asymptotic convergence, the new scheme extends this capability to allow for an exponential convergence of the estimation errors, with the ability to accommodate the selection of interval time-varying delays. Such an observer structure also allows for a user-specified rate of decay of the observers’ estimation errors, making the observers’ estimation error characteristics more predictable.
Chapter 5

An LMI-Based Functional Estimation Scheme of Large-Scale Time-Delay Systems with Strong Interconnections

In this chapter, the problem of distributed functional state observation of a large-scale interconnected system with the presence of time delays in the interconnections and the local state vectors is considered [77]. The resulting observer scheme is suitable for strongly coupled subsystems with multiple time-varying delays, and is shown to give superior results for systems with very strong interconnections which only require the satisfaction of some mild existence conditions. A set of existence conditions are derived along with a computationally simple observer constructive procedure. With the use of Lyapunov–Krasovskii functional method, the time-varying delay stability conditions are established and, upon the solution of linear matrix inequalities (LMI), the observer parameter that guarantees the exponential stability of the observer error dynamics can be obtained. All the developed results are tested and simulated with a numerical example of a three-area interconnected system.
to demonstrate the feasibility and effectiveness of the proposed method.

5.1 Introduction

The last few decades have seen a tremendous development in the design and refinement of state observers to facilitate the commissioning of various modern control techniques for interconnected systems. Recognizing the influence and importance of state observers as a driver and enabler of the deployment of sophisticated control strategies, advanced measurement methodologies, and critical fault detection and isolation techniques, it is not surprising to witness a considerable amount of research dedicated to the field of state observation (see, e.g., [85, 73, 118, 62, 54, 66, 24, 72, 116] and the references therein).

State observers have found widespread applications in various observer-based control implementation, particularly in industrial sectors [101, 21, 125], power systems control [145, 140, 75, 27, 88], parallel computing [1] and chemical process monitoring [107, 6], to name a few. In all cases, an estimation scheme is employed to accurately reconstruct the immeasurable state vector using available inputs and outputs information. Indeed, such an estimation scheme provides efficient and useful alternatives to the use of physical measuring instruments which are notorious for significantly escalating measurement costs, introducing undesired time delay and noise into the measured quantities, even proving to be difficult to be incorporated or integrated into certain industrial environments. Moreover, in conceivable situations where the state variables cannot be acquired by direct measurement, state observers are nevertheless the only option.
From the point of view of a system consisting of multiple interconnections, the task of observing or estimating the states of its individual subsystems becomes crucial as the availability of local state estimates at the subsystem level is of paramount importance to the realization of decentralised control schemes [87, 69, 16, 63], which are gaining popularity due to the abundant practical advantages they offer. It is well known that the difficulties of state reconstruction of an interconnected system lie in the coupling effects of the different subsystems, the significance of which has previously been highlighted in the literature. One such state reconstruction approach is basically motivated by the concept of an Unknown-Input Observer (UIO) or Disturbance-Decoupling Observer, as demonstrated by [113, 59, 3]. With such a method, the coupling effects amongst the subsystems are decoupled by treating these interconnections as unknown inputs or perturbations, thereby eliminating the influence of such interconnections altogether. The trade-off, however, is that such schemes have to satisfy a highly restrictive existence condition. Furthermore, due to the requirement that the number of unknown inputs (or interconnections) must be lower than that of the measurement outputs, this method is only attractive for systems with either weakly coupled subsystems or certain limited interconnection patterns. On the other hand, rather than dismissing the interconnections as unknown inputs, a different school of thought proposes the idea of counteracting the coupling effects by providing additional input information to the local observers, either in the form of the inputs and/or outputs of the other subsystems or the estimated states from the local observers of the other subsystems [129, 123, 117]. The benefit of this, as one might expect, is that the
existence condition can be made less conservative, resulting in an observation scheme that is capable of handling systems with stronger interconnections. Although this approach demands information transfer from other subsystems or observers, it still serves as a useful alternative in the event that the unknown input approach fails to perform. After all, with today's modern data communication capabilities, information exchange handling will be far superior than formerly. Taking into account these considerations, our proposed observer structure follows along the line of the latter school of thought, namely that of providing additional outputs from other subsystems to the local observer without the requirement for state estimates exchange between the set of local observers.

A fundamental question that commonly arises in the field of state observation is the state reconstruction of interconnected systems in the presence of time delay in various parts of the system. As time delay is an inevitable part of any physical system and constitutes a major contributor of instability, oscillation and degradation in systems, it is only reasonable to take into account time delay effects in the course of analysing and designing state estimation schemes. Characterised by high complexity, massive information exchange and sparse distribution amongst the subsystems, time delay presents an ever more significant issue for interconnected systems. Therefore, the problem of stability analysis and synthesis for time-delay systems, particularly the interconnected systems with delays is of crucial importance. Owing to the contribution of time-delay and stability analysis results in both the theoretical as well as the practical contexts, considerable attention has been given in numerous
publications [49, 50, 104, 106, 38]. Based on the Lyapunov-Krasovskii functional method, stability conditions are usually derived in terms of tractable linear matrix inequalities, which can be solved by various computational tools [57, 126, 56].

While the asymptotic stability of time-delay systems has been well studied and developed recently, the problem of exponential stability analysis is also of equal importance due to the fact that, on one hand, asymptotic stability is a synonym of exponential stability and, on the other hand, in many applications, it is important to determine the convergence rate of the system states or to find the estimates of the transient decaying rate of the system [58]. Based on a new weighted integral inequality, in the framework of LKF method, improved and less conservative conditions were proposed in [58] ensuring exponential stability with prescribed convergent rate for a class of linear systems with interval time-varying delay.

Although there are a number of studies devoted to stability analysis and derivation of estimation schemes specifically for systems with state delay, nonetheless the application of these ideas into a system composed of interconnections of subsystems, taking into account the interdependency of the subsystems, have not been investigated in detail. Such an interdependency adds up to the complexity involving the task of dealing with the mutual influences between these constituent subsystems as well as the varying level of time delays each individual subsystem contributes to the composite system as a whole.

Nevertheless, to the best of our knowledge, the problem of functional state
estimation for interconnected systems in the presence of the aforementioned
different types of delays has not been fully investigated in other work to date.
Although there are a number of decentralised estimation schemes available in
the literature but they are mainly dealing with interconnected systems without
the presence of all the practical types of time delays. Therefore it is the
intention of this work to address these interesting challenges by coming up
with an observer scheme suitable for such class of interconnected systems.

The remainder of this chapter is organised as follows. Section 5.2 explores
the class of interconnected time-delay system of interest and describes the dif-
ferent types of time delays commonly encountered in the system. Next, in
Section 5.3, a partially-decentralised observer structure is presented together
with the corresponding existence conditions and stability conditions of the er-
ror dynamics. The computational methods for obtaining the observer design
parameters are given and the necessary steps required to construct the observer
is detailed in a constructive procedure. In Section 5.4, a three-area intercon-
nected system comprising 21 states is carefully chosen as a numerical example
to demonstrate the observer design algorithm and the attractive features it
possesses. Simulation results of the state estimates with different prescribed
convergence rate are illustrated. Finally, the conclusion is drawn in Section
5.5.

5.2 Problem Formulation and Preliminaries

In the presence of a diverse types of time delays, the state-space representation
of a typical composite system comprising three coupled subsystems, can be
conveniently described by

\[
\begin{align*}
    \dot{x}_1(t) &= A_{11}x_1(t) + A_{d11}x_1(t - \tau_{11}(t)) + A_{12}x_2(t - \tau_{21}(t)) \\
                 &\quad + A_{13}x_3(t - \tau_{31}(t)) + B_1u_1(t), \\
    y_1(t) &= C_1x_1(t), \\
    \dot{x}_2(t) &= A_{22}x_2(t) + A_{d22}x_2(t - \tau_{22}(t)) + A_{21}x_1(t - \tau_{12}(t)) \\
                 &\quad + A_{23}x_3(t - \tau_{32}(t)) + B_2u_2(t), \\
    y_2(t) &= C_2x_2(t), \\
    \dot{x}_3(t) &= A_{33}x_3(t) + A_{d33}x_3(t - \tau_{33}(t)) + A_{31}x_1(t - \tau_{13}(t)) \\
                 &\quad + A_{32}x_2(t - \tau_{23}(t)) + B_3u_3(t), \\
    y_3(t) &= C_3x_3(t),
\end{align*}
\]  

(5.1)

(5.2)

(5.3)

(5.4)

(5.5)

(5.6)

in which the state equation of the first subsystem is a function of the local state vector, delayed local and remote state vectors, and the control input, which are sequentially designated as \(x_1(t), x_1(t - \tau_{11}(t)), x_2(t - \tau_{21}(t)), x_3(t - \tau_{31}(t))\) and \(u_1(t)\). Both instantaneous as well as delayed local state vectors, denoted respectively by \(x_1(t)\) and \(x_1(t - \tau_{11}(t))\), are characterised by time-invariant constant matrices \(A_{11}\) and \(A_{d11}\). The remote delayed state vectors given by \(x_2(t - \tau_{21}(t))\) and \(x_3(t - \tau_{31}(t))\) are governed by matrices \(A_{12}\) and \(A_{13}\). The quantity \(B_1\) is an input matrix whereas \(C_1\) in the output equation is referred to as the output matrix. Delays \(\tau_{11}(t), \tau_{21}(t)\) and \(\tau_{31}(t)\) are time-varying. The subsystems 2 and 3, described by Eqs. (5.3)-(5.6), are represented by similar notations.

In order to portray a more generic and concise representation of the aforementioned composite system, such a time-delay system can be restructured
\[
\dot{x}_i(t) = A_{ii}x_i(t) + A_{di}x_i(t - \tau_{ii}(t)) + \sum_{j=1, j\neq i}^{N} A_{ij}x_j(t - \tau_{ji}(t)) + B_iu_i(t), \quad t \geq 0,
\]
\[y_i(t) = C_i x_i(t), \quad (5.7)\]
\[x_i(\theta) = \phi_i(\theta), \quad \forall \theta \in [-\tau_{\text{max}}, 0], \quad (5.9)\]

where \(x_i(t) \in \mathbb{R}^{n_i}, x_j(t) \in \mathbb{R}^{n_j}, u_i(t) \in \mathbb{R}^{m_i}\) and \(y_i(t) \in \mathbb{R}^{p_i}\) are respectively the local state, remote state, control input and measured output, for a particular \(i\)th subsystem. The subscript \(i = 1, 2, \ldots, N\) denotes the subsystem index with \(N\) referring to the total number of subsystems the composite system possesses. Thus \(N\) will naturally be 3 when a system has three interconnections. Moreover, in the presence of time-varying delays, denoted as \(\tau_{ii}(t) > 0\) and \(\tau_{ji}(t) > 0\), the local and remote state vectors will therefore be written as \(x_i(t - \tau_{ii}(t))\) and \(x_j(t - \tau_{ji}(t))\), respectively. The system matrices \(A_{ii}, A_{di}, A_{ij}, B_i\) and \(C_i\) are constant matrices of appropriate dimensions. The vector \(\phi_i(\theta) \in \mathbb{R}^{n_i}, -\tau_{\text{max}} \leq \theta \leq 0\), corresponds to the initial condition with \(\tau_{\text{max}}\) denoting the highest amplitude achievable by the set of time-varying delay functions \(\tau_{ii}(t)\) and \(\tau_{ji}(t)\), that is

\[
\tau_{\text{max}} = \sup_{t \geq 0} \{\tau_{ii}(t), \tau_{ji}(t), \quad i, j = 1, 2, \ldots, N, j \neq i\}. \quad (5.10)
\]

Time delay presents itself in many different forms as indicated in the system's state equation. Viewing from the angle of a single subsystem, on its own, it is not uncommon for a subsystem to encounter latencies within its local process variables or state vector. This may be due partly to the nature of certain processes, such as manufacturing, chemical and many others, which
inadvertently introduce operational delays into a particular subsystem at various stages. The limitation of the deployed physical devices with slow response time, to a certain degree, can also be one of the contributing factors to this problem. Such a class of delays that appear in the internal state of a specific subsystem is denoted by $\tau_{ii}(t)$.

As these subsystems are interdependent in certain ways and do not operate in isolation, in general, interaction takes place between these subsystems, usually through the exchange of state information over a communication channel or network. Such a communication channel is hardly perfect and contributes to the delay in the arrival of information in the recipient subsystem, more so in spatially separated subsystems. This type of interconnection delay, with magnitude that is time dependent, is given by the function $\tau_{ji}(t)$ in the state equation.

Ultimately, the cumulative effects of these delays that appear in different forms present a real challenge in the design of standalone functional observers intending to reconstruct the state estimates of a single subsystem at the subsystem level.

### 5.3 Main Results

#### 5.3.1 Observer Design Scheme

In this chapter our objective is to design stand-alone linear functional observers for all the three subsystems. More specifically, the proposed scheme places no specific requirement of having information exchange amongst these
observers and the operation of these observers is independent of one another. It is understandable that such a mode of state observation is highly desirable compared to the observer scheme that requires real-time transmission of estimated states to other observers, as in the case of [117, 118, 28, 116]. Besides being able to execute state estimation process in parallel without having to rely on information to arrive from other remote observers, in the event that a fault is detected in a particular local observer which renders the unintended cessation of a corresponding observer, the rest of the observers will be unaffected. Additionally, this sort of distributed observer scheme lends itself to useful applications especially in the field of decentralised control [119, 1, 63, 3, 117, 69, 16, 128, 8, 68, 125], which relies heavily on the estimated local state information in order to implement its local control strategy.

For a particular subsystem that has a large number of state variables, it would be beneficial to estimate only the required state variables, so as to reduce the order of the observer and subsequently shorten the computation time [135, 22, 131, 134]. Very often, it is rather redundant to reconstruct the entire state vector as only a portion of or linear function of these state variables are needed in control application [100, 60, 72]. By defining the matrix $F_i \in \mathbb{R}^{q_i \times n_i}$ in the following expression, to be a linear combination of the state variables that one is interested to approximate, a linear functional observer of an order $q_i$ can be constructed corresponding to the number of linear combinations of states one has chosen.

$$z_i(t) = F_i x_i(t), \quad i = 1, 2, \ldots, N.$$  \hfill (5.11)

Having considered the above factors, an observer structure as depicted in
Figure 5.1: Block diagram of three stand-alone functional observers
Figure 5.1, is proposed. Being capable of dealing with time-varying delay, this delay-dependent observer scheme complements the recent work illustrated in [76], and its state equations are given by

\[
\hat{z}_i(t) = w_i(t) + G_{ii}y_i(t),
\]

\[
\dot{w}_i(t) = N_i w_i(t) + N_d w_i(t - \tau_{ii}(t)) + J_{ii}y_i(t) + K_{ii}y_i(t - \tau_{ii}(t)) + H_i u_i(t) \\
+ \sum_{j=1, j\neq i}^{N} J_{ij}y_j(t - \tau_{ji}(t)), \quad i = 1, 2, \ldots, N, \quad t \geq 0,
\]

with initial condition of

\[
w_i(\theta) = \rho_i(\theta), \quad \theta \in [-\tau_{\max}, 0],
\]

where \( \hat{z}_i(t) \in \mathbb{R}^{q_i} \) is the estimate of the desired linear functional \( z_i(t) \) and \( w_i(t) \in \mathbb{R}^{q_i} \) is the observer state vector. Matrices \( G_{ii} \in \mathbb{R}^{q_i \times p_i}, N_i \in \mathbb{R}^{q_i \times q_i}, N_{di} \in \mathbb{R}^{q_i \times q_i}, J_{ii} \in \mathbb{R}^{q_i \times p_i}, K_{ii} \in \mathbb{R}^{q_i \times p_i}, J_{ij} \in \mathbb{R}^{q_i \times p_j} \) and \( H_i \in \mathbb{R}^{q_i \times m_i} \) are unknown observer design parameters which must be found. Notice from (5.13) that the proposed observer feeds back some past values of its state vector \( w_i(t - \tau_{ii}(t)) \) through matrix \( N_{di} \) into its state equation which may seem to add complication to the observer structure at first sight. However, given the fact that such an additional observer state delay term has the ability to relax the observer’s existence condition, the trade-off is worthwhile. The example given in the subsequent section of this chapter illustrates that this scheme works for systems that are otherwise not feasible using the scheme in [76].

It is noteworthy that the implementation of this observer requires the local output information \( y_i(t) \) to be delayed for a period of \( \tau_{ii}(t) \). Thus, the real-time knowledge of the delay value of function \( \tau_{ii}(t) \) is needed to generate...
the term \( y_i(t - \tau_{ii}(t)) \) in Eq.(5.13). Additionally, in order to counteract the effect of delays in the interconnections, this scheme requires the transmission of the corresponding remote subsystems’ delayed output information given by \( \sum_{j=1,j \neq i}^{N} y_j(t - \tau_{ji}(t)) \). In this case, however, it is not necessary to have the real-time values of \( \tau_{ji}(t) \) known since the observer can directly obtain the delayed remote output information \( y_j(t - \tau_{ji}(t)), j = 1, 2, \ldots, N, j \neq i \), externally from its transmitters.

The principal criteria of the designed observer is to provide approximations of the state functionals \( \hat{z}_i(t) \) that are accurate representation of the system’s state functionals \( z_i(t) \). Or equivalently, the error between \( z_i(t) \) and \( \hat{z}_i(t) \), known as the estimation error \( \epsilon_i(t) \), is required to converge exponentially to zero under any arbitrary input signal \( u_i(t) \) and any initial condition stated in (5.14).

For the task of constructing an \( i \)th local subsystem observer, the values of these unknown coefficient matrices \( G_{ii}, N_i, N_{di}, J_{ii}, K_{ii}, J_{ij} \) and \( H_i \) have to be computed such that the solutions of these matrices satisfy a set of linear algebraic equations. These linear equations ensure the existence of an exponential observer with the convergence of its estimation error to zero and are given in the following theorem.

**Theorem 5.3.1.** Given any arbitrary values of \( u_i(t) \) and \( w_i(\theta) \), a local \( i \)th observer of order \( q_i \), having the structure of (5.12) and (5.13) exists, if the
following conditions are satisfied.

\[ \dot{\epsilon}_i(t) = N_i \epsilon_i(t) + N_d \epsilon_i(t - \tau_{i\tau}(t)) \text{ is exponentially stable} \quad (5.15) \]

\[ J_i C_i - N_i [G_{ii} C_i - F_i] + [G_{ii} C_i - F_i] A_{ii} = 0 \quad (5.16) \]

\[ [G_{ii} C_i - F_i] A_{dii} + K_{ii} C_i - N_{di} [G_{ii} C_i - F_i] = 0 \quad (5.17) \]

\[ J_{ij} C_j + [G_{ii} C_i - F_i] A_{ij} = 0, \ j = 1, 2, \ldots, N, \ j \neq i \quad (5.18) \]

\[ H_i + [G_{ii} C_i - F_i] B_i = 0. \quad (5.19) \]

**Proof.** Define estimation error \( \epsilon_i(t) \in \mathbb{R}^q \) as

\[ \epsilon_i(t) = \hat{z}_i(t) - z_i(t). \quad (5.20) \]

Substituting (5.11) and (5.12) into (5.20) gives

\[ \epsilon_i(t) = w_i(t) + [G_{ii} C_i - F_i] x_i(t), \quad (5.21) \]

and with simple rearrangement leads to

\[ w_i(t) = \epsilon_i(t) - [G_{ii} C_i - F_i] x_i(t). \quad (5.22) \]

Differentiating (5.21) followed by the substitution of (5.13) and (5.7) results in

\[ \dot{\epsilon}_i(t) = N_i w_i(t) + N_d w_i(t - \tau_{i\tau}(t)) + J_i C_i x_i(t) + K_{ii} y_i(t - \tau_{i\tau}(t)) \]

\[ + H_i u_i(t) + \sum_{j=1,j \neq i}^N J_{ij} y_j(t - \tau_{ji}(t)) + [G_{ii} C_i - F_i] [A_{ii} x_i(t) \]

\[ + A_{dii} x_i(t - \tau_{i\tau}(t)) + \sum_{j=1,j \neq i}^N A_{ij} x_j(t - \tau_{ji}(t)) + B_i u_i(t)]. \quad (5.23) \]

Replace \( w_i(t) \) with (5.22) into the above equation. The corresponding error dynamics equation can then be obtained by rearranging and regrouping the
relation as follows:

\[
\dot{\epsilon}_i(t) = N_i \epsilon_i(t) + N_{di} \epsilon_i(t - \tau_{ii}(t)) \\
+ [J_{ii}C_i - N_i(G_{ii}C_i - F_i) + (G_{ii}C_i - F_i)A_{ii}]x_i(t) \\
+ [(G_{ii}C_i - F_i)A_{di} + K_{ii}C_i - N_{di}(G_{ii}C_i - F_i)]x_i(t - \tau_{ii}(t)) \\
+ \sum_{j=1,j\neq i}^{N} \{(J_{ij}C_j + (G_{ii}C_i - F_i)A_{ij})x_j(t - \tau_{ji}(t))\} \\
+ [H_i + (G_{ii}C_i - F_i)B_i]u_i(t).
\] (5.24)

It is obvious that the error dynamics equation (5.24) is reduced to \(\dot{\epsilon}_i(t) = N_i \epsilon_i(t) + N_{di} \epsilon_i(t - \tau_{ii}(t))\) if conditions (5.16)-(5.19) of Theorem 5.3.1 are satisfied. Therefore, by (5.15), with any initial condition and input, \(\epsilon_i(t)\) exponentially approaches zero as \(t\) tends to infinity. The proof is completed.

The subsequent sections are devoted to, firstly, the solution of matrix equations stipulated by conditions (5.16)-(5.19) to obtain the unknown observer parameters, namely \(G_{ii}, N_i, N_{di}, J_{ii}, K_{ii}, J_{ij}\) and \(H_i\), and secondly, the acquisition of stability condition satisfying condition (5.15) in such a way that an exponential convergence of \(\epsilon_i(t)\) to zero is guaranteed for an interval of time-varying delay values. The existence of these solutions is embodied in the following theorem.

**Theorem 5.3.2.** Equations (5.16)-(5.18) of Theorem 5.3.1 are solvable if and only if

\[
\text{rank} \begin{bmatrix} \Psi_i \\ \Phi_i \end{bmatrix} = \text{rank}(\Psi_i)
\] (5.25)

where \(\Psi_i \in \mathbb{R}^{(2q_i + 3p_i + \sum_{j=1,j\neq i}^{N} p_j) \times (2m_i + \sum_{j=1,j\neq i}^{N} n_j)}\) and \(\Phi_i \in \mathbb{R}^{q_i \times (2m_i + \sum_{j=1,j\neq i}^{N} n_j)}\) are
given by

\[
\Psi_i = \begin{bmatrix}
C_i & 0 & 0 & \cdots & 0 \\
F_i & 0 & 0 & \cdots & 0 \\
C_iA_{ii} & C_iA_{d_{ii}} & C_iA_{ij_1,j_1\neq i} & \cdots & C_iA_{ij_N,j_N\neq i} \\
0 & C_i & 0 & \cdots & 0 \\
0 & F_i & 0 & \cdots & 0 \\
0 & 0 & C_{j_1,j_1\neq i} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & C_{j_N,j_N\neq i}
\end{bmatrix},
\]

\( (5.26) \)

\[
\Phi_i = \begin{bmatrix}
F_iA_{ii} \\
F_iA_{d_{ii}} \\
F_iA_{ij_1,j_1\neq i} \\
\vdots \\
F_iA_{ij_N,j_N\neq i}
\end{bmatrix}^T.
\]

\( (5.27) \)

**Proof.** To begin, let us define the matrices \( S_i \in \mathbb{R}^{q_i \times p_i} \) and \( T_i \in \mathbb{R}^{q_i \times p_i} \) such that

\[
S_i = J_{ii} - N_iG_{ii},
\]

\( (5.28) \)

\[
T_i = K_{ii} - N_{d_i}G_{ii}.
\]

\( (5.29) \)

It follows that equations (5.16) and (5.17) can be respectively rewritten in a more compact form as

\[
S_iC_i + N_iF_i + G_{ii}C_iA_{ii} = F_iA_{ii},
\]

\( (5.30) \)

\[
T_iC_i + G_{ii}C_iA_{d_{ii}} + N_{d_i}F_i = F_iA_{d_{ii}}.
\]

\( (5.31) \)

In order to solve Eqs. (5.16)-(5.18) of Theorem 5.3.1 for unknown parameters, these equations will be combined as a unified matrix of simultaneous
linear matrix equations. Accordingly, the combination of equations (5.18) and (5.30), (5.31) results in

\[
\begin{bmatrix}
S_i \\
N_i \\
G_{ii} \\
T_i \\
N_d \\
J_{ij_1,j_1 \neq i} \\
J_{ij_N,j_N \neq i}
\end{bmatrix}^T \begin{bmatrix}
C_i & 0 & 0 & \ldots & 0 \\
F_i & 0 & 0 & \ldots & 0 \\
C_i A_{ii} & C_i A_{di} & C_i A_{j_1,j_1 \neq i} & \ldots & C_i A_{ij_N,j_N \neq i} \\
0 & C_i & 0 & \ldots & 0 \\
0 & F_i & 0 & \ldots & 0 \\
0 & 0 & C_{j_1,j_1 \neq i} & \ldots & 0 \\
0 & 0 & 0 & \ldots & C_{j_N,j_N \neq i}
\end{bmatrix} = \begin{bmatrix}
F_i A_{ii} \\
F_i A_{di} \\
\vdots \\
F_i A_{ij_N,j_N \neq i}
\end{bmatrix}^T
\]  

(5.32)

where \( j = j_1, j_2, \ldots, j_N \) such that \( j_1 \neq i, j_N \neq i \) and \( j_1 \leq j \leq j_N \). Therefore when \( N = 3 \), the system of matrix equations for the first subsystem \( i = 1 \) becomes

\[
\begin{bmatrix}
S_1 \\
N_1 \\
G_{11} \\
T_1 \\
N_{d_1} \\
J_{12} \\
J_{13}
\end{bmatrix}^T \begin{bmatrix}
C_1 & 0 & 0 & 0 \\
F_1 & 0 & 0 & 0 \\
C_1 A_{11} & C_1 A_{d_1} & C_1 A_{12} & C_1 A_{13} \\
0 & C_1 & 0 & 0 \\
0 & F_1 & 0 & 0 \\
0 & 0 & C_2 & 0 \\
0 & 0 & 0 & C_3
\end{bmatrix} = \begin{bmatrix}
F_1 A_{11} \\
F_1 A_{d_1} \\
F_1 A_{12} \\
F_1 A_{13}
\end{bmatrix}^T.
\]  

(5.33)

Or putting Eq. (5.32) simply,

\[
\Omega_i \Psi_i = \Phi_i,
\]  

(5.34)
where $\Omega_i \in \mathbb{R}^{q_i \times \left(2q_i + 3p_i + \sum_{j=1,j\neq i}^{N} p_j\right)}$ contains a union of unknown parameter matrices given by

$$\Omega_i = \begin{bmatrix} S_i & N_i & G_{ii} & T_i & N_{di} & J_{ij1,j1 \neq i} & \cdots & J_{ijN,jN \neq i} \end{bmatrix}$$  \hspace{1cm} (5.35)$$

and $\Psi_i \in \mathbb{R}^{(2q_i + 3p_i + \sum_{j=1,j\neq i}^{N} p_j) \times (2n_i + \sum_{j=1,j\neq i}^{N} n_j)}$ and $\Phi_i \in \mathbb{R}^{q_i \times (2n_i + \sum_{j=1,j\neq i}^{N} n_j)}$ comprises known matrices specified by (5.26) and (5.27).

Having grouped in such a fashion, the above linear matrix relation explicitly reveals that the possibility of constructing the observer structure (5.12)-(5.13) is contingent upon the availability of non-trivial solutions for $\Omega_i$ satisfying Theorem 5.3.1. Recall that $\Omega_i$ represents a collection of matrices $S_i$, $N_i$, $G_{ii}$, $T_i$, $N_{di}$, $J_{ij1,j1 \neq i}$, ..., $J_{ijN,jN \neq i}$ which correspond with the observer parameters. This eventually leads to the formulation of Theorem 5.3.2 which establishes a condition ensuring the existence of these unique solutions. It is easy to see that if Theorem 5.3.2 is satisfied, solutions exist for matrices $S_i$, $N_i$, $G_{ii}$, $T_i$, $N_{di}$, $J_{ij1,j1 \neq i}$, ..., $J_{ijN,jN \neq i}$ and this, in turn, implies that an observer of the structure given by (5.12)-(5.13) is viable.

In Eq. (5.34), if $\Psi_i$ is a square and nonsingular matrix, a usual inverse matrix $\Psi_i^{-1}$ exists and therefore, $\Omega_i$ can be easily computed by $\Omega_i = \Phi_i \Psi_i^{-1}$. However, it is generally the case that $\Psi_i$ is not a square but a rectangular matrix, considering the dimension of $\Psi_i$, that is $(2q_i + 3p_i + \sum_{j=1,j\neq i}^{N} p_j) \times (2n_i + \sum_{j=1,j\neq i}^{N} n_j)$. In that case, the Moore-Penrose pseudoinverse [108, 11], denoted as $\Psi_i^+$, has to be applied instead. In order for a pseudoinverse to exist, it is important that the simultaneous matrix equation (5.34) be consistent. By virtue of the Rouché-Capelli Theorem [115], the consistency for (5.34) can be ascertained by evaluating and comparing the rank of the coefficient matrix.
\( \Psi_i \) to that of the rank of the augmented matrix \[
\begin{bmatrix}
\Psi_i \\
\Phi_i
\end{bmatrix}
\], which is essentially tantamount to the statement of (5.25), or for completeness,

\[
\text{rank} \begin{bmatrix}
\Psi_i \\
\Phi_i
\end{bmatrix} = \text{rank}(\Psi_i).
\]

That said, if both ranks are equal, then \( \Omega_i \Psi_i = \Phi_i \) has a solution. Since the existence of a unique solution for \( \Omega_i \Psi_i = \Phi_i \) depends upon the rank condition given in (5.25), if such a condition is satisfied, the observer parameters residing in \( \Omega_i \) of Eq. (5.34) will have unique solutions, implying that an \( i \)th observer can then be constructed.

Upon the satisfaction of Theorem 5.3.1, a general solution therefore exists for \( \Omega_i \) of Eq. (5.32) and is given by [108]

\[
\Omega_i = \Phi_i \Psi_i^+ + Z_i \left( I_{(2q_i+3p_i+\sum_{j=1,j\neq i}^N p_j)} - \Psi_i \Psi_i^+ \right) \tag{5.36}
\]

where \( \Psi_i^+ \in \mathbb{R}^{(2n_i+\sum_{j=1,j\neq i}^N p_j) \times (2q_i+3p_i+\sum_{j=1,j\neq i}^N p_j)} \) is a pseudoinverse of \( \Psi_i \) and \( Z_i \in \mathbb{R}^{q_i \times (2q_i+3p_i+\sum_{j=1,j\neq i}^N p_j)} \) is an arbitrary matrix that forms the design parameter for the observer.

If we define constant square vector \( e \in \mathbb{R}^{(2q_i+3p_i+\sum_{j=1,j\neq i}^N p_j)} \) to be

\[
e = \begin{bmatrix}
I_{p_i} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I_{q_i} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I_{p_i} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_{p_i} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_{q_i} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I_{p_j} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{p_{jN}}
\end{bmatrix} \tag{5.37}
\]
or in a compact form

\[ e = \text{diag}(I_{p_1}, I_{q_1}, I_{p_2}, I_{q_2}, I_{p_3}, I_{p_{j+1}}, \ldots, I_{p_{jN}}), \quad j \neq i, \quad j_N \neq i \]  

and that, column array selector \( e_k, \quad k = 1, 2, \ldots, M \), is the sub-vector of \( e \) by taking its column entries that span the column size of the corresponding element in \( \Omega_i \). Hence, the column dimensions of a set of \( e_k \) follow the corresponding column dimensions of the respective constituent submatrices \( S_i, N_i, G_{ii}, T_i, N_{di}, J_{ij}, J_{ijN} \) of \( \Omega_i \), which are respectively \( p_i, q_i, p_i, q_i, p_j, p_{j+1}, \ldots, p_{jN} \). Take for instance, if \( k = 1 \), column array of dimension \( p_i \) corresponding to \( S_i \) will be selected. The column selection technique also applies to \( k = 2 \) and so on.

\[
\begin{align*}
e_1 &= \begin{bmatrix} I_{p_1} \\ 0_{q_1,p_1} \\ 0_{p_1,p_1} \\ 0_{p_2,p_1} \\ 0_{q_1,p_1} \\ 0_{q_1,p_1} \\ 0_{p_j,p_1} \\ 0_{p_{j+1},p_1} \\ 0_{p_{jN},p_1} \end{bmatrix} & & \begin{bmatrix} 0_{p_1,q_1} \\ I_{q_1} \\ 0_{p_1,q_1} \\ 0_{q_1,q_1} \\ 0_{q_1,q_1} \\ 0_{p_{jN},q_1} \\ 0_{p_{jN},q_1} \end{bmatrix} & \begin{bmatrix} 0_{p_1,p_{jN}} \\ 0_{q_1,p_{jN}} \\ 0_{p_1,p_{jN}} \\ 0_{q_1,p_{jN}} \\ 0_{q_1,p_{jN}} \\ 0_{p_{jN},p_{jN}} \end{bmatrix}, \quad j \neq i, \quad j_N \neq i. \end{align*}
\]

Presented in their individual forms, the observer parameters extracted from \( \Omega_i \) can be broken down into

\[
S_i = \Gamma_i e_1, \quad N_i = \Gamma_i e_2, \quad G_{ii} = \Gamma_i e_3, \quad T_i = \Gamma_i e_4, \quad N_{di} = \Gamma_i e_5, \quad J_{ij} = \Gamma_i e_6, \ldots, J_{ijN} = \Gamma_i e_M. \tag{5.40}
\]

Taking \( \Gamma_i = \Phi_i \Psi_i^+ \) and \( \Upsilon_i = I_{(2q_i+3p_i+\sum_{j=1,j\neq i}^N p_j)} - \Psi_i \Psi_i^+ \), matrices \( S_i, N_i, G_{ii}, T_i, \)
\( N_d, J_{ij_1, j_1 \neq i}, ..., J_{ij_N, j_N \neq i} \) can be written as

\[
\begin{align*}
S_i &= (\Gamma_i + Z_i \Upsilon_i) e_1 \\
N_i &= (\Gamma_i + Z_i \Upsilon_i) e_2 \\
G_{ii} &= (\Gamma_i + Z_i \Upsilon_i) e_3 \\
T_i &= (\Gamma_i + Z_i \Upsilon_i) e_4 \\
N_{di} &= (\Gamma_i + Z_i \Upsilon_i) e_5 \\
J_{ij_1} &= (\Gamma_i + Z_i \Upsilon_i) e_6 \\
&\vdots \\
J_{ij_N} &= (\Gamma_i + Z_i \Upsilon_i) e_M.
\end{align*}
\]

Recall that part of the requirement for constructing the proposed observer scheme is to satisfy condition (5.15) of Theorem 5.3.1, which is to ensure that the error dynamics system \( \dot{\epsilon}_i(t) = N_i \epsilon_i(t) + N_d \epsilon_i(t - \tau_i(t)) \) is exponentially stable, therefore, the choice of \( Z_i \) must undoubtedly take into account such a requirement, in that the resultant \( N_i \) and \( N_{di} \) obtained from (5.42) and (5.45) respectively; have to result in a zero exponential convergence of \( \epsilon_i(t) \) under a prespecified \( \tau_i(t) \) and initial function for \( \epsilon_i(t) \). Knowing that \( \Gamma_i \) and \( \Upsilon_i \) are both constant except for \( Z_i \) that is a variable in both (5.42) and (5.45), therefore \( Z_i \) constitutes a critical design parameter for the observer, in a way that the aforementioned criteria can be satisfied by properly selecting a \( Z_i \).

Considering the importance and the challenges involved in obtaining the appropriate stability conditions for (5.15) that produces a suitable \( Z_i \), we now focus our attention on deriving an exponential stability criterion for it as will be presented in the next section.
5.3.2 Exponential Stability Criterion

In this section, a novel exponential stability criterion for a class of linear systems with interval time-varying delay is introduced. Being the main focus in this section, a scheme is proposed to synthesise the observer gain parameter $Z_i$ which satisfies the exponential convergent condition (5.15) of Theorem 5.3.1.

Let us first consider a class of linear systems with interval time-varying delay of the form

$$\dot{x}(t) = Ax(t) + A_d x(t - \tau(t)), \quad t \geq 0,$$  

(5.49)

where $A$, $A_d$ are real matrices of appropriate dimensions, $\tau(t)$ is the time-varying delay belonging to the interval $[\tau^l, \tau^u]$, where $\tau^l$ and $\tau^u$ are known constants corresponding to the upper and the lower bounds of time-varying delay. Note that no other restriction is imposed on the rate of change of the time-varying delay $\tau(t)$, and as a result of that, the derived exponential stability conditions can be applied to systems with fast-varying delays.

The following theorem gives an exponential stability criterion for system (5.49) with prescribed exponential convergent rate.

**Theorem 5.3.3 ([58]).** For a given scalar $\alpha > 0$, assume that there exists the symmetric positive definite matrices $P \in \mathbb{R}^{3n \times 3n}$, $Q_i, R_i \in \mathbb{R}^{n \times n}$, $i = 1, 2$, and a matrix $X \in \mathbb{R}^{2n \times 2n}$ such that $\Pi = \begin{bmatrix} \tilde{R}_2 & X \\ * & \tilde{R}_2 \end{bmatrix} \succeq 0$ and the following LMI holds for $\tau = \{\tau^l, \tau^u\}$

$$\Omega(\tau) = \Omega_{01}(\tau) + \Omega_{02} + \Omega_1 - \Omega_2 - e^{-\alpha \tau} \Delta^T \Pi \Delta < 0,$$  

(5.50)
$\hat{R}_2 = \text{diag}\{R_2, 3R_2\}$, $e_k = [0_{n \times (k-1)n} \ I_n \ 0_{n \times (7-k)n}]$, $k = 1, 2, \ldots, 7$, $\mathbf{A} = A_{e_1} + A_{e_2} e_3$ and

$$
\Upsilon(\tau) = [e_1^T, \tau^l e_5, (\tau - \tau^l) e_6^T + (\tau^u - \tau) e_7^T]^T, \quad \Upsilon_0 = [\mathbf{A}^T, (e_1 - e_2)^T, (e_2 - e_4)^T]^T,
$$

$$
\Upsilon_1 = [(e_1 - e_2)^T, \tau^l (e_1 - e_3)^T]^T, \quad \Upsilon_2 = [(e_2 - e_3)^T, (e_2 + e_3 - 2e_6)^T]^T,
$$

$$
\Upsilon_3 = [(e_3 - e_4)^T, (e_3 + e_4 - 2e_7)^T]^T, \quad \Delta = [\Upsilon_2^T \ U_3^T]^T,
$$

$$
\Omega_{01}(\tau) = \text{Sym}(\Upsilon(\tau)^T P \Upsilon_0) + \alpha \Upsilon(\tau)^T P \Upsilon(\tau), \quad \Omega_{02} = \mathbf{A}^T \left((\tau^l)^2 R_1 + (\tau^u)^2 e^{\alpha \tau^l} R_2\right) \mathbf{A},
$$

$$
\Omega_1 = e_1^T Q_1 e_1 - e^{-\alpha \tau^l} e_2^T Q_1 e_2 + e^{-\alpha \tau^l} e_2^T Q_2 e_2 - e^{-\alpha \tau^u} e_4^T Q_2 e_4,
$$

$$
\Omega_2 = \frac{\alpha \tau^l}{\gamma_0} (e_1 - e_2)^T R_1 (e_1 - e_2) + \frac{\alpha \tau^l}{\rho_0} \Upsilon_1^T \left(L_0^T L_0 \otimes R_1\right) \Upsilon_1, \quad L_0 = \begin{bmatrix} 1 - \frac{\alpha \gamma_0}{\gamma_1} \end{bmatrix},
$$

$$
\gamma_0 = e^{\alpha \tau^l} - 1, \quad \gamma_1 = e^{\alpha \tau^l} - \alpha \tau^l - 1, \quad \rho_0 = \frac{\gamma_0}{\gamma_1} \left[\gamma_0^2 - (\alpha \tau^l)^2 e^{\alpha \tau^l}\right], \quad \tau^r = \tau^u - \tau^l.
$$

Then system (5.49) is exponentially stable with a convergence rate $\alpha/2$.

On the basis of exponential stability conditions proposed in Theorem 5.3.3, a scheme to synthesise observer gain parameter is derived for the following system

$$
\dot{\epsilon}(t) = (N_a + Z N_b) \epsilon(t) + (N_{da} + Z N_{db}) \epsilon(t - \tau(t)), \quad t \geq 0, \quad (5.51)
$$

where $N_a, N_b, N_{da}, N_{db}$ are known real matrices of appropriate dimensions and $Z$ is the observer gain parameter which will be designed. For this purpose, we denote $\mathbf{A}_1 = N_a e_1 + N_{da} e_3$, $\mathbf{A}_2 = N_b e_1 + N_{db} e_3$ and apply (5.50) for $P = \text{diag}\{P_1, P_2\}$, $R_1 = R_2 = P_1$, where $P_1 \in \mathbb{R}^{n \times n}$ and $P_2 \in \mathbb{R}^{2n \times 2n}$ are symmetric positive matrices. Then (5.50) is reduced to the following matrix inequality

$$
\Pi_{11}(\tau) + \nu \left(P_1 \mathbf{A}_1 + \hat{Z} \mathbf{A}_2\right)^T P_1^{-1} \left(P_1 \mathbf{A}_1 + \hat{Z} \mathbf{A}_2\right) < 0 \quad (5.52)
$$
where \( \nu = (\tau^l)^2 + (\tau^r)^2 e^{a \tau^l} \), \( \Pi_{11}(\tau) = \Pi_0(\tau) + \Omega_1 - \hat{\Omega}_2 - e^{-\alpha \tau^r} \Delta^T \hat{\Phi} \Delta \) and

\[
\Pi_0(h) = \text{Sym} \left( e_1^T P_1 A_1 + e_1^T \hat{Z} A_2 + \hat{Y}(\tau)^T P_2 \hat{Y}_0 \right) + \alpha(e_1^T P_1 e_1 + \hat{Y}(\tau)^T P_2 \hat{Y}(\tau)),
\]
\[
\hat{Y}(\tau) = [\tau^l e_5, (\tau - \tau^l)e_0^T + (\tau^u - \tau)e_7^T], \quad \hat{Y}_0 = [(e_1 - e_2)^T, (e_2 - e_4)^T]^T,
\]
\[
\hat{\Omega}_2 = \frac{\alpha \tau^l}{\gamma_0} (e_1 - e_2)^T P_1 (e_1 - e_2) + \frac{\alpha \tau^l}{\rho_0} \Upsilon_1^T (L_0^T L_0 \otimes P_1) \Upsilon_1,
\]
\[
\hat{\Pi} = \begin{bmatrix} \tilde{P}_1 & X \\ X^T & \tilde{P}_1 \end{bmatrix}, \quad \tilde{P}_1 = \text{diag}\{P_1, 3P_1\}.
\]

In summary, the observer gain parameter \( Z \) of (5.51) can be designed from the following conditions.

**Proposition 5.3.1.** For a given scalar \( \alpha > 0 \), system (5.51) is exponentially stable if there exists the symmetric positive definite matrices \( P_1, Q_1, Q_2 \in \mathbb{R}^{n \times n} \), \( P_2 \in \mathbb{R}^{2n \times 2n} \), matrix \( \hat{Z} \in \mathbb{R}^{n \times m} \), and a matrix \( X \in \mathbb{R}^{2n \times 2n} \) such that \( \hat{\Pi} \geq 0 \) and the following LMI holds for \( \tau = \{\tau^l, \tau^u\} \)

\[
\Pi(\tau) = \begin{bmatrix} \Pi_{11}(\tau) & \Pi_{12} \\ * & -\frac{1}{r} P_1 \end{bmatrix} < 0 \quad (5.53)
\]

where \( \Pi_{12} = A_1^T P_1 + A_2^T \hat{Z}^T \). The observer gain parameter \( Z \) of (5.51) is then designed in the form

\[
Z = P_1^{-1} \hat{Z}. \quad (5.54)
\]

Now, by substituting

\[
N_i = N_{i(a)} + Z_i N_{i(b)} \quad (5.55)
\]

and

\[
N_{di} = N_{di(a)} + Z_i N_{di(b)} \quad (5.56)
\]

into (5.15), we get

\[
\dot{\epsilon}_i(t) = (N_{i(a)} + Z_i N_{i(b)}) \epsilon_i(t) + (N_{di(a)} + Z_i N_{di(b)}) \epsilon_i(t - \tau_i(t)) \quad (5.57)
\]
where \( N(a) = \Gamma_i e_2 \), \( N(b) = \Theta_i e_2 \), \( N_{di(a)} = \Gamma_i e_5 \) and \( N_{di(b)} = \Theta_i e_5 \) are known constant matrices, matrix \( Z_i \) is unknown and will be designed. Time-varying delays \( \tau_{ii}(t) \) are unknown but assumed to be bounded, \( 0 \leq \tau^l_{ii} \leq \tau_{ii}(t) \leq \tau^u_{ii} \), where \( \tau^l_{ii} \) and \( \tau^u_{ii} \) are known constants. By (5.54), the parameters \( Z_i \) can be obtained from

\[
Z_i = P^{-1}_i \hat{Z}_i. \tag{5.58}
\]

Substituting \( Z_i \) into (5.41)-(5.48), the values of matrices \( S_i, N_i, G_{ii}, T_i, N_d, J_{ij}, 1 \leq j \leq n \) become known. At this stage, the only unknown observer parameters are \( J_{ii} \) and \( K_{ii} \). It is easy to obtain their values by referring to (5.28) and (5.29), which is

\[
J_{ii} = S_i + N_i G_{ii}, \tag{5.59}
\]
\[
K_{ii} = T_i + N_d G_{ii}. \tag{5.60}
\]

Finally, \( H_i \) can be computed directly from (5.19) as soon as \( G_{ii} \) is available.

### 5.3.3 Construction Procedure

To construct a stand-alone observer for a particular \( i \)th subsystem, proceed as follows:

1. First, determine the existence of the observer scheme by checking the conditions of Theorem 5.3.1.

2. Select the observer’s exponential convergence rate \( \alpha_i/2 \) as well as the lower bound \( \tau^l_{ii} \) and upper bound \( \tau^u_{ii} \) delays.
3. Solve the matrix inequality to obtain LMI variables $P_i(i)$ and $\tilde{Z}_i$ by using LMI Toolbox in Matlab and then compute $Z_i$ through (5.58). Note also that, rather than picking all rows of $N_i(b)$ and $N_{di}(b)$, we may choose only one row that is full-column rank from both matrices. The rationale of doing this is that a significantly smaller $Z_i$ value can be obtained. Otherwise, a significantly large $Z_i$ would result in a numerical computation error if any of the combination of rows are linearly dependent. $Z_i$ can then be reconstructed with $Z_i = \begin{bmatrix} 0 & Z_{i(picked)} & 0 \end{bmatrix}$.

4. With $Z_i$ known, compute $S_i$, $N_i$, $G_{ii}$, $T_i$, $N_{di}$ and $J_{ij}$ according to (5.41)-(5.48).

5. Finally, $J_{ii}$, $K_{ii}$ and $H_i$ can be calculated from (5.59), (5.60) and (5.19) respectively. The procedure can then be repeated for the construction of local observers for other subsystems.

5.4 Numerical Examples

Consider a three-area interconnected system (5.1)-(5.6) comprises 21 states with the following system parameters

$$A_{11} = \begin{bmatrix} -13.1 & 0 & -9.62 & 0 & 0 & 0 & -11.6 \\ 0 & -8.87 & 0 & 12.3 & 0 & -12.2 & 6.36 \\ 9.2 & 0 & -8.06 & 0 & 9.82 & 12 & 0 \\ 0 & -10.5 & 0 & -15.7 & 0 & 13.4 & 8.94 \\ 17 & 0 & 0 & 19.9 & -16.1 & 0 & 0 \\ 0 & 9.12 & 9.51 & 0 & 0 & -14.1 & 9.5 \\ 11.2 & 7.25 & 6.89 & -10.2 & 0 & 0 & -8.77 \end{bmatrix}$$
\[
A_{12} = \begin{bmatrix}
-0.15 & 0.23 & -0.14 & -0.21 & 0.15 & 0.09 & -0.37 \\
0.42 & -0.58 & 0.37 & -0.49 & 0.41 & 0.15 & 0.31 \\
0.19 & -0.37 & 0.24 & 0.29 & -0.11 & -0.16 & -0.22 \\
-0.29 & -0.43 & -0.25 & 0.38 & -0.32 & -0.24 & 0.37 \\
-0.37 & 0.58 & -0.4 & 0.49 & -0.28 & 0.22 & -0.19 \\
0.32 & 0.46 & 0.32 & -0.54 & 0.34 & -0.23 & 0.33 \\
-0.13 & 0.14 & -0.18 & 0.11 & -0.19 & 0.17 & -0.24 \\
\end{bmatrix},
\]

\[
A_{13} = \begin{bmatrix}
0.56 & 0.35 & -0.54 & 0.31 & -0.47 & 0.34 & -0.35 \\
0.38 & -0.27 & 0.36 & -0.25 & 0.43 & -0.32 & 0.4 \\
-0.36 & 0.19 & -0.36 & 0.21 & -0.31 & 0.23 & 0.16 \\
0.52 & -0.4 & 0.45 & -0.33 & 0.41 & -0.36 & -0.18 \\
-0.63 & 0.41 & -0.64 & 0.28 & -0.27 & 0.29 & -0.39 \\
-0.33 & -0.44 & 0.58 & -0.35 & 0.46 & -0.41 & 0.23 \\
-0.14 & 0.16 & 0.17 & 0.12 & -0.25 & 0.26 & 0.14 \\
\end{bmatrix},
\]

\[
A_{21} = \begin{bmatrix}
-0.31 & 0.39 & -0.58 & 0.43 & -0.43 & -0.66 & 0.19 \\
0.23 & -0.27 & -0.41 & -0.31 & 0.21 & 0.47 & -0.31 \\
-0.2 & 0.3 & 0.46 & 0.33 & -0.22 & -0.38 & -0.21 \\
0.33 & -0.39 & 0.51 & -0.34 & 0.27 & 0.65 & 0.19 \\
-0.18 & 0.27 & -0.71 & 0.55 & -0.49 & 0.81 & 0.3 \\
0.3 & -0.45 & 0.63 & -0.41 & 0.33 & -0.67 & -0.21 \\
-0.14 & 0.15 & 0.12 & -0.25 & 0.34 & -0.22 & 0.04 \\
\end{bmatrix},
\]

\[
A_{22} = \begin{bmatrix}
-7.95 & 0 & 0 & 0 & 0 & -9.41 & -8.91 \\
0 & -10.4 & 8.07 & 9.85 & -12.2 & 0 & 7.76 \\
-11.4 & 0 & -6.92 & 0 & 9.59 & -11.4 & 9.36 \\
-13.4 & 0 & 0 & -13 & 13.2 & 14.9 & -12.6 \\
-12.8 & 8.66 & 0 & 0 & -11.1 & 13.5 & 0 \\
11.4 & 9.59 & -9.28 & 0 & 0 & -12.1 & 0 \\
4.06 & 4.7 & -3.69 & 0 & 0 & 0 & -6.58 \\
\end{bmatrix},
\]
\[ A_{23} = \begin{bmatrix}
-0.22 & 0.59 & -0.35 & 0.21 & -0.59 & 0.37 & -0.16 \\
0.45 & -0.59 & 0.4 & -0.31 & 0.64 & -0.72 & 0.31 \\
0.34 & -0.52 & 0.26 & -0.21 & 0.44 & -0.54 & 0.24 \\
-0.19 & 0.34 & -0.18 & 0.11 & -0.39 & 0.43 & -0.15 \\
-0.33 & 0.35 & -0.32 & 0.13 & -0.48 & 0.67 & 0.37 \\
0.32 & -0.38 & 0.33 & -0.21 & 0.47 & -0.48 & -0.32 \\
-0.39 & 0.21 & -0.14 & 0.28 & -0.12 & 0.27 & -0.28 \\
\end{bmatrix} ,
\]

\[ A_{31} = \begin{bmatrix}
-0.48 & -0.43 & -0.25 & 0.35 & -0.45 & 0.48 & -0.14 \\
0.48 & 0.31 & 0.3 & -0.28 & -0.41 & 0.32 & 0.17 \\
-0.37 & 0.42 & -0.29 & 0.23 & -0.29 & -0.31 & 0.24 \\
-0.14 & 0.31 & 0.22 & -0.25 & 0.34 & -0.32 & -0.19 \\
0.43 & -0.58 & 0.49 & -0.53 & 0.69 & -0.56 & 0.13 \\
0.3 & -0.31 & -0.26 & 0.32 & 0.33 & 0.19 & -0.21 \\
-0.22 & 0.29 & -0.21 & 0.15 & -0.21 & 0.21 & 0.04 \\
\end{bmatrix} ,
\]

\[ A_{32} = \begin{bmatrix}
-0.29 & 0.36 & 0.57 & 0.16 & -0.22 & 0.32 & -0.14 \\
0.35 & -0.47 & -0.38 & -0.26 & 0.32 & -0.48 & 0.37 \\
-0.32 & 0.19 & -0.34 & 0.19 & -0.22 & 0.26 & 0.24 \\
0.5 & -0.56 & 0.47 & -0.25 & 0.38 & -0.41 & -0.25 \\
-0.36 & 0.28 & -0.37 & 0.23 & -0.34 & 0.14 & -0.4 \\
0.31 & -0.16 & 0.29 & -0.17 & 0.21 & -0.12 & 0.32 \\
-0.23 & 0.36 & -0.27 & 0.13 & -0.13 & 0.32 & -0.18 \\
\end{bmatrix} ,
\]

\[ A_{33} = \begin{bmatrix}
-8.24 & 1.34 & 0 & 10.9 & 0 & 9.84 & 0 \\
0 & -10.5 & 0 & 0 & 8.21 & 0 & 9.99 \\
-8.81 & 0 & -12.6 & 7.73 & 0 & -8.29 & 10.3 \\
0 & -8.31 & -9.84 & -6.45 & 4.62 & 0 & 0 \\
-4.55 & 0 & -10.4 & 7.57 & -6.92 & 6.4 & -8.73 \\
-10.8 & 0 & 20.2 & 0 & 0 & -12.5 & 16 \\
0 & 0 & 13.2 & 9.33 & 0 & 0 & -8.49 \\
\end{bmatrix} ,
\]
\[ A_{d11} = \begin{bmatrix}
-0.86 & 0.75 & 0.99 & 0.52 & 0 & 0.51 & 0 \\
0 & -0.64 & 0 & 0 & 0.74 & 0 & 0.64 \\
-1 & 0 & -1.13 & 0.54 & 0 & 0.45 & 0.21 \\
0 & -0.56 & 0 & -0.57 & 0.36 & 0 & 0.82 \\
0 & 0 & -1.55 & 1.06 & -0.94 & 0.92 & 0.63 \\
-0.8 & 0 & 0.9 & 0 & 0 & -0.42 & 0 \\
-0.36 & -0.25 & 0 & 0.72 & 0.24 & 0.21 & -1.24 
\end{bmatrix}, \]

\[ A_{d22} = \begin{bmatrix}
-0.46 & 0 & -0.48 & 0 & 0 & 0 & -0.95 \\
0 & -1.12 & 0 & 0.53 & 0 & -0.87 & 0.28 \\
0.53 & 0 & -0.81 & 0 & 0.92 & 0.82 & 0 \\
0 & -0.84 & 0 & -0.5 & 0 & 0.69 & 0.25 \\
0.56 & 0 & 0 & 0.78 & -1.21 & 0 & 0.49 \\
0 & 1.02 & 0.51 & 0 & 0 & -0.73 & 0 \\
0.52 & 0 & 0.28 & 0.82 & 0.42 & 0 & -0.35 
\end{bmatrix}, \]

\[ A_{d33} = \begin{bmatrix}
-1 & 0.85 & 0 & 0.99 & 1.04 & -0.78 & 0.81 \\
0.7 & -0.97 & 0.5 & 1.14 & -0.68 & 0.9 & -0.83 \\
0 & 0 & -0.96 & 0 & 0 & 0 & 0.62 \\
0 & 0 & 0 & -1.24 & 1.22 & -1 & 0 \\
-0.4 & 0.37 & 0 & 0 & -0.34 & 0 & 0 \\
0.7 & 0.62 & -0.48 & 0 & 0 & -0.66 & 0.21 \\
0 & 0.72 & 0 & 0.83 & 0.25 & 0.93 & -0.72 
\end{bmatrix}, \]

\[ B_1 = \begin{bmatrix} 1 & 2 & -1 & -2 & -3 & -2 & -1 \end{bmatrix}^T, \]
\[ B_2 = \begin{bmatrix} 1 & -2 & -2 & -3 & -2 & 4 & 1 \end{bmatrix}^T, \]
\[ B_3 = \begin{bmatrix} -5 & 7 & -3 & 5 & -8 & -2 & -7 \end{bmatrix}^T, \]
\[ C_1 = C_2 = C_3 = [I_5 \ 0_{5 \times 2}], \]
\[ F_1 = F_2 = F_3 = [0_{2 \times 5} \ I_2]. \]

It is worth noting that the selection of such an example takes into account the strong coupling effects between the interconnections, in that the interconnection matrices \( A_{12}, A_{13}, A_{21}, A_{23}, A_{31} \) and \( A_{32} \) with \( 7 \times 7 \) dimensions are
all of rank 7. In addition, matrices $A_{d_{11}}$, $A_{d_{22}}$, $A_{d_{33}}$ which govern the internal delayed state vector of each corresponding subsystems are also of rank 7. The restrictive requirement of having the number of interconnection inputs that is less than or equal to that of the local outputs, as in [113], does not apply here and is therefore more attractive.

We now proceed through the construction procedure to design a stand-alone observer for a particular $i^{th}$ subsystem. We start with $i = 1$.

1. First, checking Theorem 5.3.2 reveals that $\text{rank} \begin{bmatrix} \Psi_1 \\ \Phi_1 \end{bmatrix} = \text{rank}(\Psi_1) = 28$. Thus, the proposed conditions of Theorem 5.3.2 hold and an observer can be designed. Note that the same system parameters do not satisfy the existence condition of [76].

2. Obtain LMI variables $P_{1(1)}$ and $\hat{Z}_1$ by using LMI Toolbox in Matlab and then compute $Z_1$ through (5.58). In this case, row 4 of $N_{1(b)}$ and $N_{d1(b)}$ are selected and leading to $Z_1(picked) = \begin{bmatrix} 13.443 \\ -55.71 \end{bmatrix}$ and $Z_1$ can be reconstructed as $Z_1 = \begin{bmatrix} 0_{2 \times 3} & 13.443 & 0_{2 \times 29} \\ -55.71 & \end{bmatrix}$.

3. With $Z_1$ known, compute $S_1$, $N_1$, $G_{11}$, $T_1$, $N_{d1}$, $J_{12}$ and $J_{13}$ according to (5.41)-(5.48) we then obtain

$$S_1 = \begin{bmatrix} 15.03 & 4.29 \\ 1.77 & 4.3 \\ -2.67 & 9.91 \\ 26.15 & -26.64 \\ -8.41 & 0.88 \end{bmatrix}, \quad G_{11} = \begin{bmatrix} -0.64 & 0.04 \\ -0.69 & 0.27 \\ -0.74 & 0.33 \\ -0.11 & -0.51 \\ -0.98 & 0.26 \end{bmatrix}, \quad T_1 = \begin{bmatrix} -2.1 & 0 \\ -0.02 & -0.39 \\ -0.82 & 0.73 \\ 1.7 & -0.04 \\ -0.36 & 0.46 \end{bmatrix}.$$

$$N_1 = \begin{bmatrix} -12.09 & 6.18 \\ 7.47 & -5.5 \end{bmatrix}, \quad N_{d1} = \begin{bmatrix} 1.14 & -0.19 \\ 1.31 & -1.23 \end{bmatrix}.$$
\[ J_{12} = \begin{bmatrix} 0.26 & -0.35 \\ 0.45 & 0.04 \\ 0.25 & -0.38 \\ -0.28 & 0.22 \\ 0.33 & -0.36 \end{bmatrix} \quad \text{and} \quad J_{13} = \begin{bmatrix} -0.53 & 0.28 \\ 0.09 & -0.15 \\ -0.36 & 0.61 \end{bmatrix}. \]

4. Finally, \( J_{11}, K_{11} \) and \( H_1 \) can be calculated from (5.59), (5.60) and (5.19), respectively,

\[ J_{11} = \begin{bmatrix} 23.07 & 0.12 \\ 12.18 & -1.47 \\ 8.79 & 3.5 \\ 23.72 & -24.55 \\ 5.31 & -6.56 \end{bmatrix}, \quad K_{11} = \begin{bmatrix} -2.79 & 0.08 \\ -0.46 & -0.59 \\ -1.24 & 0.47 \\ 0.91 & 0.61 \\ -1.14 & 0.33 \end{bmatrix}, \quad H_1 = \begin{bmatrix} -3.87 \\ -1.5 \end{bmatrix}. \]

5. An \( i \)th observer \((i = 1)\) can now be constructed. The same procedure will be repeated for local observers of the other subsystems, for instance, \( i = 2, 3. \)

The corresponding calculated sets of observer parameters for both subsystem 2 and 3 are as follows:

- \( N_2 = \begin{bmatrix} -1.63 & 1.7 \\ 1.66 & -9.2 \end{bmatrix}, \quad N_{d2} = \begin{bmatrix} -1.13 & 4.68 \\ 0.4 & 0.29 \end{bmatrix}, \)

\[ J_{21} = \begin{bmatrix} 0.16 & -0.1 \\ -0.25 & 0.31 \\ -0.17 & 3.04 \\ 0 & 0.16 \\ -0.05 & 0.19 \end{bmatrix}, \quad J_{22} = \begin{bmatrix} 6.4 & -63.78 \\ 16.69 & 6.04 \\ -12.52 & -1.24 \\ -0.46 & -14.53 \\ -13.73 & 66.4 \end{bmatrix}, \quad J_{23} = \begin{bmatrix} 0.1 & 0.05 \\ -0.14 & -0.36 \\ -0.12 & 0.04 \\ 0.15 & 0.04 \end{bmatrix}, \]

\[ G_{22} = \begin{bmatrix} 0.02 & 0.21 \\ -0.11 & 0.62 \\ 0.2 & -3.13 \\ 0.2 & -2.3 \\ -0.81 & -0.09 \end{bmatrix}, \quad K_{22} = \begin{bmatrix} 0.42 & 2.48 \\ 1.44 & -1.58 \\ -0.79 & -2.13 \\ -0.34 & -0.32 \\ -0.28 & -0.65 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 3.13 \\ -11.29 \end{bmatrix}. \]
• $N_3 = \begin{bmatrix} -0.75 & -13.37 \\ 0.9 & -21.83 \end{bmatrix}$, $N_{d3} = \begin{bmatrix} 0.55 & 2.6 \\ -0.62 & -2.39 \end{bmatrix}$,

$G_{33} = \begin{bmatrix} 0.19 & 1.92 \\ -0.27 & 0.7 \\ 0.72 & 1.12 \\ 0.81 & 0.81 \\ -1.2 & 0.59 \end{bmatrix}$, $K_{33} = \begin{bmatrix} -0.48 & -2.42 \\ 0.05 & -2.83 \\ 0.04 & -0.07 \\ 1.07 & -0.67 \\ -2.79 & -6.58 \end{bmatrix}$,

$H_3 = \begin{bmatrix} -10.59 & 1.71 \end{bmatrix}$,

$J_{31} = \begin{bmatrix} -0.14 & 0.25 \\ 0.29 & 0.07 \\ -0.5 & -0.88 \\ 0.07 & -0.14 \\ -0.22 & 0.21 \end{bmatrix}$, $J_{32} = \begin{bmatrix} 1.42 & 0.64 \\ -1.39 & 0.52 \\ 0.49 & -0.08 \\ -0.42 & -0.07 \\ 1.06 & 0.58 \end{bmatrix}$, $J_{33} = \begin{bmatrix} -6.76 & -16.1 \\ 2.14 & -23.19 \\ 25.27 & 7.34 \\ 6.77 & -47.95 \\ -8.34 & -2.21 \end{bmatrix}$.

### 5.4.1 Simulation Studies

Simulation of all the three local observers has been conducted to determine if the observers perform their functions as required and to confirm the validity of the developed observers. All subsystems and their observers are injected with input signal $u(t)$ as shown in Figure 5.2. As depicted in Figure 5.3, the employed time-varying delays $\tau_{11}(t)$, $\tau_{21}(t)$, $\tau_{31}(t)$, $\tau_{22}(t)$, $\tau_{12}(t)$, $\tau_{32}(t)$, $\tau_{33}(t)$, $\tau_{13}(t)$, and $\tau_{23}(t)$ are of periodic signals with different phase and varying amplitude.
The beauty of the derived observer stability criterion is the fact that it allows for the flexibility of selecting a lower bound $\tau_{ii}^l$ and an upper bound $\tau_{ii}^u$ delays as well as a prescribed exponential convergence rate $\alpha_i/2$ for the functional state estimates. Therefore for simulation purposes, $\tau_{ii}^l$ and $\tau_{ii}^u$ are chosen to be 0.1 and 1.6, respectively, while $\alpha_i/2$ is chosen to be 0.8. The fact that these parameters can be chosen as part of the design introduces flexibility into the observer structure.

As Figures 5.4, 5.5, 5.8, 5.9, 5.12 and 5.13 have spoken for themselves, all the functional state estimates converge exponentially to the corresponding actual state variables for all the three subsystems. This indicates and verifies
Figure 5.3: Time-varying delay functions
the ability of the constructed observers to produce functional state estimates that constitute correct representations of the true state values.

When a faster rate of convergence is required, a higher value of $\alpha_i/2$ can be selected. For the sake of comparison, when a convergence rate of 1.1 is applied, the response of the functional state estimates can be observed in Figures 5.6, 5.7, 5.10, 5.11, 5.14 and 5.15.

The simulation results clearly demonstrated the feasibility of the observer structure pertaining to its ability to achieve exponential stability in the face of multiple time-varying delays, be it in the interconnections, within the subsystem itself, or during the transmission of output information from distant subsystems.

![Figure 5.4: $z_1$ vs. $\hat{z}_1$ with exponential convergence rate 0.8](image)

- $Z_{1(1)}$ True
- $Z_{1(1)}$ Estimate
- $Z_{1(2)}$ True
- $Z_{1(2)}$ Estimate

Figure 5.4: $z_1$ vs. $\hat{z}_1$ with exponential convergence rate 0.8
Figure 5.5: Estimation Error of $\hat{z}_1$ with exponential convergence rate 0.8
Figure 5.6: $z_1$ vs. $\hat{z}_1$ with exponential convergence rate 1.1
Figure 5.7: Estimation Error of $\hat{z}_1$ with exponential convergence rate 1.1
Figure 5.8: $z_2$ vs. $\hat{z}_2$ with exponential convergence rate 0.8
Figure 5.9: Estimation Error of $\hat{z}_2$ with exponential convergence rate 0.8
Figure 5.10: $z_2$ vs. $\hat{z}_2$ with exponential convergence rate 1.1
Figure 5.11: Estimation Error of $\hat{z}_2$ with exponential convergence rate 1.1.
Figure 5.12: $z_3$ vs. $\hat{z}_3$ with exponential convergence rate 0.8
Figure 5.13: Estimation Error of $\hat{z}_3$ with exponential convergence rate 0.8
Figure 5.14: $z_2$ vs. $\hat{z}_2$ with exponential convergence rate 1.1
Figure 5.15: Estimation Error of $\hat{z}_2$ with exponential convergence rate 1.1
5.5 Conclusion

In this chapter, the problem of linear functional observers design for interconnected systems with the presence of various time-varying delays has been investigated. The proposed scheme is practical yet computationally simple and is also of low order which would potentially offer cost-effective solution for large-scale system state estimation. Simulation results show the reliability of the proposed functional observation scheme in estimating the desired state functionals that guarantees the exponential stability of the error.

This chapter has proposed a new observer scheme for interconnected time-delay systems. What happens if we want to tap into the readily available existing observer design methods by applying these to the design of observers for interconnected time-delay systems? As we are aware, due to the constraints imposed by time delays upon interconnected systems, the application of these existing established observer design techniques, such as the well-recognised Luenberger observer structure, is complicated. Therefore, in the succeeding chapter, a coordinate transformation approach will be utilised to overcome the limitations imposed by these time delays. Subsequently, this opens up the possibility of constructing various well-known observer structures.
Chapter 6

Design of State Observers for Interconnected Time-Delay Systems via a Coordinate Transformation Approach

This chapter considers the design of state observers for interconnected time-delay systems using a coordinate transformation method [78]. Through such a transformation, the system that has interconnection and state delays is metamorphosed into a new system that injects time-delay information into its input and output terms, before reintroducing them back into the latter system, effectively coupling the delay terms into the I/O injection terms and eliminating the delay values from the state variables. Next, full-order and reduced-order observers are designed based on the transformed system. Finally, the observed states of the transformed system that correspond to the original system is used to deduce the estimates of the original system. A numerical example is provided of an interconnected time-delay system.
6.1 Introduction

High volume information exchange of modern interconnected systems are driving the need for seamless communication capabilities, while at the same time pushing for an optimum usage of already saturated communication channels. In a world where the ideal case of instantaneous interactions between subsystems is limited by the communication capabilities, the occurrence of time delays is therefore inevitable. In the context of state estimation for these subsystems, the notion of time delays inflicted on the state variables of these subsystems, put constraints on the employment of prominent observer structures and its existence conditions.

Understanding the main constraints that come from the observability properties is an important state estimation issue to be addressed. For time-delay free systems that are observable, the design of observers is straightforward as many existing observer design techniques can be applied [85, 26, 22, 62, 59, 130, 131] and others. However, the opposite is true for time-delay systems [12, 34, 103, 23, 130, 24, 48, 61, 122, 94, 76] that put forward challenges and complications into devising effective observer structures. In this chapter, it will be shown that the design of asymptotic observers for time-delay systems can be approached in such a way that a coordinate transformation [61] is performed to convert the system to another domain or coordinate that guarantees the observability of its new system matrices. Subsequently, an observer can be designed to estimate the state vector of the transformed system that is algebraically linked to the state vector of the original system. For example,
for \( z(t) = Mx(t) \), \( M \) is the transformation matrix that formularises the transitional relationship between the state vector \( z(t) \) and \( x(t) \) of the transformed and original system, respectively. In short, the state variables of the original system is related to that of the transformed system through a coordinate transformation relationship [61]. Eventually, the prediction of the states of the former system \( \hat{x}(t) \) can be deduced from the estimates of the latter system \( \hat{z}(t) \) on the basis of the same coordinate transformation relationship.

The implementation of such a concept requires the establishment of a coordinate transformation [61] that governs the relationship between the original system and the transformed system in such a way that the transformation is bi-directional - the transformed system can be reversed-transformed to its original system. In other words, the state vector of an \( x \)-domain system can be conveniently convertible to that of the state vector of a \( z \)-domain system and vice-versa.

To the best of the author’s knowledge, the application of coordinate transformation in enabling the design of observers for interconnected time-delay systems has not been well considered. In view of the potential that such a transformation would act as a stepping stone towards the construction of observers for such a class of systems, it is the aim of this chapter to tap into the incentive such a coordinate transformation approach has to offer.

The rest of the chapter is organised as follows:

Session 6.2 provides the structure of a general time-delay system represented in the delay operator form. It further explores the underlying motivation and possibility of a coordinate transformation inspired by the work of
up on the satisfaction of the given Theorem 6.2.1. Next, a typical two-area interconnected time-delay system is given and an explanation provided on how best to take advantage of the benefits inherent in the transformed system to overcome the distinct limitations of an interconnected time-delay system in terms of state observation. In section 6.4, the construction of a coordinate transformation based on the idea of [61] is carried out. Subsequently, full-order and reduced-order observers are designed for the transformed system using well-known and straightforward techniques. Finally, conclusion is given in section 6.5.

### 6.2 System Description

To begin, assume that a time-delay system can be represented in two different coordinate systems, $x$-coordinate and $z$-coordinate. In a conventional $x$-coordinate system, the state vector is known as $x(t)$ and as $z(t)$ in the new $z$ domain.

If we define $\tau$ as a time-delay operator, a time-delay system can be represented as the following general form such that

\[
\dot{x}(t) = A(\tau)x(t) + B(\tau)u(t),
\]

\[
x(t) = \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix}, \quad \forall t \in [-\tau_{max}, 0],
\]

\[
y(t) = C(\tau)x(t),
\]

where $\tau = \{\tau_i\}$ for systems with multiple time-delay constants for $i = 1, 2, \ldots, K$ and $\tau_{max} = max \{\tau_i\}$. Vectors $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are the state, control input and output measurement, respectively. Matrices $A(\tau) \in \mathbb{R}^{n \times n}$, $B(\tau) \in \mathbb{R}^{n \times m}$ and $C(\tau) \in \mathbb{R}^{p \times n}$.
$R^{n \times n}$, $B(\tau) \in R^{n \times m}$ and $C(\tau) \in R^{p \times n}$ are known system polynomial matrices as a function of $\tau$ operator.

The purpose of the embedment of time-delay operator into the system matrices is twofold. The first reason is to simplify the representation of the structure of time-delay systems to accommodate for potentially a larger class of interconnected systems with time-delay appearing in diverse forms and values; and secondly, to standardise the time-delay system to the notational form that is conducive for the subsequent coordinate transformation to take place. To demonstrate the usage of delay operator in a particular polynomial matrix, the multiplication of the time-delay operator of an element of a polynomial matrix with a state variable, for instance

$$A(\tau)x(t) = \begin{bmatrix} 3 + 2\tau_1 - \tau_2^2 & \tau_3 \\ \cdots & \cdots \\ \tau_1 & \tau_2 & \cdots \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

produces

$$\begin{bmatrix} 3x_1(t) + 2x_1(t - \tau_1) - x_1(t - 2\tau_2) + x_2(t - \tau_3) \\ \cdots \end{bmatrix}.$$ 

According to [61], a coordinate transformation can be performed to transform the system into an equivalent $z$-coordinate system which offers distinct advantages. With the computation of a polynomial matrix $T(\tau)$ having a coordinate transformation relationship as

$$z(t) = T(\tau)x(t), \quad (6.4)$$

the system described in (6.1)-(6.3) can be transformed into the following

$$\dot{z}(t) = \bar{A}z(t) + \bar{E}(\tau)y(t) + \bar{B}(\tau)u(t), \quad (6.5)$$

$$y(t) = \bar{C}z(t), \quad (6.6)$$
where $\bar{E}(\tau)y(t)$ and $\bar{B}(\tau)u(t)$ are the output and input injection terms respectively. $\bar{A}$ and $\bar{C}$ are constant matrices of appropriate dimensions that carry the forms of

\[
\bar{A} = \begin{bmatrix}
0 & I_p \\
\cdots & \cdots \\
\cdots & I_p \\
0 & \cdots & \cdots & 0
\end{bmatrix},
\] (6.7)

\[
\bar{C} = \begin{bmatrix}
I_p & 0 & \cdots & 0
\end{bmatrix},
\] (6.8)

in which $p$ is the dimension of the system output with reference to (6.3).

Apparently, one would now realise the distinguishable benefit that the coordinate-transformed system (6.5)-(6.6) is unquestionably observable. Such a characteristic is pivotal because the conformity to the observability criteria signifies the affirmation of the existence of an observer for such a system. Additionally, one would notice that the delay values associated with the state vector $x(t)$ brought about by the system polynomial matrix $A(\tau)$ in (6.1) is no longer existent in the new system of (6.5)-(6.6) as the value of $\bar{A}$ follows (6.7). This is an indication that a shift of time-delay association has happened, that the time-delay constants are now tied to the output and input injection terms, through $\bar{E}(\tau)$ and $\bar{B}(\tau)$ polynomial matrices, rather than having a direct association with the state vector. In principal, the use of coordinate transformation has redefined the time-delay problem in the state vector into a whole new problem of having time-delay terms in the input and output which is much easier to be dealt with. In comparison, the same time-delay system in the pre-transformed form of (6.1)-(6.3) suffers from a limited viability of observer construction.
In order to obtain a transformed system that possesses system matrices that are observable, one has to construct a transform or coordinate change matrix which in this case denoted as $T(\tau)$. Now, an important question arises as to under what situation the coordinate transformation is available? The possibility of constructing such a matrix depends largely on a set of conditions. The existence condition of the transformation matrix is given in [61] and for convenience, it will be quoted as follows.

**Theorem 6.2.1.** [61] There exists a coordinate transformation that transforms an original system of (6.1)-(6.3) into an observable form of (6.5)-(6.6) if the observability matrix $Q_k(\tau)$ defined in the following is column unimodular.

The observability matrix for (6.1)-(6.3) is given by

$$Q_k(\tau) = \begin{bmatrix}
C(\tau) \\
C(\tau)A(\tau) \\
\vdots \\
C(\tau)A^{k-1}(\tau)
\end{bmatrix},$$

(6.9)

where $k \leq n$ is the smallest integer such that $\text{rank}(Q_k(\tau)) = p$ for all $\tau$.

**Proof.** The reader may refer to the proof detailed in [61].

In the sequel, the succeeding section explores the construction of observers for an interconnected time-delay system via coordinate transformation by capitalizing on the benefits of inherent observability in the matrix pair $(\bar{C}, \bar{A})$. 

6.3 Coordinate Transformation of an Interconnected Time-Delay System

Similar to that of (6.1)-(6.3) that is represented in $\tau$ time-delay operator form, a class of interconnected time-delay system having the form of

$$\dot{x}_i(t) = A_{ii}x_i(t) + \sum_{j=1, j\neq i}^N A_{ij}x_j(t - \tau_{ji}) + A_{di}x_i(t - \tau_{ii}) + B_iu_i(t)$$

(6.10)

$$y_i(t) = C_ix_i(t); \quad i = 1, 2, ..., N,$$  

(6.11)

can be conveniently expressed in the similar fashion. It is important to note that the common notation would be slightly different when the system is expressed in the general time-delay operator form. In order to illustrate the process of coordinate transformation in terms of the mapping of notations, an example of the following two-area interconnected system will be utilised.

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t - \tau_{21}) + A_{d1}x_1(t - \tau_{11}) + B_1u_1(t),$$  

(6.12)

$$y_1(t) = C_1x_1(t),$$  

(6.13)

$$\dot{x}_2(t) = A_{22}x_2(t) + A_{21}x_1(t - \tau_{12}) + A_{d2}x_2(t - \tau_{22}) + B_2u_2(t),$$  

(6.14)

$$y_2(t) = C_2x_2(t),$$  

(6.15)

where $i = 1, 2$, $x_i(t) \in \mathbb{R}^{n_i}$, $u_i(t) \in \mathbb{R}^{m_i}$ and $y_i(t) \in \mathbb{R}^{p_i}$ are the state, input and the measured output for the $i - th$ subsystem, respectively. Matrices $A_{ii} \in \mathbb{R}^{n_i \times n_i}$, $A_{di} \in \mathbb{R}^{n_i \times n_i}$, $A_{ij} \in \mathbb{R}^{n_i \times n_j}$, $B_i \in \mathbb{R}^{n_i \times m_i}$ and $C_i \in \mathbb{R}^{p_i \times n_i}$ are real known system matrices.

The system matrices for the system described in (6.12)-(6.15) are chosen
as

\[ A_{11} = \begin{bmatrix} -0.2 & -0.2 \\ 0 & -0.1 \end{bmatrix}, A_{12} = \begin{bmatrix} -0.6 & 0 \\ -0.1 & 0.3 \end{bmatrix}, A_{21} = \begin{bmatrix} 0 & 0 \\ 0 & -0.1 \end{bmatrix}, A_{22} = \begin{bmatrix} -0.6 & -0.3 \\ 0.4 & -0.2 \end{bmatrix}, \]

\[ A_{d11} = \begin{bmatrix} -0.6 & 0 \\ 0 & -0.1 \end{bmatrix}, A_{d22} = \begin{bmatrix} -0.6 & 0 \\ 0.3 & -0.3 \end{bmatrix}, \]

\[ C_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \]

For the convenience of representation, the delay constants in the system equations are mapped to \( \tau \) notation as follows:

\[ \tau_1 = \tau_{11}, \tau_2 = \tau_{21}, \tau_3 = \tau_{22} \text{ and } \tau_4 = \tau_{12}. \]

Obviously, the system is entangled with four different delay terms in its state variables. For this reason, the application of standard design techniques that requires that the observability criteria to be true becomes out of the question when it comes to the design of observer for such a system. Traditionally, this results in a state observation problem that requires a complicated solution or observer structure to account for the time-delay terms implicit in the system state variables.

It is therefore of crucial importance to have a means to overcome this observability problem and limitation of the time-delay terms in the state variables by working on a less restrictive alternative system that is backward convertible to the original system. Before a coordinate transformation is taking place, it is convenient to represent the system in the time-delay operator form as previously shown. In this case, the system polynomial matrices in the form of (6.1)-(6.3) would be easily computed as
\[ A(\tau) = \begin{bmatrix}
-0.6\tau_1 - 0.2 & -0.2 & -0.6\tau_2 & 0 \\
0 & -0.1\tau_1 - 0.1 & -0.1\tau_2 & 0.3\tau_2 \\
0 & 0 & -0.6\tau_3 - 0.6 & -0.3 \\
0 & -0.1\tau_4 & 0.3\tau_3 + 0.4 & -0.3\tau_3 - 0.2
\end{bmatrix}, \]

(6.16)

\[ B(\tau) = \begin{bmatrix}
-0.1 & 0 \\
0.1 & 0 \\
0 & 0.4 \\
0 & 0.1
\end{bmatrix}, \]

(6.17)

\[ C(\tau) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}. \]

(6.18)

Now, a coordinate transformation relationship \( T(\tau) \) can be obtained and is given as

\[ T(\tau) = \begin{bmatrix}
C(\tau) \\
C(\tau)A(\tau) - \bar{E}_1(\tau)C(\tau)
\end{bmatrix}, \]

(6.19)

such that

\[ z(t) = T(\tau)x(t), \]

(6.20)

where \( \begin{bmatrix} \bar{E}_2(\tau) & \bar{E}_1(\tau) \end{bmatrix} = C(\tau)A(\tau)^2Q_k^+(\tau) \), and that \( Q_k^+(\tau) \) is the pseudoinverse of \( Q_k(\tau) \) obtained in (6.9).

The system can now be transformed into

\[ \dot{z}(t) = \bar{A}z(t) + \bar{E}(\tau)y(t) + \bar{B}(\tau)u(t), \]

(6.21)

\[ y(t) = \bar{C}z(t), \]

(6.22)

where \( \bar{E}(\tau) = \begin{bmatrix} \bar{E}_1(\tau) \\
\bar{E}_2(\tau) \end{bmatrix} \in \mathbb{R}^{n_x \times p} \), \( \bar{B}(\tau) = T(\tau)B(\tau) \).
Or equivalently, when (6.16)-(6.18) are used, the specific transformed system obviously becomes that of

\[
\dot{z}(t) = \bar{A}z(t) + \Gamma y(t) + \Gamma_1 y(t - \tau_1) + \Gamma_2 y(t - \tau_2) \\
+ \Gamma_3 y(t - \tau_3) + \Gamma_4 y(t - \tau_4) + \Gamma_5 y(t - 2\tau_1) \\
+ \Gamma_6 y(t - 2\tau_3) + \Gamma_7 y(t - \tau_1 - \tau_2) \\
+ \Gamma_8 y(t - \tau_1 - \tau_4) + \Gamma_9 y(t - \tau_2 - \tau_3) \\
+ \Gamma_{10} y(t - \tau_2 - \tau_4) + \bar{B}_1 u(t) + \bar{B}_2 u(t - \tau_1) \\
+ \bar{B}_3 u(t - \tau_2) + \bar{B}_4 u(t - \tau_3) \\
+ \bar{B}_5 u(t - \tau_4), \quad t \geq 0, \\
\]

\[
y(t) = \bar{C}z(t), \quad (6.23)
\]

where pair \((\bar{C}, \bar{A})\) is observable, \(\sum_{i=1}^{10} \Gamma_i\) and \(\sum_{j=2}^{5} \bar{B}_j\) are the delayed output and input injection terms to be computed during the coordinate transformation process and will be covered in the next section. \(z(t) \in \mathbb{R}^{n_z}\), \(u(t) \in \mathbb{R}^m\) and \(y(t) \in \mathbb{R}^p\) are respectively the state, input and the measured output of the transformed system. Matrices \(\bar{A} \in \mathbb{R}^{n_z \times n_z}\), \(\bar{C} \in \mathbb{R}^{p \times n_z}\), \(\Gamma_i \in \mathbb{R}^{n_z \times p}\), \(1 \leq i \leq 10\) and \(\bar{B}_j \in \mathbb{R}^{n_z \times m}\), \(2 \leq j \leq 5\), are known constant matrices.

Note that the state vector \(z(t)\) has no association of delay terms in comparison to the original system. The coordinate transformation is illustrated in Figure 6.1 in which the transformed system in \(z\)-coordinate, receives the same control input information \(u(t)\) as the original \(x\)-coordinate system, and eventually produces the same output measurements \(y(t)\) as the original system. In other words, the functionality of the transformed and original system is virtually the same.
6.3.1 Design of a Full-Order Observer

The structure of a typical Luenberger observer for the transformed system is shown below.

\[
\dot{\hat{z}}(t) = (\bar{A} - L\bar{C})\hat{z}(t) + \{L + \Gamma\}y(t) + \Gamma_1 y(t - \tau_1) \\
+ \Gamma_2 y(t - \tau_2) + \Gamma_3 y(t - \tau_3) + \Gamma_4 y(t - \tau_4) \\
+ \Gamma_5 y(t - 2\tau_1) + \Gamma_6 y(t - 2\tau_3) \\
+ \Gamma_7 y(t - \tau_1 - \tau_2) + \Gamma_8 y(t - \tau_1 - \tau_4) \\
+ \Gamma_9 y(t - \tau_2 - \tau_3) + \Gamma_{10} y(t - \tau_2 - \tau_4) \\
+ \bar{B}_1 u(t) + \bar{B}_2 u(t - \tau_1) + \bar{B}_3 u(t - \tau_2) \\
+ \bar{B}_4 u(t - \tau_3) + \bar{B}_5 u(t - \tau_4), \quad t \geq 0,
\]  

(6.25)
where $L$ is a matrix of appropriate dimension. It is worth noting that part of this observer structure follows the standard Luenberger structure except for the additional delayed input and output injection terms. The use of these injection terms renders the relaxation of the observer existence condition which in this case the observability criteria.

### 6.3.2 Design of a Reduced-Order Observer

The objective of a reduced order observer is to estimate a function $f(t) \in \mathbb{R}^{(n_z-p)}$ defined as:

$$ f(t) = Lz(t), \quad (6.26) $$

where $L = \begin{bmatrix} 0_{(n_z-p)\times p} & I_{n_z-p} \end{bmatrix} \in \mathbb{R}^{(n_z-p)\times n_z}$ is a full-row rank matrix such that $\begin{bmatrix} \bar{C} \\ L \end{bmatrix}$ is non-singular.

The reduced-order observer structure that we propose for the transformed $z$-domain system is

$$ \begin{aligned} \hat{f}(t) &= w(t) + Ey(t), \\
\dot{w}(t) &= Nw(t) + Jy(t) + M_1 y(t - \tau_1) + M_2 y(t - \tau_2) \\
&\quad + M_3 y(t - \tau_3) + M_4 y(t - \tau_4) + M_5 y(t - 2\tau_1) \\
&\quad + M_6 y(t - 2\tau_3) + M_7 y(t - \tau_1 - \tau_2) \\
&\quad + M_8 y(t - \tau_1 - \tau_4) + M_9 y(t - \tau_2 - \tau_3) \\
&\quad + M_{10} y(t - \tau_2 - \tau_4) + Hu(t) + K_1 u(t - \tau_1) \\
&\quad + K_2 u(t - \tau_2) + K_3 u(t - \tau_3) \\
&\quad + K_4 u(t - \tau_4), \quad t \geq 0, \quad (6.28) \end{aligned} $$
where \( \hat{f}(t) \in \mathbb{R}^{n_x} \), \( u(t) \in \mathbb{R}^m \) and \( y(t) \in \mathbb{R}^p \) are the reduced-order state estimates, input and output respectively. Matrices \( E, N, J, M_i, 1 \leq i \leq 10 \), \( H, K_j, 1 \leq j \leq 4 \) are matrices of appropriate dimensions. The construction of a reduce-order observer requires that these unknown matrices to be obtained.

As soon as the state estimates of the \( z \)-coordinate system are obtained, the state of the \( x \)-coordinate system can be approximated by recovering them from the \( z \)-coordinate state estimates, for instance, \( \hat{x}(t) \) and \( \hat{z}(t) \) are related through the linear function of

\[
\hat{x}(t) = T^+ (\tau) \hat{z}(t),
\]

where \( T^+ (\tau) \) of appropriate dimension is the pseudoinverse of \( T(\tau) \) obtained from equation (6.19).

### 6.4 Numerical Example

Consider the previous example of an interconnected time-delay system where the system polynomial matrices have been worked out to be (6.16)-(6.18). The time-delay values are chosen to be \( \tau_1 = 5 \), \( \tau_2 = 5 \), \( \tau_3 = 5 \) and \( \tau_4 = 5 \) for simulation purposes.

According to Theorem 6.2.1, an observability matrix \( Q_k(\tau) \) has to be computed, and provided that it is column unimodular, the system can be transformed into an observable form of (6.5)-(6.6). It is found that \( \text{rank}(Q_2(\tau)) = p \), which signifies that the following \( Q_2(\tau) \) is column unimodular.
\[
Q_2(\tau) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-0.6\tau_1 - 0.2 & -0.2 & -0.6\tau_2 & 0 \\
0 & 0 & -0.6\tau_3 - 0.6 & -0.3
\end{bmatrix} .
\] (6.30)

It then follows that a left-inverse \(Q_2^+(\tau)\) exists to be

\[
Q_2^+(\tau) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
-3\tau_1 - 1 & -3\tau_2 & -5 & 0 \\
0 & 1 & 0 & 0 \\
0 & -2\tau_3 - 2 & 0 & -\frac{10}{3}
\end{bmatrix} .
\] (6.31)

By definition of (6.19), the coordinate transformation \(T(\tau)\) is derived as

\[
T(\tau) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0.1\tau_1 + 0.1 & -0.2 & -0.2\tau_2 & 0 \\
0.15\tau_4 & 0 & 0.3\tau_3 + 0.2 & -0.3
\end{bmatrix} ,
\] (6.32)

which eventually produces

\[
\bar{E}(\tau) = \begin{bmatrix}
-0.7\tau_1 - 0.3 \\
-0.15\tau_4 \\
-0.08\tau_1 - 0.06\tau_1^2 - 0.02 \\
-0.03\tau_4 - 0.09\tau_1\tau_4 \\
-0.4\tau_2 \\
-0.9\tau_3 - 0.8 \\
0.08\tau_2 - 0.06\tau_1\tau_2 + 0.12\tau_2\tau_3 \\
-0.09\tau_2\tau_4 - 0.39\tau_3 - 0.18\tau_3^2 - 0.24
\end{bmatrix} .
\] (6.33)

and

\[
\bar{B}(\tau) = \begin{bmatrix}
-0.1 & 0 \\
0 & 0.4 \\
-0.03 - 0.01\tau_1 & -0.08\tau_2 \\
-0.015\tau_4 & 0.12\tau_3 + 0.05
\end{bmatrix} .
\] (6.34)
This $z$-coordinate system can be easily converted into the form of (6.23)-(6.24) and therefore the detailed conversion process will be omitted here.

### 6.4.1 Full-Order Observer

A full-order observer is designed for such a system utilizing the structure given in (6.25). Selecting the poles to be $[-3; -4; -5; -6;]$, the observer gain matrix $L$ can be obtained through the well-known pole-assignment technique and the calculated observer parameters are given as follows.

\[
L = \begin{bmatrix}
11 & 0 \\
0 & 7 \\
30 & 0 \\
0 & 12 \\
\end{bmatrix}, \quad \Gamma = \begin{bmatrix}
-0.3 & 0 \\
0 & -0.8 \\
-0.02 & 0 \\
0 & -0.24 \\
\end{bmatrix}, \quad \Gamma_1 = \begin{bmatrix}
-0.7 & 0 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix},
\]

\[
\Gamma_2 = \begin{bmatrix}
0 & -0.4 \\
0 & 0 \\
0.08 & 0 \\
0 & 0 \\
\end{bmatrix}, \quad \Gamma_3 = \begin{bmatrix}
0 & 0 \\
0 & -0.9 \\
0 & 0 \\
0 & 0.39 \\
\end{bmatrix}, \quad \Gamma_4 = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & -0.15 \\
0 & 0 \\
\end{bmatrix},
\]

\[
\Gamma_5 = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
-0.06 & 0 \\
0 & 0 \\
\end{bmatrix}, \quad \Gamma_6 = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{bmatrix}, \quad \Gamma_7 = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & -0.06 \\
\end{bmatrix},
\]

\[
\Gamma_8 = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
-0.09 & 0 \\
\end{bmatrix}, \quad \Gamma_9 = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0.12 \\
0 & 0 \\
\end{bmatrix}, \quad \Gamma_{10} = \begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & -0.09 \\
\end{bmatrix},
\]

Subsequently, with the attainment of $\hat{z}(t)$ from the observer, state estimates of $x$-coordinate system $\hat{x}(t)$ can be deduced directly from the state estimates of $z$-coordinate system $\hat{z}(t)$ through equation (6.29) where $T^+ (\tau)$ is computed as:
\[ T^+ (\tau) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} \tau_1 + \frac{1}{2} & -\tau_2 & -5 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{2} \tau_3 & \frac{2}{3} \tau_3 & 0 & -\frac{10}{3} \end{bmatrix}. \]

Simulation of the \( z \)-domain full-order observer was carried out and comparisons between the \( x \) state variables and its estimates are plotted as below.

Figure 6.2: State vector, \( x_1(t) \) from the original system vs. \( \hat{x}_1(t) \) derived from the observer of the transformed system.
Figure 6.3: State vector, $x_2(t)$ from the original system vs. $\hat{x}_2(t)$ derived from the observer of the transformed system.
Figure 6.4: State vector, $x_3(t)$ from original system vs. $\hat{x}_3(t)$ derived from the observer of the transformed system.
Figure 6.5: State vector, $x_4(t)$ from original system vs. $\hat{x}_4(t)$ derived from the observer of the transformed system ($x_4(t) \neq z_4(t)$)
6.4.2 Reduced-Order Observer

A reduced-order observer of the structure mentioned in (6.27)-(6.28) is constructed. The computed parameters are as follows.

\[ E = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}, \quad N = \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix}, \quad J = \begin{bmatrix} -8.12 & 0 \\ 0 & -13.04 \end{bmatrix}, \]

\[ M_1 = \begin{bmatrix} 2.02 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1.28 \\ 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 0 \\ 0 & 3.21 \end{bmatrix}, \]

\[ M_4 = \begin{bmatrix} 0 & 0 \\ 0.57 & 0 \end{bmatrix}, \quad M_5 = \begin{bmatrix} -0.06 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_6 = \begin{bmatrix} 0 & 0 \\ 0 & -0.18 \end{bmatrix}, \]

\[ M_7 = \begin{bmatrix} 0 & -0.06 \\ 0 & 0 \end{bmatrix}, \quad M_8 = \begin{bmatrix} 0 & 0 \\ -0.09 & 0 \end{bmatrix}, \quad M_9 = \begin{bmatrix} 0 & 0.12 \\ 0 & 0 \end{bmatrix}, \]

\[ M_{10} = \begin{bmatrix} 0 & 0 \\ 0 & -0.09 \end{bmatrix}, \quad H = \begin{bmatrix} 0.27 & 0 \\ 0 & -1.55 \end{bmatrix}, \quad K_1 = \begin{bmatrix} -0.01 & 0 \\ 0 & 0 \end{bmatrix}, \]

\[ K_2 = \begin{bmatrix} 0 & -0.08 \\ 0 & 0 \end{bmatrix}, \quad K_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0.12 \end{bmatrix}, \quad \text{and} \quad K_4 = \begin{bmatrix} 0 & 0 \\ -0.015 & 0 \end{bmatrix}. \]

From (6.26), since essentially, \( f(t) = \begin{bmatrix} z_3(t) \\ z_4(t) \end{bmatrix} \), therefore, \( \hat{f}(t) = \begin{bmatrix} \hat{z}_3(t) \\ \hat{z}_4(t) \end{bmatrix} \).

Again, \( \hat{x}_2 \) and \( \hat{x}_4 \) can be deduced according to equation (6.29) as soon as \( \hat{f}(t) \) and the measurable output \( y(t) \) are available.

Simulation of the \( z \)-domain reduced-order observer was carried out and the comparison between the \( x \) state vector and its deduced estimates are shown below.
Figure 6.6: $x_2(t)$ from the original system vs. $\hat{x}_2(t)$ derived from the observer of the transformed system.
Figure 6.7: $x_4(t)$ from the original system vs. $\hat{x}_4(t)$ derived from the observer of the transformed system.
6.5 Conclusion

This chapter has applied a state transformation method of [61] into the design of state observers for interconnected time-delay systems. Through the use of a coordinate transformation, an equivalent system of different coordinates has been established, effectively redefining the restrictive time-delay problem in the state vector into a less complex problem of having time-delay terms in the input and output. This in turn opens up the opportunity of accommodating well-established standard observer design techniques for delay-free linear systems which have otherwise lacked viability in the original interconnected time-delay system. Numerical results show that, for the coordinate-transformed time-delay system, observers of desirable asymptotic convergence properties may be designed using estimation theory available for delay-free systems. Further work is needed in order to meet constraints imposed on the flow of information in an interconnected system. Hence some forms of distributed or decentralised observer schemes will be a possible topic for future research.
Chapter 7
Conclusion and Future Work

7.1 Conclusion

In this thesis a number of distributed functional estimation schemes have been proposed for several classes of interconnected systems in the presence of multiple time delays, including internal state and interconnection delays. The key contributions of this thesis in conjunction with the production of several research publications [79, 80, 76, 77, 78], are summarised as follows:

- The resolution of the significant problem of designing low-order distributed functional observers, capable of observing the linear functional states of large-scale interconnected systems, despite the presence of time delays in both the interconnections and state vectors;

- The derivation and relaxation of the existence conditions of functional observers for interconnected time-delay systems, leading to the successful removal of restriction in the formulation of observation schemes for interconnected systems with very strong interconnections;
• The extension of functional estimation schemes developed in the preceding chapters into a class of interconnected systems, which are affected by multiple distinctive time-varying delays. This was accompanied by the derivation of exponential stability criteria for the convergence of the functional estimates, allowing the user to prescribe the upper and lower bound time-varying delays, without placing restriction on the rate of change of the time-varying delays;

• Overcoming the severe restrictions imposed on the state observation of time-delay systems by tailoring a coordinate transformation solution to the construction of observers for large-scale interconnected systems, thus taking advantage of the pre-existing state estimation schemes, rather than revising an entirely new scheme;

• The development of estimation schemes for interconnected time-delay systems that offer the compelling features of being highly efficient yet cost effective in terms of practicality of implementation, as a result of the accomplishment of relatively low order observers, in contrast to the sheer size and complexity of interconnected systems;

• The development and provision of systematic, step-by-step observer construction procedures, which encompass the strategies for the computation of necessary observer parameters. Realistic numerical examples and simulation results are provided, demonstrating the feasibility of such design approaches. In addition, these results may be replicated in ongoing further research.
7.2 Future Work

The results presented in this thesis offer some propositions in regard to addressing estimation problems associated with interconnected time-delay systems. Further to this, a number of suggestions for potential future work are proposed below, which serve as a guide for future research endeavours, namely:

- The developed observer structures could be improved to accept additional remote input or output information, thereby further relaxing the restrictive existence conditions. As a consequence, the observer scheme would be able to support a wider class of interconnected systems possessing more complex interactions;

- As it is not uncommon for interconnected systems to be affected by unknown inputs and disturbances, enhancement could be made to the functional observer scheme by adding the capability to deal with unknown inputs and disturbances;

- As interconnected systems are constantly changing, including the addition and/or removal of subsystems, an estimation scheme could be devised to deal with such variations. Consequently, a potential further area of research would be to devise a robust estimation scheme that better responds to, or account for, structural disturbances and system parameter variations;

- Despite the fact that in the majority of cases, assumptions are made that the time delays are known and determinable, oftentimes, time-delay values can be difficult to obtain. Therefore, an observer scheme that is able
to handle unknown time-delay values or a specific range of time-delay values or functions, could be yet another potential subject for further investigation.
References


scheme for interconnected time-delay systems,” *International Journal of

scheme of large-scale time-delay systems with strong interconnections,” *Journal of the Franklin Institute*, submitted, 2015.

time-delay systems via a coordinate transformation approach,” in *The 12th International Conference on Informatics in Control, Automation


for large-scale systems with interconnection delays,” in *9th Conference
on Industrial Electronics and Applications (ICIEA)*. HangZhou, China:

linear autonomous time lag systems,” *IEEE Transactions on Automatic


