Motion/Force Transmission Analysis of Axis-Symmetric Parallel Mechanisms with Closed-Loop Sub-Chains

by

Kristan Marlow
BEng(Hons) Mechatronics and Robotics

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Deakin University
Centre for Intelligent Systems Research

December 2015
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To my family, who have not only given me their complete love and support, but also their precious time and patience, without which this thesis would not have been possible.
Abstract

Singularities are one of the most important issues affecting the performance of parallel mechanisms. They are an inherent characteristic, which affects their stiffness, dexterity, load capacity and workspace size. Therefore, analysis of the locations and causes of singularities, along with the mechanism’s closeness to a singularity, is crucial to the design and control of high performance parallel mechanisms.

Researchers have approached the problem of singularity analysis in multiple ways. A popular method involves examining a mechanism’s input-output velocities and generating measures of mechanism performance through analysis of the input-output Jacobian matrix. However, this method should not be applied to mixed degree of freedom mechanisms, due to inconsistent units within the Jacobian matrix. More importantly, this method is incapable of detecting all potential types of singularities.

Recently, researchers have applied screw theory to the problem of singularity detection, classification and closeness. In screw theory, the instantaneous kinematics and statics of rigid bodies are modelled by systems of twists and wrenches, respectively. An important concept is the reciprocal product, which represents the instantaneous work performed by a wrench on a twist. Screw theory has the advantage of providing intuitive geometrical insight and enables precise and thorough definitions of
all singularities, along with their causes and consequences. Several closeness measures have been defined based on the normalised reciprocal product, one of which is the recently proposed power coefficient, which produces a set of finite, dimensionless, frame invariant indices with values ranging from zero to unity. The power coefficient measures a mechanism's motion/force transmission characteristics, furthermore, it can be applied to purely translational, purely rotational and combined motion parallel mechanisms.

The power coefficient has been previously applied to non-redundant fully parallel mechanisms, however, an in-depth examination into its applications on parallel mechanisms with closed-loop sub-chains has not yet been performed. A chain is defined as the set of links connecting an actuator on the fixed base and the mobile platform, thus, a closed-loop sub-chain is defined as a closed-loop within a chain. The closed-loop sub-chains examined herein, are composed of four links, where the output link attaches directly to the mobile platform. This type of closed-loop sub-chain is utilised in the world's best selling parallel mechanism, the Delta, along with the H4, Orthoglide and SCARA-Tau mechanisms. The SCARA-Tau belongs to a family of parallel mechanisms, termed axis-symmetric, designed to improve the limited workspace-to-footprint ratio innate to most parallel mechanisms. A common feature in these mechanisms is the use of the previously stated closed-loop sub-chains. Additionally, many of the proposed axis-symmetric designs also possess non-parallelogram closed-loop sub-chains. The performance effects of non-parallelogram closed-loop sub-chains in parallel mechanisms have not previously been examined.

The application of existing motion/force transmission indices on these mechanisms has been found to require alternative wrench definitions for certain variants. Furthermore, the simplification techniques commonly utilised in the screw theory based analysis of closed-loop sub-chains is shown to remove key information about
the transmission performance of the closed-loop sub-chain itself, and therefore, does not completely characterise the mechanisms’ transmission abilities. To address this, I propose an additional screw theory based performance index, termed the intra-chain constraint index, that provides a measure of the motion/force transmission ability of the closed-loop sub-chain itself. The necessity for the intra-chain constraint index is demonstrated on a family of two degree of freedom planar axis-symmetric parallel mechanisms with various planar closed-loop sub-chain configurations. Thereafter, the proposed index is incorporated into the motion/force transmission analysis of the three translational degree of freedom SCARA-Tau mechanism, further highlighting its necessity. This thesis makes major contributions to research on screw theory based singularity and motion/force transmission analysis of parallel mechanisms with closed-loop sub-chains. Firstly, by developing the alternative wrench definitions that enable the utilisation of existing performance indices on these mechanisms, and secondly, through demonstrating the need, and providing a method, to monitor the motion/force transmission performance of the closed-loop sub-chain itself.

The closed-loop sub-chains in the parallel mechanisms stated above are all designed to remain planar. However, some axis-symmetric parallel mechanisms utilise closed-loop sub-chains that do not remain planar, and as such, increase the complexity of defining the equivalent wrenches. Therefore, another major contribution of this thesis is the definition of a modified screw theory based motion/force transmission analysis method, which is intuitive, systematic and detects all singularities in axis-symmetric parallel mechanisms, including variants that contain potentially non-planar closed-loop sub-chains, completely characterising their transmission performance. Its results are meaningful and determine the cause and consequence of a singularity, along with providing a clear visual representation of the motion/force transmission abilities of each chain and the overall mechanism.
Research Contributions

This research contributes to the kinematic performance analysis of parallel mechanisms by extending the use of current indices into a family of parallel mechanisms with closed-loop sub-chains, as well as through the generation of new indices. Additionally, a formal procedure has been developed for the systematic singularity and motion/force transmission analysis of such mechanisms. The contribution of this research can be summarised in the following four points:

1. Characterisation of the actuation wrenches associated with all feasible planar three and four-bar closed-loop sub-chains. This involved the definition of different actuation wrenches for the input transmission index (ITI) and output transmission index (OTI) calculations. These definitions are essential to detect singular locations and enable the systematic analysis of planar mechanisms with closed-loop sub-chains.

2. Definition of the intra-chain constraint singularity index (ICCI) to measure the closeness of a planar closed-loop sub-chain to an internal singularity. It was shown that the simplification techniques commonly utilised in the screw theory based analysis of planar closed-loop sub-chains results in key information
about the transmission performance of the closed-loop sub-chain itself being lost. The ICCI provides this information. The ICCI is a finite, dimensionless, frame invariant index, based on the power coefficient and screw theory. This index facilitates a complete characterisation of the studied lower-mobility parallel mechanisms with planar closed-loop sub-chains.

3. Proposed a systematic procedure for the singularity and motion/force transmission analysis of planar 2-DOF axis-symmetric parallel mechanisms with planar closed-loop sub-chains. The approach is based on the theory of screws and the power coefficient and identifies all possible singularities. It provides a clear and intuitive representation of a mechanism’s performance throughout its workspace in terms of its motion/force transmission characteristics. The proposed method is utilised to completely characterise the singularities and motion/force transmission abilities of a family of planar 2-DOF axis-symmetric parallel mechanisms with various planar closed-loop sub-chains.

4. Extension of the contributions one through three to support spatial 3-DOF axis-symmetric parallel mechanisms with planar, and potentially non-planar, closed-loop sub-chains, to provide a complete characterisation of their singularities and motion/force transmission abilities. This involved the development of a modified analysis method and the definition of the output performance index (OPI), which is simple to implement and produces intuitive results.
List of Publications


• M. Isaksson, C. Gosselin and K. Marlow, “An Introduction to Utilising the

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<th>Acronym</th>
<th>Definition</th>
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<tr>
<td>CLSC</td>
<td>Closed-loop sub-chain</td>
</tr>
<tr>
<td>CLSCs</td>
<td>Closed-loop sub-chains</td>
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<tr>
<td>CTI</td>
<td>Constraint transmission index</td>
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<td>CTS</td>
<td>Constraint transmission singularity</td>
</tr>
<tr>
<td>Def.</td>
<td>Definition</td>
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<tr>
<td>DOF</td>
<td>Degree of freedom</td>
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<td>Fig.</td>
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<td>Figs.</td>
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<tr>
<td>ICCI</td>
<td>Intra-chain constraint index</td>
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<td>ICCS</td>
<td>Intra-chain constraint singularity</td>
</tr>
<tr>
<td>IFToMM</td>
<td>International federation for the promotion of mechanism and machine science</td>
</tr>
<tr>
<td>IK</td>
<td>Inverse kinematics</td>
</tr>
<tr>
<td>ITI</td>
<td>Input transmission index</td>
</tr>
<tr>
<td>ITS</td>
<td>Input transmission singularity</td>
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<tr>
<td>OPI</td>
<td>Output performance index</td>
</tr>
<tr>
<td>OTI</td>
<td>Output transmission index</td>
</tr>
<tr>
<td>OTS</td>
<td>Output transmission singularity</td>
</tr>
<tr>
<td>P</td>
<td>Prismatic joint</td>
</tr>
<tr>
<td>R</td>
<td>Revolute joint</td>
</tr>
<tr>
<td>S</td>
<td>Spherical joint</td>
</tr>
<tr>
<td>SCARA</td>
<td>Selective compliance assembly robot arm</td>
</tr>
<tr>
<td>U</td>
<td>Universal joint</td>
</tr>
<tr>
<td>#</td>
<td>An actuated # joint, where # is R, U, S or P</td>
</tr>
<tr>
<td>( )₂</td>
<td>A cluster of two links</td>
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<tr>
<td>( )₃</td>
<td>A cluster of three links</td>
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Singularities are one of the most important issues affecting the performance of parallel mechanisms. They are an inherent characteristic, which impacts their stiffness, dexterity, accuracy and load capacity. Therefore, a clear understanding of the singularities present is crucial to the design and control of high performance parallel mechanisms. In a singularity, a mechanism either gains or loses one or more degrees of freedom (DOF). If an additional DOF is gained, the mechanism loses stiffness and load bearing capacity about the direction of this new DOF. If a DOF is lost, the mechanism can no-longer be actuated along the direction of this DOF. Therefore, identification of singular locations and their causes is an important part of the design and verification of all mechanisms.

Parallel mechanisms typically exhibit higher accuracy, acceleration, stiffness and payload capacity when compared to similar-sized serial mechanisms [1]. Hence, research into the design and performance of parallel mechanisms is strongly motivated.
by these benefits [2–7]. It is commonly accepted that the workspace-to-footprint ratio of parallel mechanisms is limited when compared to their serial counterparts. This issue is overcome in the axis-symmetric configuration, where infinite rotation of the mobile platform about a fixed common axis of rotation leads to a significantly increased positional workspace [8, 9]. An extensive review of the history of axis-symmetric parallel mechanisms to the current state of the art is presented in Chapter 2.

The axis-symmetric configuration features the innate characteristic of identical properties in all radial half-planes, defined by the common axis of rotation of the actuated links. This enables complete characterisation of a mechanism’s kinematic and static performance from the analysis of a single radial half-plane. Revolution of the single radial half-plane about the common axis of rotation generates the complete toroidal-shaped workspace. Hence, the computational requirements of the performance analysis can be significantly reduced.

The majority of proposed axis-symmetric parallel mechanisms are lower-mobility. A mechanism is classed as lower-mobility if it possesses fewer than 6-DOF. The chains of lower-mobility mechanism can be of serial, parallel or mixed topology. This thesis is concerned with the mixed topology chains, also known as serial-parallel chains. As the name suggests, these chains include links in both series and parallel, with the parallel portion termed a closed-loop sub-chain (CLSC).

Several axis-symmetric mechanisms incorporate closed-loop sub-chains (CLSCs) to apply constraints on the mobile platform. However, such chains can also introduce unwanted coupled motion of the mobile platform, termed parasitic motion in [10]. Axis-symmetric parallel mechanisms commonly experience this parasitic motion in the platform’s yaw angle, and is defined as an unwanted change in platform orientation during radial or vertical translations. A variation in platform yaw during tangential
translations, rotation as a whole about the common axis, is therefore not considered as parasitic.

A commonly utilised CLSC is the planar four-bar spherical joint, which has been implemented in the world’s best-selling parallel mechanism, the Delta [11–14], along with the H4 [15–17] and Orthoglide [18–20] mechanisms. This CLSC is also present in the SCARA-Tau [21–24] axis-symmetric parallel mechanism.

The singularities of axis-symmetric parallel mechanisms with CLSCs have been previously analysed in [25–31]. However, multiple methods were required to detect all singularity types, which resulted in limited physical insight into the cause or consequence of the singularities. Therefore, an intuitive and systematic analysis tool is required that delivers logical results with a physical meaning to determine the cause and consequence of a singularity. Furthermore, the performance of a mechanism is not only affected at these singular locations, but also in the regions surrounding the singularities. The additional performance effects of these CLSCs have not previously been examined in depth.

The theory of screws has attracted attention for its use in the singularity analysis of parallel mechanisms due to its intuitive implementation and results [32–39]. Screw theory based methods have been applied to many serial chain parallel mechanisms to completely characterise their singularities. Screw theory was first presented by Ball [40] in 1900 and revived by Hunt and Phillips many decades later [41, 42]. The theory has since been refined and expanded through the work of many researchers.

The application of screw theory singularity analysis techniques onto parallel mechanisms with CLSCs has only been performed on mechanisms where all CLSCs are planar parallelograms [16, 17, 43–48], as present in the Delta, H4, Orthoglide and SCARA-Tau. However, many of the proposed axis-symmetric designs also possess planar non-parallelogram [25, 31] and potentially non-planar [49] CLSCs. A planar
CLSC, is a CLSC in which all points of its links describe paths located in parallel planes, thus a planar parallelogram CLSC has a projection in these planes that forms a parallelogram shape. Likewise, a planar non-parallelogram CLSC has a projection in these planes that is not a parallelogram shape. A non-planar CLSC, is a CLSC in which some points of some of its links describe non-planar paths, or paths located in non-parallel planes. The performance effects of non-parallelogram and non-planar CLSCs have not previously been examined.

Recently, a new screw theory based technique has been proposed, which includes a set of indices to provide a measure of closeness to singular locations. The technique is based on motion/force transmissions analysis and is applicable to non-redundant parallel mechanisms, with purely translational, purely rotational or combined DOF.

A set of indices based on the power coefficient were proposed by Wang et al. [50], extended by Liu et al. [51] and further exemplified in [52–54]. These indices are finite, dimensionless and frame invariant, with values ranging from zero to unity. A value of unity represents locations furthest from a singularity and zero occurs at singular locations. The technique is said to measure the closeness to all singularities and explain their physical meaning, in terms of the consequences on the mechanism’s motion/force transmission abilities. These indices have recently been applied to a spatial 3-DOF mechanism that incorporates a planar parallelogram CLSC by Xie et al. [55]. The mechanism’s output motion is limited to two rotations and a translation, which result in very small motions of the CLSC’s output link, meaning that any motion/force transmission issues due to the CLSC itself will be negligible. However, for many other parallel mechanisms that utilise parallelogram CLSCs, including the ones analysed in this thesis, the CLSCs can undergo large displacements during the mobile platform’s motion, resulting in the potential of singularities within the CLSCs themselves [56]. Furthermore, for axis-symmetric parallel mechanisms the CLSCs can also
be non-parallelogram and non-planar. The motion/force transmission characteristics of mechanisms with these CLSCs have not previously been examined. The analysis of axis-symmetric parallel mechanisms with these CLSCs forms the core of this thesis.

1.1 Problem Statement

As described in the previous section, axis-symmetric parallel mechanisms have introduced a wide range of planar non-parallelogram CLSCs in addition to the commonly utilised parallelogram CLSC. Furthermore, some variants have also introduced CLSCs that do not remain planar. This thesis aims to answer the following questions:

1. Can current kinematic performance analysis methods be applied to parallel mechanisms with closed-loop sub-chains?

2. Do these methods completely characterise the kinematic performance of parallel mechanisms with closed-loop sub-chains?

3. How do closed-loop sub-chains affect the kinematic performance of a parallel mechanism?

With regards to the first question, the application of current kinematic performance analysis methods on planar axis-symmetric parallel mechanisms with CLSCs is investigated in Chapter 3 and Chapter 4, and further examined for spatial axis-symmetric parallel mechanisms with CLSCs in Chapter 5. For the second question, the limitations of current kinematic performance analysis methods are explored and addressed for planar axis-symmetric parallel mechanisms with CLSCs in Chapter 4, with the findings extended to spatial axis-symmetric parallel mechanisms with CLSCs in Chapter 5. Further limitations of the methods when applied to spatial mechanisms with
CLSCs are also examined in Chapter 5 and addressed in Chapter 6. The third question forms the core of this thesis and is investigated throughout.

1.2 Key Terminology and Definitions

The parallel robotics field does not yet conform to a strict set of terminologies. Some researchers have made attempts at standardising terminology [57,58], however, many terms are still utilised very loosely. Hence, a summary of the key terminology utilised throughout this thesis along with the corresponding definitions is required and thus is presented below.

The terms parallel robot, parallel manipulator, parallel mechanism and closed-loop kinematic chain have been widely used to describe the type of actuator and link arrangements examined herein. In order to determine the most suitable term, the definitions developed by the International Federation for the Promotion of Mechanism and Machine Science (IFToMM) [57] are examined:

- **Robot** - Mechanical system under automatic control that performs operations such as handling and locomotion.

- **Manipulator** - Device for gripping and the controlled movement of objects.

- **Mechanism** - 1. System of bodies designed to convert motions of, and forces on, one or several bodies into constrained motions of, and forces on, other bodies. 2. Kinematic chain with one of its components (links) taken as a frame.

- **Closed-(Loop) Kinematic Chain** - An assemblage of links and joints permitting relative motion of any one link with respect to the remaining links in which each link is connected with at least two other links, where loop is with reference to the subset of links that forms a closed circuit.

From the above definitions it becomes evident that the term parallel robot does in-fact describe the actuator and link arrangements, however a key term in this definition is control, likewise, a parallel manipulator should be capable of the gripping and
controlled movement of objects. The actuator and link arrangements analysed in this thesis could be controlled and a gripper or tool could be integrated, however, this is not the aim. The interest of this research lies in the interaction amongst joints and links, and the forces and motions transmitted between the mobile platform and fixed base, and not in the control of such devices. Hence, the term parallel mechanism is exclusively used within this thesis, unless describing an actual manipulator or robot.

**Definition 1.2.1**

Parallel Mechanism - A system of bodies with a fixed base frame and a mobile platform, where each body is connected by joints to at least two other bodies, with the purpose of converting motions of, and forces on, one or several bodies into constrained motions of, and forces on, other bodies by means of at least two kinematic chains from the fixed base towards the mobile platform.

The terms moving platform, mobile platform, manipulated platform, end-effector platform, tool platform and output platform have all been used to identify the common unfixed body terminating each kinematic chain. All of these terms are adequately descriptive, however for consistency the term mobile platform will be utilised herein.

**Definition 1.2.2**

Mobile Platform - The unfixed body that is common between all kinematic chains in a parallel mechanism.

The terms base, fixed base, base platform, central base column, central column and base column are commonly used in literature to describe the common fixed body that begins each chain, with the last three specific to axis-symmetric parallel mechanisms. The term central could potentially add confusion to descriptions due to its positional relation and the term platform has been utilised to define the mobile platform, these will therefore not be used. In order to describe this body for axis-symmetric parallel
mechanisms, the term *fixed base column* will be utilised exclusively, with the exception of multiple repetitions in the same paragraph, where just *base column* is used to improve reading flow.

**Definition 1.2.3**

*Fixed Base Column / Base Column* - The fixed body that is common between all kinematic chains in an axis-symmetric parallel mechanism.

The terms *series-parallel* and *mixed topology* have been used to describe a kinematic chain that contains additional closed-loops between the base and mobile platform. The closed-loop itself has been labelled by the author a *closed-loop sub-chain* and defined as:

**Definition 1.2.4**

*Closed-Loop Sub-Chain* - The subset of links that form a closed circuit within a kinematic chain.

The definition of a *parallel mechanism*, Def. 1.2.1, includes configurations with more than one actuator per chain, termed hybrid mechanisms. Mechanisms with a single actuator per chain are known as *fully parallel mechanisms* [58] and are defined as:

**Definition 1.2.5**

*Fully Parallel Mechanism* - A parallel mechanism with an n-DOF mobile platform connected to the base by n independent kinematic chains, each having a single actuated joint.

The parallel mechanisms analysed herein are *fully parallel* with at least one mixed topology chain. The chains are composed of revolute, universal and spherical joints, denoted by the letters R, U and S respectively. If a joint is actuated, its respective letter is underlined. If the terms ‘intersects’, ‘is coaxial to’ or ‘is parallel to’ are utilised
with reference to a joint, the description is referring to the joint’s axis of rotation for a revolute joint, or the intersection point of the joint axes for the universal or spherical joints.

The terms upper arm, input link and proximal link have been utilised interchangeably by many authors to describe the link of an RRR chain closest to the fixed base frame located at R. Due to the term upper bearing a relation to the orientation of the whole mechanism, the term proximal link is utilised herein.

**Definition 1.2.6**

Proximal Link - The first link (body) in a kinematic chain.

Likewise, the terms lower arm, coupler link and distal link have been used to describe the link between the second and third R joints of the RRR chain. Due to the term lower bearing a relation to the orientation of the whole mechanism, the term distal link is utilised herein.

**Definition 1.2.7**

Distal Link - The link (body) or links (bodies) in a kinematic chain that connect the proximal link with the mobile platform.

The key terminology of parallel robotics that has been presented above is anything but exhaustive. However, it provides the fundamental definitions required to follow and visualise the mechanisms described throughout this thesis.

### 1.3 Thesis Outline

This thesis is organised as follows:

- **Chapter 2** begins with an in-depth review of axis-symmetric parallel mechanisms, from their origins to the state of the art. This is followed by a critical analysis of the literature findings and identification of key gaps in the field. Next, a geometry
1.3 Thesis Outline

A based tool called screw theory is introduced. A brief history is presented, followed by a clear overview of its essential mathematical foundation, required to assimilate this thesis.

Chapter 3 introduces a family of planar 2-DOF axis-symmetric parallel mechanisms with various planar CLSC arrangements. Their parameters are defined and equations of motion are derived, then the configurations of the CLSCs are illustrated and examined. Next, the singularity types experienced by parallel mechanisms are described, along with a review of the methods utilised to detect them. A screw theory based singularity detection method is then detailed and adapted for use with CLSCs. Then an analysis of the singularities detected by this method is performed on the family of planar mechanisms with various planar CLSCs, along with an examination of their causes and consequences. The detected singular locations are verified against those found by the zero points of the numeric input-output Jacobian’s conditioning index. This is followed by a discussion and exemplification of some interesting singular configurations due to the use of certain CLSC arrangements.

Chapter 4 builds upon the singularity analysis methods defined in Chapter 3, and introduces a set of indices to indicate the closeness to a singularity, based on the principle of motion/force transmission. The limitation of these existing performance indices, when applied to mechanisms with CLSCs, is shown and then addressed through the development of an additional index. The necessity of this index is exemplified on the family of planar 2-DOF axis-symmetric parallel mechanisms with various CLSC arrangements. The existing and proposed performance indices are then combined to completely characterise the mechanisms’ singularities and motion/force transmission performance.

Chapter 5 extends and adapts the concepts from Chapter 4 into spatial 3-DOF mechanisms with planar CLSCs. A family of 3-DOF translational axis-symmetric
parallel mechanisms is introduced, parametrised and their equations of motion are derived. The detected singular locations using the screw theory based methods are verified against those detected by the zero points of the numeric input-output Jacobian’s conditioning index. The necessity of proposed additional performance index is further exemplified on a 3-DOF axis-symmetric parallel mechanism and its complete singularity and motion/force transmission abilities are characterised. Finally, key limitations of the presented method’s application onto mechanisms with non-planar CLSCs and a different mobile platform joint configuration is detailed.

Chapter 6 addresses the limitations highlighted in Chapter 5 through the development of a modified methodology for the singularity and motion/force transmission analysis of mechanisms with non-planar CLSCs and a different configuration of the mobile platform joints. The singular locations detected by the new methodology are verified, and the completeness of the motion/force transmission analysis is confirmed, through comparison with the results produced by the previous method. Next, the new method is exemplified by completely characterising the singularities and motion/force transmission abilities of a spatial 3-DOF axis-symmetric parallel mechanism. This mechanism utilises a different mobile platform joint arrangement and includes multiple CLSCs, where one CLSC is not always planar.

Chapter 7 presents concluding remarks and directions for future work.
2.1 Overview

This chapter presents a review of the origins to the current state of the art of axis-symmetric parallel mechanisms and introduces the key concepts and mathematics forming a foundation for the screw theory analysis utilised throughout this thesis.

2.2 Axis-Symmetric Parallel Mechanisms

The term axis-symmetric refers to a style of mechanism that exhibits identical properties in all closed radial half-planes. A closed radial half-plane is defined by the planar region consisting of all points on one side of an infinite straight line, including points on the line, and no points on the other side [59]. A line drawing of a general
2.2 Axis-Symmetric Parallel Mechanisms

The general axis-symmetric parallel mechanism, shown in Fig. 2.1(a), can be expanded to produce higher, or lower, DOF variants through the addition or removal of kinematic chains, as demonstrated in [31]. This family of mechanisms have the innate ability to achieve infinite rotation about the fixed base column. This results in a parallel mechanism with a large ratio between the positional workspace size and the mechanism’s footprint, similar to that of a serial mechanism, while retaining all the previously stated benefits of a parallel mechanism.

2.2.1 History

An in-depth review of the axis-symmetric parallel mechanism field is presented in this section, examining its origins to the current state of the art. The review covers
the analysis performed on the mechanisms, patents, mechanisms that require small modifications to become axis-symmetric, mechanisms that have been physically prototyped and mechanisms that are manufactured and available for purchase.

In order to achieve an accurate chronological review of the history, cases where multiple international patents were filed for the same mechanism, the earliest filing date was utilised. Additionally, the date-received was used for journal papers rather than the publication date, due to the sometimes lengthy delays between receiving and publication.

1995 The first truly axis-symmetric parallel mechanism appeared in the literature through a patent filed by Reboulet [8], illustrated in Fig. 2.2. Reboulet’s design incorporated cable links in a parallelogram arrangement to constrain the mobile platform’s pitch. Additionally, a vertically actuated link mounted on a passive horizontal bearing was utilised to actuate one of the mobile platform’s translations as well as constraint its roll and yaw. The passive bearing removes the over-constraints in the system, enabling infinite rotation of the mechanism around the fixed base column, while maintaining a tangentially constant yaw angle of the mobile platform.

Figure 2.2: The first axis-symmetric parallel mechanism, filed for patenting in 1995 by Reboulet [8].
Three years later Brogårdh from ABB robotics filed a patent that included an axis-symmetric parallel mechanism design [60], seen in Fig. 2.3. The major differences between Brogårdh’s and Reboulet’s designs are the removal of the cable links and the utilisation of only 5-DOF fixed-length spherical-spherical (SS) distal links to constrain the mobile platform. The distal links are arranged in multiple parallelogram CLSCs. In this design, the distal links experience only axial forces, in contrast to the top link in Reboulet’s design that is also susceptible to bending and torsion.

Then in 2000, Mitsubishi produced the RP series of precision parallel robots [61]. This mechanism is not axis-symmetric by definition, however, through a simple repositioning of the actuators on the proximal links, so they possess a common axis of rotation, the mechanism can become axis-symmetric. The Mitsubishi robot is useful for high accuracy, top down, pick and place or assembly tasks. However, due to its current actuator locations the mechanism generates a comparably smaller workspace to that of a similar-sized axis-symmetric mechanism.

Furthermore, in 2000, Brogårđh published an article entitled Design of high performance parallel arm robots for industrial applications [62], which detailed a refined version of the mechanism presented in [60]. Brogårđh was aiming to improve on the Delta mechanism’s design through link clustering and actuator repositioning. The

Figure 2.3: An axis-symmetric parallel mechanism, filed for patenting in 1998 by Brogårđh [60].
result was an axis-symmetric mechanism with SCARA motion.

2001  In 2001, Brogårdh filed a patent for the design of the axis-symmetric parallel mechanism in Fig. 2.4(a) [63]. The patent proposed a mechanism that was similar to his design from 2000, with modifications to the mobile platform’s joint positions. As seen in the figure, each mobile platform joint is now located along a single vertical axis. The mobile platform’s roll and pitch are constrained by multiple 5-DOF fixed-length SS distal links in two separate parallelogram CLSCs, while the platform’s yaw rotation, about the common mobile platform joint axis, is unconstrained.

2002  Another patent, filed in 2002 by Brogårdh et al. [64], proposed a 3-DOF axis-symmetric parallel mechanism with a rectangular mobile platform and planar parallelogram CLSCs, along with various additional links to couple the passive horizontal motion of the actuated vertical proximal link to the other two horizontally actuated proximal links. This mechanism is illustrated in Fig. 2.4(b). The kinematic and error modelling of Brogårdh’s patented design was performed soon after [23]. The paper

Figure 2.4: Two axis-symmetric parallel mechanisms filed for patenting in (a) 2001 by Brogårdh [63] and (b) 2002 by Brogårdh et al. [64].
derived the kinematic model, along with a method for calculating the input-output Jacobian of the axis-symmetric mechanism.

Also in 2002, a variation of Brogårdh’s previous patents was filed by Kock et al. [21] from ABB robotics. The patent included a clear definition of an axis-symmetric parallel mechanism, which is now known as the SCARA-Tau. The SCARA portion of the name is with reference to SCARA type workspace the mechanism produces and Tau is based on the T shape formed by the actuator axes in Brogårdh’s previous patents [60, 63, 64]. The SCARA-Tau, shown in Fig. 2.5, has replaced the passive base joint and vertically actuated proximal link from Brogårdh’s designs, with a horizontally actuated proximal link. The SCARA-Tau also utilises multiple CLSCs and remains one of the few axis-symmetric parallel mechanisms to be developed into a full scale functional industrial prototype. The SCARA-Tau mechanism has been discussed and analysed in [3, 9, 22, 62]. The patent [21] also presented several 4-DOF designs that utilised a crankshaft mechanism plus additional actuated chains to control the mobile platform’s rotation. Redundantly actuated chains were also proposed to facilitate passing through singular configurations of the mobile platform’s

![Figure 2.5](image_url): The full scale functional industrial prototype of the 3-DOF SCARA-Tau axis-symmetric parallel mechanism patented and developed by ABB [21].
orientation, this concept was further analysed in [25].

2003  Brogårðh et al. filed another patent for an axis-symmetric design in 2003 [65] in which an additional rotary actuated joint was incorporated coaxial with the link of two of the proximal links, enabling actuated mobile platform rotations. This additional actuated joint classifies this design as a hybrid mechanism.

2004  In 2004, dynamic modelling of the SCARA-Tau, utilising the principle of virtual work, was performed in 2004 by Zhu et al. [24]. This work aimed at developing an analytical rigid body dynamic model for use in the feedforward control, enabling high dynamic performance of the Tau mechanisms.

Also in 2004, Roy and Merz filed a patent for a mechanism that incorporated a three way spherical joint on its mobile platform, creating a common termination point for the three kinematic chains [66]. This joint resulted in simplified forward kinematic calculations, which were reduced to calculating the intersection of three spheres. The patent introduced many mechanism variants ranging from 3-DOF to 6-DOF. The lower-DOF variants utilised planar parallelogram CLSCs. The three way spherical joint also decoupled the position from the orientation of the mobile platform. However, the relatively complex construction of the three way spherical joint could introduce expenses in the manufacturing and assembly process, as well as possible issues with ruggedness and maintenance.

2005  Brogårðh et al. published a paper in 2005 focusing on the development of parallel manipulators for industrial applications [56]. He presented another axis-symmetric variant similar to patents [21,63,64], with modifications to the parallelogram CLSCs. The joints on the proximal links of these chains were repositioned to form a triangle, while maintaining the mobile platform joint positions, as shown in
2.2 Axis-Symmetric Parallel Mechanisms

Fig. 2.6(a). This enabled the mechanism to achieve a larger vertical workspace. A key issue identified was that most axis-symmetric parallel mechanisms suffered from coupled parasitic yaw rotation during radial and vertical translations of the mobile platform. The parasitic yaw rotation, about the vertical axis of the mobile platform, is defined as any deviation of the mobile platform’s yaw angle from being tangential to the fixed base column. Brogårdh et al. reduced this coupled parasitic yaw by proposing a variant with a triangular CLSC shown in Fig. 2.6(b). Furthermore, they also indicated that the utilisation of the triangular variant can increase the maximum reachable workspace by also removing the internal singularity within the horizontal parallelogram CLSC.

2006 This was followed in 2006 by another of Merz and Roy’s patents [67], which utilised a serial pair of planar parallelogram CLSCs along the vertically actuated proximal link to maintain the mobile platform’s pitch. The additional mass along the vertically actuated chain has the potential to introduce unfavourable dynamic effects into the mechanism. As with their other design, a three way spherical joint was used, with the benefit of decoupling the position from the orientation of the mobile platform.
platform.

2007 In 2007, a 3-DOF axis-symmetric parallel haptic interface with parallelogram CLSCs was prototyped by Kyungnam University in Korea [68]. An optimisation was performed to maximise the mechanism’s workspace and stiffness. It possesses the same link arrangement as many of the previously presented designs [8, 21, 66], with the addition of a redundant actuator actively orienting a vertically actuated chain in the horizontal plane. This redundant actuation overcomes the issue associated with the passive moving mass of the vertical chain. However, in addition to increased cost due to the additional actuator, this introduces the requirement for a high frequency and accurate control system in order to position the vertical chain without introducing stresses into the system.

2008 In a book chapter in 2008, Brogårdh discussed the Tau parallel kinematic structures [9]. He highlighted how scaling up the Delta mechanism becomes impractical after a certain point, due to the physical size of the Delta’s base. This justifies the design of a mechanism with the benefits of the Delta and the workspace and footprint of a serial mechanism. Brogårdh suggests one possible solution is the SCARA-Tau mechanism discussed earlier. This design scales similarly to a serial mechanism, in terms of positional workspace gain versus footprint increase. He concludes by reiterating that there are many practical applications that make use of the Tau family of mechanisms unique combination of properties. Namely, high speed, high stiffness, high accuracy, a small footprint, large positional workspace and infinite rotation about the fixed base column.

The patents [66, 67] were constructed into a physical prototype by Pentec Robotics in 2008. A computer model and the physical prototype is shown in Fig. 2.7(a) and (b),
2.2 Axis-Symmetric Parallel Mechanisms

Figure 2.7: A computer model (a) and physical prototype (b) of two axis-symmetric parallel robots developed by Pentec Robotics in 2008 [69], based on the patents [66, 67].

respectively. Pentec Robotics state on their website that they have closed a licensing agreement to begin manufacturing [69]. A video showing the robots in motion is available from the Pentec Robotics website. It demonstrates the mechanisms’ ability to produce high accelerations and large orientations of the mobile platform.

2009 The following year, in 2009, Quanser manufactured a 2-DOF and 5-DOF parallel haptic device. These devices can be simply converted to axis-symmetric through the same modifications discussed for [61]. Quanser is currently selling versions of these devices as research and learning platforms aimed at universities and research centres [70].

Also in 2009, a patent for a planar 3-DOF axis-symmetric parallel mechanism, called the V3, was filed by Lou et al. [71]. This mechanism is capable of planar translation and infinite rotation of the mobile platform about the axis normal to this plane. The infinite rotation is possible due to a crank style design of the mobile platform. The mechanism, as designed, is not capable of infinite rotation about the common axis of the proximal links, due to interference with a rigid supporting structure. However, by redesigning this supporting structure, while maintaining the
common axis of rotation of the proximal links, the mechanism can achieve this infinite rotation.

2010 In 2010, the kinematic performance of the SCARA-Tau was analysed [22]. The triangular link structure shown in Fig 2.6(b) was analytically determined to significantly reduce the parasitic mobile platform yaw rotation inherent in the SCARA-Tau structure, as was proposed in [56]. The parasitic yaw angle of the SCARA-Tau is a function of the radial distance from the fixed base column and the vertical $z$-position.

2011 In 2011, a family of planar axis-symmetric mechanisms were presented and analysed [25]. These mechanisms included variants with non-parallelogram CLSCs. Methods of utilising redundant actuators to pass through singularities were also proposed. This enabled infinite rotation of the mobile platform itself, through techniques similar to [71], in addition to the infinite rotation around the fixed base column. This increases the industrial potential of axis-symmetric parallel mechanisms.

Also in 2011, Isaksson et al. presented a new 6-DOF parallel mechanism, the Octahedral Hexarot [72]. The design was an expansion of the SCARA-Tau mechanism from three to six actuated proximal links, each with a single 5-DOF SS distal link. The fundamental concept of the Octahedral Hexarot can be seen in a patent from ABB in 2003 [21]. The major differences are in the optimal proximal link spacing and mobile platform design. The Octahedral Hexarot possesses a large positional workspace in relation to its footprint and can achieve considerable mobile platform roll, pitch and yaw rotations. The main limitations to these rotations are collisions between the links and singularities.

In the same year, [49] discussed 3 and 4-DOF parallel mechanisms with axis-symmetric link systems. The mechanisms belonging to an axis-symmetric subclass
that solely utilised 5-DOF distal links and CLSCs. Some presented mechanism variants experience non-planar and non-parallelogram CLSCs. The main argument presented was the use of only 5-DOF distal links results in no torsion or bending forces in these links, enabling the use of well engineered lower mass materials in the construction of the links, such as carbon fibre.

**2012** The following year in 2012 Isaksson et al. analysed a 5-DOF axis-symmetric variant with collinear platform joints, called the Pentarot-5 [73], based on the patents [21, 63]. The proposed mechanism utilises 5-DOF distal links. Singularity, collision and workspace size analysis was performed. The mechanism was shown to achieve a large workspace and considerable mobile platform roll and pitch, with an unconstrained yaw angle, by design. It was stated that future work should include geometry optimisation with respect to workspace, isotropy, force distribution, singularity and stiffness.

The design in [73] was expanded to a 6-DOF mechanism, called the Hexarot-5 [74]. This mechanism utilises the same structure as the Pentarot-5 with an additional kinematic chain for controlling the yaw rotation. This arrangement possesses a larger rotational workspace compared to the Octahedral Hexarot [72]. However, as with the Octahedral Hexarot, the Hexarot-5’s workspace is limited by collisions between links.

The Octahedral Hexarot was prototyped in 2012 as a haptic input device by Marlow [75]. A computer model of the device and its physical prototype are shown in Figs. 2.8(a) and (b).

**2013** An in-depth investigation into the synthesis of axis-symmetric parallel mechanism and their kinematic analysis can be found in [26], with a focus on the utilisation of only 5-DOF distal links and CLSCs. It proceeds from the synthesis of 2-DOF
variants to 6-DOF variants. A method for extending planar axis-symmetric parallel mechanisms to spatial mechanisms was proposed in [31]. Planar mechanisms employing 2-DOF distal links were shown to possess similarities with lower-mobility spatial mechanisms utilising 5-DOF distal links. A link substitution scheme was proposed and a variety of spatial axis-symmetric parallel mechanisms were derived with CLSCs, some of which included non-parallelogram CLSCs.

Also in 2013, the planar V3 axis-symmetric parallel mechanism [71] was analysed [29]. The inverse kinematics, workspace and singularities were briefly examined. Through optimisation against the average extreme velocity and the average extreme error, an optimal design for the mechanism was generated, with improved performance in workspace, velocity and accuracy.

In the same year, an experimental study on the motion error associated with the Octahedral Hexarot mechanism using constant actuator speeds was performed by Qazani et al. [76]. The non-linear motion of the mobile platform, while moving from one point to another, was calculated then experimentally verified through image
processing techniques. They concluded that the error increased with larger displacements, and that the error is larger when the mobile platform is closer to the fixed base column.

2014 In 2014, Isaksson et al. [77] analysed the inverse kinematic problem for 3-DOF axis-symmetric parallel mechanisms utilising only 5-DOF distal links. The parasitic yaw rotation of the mobile platform was found to increase the complexity of solving the inverse kinematic equations. The general solution to the inverse kinematics was reduced to solving a univariate equation, resulting in a semi-numerical algorithm. The process used to obtain the general solution can be applied to all axis-symmetric parallel mechanisms utilising 5-DOF distal links. Analytical inverse kinematic solutions for some variants were also presented.

In the same year additional error analysis of the Octahedral Hexarot was performed by Pedrammehr et al. [78]. They also examined the motion curvature of the mobile platform between two points, using constant actuator speeds, and approached the same conclusion. That is an increase in the point spacing produced larger errors. The two examinations of the Octahedral Hexarot’s motion error have marginal practical use, due to the fact that in nearly all mechanisms, constant actuator speed rarely results in a linear mobile platform motion, a control system and path planning algorithm is utilised to account for this.

Later that year Marlow et al. [27] analysed the workspace and singularities of two axis-symmetric parallel mechanisms with CLSCs, which were derived in [26] and based on the patents [66,67]. The workspace was analysed with respect to size, singularities, and dexterity distribution. Singularities were detected using the determinate of the input-output Jacobian matrices, as proposed by Gosselin and Angeles [79]. However,
this method has the major limitation of not being able to detect constraint singularities, a type of singularity identified by Zlatanov et al. in 2002 [80], which can occur in lower-DOF parallel mechanisms. The paper illustrated the singular configurations of the mechanisms and implemented genetic algorithm techniques to determine a parameter set that maximises the singularity-free workspace.

The V3 axis-symmetric parallel mechanism [29, 71] was extended to produce 4-DOF Schönflies-motion by Liao et al. [30] in late 2014, called the T4. This extension involved the integration of multiple parallelogram CLSCs into the V3’s design, along with an additional actuator and a passive base joint, similar to [8, 66, 67]. They presented the inverse and forward kinematics of the mechanism and examined its singularities through analysis of the input-output Jacobian. An algebraic derivation of the dextrous workspace was also presented.

Also in 2014, a 4-DOF axis-symmetric mechanism was proposed with the aim of producing both a large positional and rotational workspace, comparable to that of a serial mechanism [28]. The described mechanism incorporated CLSCs and utilized a gearing system on the mobile platform to enable 360-degree controllable yaw rotation actuated by an additional kinematic chain. However, as highlighted by the author, the cost and performance of such a solution should be carefully weighed against the simplicity of mounting an actuator on the mobile platform of a 3-DOF mechanism to control the yaw.

2.2.2 Findings from the Literature

The axis-symmetric style of parallel mechanism is only 20 years old, with the majority of the fundamental research and prototype development occurring in the last 10 years.
2.2 Axis-Symmetric Parallel Mechanisms

Axis-symmetric parallel mechanism exhibit the benefits of other parallel mechanisms, including the potential for high stiffness, high acceleration, high load capacity and high accuracy. Furthermore, axis-symmetric parallel mechanisms also possess the additional benefits of a small footprint, large positional workspace and infinite rotation about the fixed base column. All of these benefits combine to form a family of mechanisms with a significant industrial potential.

A feature common to many lower-mobility axis-symmetric parallel mechanisms is the use of CLSCs to provide constraints on the output motion of the mobile platform. Many researchers have stated the benefits of these CLSCs, however, no research has been performed on examining their potential negative effects. Furthermore, the type of CLSCs utilised are also present in the world’s best-selling parallel mechanism, the Delta [11–14], along with other mechanisms such as the H4 [15–17] and Orthoglide [18–20]. Additionally, many of the proposed axis-symmetric designs also possess non-parallelogram [25, 31] and potentially non-planar CLSCs [49]. The performance effects of non-parallelogram and non-planar CLSCs in parallel mechanisms have not previously been examined. Furthermore, the methods utilised for the singularity analysis of axis-symmetric parallel mechanisms required multiple techniques to detect all singularity types, which results in limited physical insight into the cause or consequence of the singularities.

Singularities are one of the most critical factors affecting the performance of parallel mechanisms. Hence, analysis of the locations and causes of singularities is essential for the development of a usable mechanism. As indicated by Brogårdh et al. [56], singularities can also exist within CLSCs themselves. This motivates the development of a method to completely characterise all singularities present within axis-symmetric parallel mechanisms, including those present in parallel, non-parallel and non-planar
CLSCs. However, the performance of a mechanism is not only affected at these singular locations, but also in the regions surrounding the singularities. Therefore, a type of closeness measure to these singular locations should also be considered.

To truly understand the causes of singularities and the factors effecting the closeness to singular configurations, a geometric approach to the study of kinematic performance is utilised. This approach brings comprehensive insight into the principles of motion, enabling a researcher to visualise the mechanism’s motion and configuration, rather than examining complex algebraic equations [81]. The main tool in this geometric approach is the theory of screws.

### 2.3 Screw Theory

Screw theory is a mathematical tool used for the analysis of mechanism kinematics, statics and dynamics. It produces meaningful results with a physical interpretation, expressed clearly and intuitively using geometrical concepts and common algebraic calculations.

Throughout the nineteenth century the theory of screws began to form. In 1806, Poinsot introduced the concept of geometric mechanics [82], which stated that a system of forces acting on a rigid body can be resolved into a single force and a couple, later known in screw theory as a wrench on a screw. Then in 1830, Chasles proposed the concept of resolving the displacement of a rigid body into a translation along an axis and a rotation about that axis [83], termed the twist of a rigid body. Later that century, Plücker proposed his six component coordinates of lines in space, called Plücker line coordinates [84]. Plücker coordinates are utilised in screw theory to describe the screws, twists and wrenches of a rigid body. In 1872, Ball published *The Theory of Screws - Geometrical Study of the Kinematics, Equilibrium and Small*
Oscillations of a Rigid Body that contained concepts in common to Plücker’s research [85]. In 1900, Ball amalgamated his research into the theory of screws describing the kinematics and dynamics of a rigid body subjected to complex constraints [40], reprinted in 1998 [86]. Up to the 1960’s, screw theory received minimal attention, until Hunt and Phillips rediscovered Ball’s results and revived interest in screw theory through application onto spatial mechanisms [85].

Over the last five and a half decades, many researchers have continued to make important contributions to screw theory, including Hunt [41], Phillips [87], Duffy [88], Roth [89], Angeles [90], Lipkin [91], Zlatanov [80], Bonev and Gosselin [92], Dai [93] and Davidson [94]. Next, an introduction to screw theory is presented, providing the fundamental mathematics and descriptions utilised throughout this thesis.

2.3.1 Mathematical Foundation

The Plücker coordinates of a unit screw is as a six-dimensional vector represented by

$$\hat{s} = \begin{bmatrix} \hat{s} \\ s_O \end{bmatrix}.$$  \hspace{1cm} (2.1)

It is composed of two three-dimensional vectors $\hat{s}$ and $s_O$, where $\hat{s}$ is a unit vector directed along the axis of the screw and $s_O$ is the sum of the parallel and perpendicular components of the screw’s effect on a point coincident with the origin $O$ fixed in a reference frame. The physical meaning of the parallel and perpendicular components vary with what the screw is representing. The vector $s_O$ is defined as

$$s_O = h\hat{s} + r_O \times \hat{s},$$  \hspace{1cm} (2.2)
given \( h \) is the screw’s pitch and \( \mathbf{r}_O \) is a position vector to any point on the screw’s axis directed from \( O \).

**Twist**

A common example of a screw is the twist associated with the instantaneous velocity of a rigid body. The twist of a moving body is defined as,

\[
\mathbf{t} = \mathbf{\omega} \hat{s} = \begin{bmatrix} \mathbf{\omega} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{\omega} \\ h\mathbf{\omega} + \mathbf{r}_O \times \mathbf{\omega} \end{bmatrix}
\]  

(2.3)

where \( \mathbf{\omega} \) represents the amplitude of the body’s angular velocity. This can then be simplified to,

\[
\mathbf{t} = \begin{bmatrix} \mathbf{\omega} \\ \mathbf{v} + \mathbf{r}_O \times \mathbf{\omega} \end{bmatrix} = \begin{bmatrix} \mathbf{\omega} \\ \mathbf{v}_O \end{bmatrix}
\]  

(2.4)

where, \( \mathbf{v} \) is the linear velocity in \( \hat{s} \) due to the pitch of the screw, \( \mathbf{r}_O \times \mathbf{\omega} \) is the linear velocity of a point coincident with \( O \) due to the rotation \( \mathbf{\omega} \) about the axis of \( \hat{s} \), and \( \mathbf{v}_O \) is the resultant linear velocity of the body. The twist is shown graphically in Fig. 2.9.

Figure 2.9: Graphical definition of the components of a twist.
Wrench

A screw can also represent a wrench associated with the force and moment acting on a rigid body. A wrench acting on a body is

$$w = f \hat{s} = \begin{bmatrix} f \hat{s} \\ f s_O \end{bmatrix} = \begin{bmatrix} f \\ h f + r_O \times f \end{bmatrix}$$ (2.5)

where, \( f \) is the intensity of the force acting on the body. This can then be simplified to,

$$w = \begin{bmatrix} f \\ m + r_O \times f \end{bmatrix} = \begin{bmatrix} f \\ m_O \end{bmatrix}$$ (2.6)

where, \( m \) is the moment due to the pitch of the screw around \( \hat{s} \), \( r_O \times f \) is the moment at a point coincident with \( O \) due to the force \( f \) along the axis of \( \hat{s} \), and \( m_O \) is the resultant moment. The wrench is shown graphically in Fig. 2.10.

As can be seen from Eqns. (2.3) and (2.5), twists and wrenches are formed by assigning an amplitude \( \omega \) or an intensity \( f \) to a unit screw, respectively. However, the amplitude and intensity are not important for the analysis performed in this thesis, hence, the notation is simplified by representing both twists and wrenches as unit screws, with each identified by specific subscripts.

Figure 2.10: Graphical definition of the components of a wrench.
Zero and Infinite Pitch Screws

Two special cases exist for twists and wrenches, one when the pitch $h = 0$ and the other when $h = \infty$. A twist of zero pitch represents a pure rotational motion, analogous to a revolute joint’s motion, while a wrench of zero pitch is a pure force. An infinite pitch twist is a pure translation, corresponding to a prismatic joint’s motion, whereas an infinite pitch wrench is a pure moment. If a unit screw has zero or infinite pitch, the dot product between its two three dimensional components must equal zero. That is

$$\mathbf{s} \cdot \mathbf{s}_O = 0, \quad (2.7)$$

for zero or infinite pitch screws. The physical interpretations of the two cases are summarised in Table 2.1 along with the related unit screw definitions. An infinite pitch screw is termed a free vector, this means it only has an associated direction and its application onto any point on a rigid body generates the same result.

Lower Kinematic Pairs

For screw theory analysis, each joint, or lower kinematic pair, in a mechanism is represented by a twist, or a set of intersecting twists. The lower kinematic pairs are the revolute (R), universal (U), spherical (S) and prismatic (P) joints. As mentioned above, an R joint is modelled as a zero pitch twist and a P joint as an infinite pitch twist. A U joint is composed of two intersecting non-coaxial zero pitch twits and an S joint consists of three intersecting non-coplanar zero pitch twits. These lower

<table>
<thead>
<tr>
<th>Table 2.1: The two special cases for twists and wrenches.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h = 0$</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>Twist</td>
</tr>
<tr>
<td>Wrench</td>
</tr>
<tr>
<td>Unit Screw</td>
</tr>
</tbody>
</table>
kinematic pairs are illustrated in Table 2.2, along with their twists shown as black arrows, the joint output indicated by an orange dot and a fixed frame indicated by the ground symbol.

**Axis- and Ray-Coordinates**

A screw can be expressed in the axis- or ray-coordinate order, as outlined in Table 2.3. These coordinate formulations were proposed by Plücker [84] to distinguish between a line described as the join of two points, versus the meet of two planes. It is common for both orders to manifest themselves in expressions. The screw in Eqn. (2.1) is expressed in ray-coordinate order; in axis-coordinate order the terms $\mathbf{s}$ and $s_O$ interchange. Therefore, to reduce confusion this thesis maintains the ray-coordinate representation.

<table>
<thead>
<tr>
<th>Lower Kinematic Pair</th>
<th>Screw Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Revolute Joint</td>
<td>Single zero pitch twist along rotation axis</td>
</tr>
<tr>
<td>Universal Joint</td>
<td>Two intersecting non-coaxial zero pitch twists</td>
</tr>
<tr>
<td>Spherical Joint</td>
<td>Three intersecting non-coplanar zero pitch twists</td>
</tr>
<tr>
<td>Prismatic Joint</td>
<td>Single infinite pitch twist along translation axis</td>
</tr>
</tbody>
</table>
2.3 Screw Theory

Table 2.3: Axis- and ray-coordinate orders of screws.

<table>
<thead>
<tr>
<th>Coordinate Order</th>
<th>[ \dot{\hat{s}} = \begin{bmatrix} s_O \ \hat{s} \end{bmatrix} ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Axis-</td>
<td>[ \dot{\hat{s}} = \begin{bmatrix} \hat{s} \ s_O \end{bmatrix} ]</td>
</tr>
<tr>
<td>Ray-</td>
<td></td>
</tr>
</tbody>
</table>

order for screws, twists and wrenches unless stated or symbolised. Screws in axis-coordinate order will be identified by the addition of a tilde, (\(\tilde{}\)) to their symbol, as seen in Table 2.3.

The Reciprocal Product

An important concept in the theory of screws is the reciprocal product [40]. If one screw represents a twist \(\hat{\delta}_t\) and the other a wrench \(\hat{s}_w\), the reciprocal product is the instantaneous work done by the wrench on the twist [95]. If this product equals zero, the screws are said to be reciprocal, that is, the wrench does not perform any work on the twist. This concept forms the foundation of the singularity and motion/force transmission analysis in this thesis. The reciprocal product is defined as

\[
\hat{\delta}_t \circ \hat{s}_w = \hat{s}_t \cdot s_{O_w} + s_{O_t} \cdot \hat{s}_w,
\]

where \(\circ\) is the reciprocal product operator. As stated, all screws are given in ray-coordinate order [84] unless otherwise identified.

A system of twists can represent the DOF of a rigid body and the reciprocal system contains the wrenches constraining the body to these DOF. The opposite is also true; if the wrench system is provided, the reciprocal system of twists represents a body’s permissible motion [93,96]. This principle is fundamental in the screw theory analysis of mechanisms and is utilised extensively in singularity and motion/force
transmission performance analysis.

Calculating the Reciprocal Screw System

A screw system composed of \( k \leq 6 \) linearly independent twists (wrenches) has a reciprocal screw system of \( 6 - k \) linearly independent (wrenches) twists. By definition, each individual screw in a given screw system is reciprocal to all the screws in the determined reciprocal screw system. An analytical method for determining the reciprocal screws of a system is presented in [95] and involves four main steps:

1. Assign a coordinate frame.
2. Form the Plücker coordinates for the \( k \) screws in the chain.
3. Check the linear dependence of the screws, through linear algebra, and find a linearly independent set.
4. Use the reciprocal condition and Eqn. (2.8) to generate a set of linear equations, then solve for the reciprocal screws.

It should be noted that the substitution of any CLSC by its generalised kinematic pair [43, 97, 98] is required before this analysis can be performed. The procedure for this is discussed in Section 3.3.

The coordinate system in Step 1 can be selected to produce the simplest Plücker coordinates in Step 2, by positioning its origin to intersect the maximum number of screw axes, resulting in \( \mathbf{r}_O = \mathbf{0} \), and directing its axes along the most screw axes, resulting in a single \( x, y \) or \( z \) component in \( \hat{s} \), with reference to Eqns. (2.1) and (2.2). The use of computer algebra systems, such as MATLAB, enables the user to arbitrarily assign a coordinate system, however, if a coordinate system is appropriately selected the screws can be more intuitively understood. Step 3 can be
performed using the inbuilt commands from a computer mathematical software, such as MATLAB. The linear dependence of screws in a system can be checked using the \texttt{rank}(M) command, where the rows of the matrix \(M\) are formed by the transpose of the screws. If the rank is lower than the number of rows in \(M\), then \(M\) contains linearly dependent screws. The set of linearly independent set of screws can be determined by inspection or by applying the \texttt{rref}(M) command in MATLAB and then removing the dependent rows. Step 4 can also be performed using mathematical software. In MATLAB, the reciprocal screws can be determined using the \texttt{null}(\tilde{M}) command. Where \(\tilde{M}\) is the matrix \(M\) with the screws in axis-coordinate order. This results in the null-space of \(\tilde{M}\), which is the reciprocal screws of \(M\), in ray-coordinate order.

The method can calculate the reciprocal twist system from a system of wrenches or the reciprocal wrench system from a system of twists. The resultant reciprocal screws may possess zero, finite or infinite pitches. This method is utilised throughout this thesis and within the motion/force transmission analysis to calculate the resultant instantaneous output twist of the mobile platform given different sets of constraining wrenches.

For certain sets of screw systems, it is possible to avoid the above algebraic derivations and determine the reciprocal twists or wrenches through inspection of the system’s geometry. Understanding the geometrical relations also enables the user to visualise the twist and wrench systems and their interactions. Hence, a geometrical method for determining the reciprocal screws is outlined below.

**Geometrical Method for Determining the Reciprocal Screws**

An intuitive geometrical approach to determine the reciprocal screws for common screw systems was presented by Zhao et al. [95]. They proposed a set of three geometrical observations that enable the identification of reciprocal screws without the
use of complex algebraic derivations. The observations are only applicable to zero and infinite pitch screws, however, this limitation does not affect its usefulness with respect to analysing the majority of parallel mechanisms.

The three observations are summarised as follows:

**Observation 1:** If the screw axis of a pure force (pure rotational motion) $\mathbf{s}_1$ is coplanar with the screw axis of a pure rotational motion (pure force) $\mathbf{s}_2$ they are reciprocal.

**Proof:** According to the observation, the two screws have zero pitch and can therefore be expressed as,

$$\mathbf{s}_1 = p_1 \begin{bmatrix} \hat{s}_1 \\ r_1 \times \hat{s}_1 \end{bmatrix} \quad \text{and} \quad \mathbf{s}_2 = p_2 \begin{bmatrix} \hat{s}_2 \\ r_2 \times \hat{s}_2 \end{bmatrix}, \quad (2.9)$$

where, $p_1$ and $p_2$ are the respective magnitudes of the screws. Then their reciprocal product is

$$\mathbf{s}_1 \circ \mathbf{s}_2 = p_1 p_2 (\hat{s}_1 \cdot (r_2 \times \hat{s}_2) + \hat{s}_2 \cdot (r_1 \times \hat{s}_1))
= p_1 p_2 (r_2 \cdot (\hat{s}_1 \times \hat{s}_2) + r_1 \cdot (\hat{s}_1 \times \hat{s}_2))
= p_1 p_2 (r_1 \cdot (\hat{s}_1 \times \hat{s}_2) - (\hat{s}_1 \times \hat{s}_2) \cdot r_2)
= p_1 p_2 ((r_1 - r_2) \cdot (\hat{n}) (\sin(\alpha)))
= p_1 p_2 a \sin(\alpha) \quad (2.10)$$

where, $a$ and $\hat{n}$ are the length and unit direction vector of the common perpendicular between the two screws, and $\alpha$ is the twist angle between the two screws, as illustrated on a local coordinate system in Fig. 2.11.
Therefore, as per observation one, for two zero pitch screws to be reciprocal they must be coplanar, that means, they either intersect at a common point or are parallel to each other. If they are parallel $\alpha = 0$, while if they intersect $a = 0$. Both cases result in Eqn. (2.10) equalling zero, proving their reciprocity. Furthermore, as stated earlier, the magnitude of the two screws do not contribute to the measure of reciprocity, unless one is zero, in which case no work can be done anyway.

To aid the reader, this observation can be visualised as trying to close a hinged door by pushing with a pure force, $s_1$, directly along a door’s edge towards its hinge, $s_2$. The axis of your force and the axis of the door’s hinge are intersecting. Therefore, your force passes directly through the door and the hinge, performing no work on the door. Hence, your zero pitch wrench is reciprocal to the zero pitch twist of the door’s hinge.

**Observation 2:** A pure moment (pure translational motion) is always reciprocal to a pure translational motion (pure moment).

It can be seen in Table 2.1 that the first three components of an infinite pitch screw are zero. Therefore, the reciprocal product of these two screws will always be zero. This can be visualised as trying to open a sliding door by applying a moment anywhere on its body. Your infinite pitch wrench is reciprocal to the infinite pitch twist of the sliding door and therefore performs no work.
Observation 3: If the axis of a pure force (pure rotational motion) $s_1$ is perpendicular to the axis of a pure translational motion (pure moment) $s_2$ they are reciprocal.

**Proof:** Given the two screws,

$$s_1 = p_1 \begin{bmatrix} \hat{s}_1 \\ r_1 \times \hat{s}_1 \end{bmatrix} \quad \text{and} \quad s_2 = p_2 \begin{bmatrix} 0_{(3 \times 1)} \\ \hat{s}_2 \end{bmatrix} ,$$

(2.11)

where, $p_1$ and $p_2$ are the respective magnitudes of the screws, their reciprocal product is,

$$s_1 \circ s_2 = p_1 p_2 (\hat{s}_1 \cdot \hat{s}_2) + 0_{(3 \times 1)} \cdot (r_1 \times \hat{s}_1)$$

(2.12)

Therefore, the reciprocal product equals zero if and only if the infinite pitch screw and the zero pitch screw are perpendicular.

This can be visualised by trying to open or close a sliding door by pushing directly onto its front surface, all forces are perpendicular to the door’s sliding axis and therefore no work is done on the door. Hence, your zero pitch force and the doors infinite pitch twist are reciprocal and you perform no work on the door.

The three observations can be reworded to provide a more intuitive understanding for their application onto serial chains of joints in mechanisms.

For reciprocity,

- **An infinite pitch wrench, pure moment,**
  - is always reciprocal to any prismatic joint in a serial chain. (Observation 2)
  - must be perpendicular to any revolute joint in a serial chain. (Observation 3)
• The axis of a zero pitch wrench, pure force, must be
  - coplanar with the axis of any revolute joint in a serial chain. (Observation 1)
  - perpendicular to the direction of any prismatic joint in a serial chain. (Observation 3)

The above observations are true for a serial chain of joints, hence any CLSCs within a chain are required to be replaced by their generalised kinematic pair [43, 97, 98], forming the equivalent serial chain, before the above observations can be implemented. This procedure is further discussed in Section 3.3.

2.4 Chapter Summary

This chapter provided the background of the mechanisms and mathematical concepts utilised throughout this thesis. An in-depth review of the history of axis-symmetric parallel mechanisms was provided, highlighting a clear gap in the field. It was found that the methods utilised for the singularity analysis of axis-symmetric parallel mechanisms required multiple techniques to detect all singularity types, which results in limited physical insight into the cause or consequence of the singularities. Furthermore, many of the proposed axis-symmetric designs were found to possess non-parallelogram and potentially non-planar CLSCs. The performance effects of non-parallelogram and non-planar CLSCs in parallel mechanisms have not previously been examined.

A geometric approach to the study of kinematic performance was selected. The approach is based on screw theory and brings comprehensive insight into the principles of motion, enabling a researcher to visualise the mechanism’s motion and configuration, rather than examining complex algebraic equations. The origins and mathematical foundation of screw theory was presented.
Some of the following chapters also include additional reviews literature that are targeted at the chapter’s content. In the next chapter, screw theory based techniques for singularity analysis are introduced and applied to a family of planar 2-DOF axis-symmetric parallel mechanisms with various CLSCs.
3

Singularity Analysis of Planar 2-DOF Axis-Symmetric Parallel Mechanisms with Closed-Loop Sub-Chains

3.1 Overview

In this chapter a screw theory based singularity analysis method is introduced and applied to a family of planar 2-DOF axis-symmetric parallel mechanisms with different planar closed-loop sub-chains (CLSCs). First, the mechanism anatomy is introduced along with a general procedure for determining its inverse kinematic solutions. Then a systematic method of defining the twists and wrenches utilised in singularity analysis for both the serial chains and chains with CLSCs is detailed. The latter case requires modified wrench definitions, which are proposed herein.

The studied closed-loop sub-chain (CLSC) configurations are then illustrated,
along with the definitions of their appropriate wrenches for singularity analysis. This is followed by a description of the singularity types experienced by parallel mechanisms and the corresponding screw theory based methods to discover their locations. The singular locations detected by the screw theory based method, utilising the proposed wrench definitions, are verified against the zero points of the conditioning index of the numerical input-output Jacobian matrix. An examination of the singular configurations associated with these mechanisms is also performed.

The presented screw theory based analysis detects all singular locations and provides physical meaning about the cause of a singularity and the direction of the motion gained or lost for the planar 2-DOF axis-symmetric parallel mechanisms with a CLSC.

3.2 The Mechanism Anatomy

A planar mechanism is classed as lower-mobility due to possessing fewer than 6-DOF. The chains of lower-mobility mechanisms can be of serial, parallel or mixed topology. This thesis is concerned with mechanisms that include both serial and mixed topology chains, also known as serial-parallel chains. As the name suggests, these chains include links in both series and parallel, with the parallel portion termed a CLSC.

The CLSCs examined herein are composed of three or four coplanar links, where the output link attaches directly to the mobile platform. Examples of such CLSCs are seen in the world’s best-selling parallel mechanism, the Delta [11], along with the H4 [15], SCARA-Tau [21–24] and Orthoglide [18] mechanisms.

The SCARA-Tau is part of a family of parallel mechanisms, termed axis-symmetric, designed to improve the limited workspace-to-footprint ratio innate to most parallel mechanisms. A common feature in many of these lower mobility axis-symmetric
3.2 The Mechanism Anatomy

Parallel mechanisms is the use of CLSCs. These mechanisms commonly experience coupled motions between the mobile platform’s yaw angle and radial translations. This coupling increases the difficulty of generating the inverse kinematic solution.

Two planar 2-DOF axis-symmetric parallel mechanisms with a CLSC, are illustrated in Fig. 3.1. These mechanisms consist of a fixed base column centred along the common axis of rotation of the two actuated proximal links. One proximal link is connected to the mobile platform through a single 2-DOF distal link and the other through two 2-DOF distal links, the latter pair forming the CLSC. The distal links are composed of a fixed-length link and a revolute joint on each end with parallel rotation axes.

The axis-symmetric configuration leads to a large reachable workspace with the ability of infinite rotation of the mechanism’s link system about the fixed base column. Hence, axis-symmetric parallel mechanisms are typically able to utilise an optimal path between two programmed positions.

Figure 3.1: Two examples of planar 2-DOF axis-symmetric parallel mechanisms with a CLSC. (a) The obtuse trapezium and (b) the obtuse triangular variants.
3.2 The Mechanism Anatomy

3.2.1 Kinematic Parameters

To develop a mathematical model of the mechanism, the locations of all joints and platforms must be clearly defined. Hence, the first step in the kinematic analysis is the parameterisation of the mechanism. This can be completed in several ways, normally with the aim to produce the smallest set of kinematic parameters. A general description of the studied 2-DOF axis-symmetric parallel mechanisms’ kinematic parameters is shown in Fig. 3.2. A fixed coordinate system $O$ is defined with its $z$-axis coincident with the common rotation-axis of the actuated proximal links. Due to the axis-symmetry, the $x$-axis can be selected arbitrarily provided that it is perpendicular to the $z$-axis, while the $y$-axis is defined according to the right-hand rule. The joint $B_{i,j}$ connects the proximal and distal links, $C_{i,j}$ are the mobile platform joints, a proximal link is labelled $L_{a_{i,j}}$ where $l_{a_{i,j}}$ is its horizontal kinematic length from the $z$-axis of $O$ to joint $B_{i,j}$, and a distal link is labelled $L_{b_{i,j}}$ with its horizontal kinematic length $l_{b_{i,j}}$, for $i = 1$ and 2, representing the two kinematic chains of the mechanism. The subscript $j$ identifies the individual distal links of a chain.

A coordinate system $O'$ is attached to the point of analysis, X, on the mobile platform, as shown in Fig. 3.2. The $y$-axis of $O'$ is defined by the direction between $C_{1,1}$ and $C_{2,1}$, with the $x$-axis orthogonal as shown and the $z$-axis as per the right-hand rule. Additionally, the actuated joint angles $q_i$ are defined, along with the mobile platform’s yaw rotation about the fixed base column $\beta$ and the parasitic platform yaw angle $\alpha$, which is the deviation of the mobile platform’s orientation from $\beta$. The terms roll, pitch and yaw are assigned to rotations of the mobile platform about the $x$, $y$ and $z$ axes of $O'$, respectively. Without loss of generality, all $z$-offsets are assumed to be zero, as modifications to these offsets does not affect the mechanism from a kinematic perspective. The listed parameters fully define the geometric structure of
The positions of joint \( B_{i,j} \) and \( C_{i,j} \) are defined as \( b_{i,j} \) and \( c_{i,j} \) respectively, and are calculated relative to the fixed reference frame \( O \), as

\[
b_{i,j} = \begin{bmatrix} b_{x_{i,j}} & b_{y_{i,j}} & 0 \end{bmatrix}^T = \begin{bmatrix} l_{a_{i,j}} \cos q_i & l_{a_{i,j}} \sin q_i & 0 \end{bmatrix}^T \tag{3.1}
\]

\[
c_{i,j} = \begin{bmatrix} c_{x_{i,j}} & c_{y_{i,j}} & 0 \end{bmatrix}^T = x + R_p R_{zyx} c'_{i,j}, \tag{3.2}
\]

where \( x = \begin{bmatrix} P_x & P_y & 0 \end{bmatrix}^T \) represents the position vector of X when described in \( O \), \( c'_{i,j} \) is the position vector of joint \( C_{i,j} \) described in \( O' \), and \( R_p \) and \( R_{zyx} \) are the rotation matrices of the unwanted parasitic and desired orientation of the mobile platform.
respectively. For the 2-DOF mechanisms studied herein,

\[
R_p = \begin{bmatrix}
    c_\alpha & -s_\alpha & 0 \\
    s_\alpha & c_\alpha & 0 \\
    0 & 0 & 1
\end{bmatrix}
\]  

(3.3)

and

\[
R_{xy} = \begin{bmatrix}
    c_\beta & -s_\beta & 0 \\
    s_\beta & c_\beta & 0 \\
    0 & 0 & 1
\end{bmatrix},
\]

(3.4)

where \( \beta = \text{atan2}(P_y, P_x) \), \( c_\alpha = \cos \alpha \), \( s_\alpha = \sin \alpha \), \( c_\beta = \cos \beta \) and \( s_\beta = \sin \beta \). Three dimensional vectors and matrices are used in the description of these 2-DOF mechanisms in order to simplify the generation of the mechanisms’ screws in Section 3.3.

### 3.2.2 Kinematic Analysis

Kinematic analysis is essential in the design of a robotic mechanism. It involves examining the properties of motion and developing a mathematical model of the mechanism. A model describing the actuated joint angles with respect to the mobile platform position and orientation is termed the inverse kinematic model, while the forward kinematic model relates the same two variable sets in the opposite direction. Unlike serial kinematic mechanisms, the inverse kinematic models of parallel kinematic mechanisms are in general relatively straightforward to determine and solve due to their closed-loop architecture. However, the complexity of solving the inverse kinematic problem is typically increased if the mechanism exhibits coupled, parasitic, motion of the mobile platform [77]. The studied mechanisms exhibit parasitic motion due to the utilisation of CLSCs.
3.2 The Mechanism Anatomy

The presence of parasitic motion results in an unknown variable of the mobile platform. That is, for the 2-DOF translational mechanisms, when \( x \) is given there remains an unknown parasitic yaw angle \( \alpha \), plus the two unknown actuated joint angles, \( q_1 \) and \( q_2 \). A general solution for 2-DOF axis-symmetric parallel mechanisms with parasitic yaw rotation can be obtained by first numerically solving for the parasitic yaw angle, then algebraically calculating the actuated joint angles. Such a method was detailed in [77]. A modified version of this method is presented below.

The inverse kinematics of axis-symmetric parallel mechanisms can be derived using the traditional loop closure method. This method equates the distance between joints \( B_{i,j} \) and \( C_{i,j} \) to the length of the distal link \( L_{b_{i,j}} \),

\[
\| c_{i,j} - b_{i,j} \| = l_{b_{i,j}}
\]  

(3.5)

given the \( j \)th distal limb of the \( i \)th chain, resulting in the length equations for the distal links \( l_{b_{i,j}} \) for each chain.

Substituting the kinematic parameters into Eqn. (3.5) produces,

\[
e_{a_i} + e_{b_i} \sin q_i + e_{c_i} \cos q_i = 0,
\]  

(3.6)

where,

\[
e_{a_i} = c_{x_{i,j}}^2 + c_{y_{i,j}}^2 + l_{a_{i,j}}^2 - l_{b_{i,j}}^2,
\]

\[
e_{b_i} = -2l_{a_{i,j}} c_{y_{i,j}} \] and 

\[
e_{c_i} = -2l_{a_{i,j}} c_{x_{i,j}}.
\]  

(3.7)

Each length equation is then solved for the actuated joint angles \( q_i \). There are two solutions available for the angle \( q_i \), one if the proximal link is actuated from the right hand side and another if actuated from the left. The right and left solutions are
3.2 The Mechanism Anatomy

labelled in Fig. 3.2 as \( q_R \) and \( q_L \), respectively. The solution for \( q_R \) and \( q_L \) presented in [77], has issues at locations where \( e_{a_i} = e_{c_i} \) resulting in a zero denominator, requiring further logic to derive the correct answer.

An alternative inverse kinematic solution was developed herein and published in [27]. This solution overcomes the above limitation by utilising the four-quadrant inverse tangent, enabling a full representation of the inverse kinematics throughout the mechanisms reachable workspace. The solution for the right hand side is,

\[
q_R = \text{atan2}(k_a, k_b),
\]

where,

\[
k_a = \frac{-e_{a_i}e_{b_i} + e_{c_i}\sqrt{-e_{a_i}^2 + e_{b_i}^2 + e_{c_i}^2}}{e_{b_i}^2 + e_{c_i}^2}
\]

\[
k_b = \frac{-e_{a_i}e_{c_i} + e_{b_i}\sqrt{-e_{a_i}^2 + e_{b_i}^2 + e_{c_i}^2}}{e_{b_i}^2 + e_{c_i}^2},
\]

Likewise, for the left hand side

\[
q_L = \text{atan2}(k_c, k_d),
\]

where,

\[
k_c = \frac{-e_{a_i}e_{b_i} + e_{c_i}\sqrt{-e_{a_i}^2 + e_{b_i}^2 + e_{c_i}^2}}{e_{b_i}^2 + e_{c_i}^2}
\]

\[
k_d = \frac{-e_{a_i}e_{c_i} + e_{b_i}\sqrt{-e_{a_i}^2 + e_{b_i}^2 + e_{c_i}^2}}{e_{b_i}^2 + e_{c_i}^2},
\]

given, \( e_{a_i}, e_{b_i} \) and \( e_{c_i} \) are per Eqn. (3.7).

For the 2-DOF mechanisms studied herein, Eqns. (3.8) and (3.10) have unknown inputs of \( P_x \) and \( P_y \), as well as the parasitic platform yaw \( \alpha \), which are all components of the variables \( c_{x_{i,j}} \) and \( c_{y_{i,j}} \). The parasitic platform yaw can be determined
numerically by first calculating the actuated joint angle using the two different mobile
platform joints of the CLSC, C_{2,1} and C_{2,2} in Fig. 3.2. At a feasible parasitic yaw
angle, the two calculations must give identical results, that is, \( q_{R_{2,1}} = q_{R_{2,2}} \), where the
second subscript identifies the mobile platform joint used. Equating the two equa-
tions leave the only unknown as \( \alpha \). Solving this equation produces two solutions
for \( \alpha \), with only one being in the current working mode. Once \( \alpha \) is determined, the
actuated joint angles can be calculated algebraically through Eqns. (3.8) and (3.10).

3.3 Screws for Singularity Analysis

Employing screw theory for singularity analysis requires the definition of a mecha-
nism’s motions and constraints in terms of twists and wrenches, respectively. The
following sections outline the process of determining these screws.

3.3.1 Mechanism Twists

This section first introduces the twists associated with the RRR serial chain and then
defines the twists of a chain containing a four-bar CLSC, R(RR)_2, where the (RR)_2 is
the CLSC. All mechanisms analysed in this chapter exclusively utilise revolute joints
with parallel rotation axes and are actuated by the revolute joint on the proximal
link connected to the fixed base column. The twist of this actuated joint is termed
the input twist [50]. For each mechanism variant, the input twist of all chains is a
zero pitch unit twist coaxial with the \( z \)-axis of \( O \) and is defined by

\[
\mathbf{s}_I = \begin{bmatrix}
\hat{\mathbf{s}}_I \\
\mathbf{r} \times \hat{\mathbf{s}}_I
\end{bmatrix} = \begin{bmatrix}
\hat{\mathbf{s}}_I \\
0_{(3 \times 1)}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 1 & 0 & 0
\end{bmatrix}^T,
\]  

(3.12)
3.3 Screws for Singularity Analysis

given \( \mathbf{r} = \mathbf{0}_{(3 \times 1)} \), due to the axis of \( \hat{s}_i \) intersecting \( O \).

Serial Chain Twists

For a serial chain, each joint is deconstructed into its equivalent 1-DOF joints, where a spherical joint (universal joint) is represented by a serial chain of three intersecting non-coplanar (two intersecting non-coaxial) revolute joints. For the \( i \)th chain, this results in a total of \( m_i \) equivalent 1-DOF joints. Determining the linearly independent set of twists from the \( m_i \) joints results in the set of \( n_i \) twists for that chain, where \( n_i \) represents the chain’s DOF. In the planar RRR serial chain is illustrated in Fig. 3.3. Each joint has a zero pitch twist directed along its axis, defined by

\[
\hat{s}_{i,j} = \begin{bmatrix} \hat{s}_{i,j} \\ r_{i,j} \times \hat{s}_{i,j} \end{bmatrix}
\]  

(3.13)

where \( \hat{s}_{i,j} \) is the unit vector along the axis \( j \)th joint in the \( i \)th chain and \( r_{i,j} \) is the vector from the origin of the coordinate system \( O \) to any point along this axis.

![Figure 3.3: Serial RRR chain twists illustrated as red arrows coaxial with each joint axis. A fixed coordinate system \( O \) is centred with its \( z \)-axis along the blue actuated joint’s axis, and the point of analysis \( X \) is indicated on the mobile platform.](image)
As long as the three joint twists remain linearly independent, the chain’s overall DOF is \( n_i = 3 \). The mobile platform is free to translate in the \( xy \)-plane and rotate about the normal to this plane.

**Closed-Loop Sub-Chain Twists**

In order to determine the twist system for a chain that includes a CLSC, the sub-chain must first be substituted by its generalised kinematic pair \([43, 97, 98]\). This is achieved through a process of linear transformation and reciprocity \([44]\). The final result is an equivalent twist system describing the motion of the output link of the CLSC. Hence, the CLSC can now be thought of as a complex joint with its DOF represented by the equivalent twist system. The complete twist system of the chain is then generated by merging the calculated equivalent twists with the twists of the non-closed-loop portion of the chain and calculating the linearly independent system of \( n_i \) twists, where \( n_i \) is the overall DOF of the chain.

### 3.3.2 Mechanism Wrenches

This section firstly describes the wrenches associated with the \( RRR \) serial chain and then examines the \( R(RR)_2 \) chain with a four-bar CLSC.

**Constraint Wrenches**

The constraint wrenches span the generalised forces that a chain can transmit from the mobile platform to the base when all joints are free to move. That is, they are reciprocal to all joint twists, or equivalent twists in the case of a CLSC. When \( n_i < 6 \), a chain produces \( (6 - n_i) \) constraint wrenches. These are denoted \( \hat{s}_{C_i,k} \) for the \( k \)th constraint wrench of the \( i \)th chain.


3.3 Screws for Singularity Analysis

Serial Chain Constraint Wrenches  The serial RRR chain’s constraints can be obtained through inspection. It was determined that the chain has \( n_i = 3 \) DOF, therefore it must produce \( (6 - n_i) = 3 \) constraint wrenches. Through examination of Fig. 3.3, it can be visualised that the mobile platform can transmit a force in the \( z \)-direction as well as a moment about the \( x \) and \( y \) axes to the base when all joints are free to move. The resultant wrenches are defined as

\[
\hat{s}_{C_i,1} = \begin{bmatrix} \hat{z} \\ c_{i,3} \times \hat{z} \end{bmatrix}, \quad \hat{s}_{C_i,2} = \begin{bmatrix} 0_{(3 \times 1)} \\ \hat{x} \end{bmatrix} \quad \text{and} \quad \hat{s}_{C_i,3} = \begin{bmatrix} 0_{(3 \times 1)} \\ \hat{y} \end{bmatrix},
\]

where, \( \hat{x}, \hat{y} \) and \( \hat{z} \) are unit vectors along the \( x, y \) and \( z \) axes of \( O \), and \( c_{i,3} \) is the position vector of the chain’s mobile platform joint.

The wrenches in Eqn. (3.14) are termed the planar system of constraints and always hold for the studied chain. The constraints are verified by confirming a zero result of the reciprocal product between each twist and constraint wrench, that is, they are reciprocal.

Closed-Loop Sub-Chain Constraint Wrenches  The \( R(RR)_2 \) chain is illustrated in Fig. 3.4. Through inspection it also produces the planar system of constraints, along with an additional constraint on the mobile platform’s yaw orientation, produced by interactions between the links of the CLSC.

The constraints generating the yaw constraint, pictured in Fig. 3.5(a), are two zero pitch screws along the distal links of the CLSC

\[
\hat{s}_{C_{i,1}}^1 = \begin{bmatrix} \hat{s}_1 \\ b_{i,1} \times \hat{s}_1 \end{bmatrix} \quad \text{and} \quad \hat{s}_{C_{i,1}}^2 = \begin{bmatrix} \hat{s}_2 \\ b_{i,2} \times \hat{s}_2 \end{bmatrix},
\]

(3.15)
3.3 Screws for Singularity Analysis

Figure 3.4: The $R(RR)_2$ chain showing the direction vectors along both RR sub-chains. A fixed coordinate system $O$ is centred with its $z$-axis along the blue actuated joint’s axis, and the point of analysis $X$ is indicated on the mobile platform.

where

$$\hat{s}_1 = \frac{c_{i,1} - b_{i,1}}{|c_{i,1} - b_{i,1}|} \text{ and } \hat{s}_2 = \frac{c_{i,2} - b_{i,2}}{|c_{i,2} - b_{i,2}|}$$

for the $i$th planar four-bar CLSC.

The equivalent constraint produced by these two screws is an infinite pitch screw normal to the plane,

$$\hat{s}_{C_i,1} = \begin{bmatrix} 0_{(3 \times 1)} \\ \hat{n}_1 \end{bmatrix},$$

where, * signifies that it is an equivalent wrench and $\hat{n}_1$ is the unit vector normal to the plane, as illustrated in Fig. 3.5(b). Given $\hat{s}_1$ and $\hat{s}_2$ are in the $xy$-plane, Eqn. (3.17) can be rewritten as

$$\hat{s}_{C_i,1} = \begin{bmatrix} 0_{(3 \times 1)} \\ \hat{z} \end{bmatrix},$$

where, $\hat{z}$ is the unit direction vector along the $z$-axis of $O$. Equations (3.17) and (3.18) are true as long as the links $B_{i,j}C_{i,j}$ and $C_{i,1}C_{i,2}$ of the CLSC do not become
3.3 Screws for Singularity Analysis

Figure 3.5: The planar four-bar CLSC constraints. (a) The two constraint generating wrenches positioned coaxial with the CLSC’s distal links and intersecting the respective proximal \( B_i,j \) and mobile platform \( C_i,j \) joints. (b) The equivalent constraint wrench, normal to the two constraint generating wrenches.

collinear. If they are collinear, the constraint disappears resulting in a singularity. This case is further discussed in Section 4.4.

**Actuation Wrenches**

A chain’s actuation wrench is the generalised force, additional to the constraint wrenches, that can be transmitted from the mobile platform to the base with the actuated joint’s input twist \( \hat{\mathbf{I}}_i \) locked. Away from singular locations, the actuation wrench is reciprocal to all passive joint twists, or equivalent passive twists in the case of a CLSC.

**Serial Chain Actuation Wrenches** For the serial RRR chain, the generalised force is directed along the distal link and is independent of the choice of the point of analysis, \( X \), on the mobile platform, as illustrated in Fig. 3.6.

The actuation wrench has zero pitch and intersects the axes of joints \( B_{i,1} \) and \( C_{i,1} \). It can be expressed as

\[
\hat{s}_{A_i} = \begin{bmatrix} \hat{s}_i \\ r_i \times \hat{s}_i \end{bmatrix}, \tag{3.19}
\]
3.3 Screws for Singularity Analysis

Figure 3.6: Actuation wrench of a serial RRR chain, positioned along the axis of the chain’s distal link and intersecting the chain’s two passive joint twists.

where \( \hat{s}_i \) is the unit vector along the distal link and \( r_i \) is any vector from the origin to the screw axis, typically selected to be \( b_{i,1} \) or \( c_{i,1} \), which respectively are the position vectors of joints \( B_{i,1} \) and \( C_{i,1} \) labelled in Fig. 3.6.

Closed-Loop Sub-Chain Actuation Wrenches  One of the key tools in determining the actuation wrench of a planar CLSC is the concept of the instantaneous centre of velocity (IC). At any instant the planar motion of a rigid body is equivalent to a rotation of that body, as a whole, about a fixed centre in space [99]. This fixed centre is known as the IC and is also equal to the equivalent twist calculated in Section 3.3.1.

Figure 3.7 illustrates a general planar four-bar closed-loop in the \( xy \)-plane. For compactness, the chain identifier \( i \) has been removed from the notion in this section. Two vectors are directed from \( B_1 \) to \( C_1 \) and \( B_2 \) to \( C_2 \), and labelled \( s_1 \) and \( s_2 \), respectively. The IC of the output link P is determined by calculating the intersection of these two vectors.
The first step in determining the IC is to compute,

\[ d = s_1x s_2y - s_1y s_2x \]  \quad \text{and}  \quad (3.20a)

\[ n = s_2x (b_{1y} - b_{2y}) - s_2y (b_{1x} - b_{2x}) \]  \quad (3.20b)

where, \( s_{kx} \) and \( s_{ky} \) are the \( x \) and \( y \) components of vector \( \mathbf{s}_k \) for \( k = 1 \) and \( 2 \), and \( b_{kx} \) and \( b_{ky} \) are the \( x \) and \( y \) components of joint \( B_k \), for \( k = 1 \) and \( 2 \).

The intersection between vectors \( \mathbf{s}_1 \) and \( \mathbf{s}_2 \) is then determined as

\[
\mathbf{v} = \begin{bmatrix} b_{1x} + \frac{n}{d} s_{1x} \\ b_{1y} + \frac{n}{d} s_{1y} \\ 0 \end{bmatrix},
\]

\[
(3.21)
\]

where \( \mathbf{v} \) is the position vector of the IC in the base coordinate system \( O \). A zero has been appended to \( \mathbf{v} \) in order to produce a \( 3 \times 1 \) vector for consistency with the other vectors.

According to the fraction in Eqn. 3.21 there are four possible solutions for the IC's location, these being...
3.3 Screws for Singularity Analysis

1. If $d = 0$ and $n \neq 0$, the two vectors are parallel and the IC is located at infinity.

2. If $d \neq 0$ and $n \neq 0$, the IC is a finite point and not located at joint $B_1$ or $B_2$.

3. If $d \neq 0$ and $n = 0$, the IC is located either at joint $B_1$, $B_2$ or both if they are coaxial.

4. If $d = 0$ and $n = 0$, the two vectors are collinear and there are infinite possible IC’s.

All points in the output link $P$ of the planar four-bar closed-loop instantaneously rotate about a virtual revolute joint located at the IC. Therefore, according to observation one in Section 2.3.1, any force applied to $P$ that intersects the IC is reciprocal to the closed-loop’s motion, and thus, cannot perform work on the closed-loop. This concept is utilised in Section 3.6 to determine the actuation wrenches associated with the various CLSC configurations present in the family of planar mechanisms under examination.

As mentioned, a chain’s actuation wrench is the generalised force, additional to the constraint wrenches, that can be transmitted from the mobile platform to the base with the actuated joint’s input twist $\dot{s}_1$ locked. For a chain containing a planar CLSC, this generalised force is not unique. If the distal links are parallel, the possible forces are coplanar and parallel with the distal links, intersecting the IC at infinity, as shown in Fig. 3.8(a). If the distal links are non-parallel then a planar pencil of lines intersecting the IC of the closed-loop’s output link is generated, illustrated in Fig. 3.8(b). Therefore, by the above definition, an infinite number of actuation wrenches are possible in both cases.

To uniquely determine the actuation wrenches, an understanding of the types of singularities present in parallel mechanisms is required. Hence, the following sections introduce the singularity types followed by a derivation of the appropriate actuation wrenches for use in the singularity analysis of CLSCs.
3.4 Singularities of Parallel Mechanisms

Singularities are one of the most important issues affecting the performance of parallel mechanisms. They are an inherent characteristic, which impacts their stiffness, dexterity, load capacity and workspace size. Therefore, a clear understanding of the singularities present is crucial to the design and control of high performance parallel mechanisms.

In a singularity, mechanisms either gain or lose one or more DOF. If an additional DOF is gained, the mechanism loses stiffness and load bearing capacity in this new DOF direction. If a DOF is lost, the mechanisms can no-longer be actuated along the direction of this DOF. Therefore, identification of singular locations and configurations is an important part of the design and verification of any mechanism.

The methods utilised to examine singularities in parallel mechanisms can be typically grouped into two main mathematical approaches, namely analytical and geometrical methods. The analytical method involves examining the input-output velocity equations of parallel mechanism’s. Deriving the input-output velocity equations for a parallel mechanism involves differentiating the inverse kinematic equations. It results
3.4 Singularities of Parallel Mechanisms

in an $N \times N$ Jacobian matrix, where $N$ is the DOF of the mechanism. For lower-mobility mechanisms, with $(N < 6)$-DOF, analysis of this $N \times N$ Jacobian matrix does not always completely characterise their singularities, potentially leaving certain singularities undetected. Rank deficiency in the $N \times N$ Jacobian matrix occurs in the presence of singularities. Using this principle, Gosselin and Angeles [79] classified singularities into type one, type two and type three based on the determinates of this matrix and the matrices it consists of.

Geometrical methods have utilised screw theory [41] and line geometry [100] to detect the singular configurations of parallel mechanisms. The instantaneous motions of the joints, links and mobile platform are described using twists, which deliver intuitive geometrical insight into the behaviour of the mechanism. Kumar [101] implemented the concept of reciprocity of screws and classified the singularities of a five-bar planar mechanism into four types based on the rank of certain reciprocal screw systems. He also highlighted the duality between instantaneous kinematics and statics base singularity analysis approaches, which in an important concept in the motion/force transmission analysis discussed in Chapter 4. Park and Kim [102] presented a coordinate-invariant screw theory based differential geometric analysis of singularities for parallel mechanisms. Zlatanov et al. [80] defined a singularity that can only occur in lower-mobility mechanisms and involves the mechanism’s constraints. This singularity, termed a constraint singularity, is not detected through analysis of the input-output Jacobian. Zlatanov et al. utilised the theory of screws and the geometrical properties of screw systems [41] to define the constraint wrench system of a mechanism, $C$. The rank of this constraint system is then determined, if $\text{rank}(C) = 6 - N$ the mechanism is fully constrained and away from a singular configuration. However, if $\text{rank}(C) < 6 - N$ there is linear dependence amongst the constraint system, signifying the mechanism is in a constraint singularity.
Soon after, Joshi and Tsai [33] presented a screw theory based method capable of detecting all singularity types. In this method a $6 \times 6$ Jacobian matrix is formed based on the constraint and actuation wrenches of the mechanism. Singularities were grouped into architecture and constraint singularities depending on the rank of the Jacobian and its sub-matrices.

Recently, Chen et al. [103] highlighted a set of indices for determining the locations of singularities, based on the reciprocal product between particular twists and wrenches of a mechanism. The method utilises the numerators of existing screw theory based performance indices that were proposed in [50, 51]. These performance indices are discussed in Chapter 4. The fundamental requirement of a parallel mechanism is to transmit and constrain motions and forces between its actuated inputs and mobile platform. If a singularity occurs, these motions and forces may no-longer be transmitted or constrained. The indices highlighted by Chen et al. have a range from zero to infinity, with a value of zero indicating a singular location. The purpose of these indices is to detect the singularities of a parallel mechanism, hence, only index values of zero are of interest.

The three classifications of singularities adhered to throughout this thesis are the input transmission singularity (ITS), output transmission singularity (OTS) and constraint transmission singularity (CTS), as proposed in [50, 51]. The definitions of these indices are provided in the following sections.
3.4 Singularities of Parallel Mechanisms

3.4.1 Input Transmission Singularity

An input transmission singularity (ITS) is detected by examining the reciprocal product between the actuation wrench \( \hat{\mathbf{A}}_i \) of the \( i \)th chain and its input twist \( \hat{\mathbf{f}}_i \),

\[
\text{ITS}_i = \left| \hat{\mathbf{A}}_i \circ \hat{\mathbf{f}}_i \right|.
\] (3.22)

An ITS occurs when Eqn. (3.22) equals zero. The location and direction of \( \hat{\mathbf{A}}_i \) and \( \hat{\mathbf{f}}_i \) vary for different mechanism designs and actuation schemes. For the actuation scheme considered herein, seen in Fig. 3.1, an ITS occurs if \( \hat{\mathbf{A}}_i \) becomes coplanar with the axis of its input twist \( \hat{\mathbf{f}}_i \). This singularity has been defined as a type one singularity by Gosselin and Angeles [79]. Without consideration of other limiting factors, an ITS generally occurs at the workspace boundary, when one or more of the mechanism’s chains are no longer capable of generating motion/force of the analysed point on the mobile platform in the \( \hat{\mathbf{A}}_i \) direction.

The overall ITS distribution of a mechanism is calculated by determining the minimum of the ITS\(_i\) for all chains at each workspace location. For the 2-DOF mechanism studied in this chapter, this equates to,

\[
\text{ITS} = \min(\text{ITS}_1, \text{ITS}_2).
\] (3.23)

3.4.2 Output Transmission Singularity

The output transmission singularity (OTS) is calculated using the reciprocal product of the \( i \)th chain’s actuation wrench \( \hat{\mathbf{A}}_i \) and its output twist \( \hat{\mathbf{O}}_i \),

\[
\text{OTS}_i = \left| \hat{\mathbf{A}}_i \circ \hat{\mathbf{O}}_i \right|.
\] (3.24)
where an OTS occurs when Eqn. (3.24) equals zero. The output twist is the free motion of the mobile platform when all wrenches are locked except the one under examination. Each wrench therefore has its own related output twist. The method for calculating the output twist is discussed in Section 3.4.4. The OTS is equivalent to a type two singularity also proposed by Gosselin and Angeles [79], and can occur within the workspace of a parallel mechanism. In an OTS the actuators can no-longer resist an external moment or force along its calculated output twist, resulting in an uncontrollable DOF.

As with the ITS, the overall OTS distribution is obtained by calculating the minimum of the $\text{OTS}_i$ for all chains at each workspace location, resulting in

$$\text{OTS} = \min(\text{OTS}_1, \text{OTS}_2), \quad (3.25)$$

for the 2-DOF mechanism studied in this chapter.

### 3.4.3 Constraint Transmission Singularity

The reciprocal product between the $k$th constraint wrench $\hat{s}_{Ck}$, or equivalent constraint wrench in the case of a CLSC $\hat{s}_{Ck^*}$, and its output twist $\hat{s}_{\mathcal{O}k}$ is utilised to detect a constraint transmission singularity (CTS),

$$\text{CTS}_k = \left| \hat{s}_{Ck} \circ \hat{s}_{\mathcal{O}k} \right|. \quad (3.26)$$

The method for calculating the output twist is discussed in the next section. As with the other measures, an CTS occurs when Eqn. (3.26) equals zero. The overall distribution of CTS locations is determined by taking the minimum of all $\text{CTS}_k$ throughout the mechanism’s workspace.
3.4 Singularities of Parallel Mechanisms

In a CTS the constraining forces produced by the mechanism’s chains no longer act in the appropriate directions and does not fully constrain the mobile platform. When considering chains with CLSCs, a constraint can also be lost due to singularities within the chain itself, this case is discussed further in Section 4.4.

3.4.4 Output Twist

The output twist for a wrench is defined as the free motion of the mobile platform when all wrenches are fixed, except the wrench under examination. Likewise, the output twist for a constraint is defined as the free motion of the platform when all wrenches are fixed and a virtual joint at the location of the constraint under examination is released, in turn removing the effect of the constraint. The mobile platform’s free motion is then determined through reciprocity of the remaining wrench system, as presented in Section 2.3.1. For a non-redundant, non-over-constrained mechanism, the process of calculating a wrench’s output twist can be summarised as,

1. Group all unit actuation and constraint wrenches to form a $6 \times 6$ matrix $W$, representing the wrench system of the mechanism, with a wrench down each column.

2. Take the transpose of $W$ and represent the wrenches in axis-coordinate order, producing $\tilde{W}^T$, where the axis-coordinate order of a screw is defined in Table 2.3.

3. Remove the $i$th wrench from $\tilde{W}^T$, leaving the $5 \times 6$ matrix $\tilde{W}_i^T$.

4. Calculate the nullspace of the matrix $\tilde{W}_i^T$, using the \texttt{null()} function in MATLAB, or other methods, then normalise. The result is the unit output twist associated with the $i$th wrench, represented in the ray-coordinate order of a screw, which is utilised throughout this thesis and also defined in Table 2.3.
5. Repeat Steps 3 and 4 for the remaining wrenches of interest.

When away from a singular location, the rank of the wrench system $\text{rank}(W) = 6$, therefore, performing the above procedure results in a single twist. However, if the mechanism is at an exact singular location the rank of the wrench system $\text{rank}(W) < 6$, thus applying the above procedure will result in either a twist system or a twist reciprocal to the wrench removed. In the case of the twist system, at lease one of the reciprocal products between the wrench and each of the twists equals zero. Therefore, in both cases if $\text{rank}(W) < 6$ then the mechanism is in an OTS or CTS.

### 3.5 Actuation Wrench of a Planar Closed-Loop Sub-Chain

In the following section, the actuation wrenches for use in the singularity analysis of mechanisms with planar CLSCs are derived. The derivation has been performed on a general 2-DOF axis-symmetric parallel mechanism with a CLSC, as illustrated in Fig. 3.1(a). Without considering other limiting factors, such as maximum joint range or collisions, the ITS of this mechanism is known to occur at the workspace boundary. As such, Eqn. (3.22) must equal zero at these locations.

The generalised force seen in Fig. 3.8(b) that intersects the point of analysis X is utilised to form the actuation wrench, as illustrated in Fig. 3.9(a). Fig. 3.9(b) shows the maximum distance point X can achieve from the fixed base column, signifying the workspace boundary. It can be seen that by utilising this actuation wrench, Eqn. (3.22) equals zero, detecting the singular location, as required. In this location, the actuation wrench $\hat{S}_{A_{2r}}$ and its input twist $\hat{S}_I$ are reciprocal. Furthermore, this location is also coincident with the type one singularity locus [79].
3.5 Actuation Wrench of a Planar Closed-Loop Sub-Chain

Figure 3.9: A general planar 2-DOF axis-symmetric parallel mechanism (a) showing the ITS actuation wrench and (b) at the boundary of the workspace with Eqn. (3.22) equalling zero.

However, as shown in Fig. 3.10, the singular locations and mechanism configuration obtained when applying the ITS actuation wrench to the OTS calculation does not correlate with the locations and mechanism configuration of type two singularities [79]. In Fig. 3.10(a), the ITS actuation wrench of the CLSC chain, $\hat{S}_{A2i}$, is parallel with the actuation wrench of the serial chain, $\hat{S}_{A1}$, and therefore detected as an OTS by Eqn. (3.24). However, due to the constraints provided by the CLSC, the mechanism is still completely constrained in this configuration. Furthermore, it is not in the known configuration of a type two singularity, as illustrated in Fig. 3.10(b). Therefore, the ITS actuation wrench cannot be utilised to determine the OTS locations. Through further examination of Fig. 3.10(b), it becomes evident that in a type two singularity the serial chain actuation wrench intersects the IC of the CLSC. Hence, for the OTS to equal zero at type two singularity locations, the actuation wrench of the CLSC must pass through the IC and the mobile platform joint of the serial chain. The reason for the different definitions of actuation wrenches becomes
3.5 Actuation Wrench of a Planar Closed-Loop Sub-Chain

Figure 3.10: A general planar 2-DOF axis-symmetric parallel mechanism in (a) the detected OTS configuration calculated using the ITS actuation wrench of the CLSC and (b) the known OTS configuration with the proposed OTS actuation wrench of the CLSC overlaid.

Figures 3.11(a) to (c) illustrate the ITS and OTS actuation wrenches, red and blue arrows respectively, of a general planar 2-DOF axis-symmetric parallel mechanism with a CLSC, given three different positions for the point of analysis X. The utilisation of a CLSC in chain two results in the motion of the mobile platform being completely coupled to the output motion of the CLSC. Hence, in essence, the mobile platform has become part of chain two and the point of analysis X is embedded within chain two, illustrated in Fig. 3.11(a). The configuration of chain two in an ITS is therefore dependent the location of X on the mobile platform within chain two. Chain one’s configuration in an ITS is independent of the position of X in chain two, resulting in its ITS and OTS wrenches being identical. Considering the mobile platform as part of chain two, it is clear that the interaction point between the two chains is joint.
3.5 Actuation Wrench of a Planar Closed-Loop Sub-Chain

Figure 3.11: The ITS and OTS actuation wrenches, red and blue arrows respectively, of a general planar 2-DOF axis-symmetric parallel mechanism with a CLSC, where the point of analysis X is embedded (a) in the mobile platform, (b) in the distal link of chain one and (c) placed at joint C_{1,1}. Therefore, all forces between the two chains must transfer through this joint. Thus, the OTS actuation wrench for chain two must also pass through this joint while intersecting the IC of the CLSC.

If X was rigidly attached to the distal link of chain one, as illustrated in Fig. 3.11(b), then the ITS configuration of chain one would now be dependent on the selected X location. Different definitions would now be required for the ITS and OTS actuation wrenches for chain one, as shown. Furthermore, the ITS configuration of chain two is now independent of the position of X, and its ITS and OTS wrenches are identical, intersecting the IC and C_{1,1}. Additionally, if X is positioned at joint C_{1,1}, as illustrated in Fig. 3.11(c), then the ITS and OTS actuation wrenches within each chain are identical. Therefore, the definition of the ITS and OTS actuation wrenches for each chain in the studied mechanisms depend on the selected location of the point of analysis X.

For the mechanism configurations analysed herein, X is positioned as in Fig. 3.11(a).
Therefore, the actuation wrenches for the serial chain, chain one, are defined as,

\[ \hat{\mathbf{s}}_{A1r} = \hat{\mathbf{s}}_{A1o} = \begin{bmatrix} \hat{s}_1 \\ b_{1,1} \times \hat{s}_1 \end{bmatrix}, \quad (3.27) \]

where, \( \hat{s}_1 = \frac{c_{1,1} - b_{1,1}}{|c_{1,1} - b_{1,1}|} \), \( b_{1,1} \) is the position vector of joint \( B_{1,1} \) and \( c_{1,1} \) is the position vector of joint \( C_{1,1} \). The actuation wrenches for the CLSC chain, chain two, are defined as,

\[ \hat{\mathbf{s}}_{A2r} = \begin{bmatrix} \hat{s}_2 \\ x \times \hat{s}_2 \end{bmatrix} \quad \text{and} \quad (3.28) \]
\[ \hat{\mathbf{s}}_{A2o} = \begin{bmatrix} \hat{s}_3 \\ a \times \hat{s}_3 \end{bmatrix}, \]

where, \( x \) is the position vector of the point of analysis \( X \), \( \hat{s}_2 = \frac{x - v}{|x - v|} \), \( v \) is the position vector of the CLSC’s IC as defined in Eqn. (3.21), \( a \) is the position vector of the interaction point between the two chains at joint \( C_{1,1} \) and \( \hat{s}_3 = \frac{a - v}{|a - v|} \).

It should be noted that in all cases the actuation wrenches for the ITS and the OTS are reciprocal to all the involved twists of the serial chain or the equivalent twists of the CLSC, except the input twists, as required by the definition of the actuation wrench.

### 3.6 Planar Closed-Loop Sub-Chain Configurations

Considering that the mechanisms exhibit planar motion, the set of all feasible planar three and four link CLSCs are shown in Fig. 3.12. As discussed earlier, when determining an actuation wrench, the chain’s actuated revolute joint \( R \) is locked, which in turn fixes the location of the corresponding proximal link. Therefore, to increase
3.6 Planar Closed-Loop Sub-Chain Configurations

Figure 3.12: The CLSC variants used in the analysis, with the ITI and OTI actuation wrenches overlaid, red and blue arrows respectively. The sub-chain arrangements are termed (a) Parallelogram, (b) Obtuse Trapezium, (c) Acute Trapezium, (d) Equal Crossed, (e) Obtuse Crossed, (f) Acute Crossed, (g) Obtuse Triangular and (h) Acute Triangular. The point of analysis X is indicated and the interaction point between the two chains a is also shown. The actuated revolute joint and proximal link have been replaced with a fixed frame, illustrated in purple, and the R joint’s axis is normal to the page.

In order to improve the readability of the figures, the actuated revolute joint R and proximal link have been substituted by a fixed frame, illustrated in purple. The sub-chains are illustrated normal to the actuated joint’s axis. The point under examination on the mobile platform is labelled X, as in Eqn. (3.2), and the interaction point between the two chains is
3.6 Planar Closed-Loop Sub-Chain Configurations

labelled \( a \), as in Eqn. (3.28). An illustration of the ITS actuation wrench for each variant, \( \hat{\mathbf{s}}_{A_{\text{IT}}} \), has also been overlaid onto the figures using a red arrow and the OTS actuation wrenches, \( \hat{\mathbf{s}}_{A_{\text{OT}}} \), are shown by a blue arrow.

Figure 3.12(a) shows the parallelogram arrangement of the CLSC, where the two supporting links are parallel. The IC of the output link is located at infinity, verified by Eqn. (3.20). Therefore, \( \hat{\mathbf{s}}_{A_{\text{IT}}} \) intersects \( X \) and the IC, while \( \hat{\mathbf{s}}_{A_{\text{OT}}} \) intersects \( a \) and the IC. Both wrenches are coplanar and parallel to the distal links of the CLSC.

The link arrangements in Figs. 3.12(b) and (c) are the obtuse and acute trapeziums, respectively. Obtuse and acute are with reference to the relative distance between the two joints on the proximal link and the two on the mobile platform, when the larger distance is on the mobile platform it is termed obtuse. In both cases, \( \hat{\mathbf{s}}_{A_{\text{IT}}} \) passes through the IC and intersects \( X \), while being coplanar with the CLSC. Likewise, \( \hat{\mathbf{s}}_{A_{\text{OT}}} \) passes through the IC and the chain intersection point \( a \).

Figures 3.12(d), (e) and (f) show the equal, obtuse and acute crossed arrangements, where the IC is located at the cross point of the distal links. As with the trapezium variants, equal, obtuse and acute are with reference to the difference in distance between the two joints on the proximal link and the two on the mobile platform. For the three variants, the \( \hat{\mathbf{s}}_{A_{\text{IT}}} \) and \( \hat{\mathbf{s}}_{A_{\text{OT}}} \) actuation wrenches are positioned in the same manner as in the other link arrangements.

The obtuse and acute triangular arrangements are illustrated in Figs. 3.12(g) and (h), respectively. In both of these variants, two of the joint axes are coaxial. For the obtuse variant the coaxial joint are the proximal link joints and for the acute variant they are the mobile platform joints. The IC is located at these points. As with the other variants, \( \hat{\mathbf{s}}_{A_{\text{IT}}} \) intersects \( X \) and IC, while \( \hat{\mathbf{s}}_{A_{\text{OT}}} \) intersects \( a \) and IC, with both wrenches coplanar to the CLSC.
3.7 Singularity Analysis of Closed-Loop Sub-Chains

The presented link configurations can be substituted into various planar mechanisms to apply constraints on the mobile platform.

3.7 Singularity Analysis of Closed-Loop Sub-Chains

In order to verify the wrench definitions in the above CLSCs, the detected ITS and OTS singularity locations using the screw theory based method are compared with the zero points of the inverse condition number of the numeric input-output Jacobian matrix for each mechanism variant. The condition number represents the error amplification factor between the actuated joint rates and task space [104] and is defined as,

$$\kappa = \frac{1}{\|J^{-1}\|_2 \|J\|_2}, \quad (3.29)$$

where $J$ is the input-output Jacobian matrix. The Jacobian is generated through numerical differentiation of the inverse kinematic solutions in Eqns. (3.8) and (3.10), utilising a step size of $10^{-9}$. The condition number has been utilised as a performance index by Gosselin and Angeles in the analysis and optimization of a planar mechanism [105]. In singular configurations a mechanisms’s condition number approaches infinity, while in isotropic configurations it equals one. Therefore, to bound the measure, the inverse of the condition number, $1/\kappa$, is utilised. This is termed the conditioning index.

Computer models of two of the mechanism variants are shown in Fig. 3.1. The mechanisms are composed of a $\underline{RRR}$ serial chain and a $\underline{R}R(R)\underline{2}$ chain with a CLSC. The axis-symmetric configuration features the innate characteristic of identical kinematic and static properties in all radial half-planes, defined by the common axis of rotation. This allows complete characterisation of the mechanism’s kinematic and
static performance from the analysis of a single radial half-plane. Revolution of the single radial half-plane about the common axis of rotation generates the complete toroidal-shaped workspace. Hence, the computational requirements of the analysis can be significantly reduced. For these mechanisms exhibiting planar motion, the radial half-plane degenerates to a line and can be define by \( y = 0 \) and \( x \geq 0 \). A one dimensional quasi-random set of \( 10^5 \) points was generated within the bound \( 0 \leq x \leq (l_{a1,1} + l_{b1,1}) \). At each location, the ITS, OTS and conditioning index are evaluated. To improve computational speed, MATLAB’s parallel computing toolbox was utilised with a pool of eight workers. Collisions between the distal links and the base column can result in a thin section of unreachable workspace close to the base column. As the section’s size is dependent on the radius of both the base column and the distal links, these collisions are not examined herein.

The parameters utilised for this analysis have been selected to demonstrate the range of singularities of these mechanisms and as such may not be optimal in terms of workspace and performance. The parameters are listed in Table 3.1. All parameters are identical for each mechanism variant, except the proximal-distal connection point of joint \( B_{2,2} \), which is modified by a distance \( w \) along the proximal link, as listed in Table 3.2. A special case is the acute triangular variant, where joint \( C_{2,2} \) becomes

**Table 3.1:** Parameters for a family of planar 2-DOF axis-symmetric parallel mechanism with various CLSCs defined by \( w \) in Table 3.2.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proximal</td>
<td>( l_{a1,1} ) ( 0.900 ) | ( l_{a2,1} ) ( 0.900 ) | ( l_{a2,2} ) ( 0.900 - w )</td>
</tr>
<tr>
<td>Distal</td>
<td>( l_{b1,1} ) ( 0.500 ) | ( l_{b2,1} ) ( 0.500 ) | ( l_{b2,2} ) ( 0.500 )</td>
</tr>
<tr>
<td>Mobile Platform</td>
<td>( e'<em>{1,1} ) ( [x \ y \ z]^T ) | ( e'</em>{2,1} ) ( [0 \ 0.100 \ 0]^T ) | ( e'_{2,2} ) ( [-0.07071 \ -0.02929 \ 0]^T )</td>
</tr>
</tbody>
</table>
Table 3.2: Parameter modification between variants.

<table>
<thead>
<tr>
<th>Variant</th>
<th>(w) (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parallelogram</td>
<td>0.100</td>
</tr>
<tr>
<td>Obtuse Trapezium</td>
<td>0.050</td>
</tr>
<tr>
<td>Acute Trapezium</td>
<td>0.150</td>
</tr>
<tr>
<td>Equal Crossed</td>
<td>-0.100</td>
</tr>
<tr>
<td>Obtuse Crossed</td>
<td>-0.050</td>
</tr>
<tr>
<td>Acute Crossed</td>
<td>-0.150</td>
</tr>
<tr>
<td>Obtuse Triangular</td>
<td>0.000</td>
</tr>
<tr>
<td>Acute Triangular(#)</td>
<td>0.100</td>
</tr>
</tbody>
</table>

\(\#\) Additionally \(c_{2,2}' = c_{2,1}'\)

The results of the analysis are shown in Table 3.3. The values in the table represent the \(x\)-coordinate of the detected singular location, while \(y = 0\). There are three singular locations in the workspace for each variant, one at the both extremes of the workspace and one central. The central singularity is an OTS and other two are ITS. As will be discussed in Section 3.7.1, the parallelogram and equal crossed variants actually experience three independent ITS configurations towards the front of their workspace. However, for the purposes of this verification, one of these ITS locations is listed in Table 3.3 for each of the two variants.

It is evident that the detected ITS and OTS singular locations using screw theory based method, with the new actuation wrench definitions, are consistent with those detected by the input-output Jacobian analysis based method. The largest error between the numerical Jacobian and screw theory based approaches was \(1.4 \times 10^{-5}\).

Table 3.3: Numeric conditioning index versus the ITS and OTS locations for all planar variants. Each value represents the \(x\)-coordinate, while \(y = 0\), of the detected singular locations within the mechanisms' workspace.

<table>
<thead>
<tr>
<th>Mechanism Variant</th>
<th>1/(\kappa) (\to 0)</th>
<th>(\min(\text{ITS}, \text{OTS})) (\to 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parallelogram</td>
<td>0.4114 0.7050</td>
<td>0.6710 1.3872</td>
</tr>
<tr>
<td>Obtuse Trapezium</td>
<td>0.4643 0.7373</td>
<td>0.6857 1.1691</td>
</tr>
<tr>
<td>Acute Trapezium</td>
<td>0.6857 0.7373</td>
<td>0.6857 1.1691</td>
</tr>
<tr>
<td>Equal Crossed</td>
<td>0.4520 0.8596</td>
<td>0.4520 1.3381</td>
</tr>
<tr>
<td>Obtuse Crossed</td>
<td>0.4989 0.8070</td>
<td>0.4989 1.3482</td>
</tr>
<tr>
<td>Acute Crossed</td>
<td>0.6772 0.9088</td>
<td>0.6772 1.3518</td>
</tr>
<tr>
<td>Obtuse Triangular</td>
<td>0.4918 0.7543</td>
<td>0.4918 1.3636</td>
</tr>
<tr>
<td>Acute Triangular</td>
<td>0.8849 0.9419</td>
<td>0.8849 1.0849</td>
</tr>
</tbody>
</table>
for the acute trapezium variant at the front of the workspace. Motions of the mobile platform near this workspace extreme require large actuator rotations to output small platform motions, this along with the numerical methods utilised, can lead to small differences in the obtained singular locations near these extremes.

Figure 3.13 illustrates the parallelogram variant in three of its singular configurations. These configuration illustrations are direct outputs from the MATLAB code developed for this research. The fixed base column is dark blue, proximal links are black, distal links are grey, mobile platform is red and the point of analysis X is a light blue dot. All revolute, universal and spherical joints are modelled by points. This enables the direct visualisation of the joint centre and aids in identifying the exact connection points of the distal and proximal links. The type of joint at each point dependent on the mechanism and is clearly shown in their kinematic diagram and in the text. All bodies in these illustrations are slightly transparent to allow viewing of overlapping links, as seen in Fig. 3.13(a). These mechanism configuration illustrations

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3_13}
\caption{Planar parallelogram variant in (a) an ITS due to chain one near the base column, (b) an OTS in the central workspace with the arrow showing its unconstrained direction and (c) an ITS due to chain two at the front extreme of the workspace. The fixed base column is dark blue, proximal links are black, distal links are grey, mobile platform is red and the point of analysis X is a light blue dot. All revolute, universal and spherical joints are modelled by points, the type of joint at each point is clearly described in the mechanism’s kinematic diagram and in the text. The kinematic diagram of this family of mechanism is shown in Fig. 3.2.}
\end{figure}
are utilised throughout this thesis and follow the convention stated above.

The mechanism is in an ITS near the base column in Fig. 3.13(a) and near the front of its workspace in Fig. 3.13(c). The former ITS is due to the input twist and ITS actuation wrench of chain one being reciprocal, while the latter ITS is due to the reciprocity between the input twist and ITS actuation wrench of chain two with the CLSC. Figure 3.13(b) illustrates the OTS configuration. It can be clearly visualised that in this configuration the mechanism cannot withstand a force applied to the mobile platform along the vector illustrated in the figure. Additionally, the actuators cannot reposition the mobile platform out of this location without external assistance.

To further clarify the validity of the screw theory based singularity detection method and the new wrench definitions, the singular configurations of three other variants are illustrated in Fig. 3.14(a) to (c) for the obtuse trapezium, (d) to (f) for the acute trapezium and (g) to (i) for the obtuse crossed. The set of three illustrations for each variant are ordered as, an ITS due to chain one near the fixed base column, an OTS in the central workspace and an ITS due to chain one at the front edge of the workspace. The direction of the unconstrained DOF along the output twist of each OTS is indicated by a black arrow.

Through examination of the mechanisms’ configurations, it is clear that in all cases the mechanisms are either at locations where the point of analysis, blue dot on the mobile platform, is at is maximum reach, or at a location where the mobile platform cannot withstand an externally applied force in the direction given by the calculated output twists, $\hat{\mathbf{S}}_{O1}$ and $\hat{\mathbf{S}}_{O2}$. Analysis of the singular configuration of the remaining mechanisms produce the same conclusions. This further verifies the definitions of the ITS and OTS actuation wrenches, and emphasises the intuitive understanding gained by implementing screw theory methods.
3.7 Singularity Analysis of Closed-Loop Sub-Chains

Figure 3.14: The singular configurations of the (a) to (c) obtuse trapezium, (d) to (f) acute trapezium and (g) to (i) obtuse crossed variants. The set of three illustrations for each variant are ordered as, an ITS due to chain one, an OTS and an ITS due to chain one, except for (d) and (f) in which the ITS is due to chain two. The direction of the unconstrained DOF in each OTS is indicated by a black arrow.
3.7.1 Input Transmission Singularities of the Parallelogram Closed-Loop Sub-Chain

Interesting singular locations occur near the front of the workspace for the mechanism variant with the parallelogram CLSC. These locations correspond to ITS, where the mechanism loses a DOF due to the input twist becoming reciprocal to the ITS actuation wrench. At an ITS location, the number of solutions to the inverse kinematic equations is reduced, resulting in the loss of a DOF. In a singularity-free configuration, there are four solutions to the inverse kinematics of the chain with the parallelogram CLSC, chain two in the mechanism under analysis. These solutions result in different combinations of actuator and mobile platform yaw angles, the $xy$-projection of which, are illustrated in Fig. 3.15(a), with each solution coloured differently. Unlike many other ITS associated with this kind of actuation scheme, two of the three ITS do not occur at the workspace boundary. There are three separate bands of ITS locations, with the first listed in Table 3.3 and illustrated in Fig. 3.13(c) towards the front of the workspace. A clearer illustration of the configuration of chain two at this ITS band is shown in Fig. 3.15(b). To improve readability, the distal links of the CLSC are coloured orange and purple. Furthermore, the axis of the ITS actuation wrench is indicated by the red dotted line. In this location, the number of solutions to the inverse kinematic equations reduces to three and the point of analysis X cannot actuate any further in the positive $x$-direction without first moving in the negative $x$-direction, for the reasons detailed next. This location occurs at point $\{b\}$ in Fig. 3.16. The figure illustrates the relative $x$-positions of the three ITS locations at the front of the mechanism’s workspace, along $y = 0$. The dashed green, red, cyan and black lines each represent a different chain configuration, matching the four coloured configurations in Fig. 3.15(a). The path of these lines represent the relative $x$-displacement that the
Figure 3.15: Chain two of the planar mechanism with a parallelogram CLSC. (a) Illustrates the four solutions to the inverse kinematic equations in a general singularity-free location, (b) shows the first ITS configuration, (c) shows the second ITS, which is also simultaneously a CTS configuration and (d) illustrates the third ITS singularity occurring at the workspace boundary. To improve readability of (b) to (d), one distal link is coloured orange and the other cyan. The input twist axis is identified and the red dotted line represents the axis of the ITS actuation wrench.
3.7 Singularity Analysis of Closed-Loop Sub-Chains

Figure 3.16: The relative $x$-positions of the three ITS locations at the front of the workspace of the chain with a parallelogram CLSC. The ITS locations are marked in orange and labelled \{b\}, \{c\} and \{d\}, with respect to the singular configurations illustrated in Figs. 3.15(b), (c) and (d). At location \{c\} a CTS also occurs. The dashed green, red, cyan and black lines each represent a different chain configuration, matching the four coloured configurations in Fig. 3.15(a). The path of these lines represent the relative $x$-displacement that the point of analysis X requires in order to move between each singular configuration. The number of solutions to the inverse kinematics (IK) of the chain is indicated between and at each singular location.

The singular locations are marked in orange and labelled \{b\}, \{c\} and \{d\}, with respect to the singular configurations illustrated in Figs. 3.15(b), (c) and (d). The number of inverse kinematic (IK) solutions between and at each singular location is also indicated. When translating in the positive $x$-direction from a point in the central workspace, depending on the colour configuration the chain is in, determines which singular configuration will be encountered first. This can be intuitively understood by tracing the different coloured paths in Fig. 3.16. The mechanism incorporating this CLSC is designed to be in the green configuration, illustrated in Fig. 3.15(a), in its central workspace and therefore first encounters singular location \{b\}. From this location, if the proximal link continues to actuate in an anticlockwise direction, the point of analysis X begins to move back towards negative $x$ until reaching the second ITS labelled a point \{c\}, with the chain’s configuration illustrated in Fig. 3.15(c).
3.7 Singularity Analysis of Closed-Loop Sub-Chains

Here, the number of inverse kinematic solution are again reduced to three. The proximal and distal links are collinear in this configuration. Furthermore, at this instant the mechanism is also in a CTS, due to the collapse of the yaw constraining CLSC. This results in the ITS and OTS actuation wrenches being equal and collinear with the chain’s distal links, producing an uncontrollable DOF of the mobile platform. The axis of the wrench is illustrated by the red dotted line in Fig. 3.15(c).

From singular location \{c\}, the chain can theoretically transition into any of the four configurations coloured in Fig. 3.15(a). If the proximal link is actuated in the opposite, clockwise, direction, the red and green configurations may be entered. Alternatively, if the chain continues to actuate in the anticlockwise direction it can enter either the cyan or black configurations. As seen in Fig. 3.16, if the chain enters the cyan configuration, where its distal links remain parallel, point X begins to move back in the negative \(x\)-direction. However, if the black configuration is entered, where the distal links become crossed, point X begins to move towards the right. It continues to the right until reaching the third ITS band at point \{d\}, with its configuration illustrated in Fig. 3.15(d). In this configuration, the number of solutions to the inverse kinematics is reduced to one, thus, this point marks the absolute maximum reach of the chain, and therefore its workspace boundary. However, as was discussed above, the CTS occurring at point \{c\} must be passed through, after a backward motion away from point \{b\}, to reach the workspace boundary at point \{d\}. The mechanism has no control over which configuration it enters when passing through point \{c\}, therefore, the boundary of the usable workspace is defined as the first ITS at point \{b\}, as illustrated in Figs. 3.13(c) and (b). This singular location was utilised to verify the screw theory based singularity analysis method, with the defined actuation wrenches, in the previous section.

The above analysis also applies to the equal crossed CLSC variant, which is a
version of the parallelogram variant, where the mobile platform joints $C_{1,1}$ and $C_{2,2}$ are swapped, creating the crossed distal links of the CLSC, as seen in Fig. 3.12(d). Therefore, in order to avoid repetition, a brief summary about these singularities in the equal crossed CLSC variant is given below. The above observations about the gain and loss of DOF at the singularity points \{b\}, \{c\} and \{d\} are identical. The equal crossed CLSC results in the chain initially moving from the central workspace along the black path in Fig. 3.16 and encountering the ITS at point \{d\} first. This point marks the absolute maximum reach of the chain, and therefore its workspace boundary. As per the above observations, point \{c\} is a combined ITS and CTS, and thus results in an uncontrollable DOF. Therefore, controlled access from point \{c\} to \{b\}, or any other configuration, is not possible.

## 3.8 Chapter Conclusion

This chapter introduced and applied a screw theory based singularity analysis method to a family of planar 2-DOF axis-symmetric parallel mechanisms with CLSCs. A general procedure for determining the inverse kinematic solutions of these mechanisms was first defined. Then a systematic method of determining the twists and wrenches used in the singularity analysis for both the serial chains and chains with CLSCs was detailed. The latter case was shown to require modified wrench definitions, which were proposed herein. The consistency of the singularity identification between the presented screw theory based method, which utilised the new wrench definitions, and the zero points of the conditioning index of the numerical input-output Jacobian matrix has been demonstrated. The presented screw theory based method with the new wrench definitions were shown to also provide intuitive physical meaning to the causes and consequences of the singularities for the planar 2-DOF axis-symmetric
parallel mechanisms with a CLSC. Furthermore, the analysis uncovered three ITS configurations for the parallelogram and equal crossed CLSC variants towards the front of their workspace. These ITS locations and configurations were examined in depth for the parallelogram CLSC variant, providing a clear and intuitive explanation of their sources.

Knowing the exact workspace location of a singularity and the mechanism configuration in a singularity is important. However, the performance of a mechanism is affected both at singular locations and in the regions surrounding a singularity. Therefore, a measure of closeness to these singular locations is essential in the design of a high performance and practical mechanism. This concept is explored in the following chapter.
4.1 Overview

In this chapter the concept of motion/force transmission analysis is introduced and applied to the family of planar 2-DOF axis-symmetric parallel mechanisms with the various planar CLSCs that were defined in Chapter 3. A set of recently proposed indices to measure the closeness to a singularity the motion/force transmission performance of a mechanism are first described and adapted for use in mechanisms with CLSCs. It is demonstrated that the set of existing performance indices is incomplete and requires an additional measure to fully characterise the motion/force transmission abilities of mechanisms with CLSCs. Hence, a new performance index is
4.2 A Measure of Closeness to Singularities

As singularities are one of the most important factors limiting the performance of parallel mechanisms, knowledge of the closeness to a singularity is crucial to the design and control of high performance parallel mechanisms. This has therefore motivated extensive research over the past decades. The research commenced with the introduction of the transmission angle by Alt in 1932 [106], for the measurement of the quality of motion transmission in planar kinematic chains. For spatial mechanisms Yuan et al. [107] implemented screw theory and the reciprocal product as a transmission factor, however this index was unbounded. Then in 1973, Sutherland and Roth proposed a finite transmission index based on screw theory and the reciprocal product [108], which formed the foundation of the transmissivity proposed by Tsai and Lee in 1994 [109] and the generalised transmission index introduced by Chen and Angeles in 2007 [110]. In 2010, Wang et al. [50] defined the power coefficient also based on Sutherland and Roth’s work. Wang et al. also proposed three performance indices termed the input transmission index (ITI) based on the closeness to an ITS, the output transmission index (OTI) based on the closeness to an OTS and the local transmission index, which is the minimum of the ITI and OTI. These indices provide a measure of the motion/force transmission ability of a mechanism. Liu et al. [51], then introduced the constraint transmission index (CTI) in 2012 to
measure the closeness to a CTS. The power coefficient has since been applied to a variety parallel mechanisms in [52–54], however, it has not yet been applied and tested on parallel mechanisms with CLSCs in the configurations introduced in the previous chapter. Utilising the theory of reciprocal screws enables the generation of a singularity closeness measure that can be applied to purely translational, purely rotational and combined motion parallel mechanisms. These measures are finite, dimensionless and frame invariant, with values ranging from zero to unity.

### 4.2.1 The Power Coefficient

Given the twist and wrench

\[
\hat{\mathbf{s}}_t = \begin{bmatrix} \mathbf{s}_t \\ \mathbf{s}_{O_t} \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{s}}_w = \begin{bmatrix} \mathbf{s}_w \\ \mathbf{s}_{O_w} \end{bmatrix},
\]

(4.1)

the power coefficient is defined as

\[
\rho = \frac{\left| \hat{\mathbf{s}}_w \circ \hat{\mathbf{s}}_t \right|}{\left| \hat{\mathbf{s}}_w \circ \hat{\mathbf{s}}_t \right|_{\text{max}}},
\]

(4.2)

where the numerator is the reciprocal product between a unit wrench $\hat{\mathbf{s}}_w$ and unit twist $\hat{\mathbf{s}}_t$. The denominator is defined as,

\[
\left| \hat{\mathbf{s}}_w \circ \hat{\mathbf{s}}_t \right|_{\text{max}} = \sqrt{(h_t + h_w)^2 + d_{\text{max}}^2},
\]

(4.3)

where $h_t$ and $h_w$ are the pitch of the twist and wrench, respectively and $d_{\text{max}}$ represents the potential maximal length of the common normal $d$ between the axes of $\hat{\mathbf{s}}_t$ and $\hat{\mathbf{s}}_w$. 
4.2 A Measure of Closeness to Singularities

As will be shown in Section 4.2.2, the method of calculating \( d_{\text{max}} \) is different for the ITI and OTI.

Three special cases exist for the power coefficient [110]

1. If one of the screws has infinite pitch, the power coefficient is reduced to
   \[
   \rho = \left| \hat{s}_w \circ \hat{s}_t \right|
   \]

2. If both screws have infinite pitch the power coefficient equals zero

3. If \( h_w + h_t = 0 \) and \( d_{\text{max}} = 0 \), the mechanism is at a singular location and by definition the power coefficient should equal zero. However, due to the numerator and denominator both equalling zero, an infinite solution is obtained. To account for this, if the numerator is ever equal to zero then the power coefficient is equal to zero.

It should be noted that the ‘closeness’ measure produced by the power coefficient does not indicate the physical distance from a point to the singular locus in the workspace. It measures the mechanism’s motion/force transmission performance [111]. The larger the value, the better the performance.

4.2.2 Performance Indices

The power coefficient has been implemented by Wang et al. [50] and Liu et al. [51] to form a set of dimensionless, finite, frame invariant indices with values ranging from zero to unity. A value of unity signifies the optimum motion/force transmission between the unit wrench \( \hat{s}_w \) and the unit twist \( \hat{s}_t \), whereas a zero occurs at a singular location when \( \hat{s}_w \) can no-longer perform work on \( \hat{s}_t \). As will be seen, the numerators of the indices are the ITS, OTS and CTS equations introduced in Section 3.4. As briefly listed before, these indices are the input transmission index (ITI), output transmission
index (OTI) and the constraint transmission index (CTI). The definitions of these performances indices are outlined below.

**Input Transmission Index**

The ITI measures the closeness to an input transmission singularity in terms of the efficiency of motion/force transmission between a chain’s ITS actuation wrench \( \hat{\mathbf{s}}_{A_{IT}} \) and its input twist \( \hat{\mathbf{s}}_I \). It is defined as

\[
\text{ITI}_i = \left| \frac{\hat{\mathbf{s}}_{A_{IT}} \circ \hat{\mathbf{s}}_I}{\max} \right|,
\]

for the \( i \)th chain, with the denominator as defined in Eqn. (4.3). For a serial chain the \( d_{\text{max}} \) of the ITI can be obtained by rotating the axis of \( \hat{\mathbf{s}}_w \) about the joint \( B_{i,1} \) on the proximal link, as illustrated in Fig. 4.1(a). This rotation is continued until the maximum orthogonal distance between the input twist \( \hat{\mathbf{s}}_I \) and the joint \( B_{i,1} \) is achieved. This distance represents \( d_{\text{max}} \). For the serial chain, the axis of the ITS actuation wrench always intersects the joint \( B_{i,1} \), therefore, this joint is utilised as the rotation point in the \( d_{\text{max}} \) calculation. However, determining the \( d_{\text{max}} \) for a chain with a CLSC is more complex, as there is typically no point in space that the wrench always passes through. Therefore, a modified method for calculating \( d_{\text{max}} \) is required. Fig. 4.1(b) illustrates the method I have developed for calculating the \( d_{\text{max}} \) of chains with these CLSCs.

The method follows a similar procedure to the traditional approach and aims to determine the maximum common perpendicular between the twist and wrench. As can be seen in Fig. 4.1(b), the whole CLSC is rotated about joints \( B_{i,1} \) and \( B_{i,2} \). The rotation is continued until the length of \( d \) becomes maximum, where \( d \) is the current
4.2 A Measure of Closeness to Singularities

common perpendicular distance. This maximum value of $d$ is equal to $d_{\text{max}}$.

The overall ITI is determined by taking the minimum of all ITI$_i$ at each point in the workspace. The ITI provides a measure of how effectively the input twists produces motion/force along the respective actuation wrenches.

Output Transmission Index

The OTI indicates the closeness to an OTS in terms of the efficiency of motion/force transmission between a chain’s OTS actuation wrench $\hat{S}_{\text{AOT}}$ and its output twist $\hat{S}_O$,

$$\text{OTI}_i = \frac{|\hat{S}_{\text{AOT}} \circ \hat{S}_O^i|}{|\hat{S}_{\text{AOT}} \circ \hat{S}_O^i|_{\text{max}}}.$$  (4.5)
for the $i$th chain. The output twist was defined in Section 3.4.4 as the free motion of the mobile platform when all actuators, except the actuator of the chain under examination, are locked and is calculated through the reciprocal relation.

For the OTI, $d_{\text{max}}$ is obtained by rotating the screw axis of $\hat{s}_w$ about the application point $C_{i,j}$ on the mobile platform, as illustrated in Fig. 4.2. The value of $d_{\text{max}}$ can be calculated as

$$d_{\text{max}} = \|\hat{s}_t \times (e - p_t)\|$$

(4.6)

where, $p_t = \hat{s}_t \times s_{Ot}$ and $e$ is the position vector to the application joint $C_{i,j}$. In the above derivation, all the screws and vectors are defined with respect to the mobile platform reference frame $O'$, however due to the frame invariant nature of the index another frame, such as the fixed base frame $O$ could also be used.

The OTI analyses the interactions between the chains of a mechanism, equalling unity when $\hat{s}_{AOT}$ optimally constrains the motion that is not constrained by the other wrenches, and zero when a singularity occurs. The latter is equivalent to a type

\[\text{Figure 4.2: Definition of } d_{\text{max}} \text{ of the OTI calculation between OTS actuation wrench } \hat{s}_w \text{ and its output twist } \hat{s}_{O}, \text{ where } d \text{ is the current perpendicular distance between the two screws.}\]
two singularity [79] and results in an uncontrollable motion of the mobile platform. The overall OTI is determined by taking the minimum of all $\text{OTI}_i$ throughout the workspace.

**Constraint Transmission Index**

The CTI represents the closeness to a CTS through examination of the efficiency of motion/force transmission between a constraint wrench $\hat{\mathbf{C}}$, or equivalent constraint wrench in the case of a CLSC $\hat{\mathbf{C}}^*$, and its output twist $\hat{\mathbf{O}}$. The CTI for the $k$th constraint is defined as

$$\text{CTI}_k = \frac{|\hat{\mathbf{C}}_k \circ \hat{\mathbf{O}}_k|}{|\hat{\mathbf{C}}_k \circ \hat{\mathbf{O}}_k|_{\text{max}}}.$$  \hspace{1cm} (4.7)

As discussed in Section 3.4.4, the output twist is determined by locking all actuators and suppressing the constraint under analysis, followed by the calculation of the mobile platform’s free motion through reciprocity of the remaining wrench system. The overall CTI is determined by taking the minimum of all $\text{CTI}_k$ throughout the workspace. The equivalent constraint wrench of the CLSC in the studied mechanisms have infinite pitch, as seen in Eqn. (3.17). Therefore, as listed in special case one in Section 4.2.1, Eqn. (4.7) becomes,

$$\text{CTI}_k = |\hat{\mathbf{C}}_k \circ \hat{\mathbf{O}}_k|,$$  \hspace{1cm} (4.8)

removing the need to determine $d_{\text{max}}$.

For the screw theory based analysis of mechanisms with CLSCs, many techniques suggest utilising the equivalent constraint wrench obtained after the chain’s treatment as a generalised kinematic pair, as in Eqn. (3.17). However, as will be highlighted in the following section, and discussed in Section 4.4, I have found that this removes
key information about the performance of the constraint produced by a CLSC when utilised in motion/force transmission analysis, and therefore, leads to incomplete results.

### 4.3 Limitations of Current Transmission Indices

In order to demonstrate the limitations of the current transmission indices, the planar 2-DOF axis-symmetric parallel mechanism with a parallelogram CLSC is utilised. The analysis is set up identical to the singularity analysis in Section 3.7. The parameters utilised are based on the kinematic parameters of the physical prototype of the SCARA-Tau parallel mechanism [21] and are listed in Table 4.1.

The ITI is related to how effectively the actuator’s input motion/force is transmitted to the mobile platform, with a zero value corresponding to the loss of a DOF. In contrast, both the OTI and CTI are associated with the mechanism’s output characteristics, and when either equal zero, the mobile platform can no-longer resist a certain moment or force. In order to highlight the limitations of these indices, the OTI and CTI, from Eqns. (4.5) and (4.7), are evaluated throughout the mechanism’s workspace. Fig. 4.3 shows the distribution plots for the OTI of chains one and two.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value (m)</th>
</tr>
</thead>
<tbody>
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</tr>
<tr>
<td>$l_{a2,1}$</td>
<td>0.900</td>
</tr>
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<td>$l_{a2,2}$</td>
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</tr>
<tr>
<td>$l_{b1,1}$</td>
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<td>1.100</td>
</tr>
<tr>
<td>$l_{b2,2}$</td>
<td>1.100</td>
</tr>
</tbody>
</table>

Table 4.1: Parameters for the planar 2-DOF axis-symmetric parallel mechanism with a parallelogram CLSC.
4.3 Limitations of Current Transmission Indices

Figure 4.3: OTI and CTI distribution plots for the planar 2-DOF axis-symmetric parallel mechanism with a parallelogram CLSC in Fig. 4.4.

and the CTI of the CLSC in chain two. To improve readability of the figure, two of the three singularity bands that occur near the workspace front for the parallelogram CLSC, as described in Section 3.7.1, have not been shown. The results up to the first ITS location at point \(b\), as labelled in Fig. 3.16, are illustrated.

The OTI distribution remains high throughout most of the workspace and gradually reduces towards the workspace front. The two OTI results are identical due to the actuator and link scheme implemented, which leads to the same relative difference between the output twist and its respective wrench for the two chains. The CTI remains a constant unity throughout the whole workspace, only instantaneously equalling zero at \(x = 0.27974\) m as a result of linear dependence amongst the links of the CLSC. The approach to this linear dependence does not modify the CTI due to the normalisation of the constraint wrench and its output twist utilised.

Therefore, according to the OTI and CTI, the mechanism should be able to resist any moment or force applied to the mobile platform throughout its whole workspace. However, examining the mechanism’s configuration as it nears the fixed base column and the front of the workspace, illustrated in Figs. 4.4(a) and (b) respectively, it
4.4 Intra-Chain Constraint Singularity

Closed-loop sub-chains are intended to provide constraints on a chain's output motion. Therefore, when utilising CLSCs such as the parallelogram examined in the previous
section, it is important to examine how effectively the CLSC produces its intended constraints. The planar four-bar closed-loop, illustrated in Fig. 4.5(a), is the most commonly used CLSC. This type of sub-chain has been incorporated into the world’s best-selling parallel mechanism, the Delta [13], along with the H4 [15], SCARA-Tau [21] and Orthoglide [18]. The main drawback of utilising these CLSCs is the possibility of singular configurations within the sub-chain itself. Herein referred to as intra-chain constraint singularities (ICCS).

The ICCS can be determined for each chain independently, due to occurring within a CLSC itself. These singularities occur when the constraint wrench $\hat{s}_{C_{i,k}}$ becomes reciprocal to its respective local restricted output twist $\hat{s}_{O_{i,k}}$.

$$\hat{s}_{C_{i,k}} \circ \hat{s}_{O_{i,k}} = 0. \quad (4.9)$$

The local restricted output twist $\hat{s}_{O_{i,k}}$ differs from the output twist described in Section 3.4.4. In this case, it is the additional twist gained in the chain’s twist system.

**Figure 4.5:** (a) The vectors associated with the wrenches of the $(RR)_2$ four-bar closed-loop, with $\hat{s}_1$ and $\hat{s}_2$ directed along the distal links and $\hat{n}_1$ normal to the plane of $\hat{s}_1$ and $\hat{s}_2$. (b) The $R(\text{RR})_2$ chain incorporating the CLSC in (a).
when the $\hat{\mathbf{S}}_{C_{i,k}}$ constraint is suppressed. However, this constraint has two forms, one obtained before and the other after the chain’s treatment as a generalised kinematic pair.

If the latter, equivalent, constraint wrenches are utilised to calculate the ICCS, the result equals constant unity, as seen with the CTI in Section 4.3. According to [53], this indicates that the chain is properly constrained. However, because the equivalent constraint wrenches are produced by the individual limbs of the CLSC, the relative orientation between these limbs can change during motion of the mechanism and therefore affect the constraint’s efficiency. Hence, the chain may not be properly constrained. To generate a valid measure, I propose to use the constraint wrenches of the CLSC, prior to its treatment as a generalised kinematic pair, in the ICCS calculation.

To demonstrate the two cases, both constraint wrench definitions and calculations are implemented on the planar $R(RR)_2$ chain, illustrated in Fig. 4.5(b). For increased generality the $(RR)_2$ closed-loop is not considered to necessarily be in the common parallelogram configuration. The output link of the $(RR)_2$ closed-loop is the link $BC$.

As highlighted in Section 3.3.2, the $R(RR)_2$ chain possesses four constraints and therefore, has two freedoms. Three of the constraints are the permanent planar constraints. The remaining constraint is produced by the interactions between the two distal links of the CLSC, defined in Eqn. (3.15). The equivalent screw system is determined through a linear transformation [44] and results in Eqn. (3.17).

The local restricted output twist $\hat{\mathbf{S}}_{O_{i,1}^*}$ of the equivalent constraint wrench is determined by suppressing the equivalent constraint $\hat{\mathbf{S}}_{C_{i,1}^*}$ and examining the additional
4.4 Intra-Chain Constraint Singularity

Motion now available to the output link. This produces the zero pitch twist

\[ \hat{\mathbf{s}}_{\hat{O}_i,k^*} = \begin{bmatrix} \hat{\mathbf{n}}_1 \\ \mathbf{x} \times \hat{\mathbf{n}}_1 \end{bmatrix}, \] (4.10)

which corresponds to a rotational motion at the point of analysis \( X \) in the \( \hat{\mathbf{n}}_1 \) direction.

Calculating the ICCS using the equivalent constraint wrench and output twist results in

\[ \left| \hat{\mathbf{s}}_{\hat{C}_i,k^*} \circ \hat{\mathbf{s}}_{\hat{O}_i,k^*} \right| = \left| \mathbf{0}_{(3 \times 1)} \cdot (\mathbf{x} \times \hat{\mathbf{n}}_1) + \hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_1 \right| = |\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_1| = 1 \] (4.11)

as previously described. This reinforces that these wrenches cannot be used to detect the reduction in the constraint performance produced by the relative positions of the links within a CLSC.

Recalculation of the ICCS utilising the constraint wrenches of the CLSC in Eqn. (3.15) is now performed. This process can be visualised as the removal of a single distal link from the CLSC in Fig. 4.5(b) and determining the additional local output twist gained. This local restricted output twist equals

\[ \hat{\mathbf{s}}^q_{\hat{O}_i,k} = \begin{bmatrix} \hat{\mathbf{n}}_1 \\ \mathbf{e} \times \hat{\mathbf{n}}_1 \end{bmatrix}, \] (4.12)

where \( \mathbf{e} \) is the vector to a point on the output twist’s axis. If \( q = 1 \) then \( \mathbf{e} = \mathbf{c}_{i,2} \) and when \( q = 2 \) then \( \mathbf{e} = \mathbf{c}_{i,1} \). Given, \( \mathbf{c}_{i,1} \) and \( \mathbf{c}_{i,2} \) are the position vectors of joints \( C_{i,1} \)

4.4 Intra-Chain Constraint Singularity

and C, respectively. Calculating the ICCS for \( \hat{S}_{O_{i,k}}^1 \) and \( \hat{S}_{O_{i,k}}^2 \) produces

\[
\left| \hat{S}_{C_{i,k}} \circ \hat{S}_{O_{i,k}}^1 \right| = \left| \mathbf{s}_1 \cdot (\mathbf{c}_{i,2} \times \hat{\mathbf{n}}_1) + (\mathbf{c}_{i,1} \times \hat{\mathbf{s}}_1) \cdot \hat{\mathbf{n}}_1 \right|
\]

\[
= \left| \mathbf{c}_{i,2} \cdot (\hat{\mathbf{n}}_1 \times \hat{\mathbf{s}}_1) + (\mathbf{c}_{i,1} \cdot (\hat{\mathbf{s}}_1 \times \hat{\mathbf{n}}_1) \right| \quad (4.13a)
\]

\[
\left| \hat{S}_{C_{i,k}} \circ \hat{S}_{O_{i,k}}^2 \right| = \left| (\mathbf{c}_{i,1} - \mathbf{c}_{i,2}) \cdot (\hat{\mathbf{n}}_1 \times \hat{\mathbf{s}}_2) \right|. \quad (4.13b)
\]

The two ICCS calculations now possess non-constant values and equal zero when in singular locations. They can therefore be used as a measure to detect when an ICCS occurs.

4.4.1 Intra-Chain Constraint Index

The power coefficient, defined in Eqn. (4.2), can be applied to the ICCS measure to provide an index of closeness to an ICCS. The intra-chain constraint index (ICCI) for the qth constraining leg of the kth constraint in the ith chain is defined as

\[
\text{ICCI}_{i,k}^q = \frac{\left| \hat{S}_{C_{i,k}}^q \circ \hat{S}_{O_{i,k}}^q \right|}{\left| \hat{S}_{C_{i,k}}^q \circ \hat{S}_{O_{i,k}}^q \right|_{\text{max}}}. \quad (4.14)
\]

The ICCI provides a method to incorporate the performance effects of these CLSCs into the overall motion/force transmission evaluation of a mechanism. As the ICCI approaches zero, an applied force or torque on the CLSC’s output link cannot be effectively resisted due to nearing linear dependence amongst the limbs of the CLSC, and when equal to zero a singularity occurs.
4.4 Intra-Chain Constraint Singularity

The ICCI is related to the transmission angle [112]. The commonly accepted range for the transmission angle in planar four-bar closed-loops is $40^\circ$ to $140^\circ$ [106]. Mapping this to the ICCI results in a minimum acceptable value of 0.64. Lower values than this can lead to poor operational characteristics and an increased sensitivity to manufacturing tolerances in the link lengths and joint clearances, as well as to the effects of thermal expansion and contraction [113]. Therefore, a cropping can be applied to the workspace with respect to this minimum value for the ICCI. However, 0.64 is not strictly the only lower limit of the index. The selected limit depends on the overall requirements of the mechanism’s practical applications, and as such should be designated on a case-by-case basis.

4.4.2 Justification for the Intra-Chain Constraint Index

In order to justify and exemplify the necessity of the ICCI, the following section applies the ICCI to the same 2-DOF axis-symmetric parallel mechanism with a parallelogram CLSC from Section 4.3. The simulation process is also identical.

Figure 4.6(a) displays the results for the mechanism’s previous analysis, combined with the new ICCI result. As highlighted previously, the OTI for the two chains are identical due to the actuator and link scheme implemented and the CTI remains a constant unity throughout the whole workspace, only instantaneously equalling zero at $x = 0.27974\text{m}$ as a result of linear dependence amongst the links of the CLSC. Through further examination of the figure, the two ICCI distributions are identical. This is a result of the parallelogram CLSC and if an alternative planar four-bar CLSC is utilised, the values of the two indices will be different. Furthermore, it becomes evident that the ICCI = 0 at location $x = 0.27974\text{m}$, signifying the linear dependence amongst the links of the CLSC. The mechanism’s configuration at this point where
4.4 Intra-Chain Constraint Singularity

Figure 4.6: The output transmission performance for the planar 2-DOF axis-symmetric parallel mechanism with a parallelogram CLSC showing (a) the OTI, CTI and ICCI distributions; (b) the mechanism’s configuration where the ICCI = 0 and (c) where ICCI = 1.
ICCI = 0 and where ICCI = 1 are illustrated in Fig. 4.6(b) and (c), respectively.

From these two configurations, it can be seen that the parallelogram is completely collapsed when ICCI = 0 and has internal angles of 90° when ICCI = 1. However, as ICCI → 0 both the OTI and CTI have values near or at unity. This reinforces that the existing motion/force transmission indices do not provide information about the performance of the constraint produced by the CLSC itself. Therefore, the results obtained in this section clearly justify and exemplify the requirement for the ICCI to monitor the constraint performance of a CLSC.

4.5 Motion/Force Transmission Analysis of Planar Variants

This section characterises the motion/force transmission abilities of the planar 2-DOF axis-symmetric parallel mechanism variants that were analysed for singularities in Section 3.7. The initial kinematic parameters utilised in this analysis are shown in Table 3.1 and the modification variable between variants is given in Table 3.2. A special case exists for the acute triangular variant, where joint C_{2,2} becomes coincident with joint C_{2,1}, forming the acute triangle. Hence, this case is parameterised slightly differently, with c'_{2,2} = c'_{2,1}.

A one dimensional quasi-random set of 10^5 points is generated within the bound 0 ≤ x ≤ (l_{a1,1} + l_{b1,1}). At each location, the ITI, OTI and ICCI are evaluated. As evident from Section 4.3, the CTI for these planar mechanisms results in a constant unity, except in the exact locations where the links of the CLSC become linearly dependent. Therefore, to improve readability the CTI is not illustrated in the results. Additionally, the results up to the first ITS location at point {b}, as labelled in
Fig. 3.16, are illustrated for the parallelogram variant. For all other variants, their ITS mark the maximum reachable workspace boundary.

To improve computational speed, a pool of eight workers was implemented on MATLAB’s parallel computing toolbox. As with the previous analyses, collisions between the distal links and the fixed base column result in a thin section of unreachable workspace close to the base column. As the section’s size is dependent on the radius of both the base column and the distal links, these collisions are not examined herein. Friction and gravity are not considered in this analysis.

Figures 4.7(a) to (h) show the performance distribution plots for the eight planar 2-DOF axis-symmetric parallel mechanism variants incorporating the CLSCs presented in Fig. 3.12(a) to (h), respectively. The vertical axis $\rho$ of each plot represents the power coefficient and gives the value of the indices ranging from zero to unity. The legend for these distribution plots is given in Fig. 4.7(i).

The index plots are bounded by locations where the ITI of chain one or two equals zero, signifying an input transmission, type one, singularity. The $\text{ITI}_1 = 0$ at both the front (right end of the plot) and rear (left end of the plot) of the workspace for the obtuse trapezium, obtuse crossed and obtuse triangular variants, with the $\text{ITI}_2 = 0$ at both the front and rear of the workspace for the acute trapezium and acute triangular mechanisms. For the parallelogram variant, the $\text{ITI}_1 = 0$ at the rear while the $\text{ITI}_2 = 0$ at the front of the workspace. The equal and acute crossed variants both experience $\text{ITI}_1 = 0$ at the front of their workspaces, while the rear is limited by $\text{ITI}_2 = 0$. It is also evident from the ITI distribution that the acute trapezium and acute triangular variants result in a significant decrease in the size of the reachable workspace.

The OTI for chain one and two of each mechanism variant are coloured purple and green, respectively, in Fig. 4.7(a) to (h). For each variant, the $\text{OTI}_1 = \text{OTI}_2$
Figure 4.7: The ITI, OTI and ICCI distribution plots at $y = 0 \ x \geq 0$ for the planar 2-DOF axis-symmetric parallel mechanisms with various CLSCs, using the parameters in Tables 3.1 and 3.2. The vertical axis $\rho$ specifies the value of the indices from zero to unity. The distribution plots are for the: (a) Parallelogram, (b) Obtuse Trapezium, (c) Acute Trapezium, (d) Equal Crossed, (e) Obtuse Crossed, (f) Acute Crossed, (g) Obtuse Triangular and (h) Acute Triangular variants. The legend for the plots is shown in (i).
and converges to zero at a location in the central workspace region. This corresponds to a type two singularity, where the mobile platform can no-longer resist an applied moment or force along the respective output twists, as was illustrated in Figs. 3.13(b), 3.14(b), 3.14(e) and 3.14(h) for the parallelogram, obtuse trapezium, acute trapezium and obtuse crossed variants, respectively. The OTS splits the workspace into two assembly modes, either side of where \( OT_i = 0 \). Typically, only one assembly mode is used, however, both can be accessed by exploiting the mechanism’s inertia to pass through the singular location or by incorporating redundant actuation into the design. The pose associated with the OTS for the acute trapezium and acute crossed variants is shown in Fig. 4.8. These singular configurations are not obvious at first, however after examining the mechanisms’ wrenches and twists, as well as their interactions, visual identification of these OTS configurations becomes intuitive. The OTS actuation wrench of both chains are illustrated as red arrows and all pass through the IC of the CLSC, resulting in linear dependence amongst the wrench system. The

\[ IC_{\hat{\theta}_A} = IC_{\hat{\theta}_B} \]

![Figure 4.8](image)

**Figure 4.8:** The (a) acute trapezium and (b) acute crossed planar mechanism variants in an OTS configuration, with the OTS actuation wrenches shown in red and the direction of the infinitesimal translation, allowed by the output twist, at the interaction point between the two chains shown in blue.
output twists are directed out of the page along $z$, have zero pitch and pass through the CLSC’s IC for chain one and joint $B_{1,1}$ for chain two. Visualisation becomes more intuitive by imagining the motion at the intersection point between the two chains, $C_{1,1}$, caused by the respective output twists. For these mechanisms, this motion is an instantaneous translation, tangential to the output twist under examination. Hence, the direction vectors of these translations are illustrated as blue arrows in the figures.

The workspace locations where the $\mathrm{OTI}_i = 1$ correspond to configurations where the $i$th OTS actuation wrench optimally constrains the motion $\hat{\mathbf{s}}_{O_i}$ not constrained by the other distal links. These locations for the acute trapezium and acute crossed variants are shown in Fig. 4.9. The motion at the intersection point between the two chains, $C_{1,1}$, about the respective output twists are indicated by the blue arrow for chain one and the purple arrow for chain two. Examining the two figures, it is evident that in these optimal OTI configurations an actuation wrench and the motion produced by its output twist are collinear. Additionally, the two actuation wrenches are perpendicular. These observations are consistent with the optimal motion/force output transmission configurations of other planar parallel mechanisms [106, 114], further verifying the results obtained herein.

**Figure 4.9:** The (a) acute trapezium and (b) acute crossed variants in configurations with optimal output transmission performance, with the OTS actuation wrenches shown in red and the direction of the infinitesimal translation, allowed by the output twists, at the interaction point between the two chains are shown in blue and purple, for chains one and two respectively.
The distribution of the ICCI clearly demonstrates a strong correlation between the CLSC variant and the resultant ICCI values. For the parallelogram and acute trapezium variants in Fig. 4.7(a) and (c), the ICCI tends to zero at both workspace extremes. The ICCI of the acute crossed variant, Fig. 4.7(f), approaches zero toward the workspace rear and remains high throughout the rest of the workspace. For the acute and obtuse triangular variants, seen in Fig. 4.7(g) and (h), the CLSCs form rigid structures and therefore the ICCI is not required in their analysis. The sub-chains can be replaced by an equivalent rigid serial link without affecting the mechanisms motion throughout its workspace. The CLSC of the acute triangular variant acts as a rigid extension of the chains proximal link. That is, the chain can be substituted by a rigid link between the actuated revolute joint $R$ and joint $C_{2,1}$, which is equal to $C_{2,2}$, on the mobile platform, without chaining the analysis results. Therefore, this variant simplifies to a mechanism with two serial $RRR$ chains. The CLSC of the obtuse triangular variant forms a structure with the mobile platform and could therefore be replaced by a rigid link with an $R$ joint on the proximal end and a fixed connection to the mobile platform on the other. This would provide identical motion throughout the workspace in terms of kinematics, however, force distribution within the link will be different. Hence, the ICCI is not evaluated for the triangular CLSC variants. The ICCI of all other variants remains high and has no significant effect on their performance.

To further exemplify the ability of the indices to completely characterise a mechanism’s motion/force transmission abilities and provide intuitive visual results, the indices are applied to eight additional variants that utilise the parameter set defined in Table 4.2, with their CLSC determined by $w$ in Table 3.2. Figure 4.10 illustrates the analysis results for the eight mechanism variants with the modified parameter sets.
4.5 Motion/Force Transmission Analysis of Planar Variants

Table 4.2: Modified parameters for a family of planar 2-DOF axis-symmetric parallel mechanism with various CLSCs defined by $w$ in Table 3.2.

<table>
<thead>
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<th>Parameter</th>
<th>Value (m)</th>
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<td>$l_{a2,1}$</td>
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<td>$l_{a2,2}$</td>
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</tr>
<tr>
<td>$l_{b1,1}$</td>
<td>1.100</td>
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<tr>
<td>$l_{b2,1}$</td>
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<td>$c'_{1,1}$</td>
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</tr>
<tr>
<td>$c'_{2,1}$</td>
<td>$[0 \ 0.100 \ 0]^T$</td>
</tr>
<tr>
<td>$c'_{2,2}$</td>
<td>$[-0.07071 \ -0.02929 \ 0]^T$</td>
</tr>
</tbody>
</table>

Through visual inspection, the increased distal link lengths of the new parameters have shifted the OTS towards the workspace rear, thus it is no-longer located in the central workspace of each variant, as was seen in Fig. 4.7. Additionally, the distance between ITS locations has also become larger, resulting in an overall larger workspace compared to the previous parameter set.

The shape of the ICCI distribution for these mechanisms is similar to that obtained by the previous parameters. However, due to the shift in the OTI distribution, the effects of the ICCI are more evident with this parameter set, as can be seen through comparison of Figs. 4.7 and 4.10. The ICCI significantly impacts the motion/force transmission ability of the parallelogram, acute trapezium and acute crossed variants, due to it possessing the lowest value of all indices throughout large portions of the workspaces. As with the previous analysis, the CLSCs of the two triangular variants form rigid structures, hence, the ICCI is not required in their analysis.

As discussed in Section 4.4.1, the ICCI’s relation to the transmission angle results in a lower acceptable bound on its value of 0.64. Going beyond this can lead to poor operational characteristics of the constraint produced by the CLSC. The same argument has been raised for the other performance indices, however their threshold
4.5 Motion/Force Transmission Analysis of Planar Variants

Figure 4.10: The ITI, OTI and ICCI distribution plots at $y = 0$, $x \geq 0$ for the planar 2-DOF axisymmetric parallel mechanisms with the various CLSC arrangements utilising kinematic parameters defined in Tables 4.2 and 3.2. The vertical axis $\rho$ specifies the value of the indices from zero to unity. The distribution plots are for the: (a) Parallelogram, (b) Obtuse Trapezium, (c) Acute Trapezium, (d) Equal Crossed, (e) Obtuse Crossed, (f) Acute Crossed, (g) Obtuse Triangular and (h) Acute Triangular variants. The legend for the index plots is shown in (i).
values should be determined considering application requirements and therefore, selected on a case-by-case basis. Whereas, the ICCI is focused on the CLSC’s ability to produce a known constraint, hence, the threshold is only applied to the ICCI.

The parallelogram, acute trapezium, equal crossed and acute crossed variants all experience ICCI values below this threshold. Therefore, the workspace regions with ICCI < 0.64 should be removed to avoid performance issues due to the planar CLSC itself. Figs. 4.11(a) to (f) illustrate the minimum of the ITI, OTI and ICCI, with the threshold applied on the ICCI value, for the parallelogram, obtuse trapezium, acute trapezium, equal crossed, obtuse crossed and acute crossed variants, respectively. The threshold on the ICCI significantly reduces the usable workspace size for the parallelogram, acute trapezium, equal crossed and acute crossed variants, as seen in their respective plots. The obtuse trapezium and obtuse crossed variants are not affected by the threshold and both remain with large usable workspaces. In the workspace region that remains for all variants, the constraint produced by the planar CLSC is known to effectively constrain the yaw rotation of the mobile platform, hence, performance issues due to the transmission behaviour of the planar CLSC itself are avoided.

The differences between the index distribution plots in Figs. 4.7, 4.10 and 4.11 highlights the ability of this methodology to enable a user to modify kinematic parameters and obtain a visual and numerical indication of improvements or reductions in a mechanism’s performance.

4.6 Chapter Conclusion

This chapter introduced and applied the concept of motion/force transmission analysis to a family of planar 2-DOF axis-symmetric parallel mechanisms with CLSCs.
The existing ITI measure, based on the power coefficient, was shown to require modification when applied to the studied mechanisms. Therefore, a new method for calculating the ITI of chains with CLSCs was developed, tested and verified. Furthermore, the set of existing motion/force transmission indices were shown to lack key information about the performance of the CLSC itself, leading to the proposal of the ICCI. The ICCI was shown to provide important information about the efficiency...
of the constraint produced by a CLSC.

Utilising the existing indices along with the proposed ICCI enabled, for the first time, the complete motion/force transmission analysis of eight planar 2-DOF axisymmetric parallel mechanism with various CLSCs, for two different parameter sets. The analysis of these planar mechanisms delivers an intuitive understanding of the motion/force transmission effects in the presence of CLSCs and highlights the need for the proposed ICCI.

The optimisation of these mechanisms is out of the scope of this research, thus, strict conclusions cannot be made about the absolute motion/force transmission performance of each mechanism studied. However, from the analysis it can be concluded that overall the acute variants result in substantially smaller workspace sizes compared to the other variants, and would therefore be of limited practical use. The parallelogram and equal crossed variants both resulted in a large singularity free workspace, however, their usable workspace was reduced by the transmission performance of their CLSC. In contrast, the obtuse trapezium, obtuse crossed and obtuse triangular mechanisms all demonstrated large singularity-free workspaces, which were unaffected by the transmission performance of their CLSC. Therefore, all variants, except the acute variants, warrant further investigation.

The application of the ICCI onto spatial axis-symmetric parallel mechanisms with CLSC is explored in the following chapter.
5

Singularity and Motion/Force Transmission
Analysis of Spatial 3-DOF Axis-Symmetric
Parallel Mechanisms

5.1 Overview

This chapter explores the application of the motion/force transmission indices on spa-
tial parallel mechanisms, exemplified on the SCARA-Tau parallel mechanism. Pre-
vous researchers who utilised screw theory techniques to analyse the singularities of
spatial parallel mechanisms with closed-loop sub-chains (CLSCs), such as the Delta
mechanism [11], simplified the actuation and constraint wrenches produced by the
CLSCs. As examined in the previous chapter, this results in key information about
the performance of the closed-loop sub-chain (CLSC) itself being lost, when applied
to motion/force transmission analysis. Therefore, the ICCI was introduced to enable
the complete characterisation of the motion/force transmission abilities of a family of planar mechanisms with CLSCs. In this chapter, the previous analysis is extended to spatial parallel mechanisms and it is shown that the introduction of the ICCI is essential to completely characterise the motion/force transmission analysis of such mechanisms.

The kinematic equations describing translational 3-DOF axis-symmetric parallel mechanisms with CLSCs is firstly presented. Next, the mechanisms’ twists and wrenches required for the singularity and motion/force transmission analysis are defined. This is followed by an analysis of the singular locations detected using the screw theory based method from Section 3.4, with the defined constraint and actuation wrenches. The detected locations are verified against the singular locations detected by the zero points of the numeric input-output Jacobian’s conditioning index. Thereafter, the ITI, OTI, CTI and ICCI are applied to the SCARA-Tau parallel mechanism, completely characterising its motion/force transmission performance. The chapter is finalised by a discussion of the limitations of the presented analysis methodology, followed by concluding remarks.

5.2 The Mechanism Anatomy

A common feature in many spatial lower-mobility axis-symmetric parallel mechanism variants is the inclusion of multiple CLSCs. These sub-chains generally utilise spherical-spherical (SS) joint links in a four-bar closed-loop arrangement (SS)$_2$, where the output link attaches directly to the mobile platform, in turn applying constraints on its allowable motions. Through interactions amongst the constraints, the CLSCs generally remain planar. As highlighted previously, this type of planar CLSC is utilised in the world’s best-selling parallel mechanism, the Delta [11], along with the
5.2 The Mechanism Anatomy

H4 [15], Orthoglide [18] and SCARA-Tau [21] mechanisms.

For 3-DOF and 4-DOF axis-symmetric mechanisms, two planar CLSCs are frequently integrated with their planes parallel to the common axis of rotation, as seen in the SCARA-Tau mechanism’s physical prototype and kinematic model in Figs. 5.1(a) and (b), respectively. In this configuration, the two closed-loops remain planar and apply rotational constraints on the mobile platform, directed normal to the two planes, in turn, constraining the platforms roll and pitch. The orientation of these two CLSCs is termed vertical.

As well as the two vertical parallelograms, an additional link is added to one of these chains to constraint the mobile platform’s yaw. A separate RSS chain is also included to complete the constraints on the mobile platform’s spatial position. The three chains include a total of six distal links to completely constrain the mobile

![Figure 5.1: The SCARA-Tau axis-symmetric parallel mechanism (a) physical prototype and (b) kinematic model.](image)
platform’s three rotational motions and actuate its three translational motions.

5.2 The Mechanism Anatomy

5.2.1 Kinematic Parameters

The position of all joints and the mobile platform of a spatial 3-DOF axis-symmetric parallel mechanism, the SCARA-Tau, is defined in this section. The kinematic parameters of the SCARA-Tau are shown in Fig. 5.1(b). A fixed coordinate system $O$ is defined with its $z$-axis coincident with the common rotation-axis of the actuated proximal links. The $x$-axis can be selected arbitrarily provided that it is perpendicular to the $z$-axis, while the $y$-axis is defined according to the right-hand rule. The joint $B_{i,j}$ connects the proximal and distal links. The $z$-heights of these joints are given by $h_{i,j}$. Without loss of generality, the height of joints $B_{1,1}$ and $B_{2,1}$ equal zero and are therefore in the $xy$-plane of $O$. The mobile platform joint are labelled $C_{i,j}$, the proximal links are labelled $L_{ai,j}$, where $l_{ai,j}$ is its horizontal kinematic length from the $z$-axis of $O$ to joint $B_{i,j}$, and the $j$th distal link is $L_{bi,j}$ with length $l_{bi,j}$, for the $i = 1, 2$ and 3 kinematic chains of the mechanism. The position vectors of joints $B_{i,j}$ and $C_{i,j}$, are defined as $b_{i,j} = \begin{bmatrix} b_{x_{i,j}} & b_{y_{i,j}} & b_{z_{i,j}} \end{bmatrix}^T$ and $c_{i,j} = \begin{bmatrix} c_{x_{i,j}} & c_{y_{i,j}} & c_{z_{i,j}} \end{bmatrix}^T$, respectively. Furthermore, the joint height difference $b_{22,1} - b_{22,3} = c_{22,1} - c_{22,3} = h_{2,3}$.

A coordinate system $O'$ is attached to the central point of the joints $C_{1,1}$, $C_{1,2}$, $C_{2,1}$ and $C_{2,2}$ on the mobile platform, as shown in Fig. 5.1(b). This point will be referred to as the point of analysis $X$. The $y$-axis of $O'$ is defined by the direction between $C_{1,1}$ and $C_{2,1}$, with the $x$-axis perpendicular to the plane formed by $C_{1,1}$, $C_{1,2}$, $C_{2,1}$ and $C_{2,2}$ and the $z$-axis is parallel to the $z$-direction of $O$. The actuated joint angles $q_i$ are measured from the positive $x$-axis of $O$ to the $xy$-projection of the $i$th proximal link, as seen in Fig. 5.2. With a positive angle measured towards the positive $y$-axis. The mobile platform’s intended yaw rotation is labelled $\beta$ and equals $\text{atan2}(p_y, p_x)$, where $p_x$ and $p_y$ are the $x$ and $y$ components of point $X$ in $O$. Due to
coupling between different chains and limbs, the mobile platform rotation does not typically remain at $\beta$. The deviation from this angle is known as parasitic rotation and is defined by $\alpha$, as shown in Fig. 5.2. The listed parameters completely define the mechanism’s geometry.

The joint positions $b_{i,j}$ can be obtained through trigonometry,

$$b_{i,j} = \begin{bmatrix} l_{ai,j}\cos q_{i,j} & l_{ai,j}\sin q_{i,j} & h_{i,j} \end{bmatrix}^T. \tag{5.1}$$

The locations of joints $C_{i,j}$ require transformations to be described in the $O$ frame

$$c_{i,j} = x + R_p R_{zyz} c'_{i,j} \tag{5.2}$$

where, where $x = \begin{bmatrix} p_x & p_y & p_z \end{bmatrix}^T$ represents the position vector of $X$ when described in $O$, $c'_{i,j}$ is the position vector of joint $C_{i,j}$ described in $O'$, and $R_p$ and $R_{zyz}$ are the rotation matrices of the unwanted parasitic and desired orientation of the mobile platform, respectively. For these 3-DOF translational mechanisms with parasitic yaw
rotation,

\[ R_p = \begin{bmatrix} c_\alpha & -s_\alpha & 0 \\ s_\alpha & c_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

(5.3)

and

\[ R_{zyx} = \begin{bmatrix} c_\beta & -s_\beta & 0 \\ s_\beta & c_\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

(5.4)

where \( \beta = \text{atan2}(p_y, p_x) \), \( c_\alpha = \cos \alpha \), \( s_\alpha = \sin \alpha \), \( c_\beta = \cos \beta \) and \( s_\beta = \sin \beta \).

Populating Eqns. (5.1) and (5.2) with the mechanism’s physical parameters, provides a complete definition of all joint positions. With slight modifications, the described parametrisation is applicable to all axis-symmetric parallel mechanisms.

### 5.2.2 Kinematic Analysis

The kinematic equations for the 3-DOF SCARA-Tau mechanism have been derived by Isaksson et al. [22, 77]. The derivation follows the same procedure as for the 2-DOF mechanisms in Chapter 3. The inverse kinematic equations are summarised below for completeness. Using the loop closure method, the distance between \( B_{i,j} \) and \( C_{i,j} \) are equated to the distal link length

\[ ||c_{i,j} - b_{i,j}|| = l_{bi,j}. \]

(5.5)

Substituting Eqns. (5.1) and (5.2) into Eqn. (5.5) reduces to,

\[ e_{a_i} + e_{b_i} \sin q_i + e_{c_i} \cos q_i = 0, \]

(5.6)
where,

\[ e_{a_i} = c_{x_{i,j}}^2 + c_{y_{i,j}}^2 + l_{a_{i,j}}^2 + (c_{z_{i,j}} - h_{i,j})^2 - l_{b_{i,j}}^2, \]

\[ e_{b_i} = -2l_{a_{i,j}}c_{y_{i,j}}, \]

\[ e_{c_i} = -2l_{a_{i,j}}c_{x_{i,j}}. \]

The above equations are almost identical to those for the planar mechanisms in Eqn. 3.7, the key difference being the additional component, \((c_{z_{i,j}} - h_{i,j})^2\), which takes into consideration the \(z\)-displacements.

Because the spatial length Eqn. (5.5) exactly equals the planar length Eqn. (3.5), solving the length equations for the actuated joint angle \(q_i\), also produces the same equations for the respective right and left solutions of the inverse kinematics, labelled \(q_{R_i}\) and \(q_{L_i}\) respectively in Fig. 5.2. The solution for the right hand side is therefore,

\[ q_{R_i} = \text{atan2} (k_a, k_b), \]  
(5.8)

where,

\[ k_a = \frac{-e_{a_i}e_{b_i} + e_{c_i}\sqrt{-e_{a_i}^2 + e_{b_i}^2 + e_{c_i}^2}}{e_{b_i}^2 + e_{c_i}^2} \]

and

\[ k_b = \frac{-e_{a_i}e_{c_i} + e_{b_i}\sqrt{-e_{a_i}^2 + e_{b_i}^2 + e_{c_i}^2}}{e_{b_i}^2 + e_{c_i}^2}. \]  
(5.9)

Likewise, for the left hand side

\[ q_{L_i} = \text{atan2} (k_c, k_d), \]  
(5.10)

where,

\[ k_c = \frac{-e_{a_i}e_{b_i} + e_{c_i}\sqrt{-e_{a_i}^2 + e_{b_i}^2 + e_{c_i}^2}}{e_{b_i}^2 + e_{c_i}^2} \]

and

\[ k_d = \frac{-e_{a_i}e_{c_i} + e_{b_i}\sqrt{-e_{a_i}^2 + e_{b_i}^2 + e_{c_i}^2}}{e_{b_i}^2 + e_{c_i}^2}. \]  
(5.11)

given, \(e_{a_i}\), \(e_{b_i}\) and \(e_{c_i}\) are per Eqn. (5.7), where the subscripts \(a, b, c\) and \(d\) are to differentiate the components of each equation.
Equations (5.8) and (5.10) have unknown inputs of the parasitic platform yaw \( \alpha \), along with \( p_x \), \( p_y \) and \( p_z \), which are the components of the vector to the point of analysis X. The unknown variables are all components of \( \mathbf{c}_{i,j} \).

The parasitic platform yaw can be calculated numerically by equating the actuated joint angle calculated using the two different mobile platform joints of the yaw constraining CLSC [77]. These joint are \( C_{2,1} \) and \( C_{2,3} \) or \( C_{2,2} \) and \( C_{2,3} \) for the SCARA-Tau mechanism. The only unknown in the equation generated is \( \alpha \). Solving this equation produces two solutions for \( \alpha \), with only one being in the current working mode. As for the planar analysis, once \( \alpha \) is determined, the actuated joint angles can be calculated algebraically through Eqns. (5.8) and (5.10). The derivation of the inverse kinematic solutions for all other axis-symmetric parallel mechanisms follow a similar process as shown above.

5.3 Screws for Singularity and Transmission Analysis

The method for determining the twists and wrenches for a spatial axis-symmetric parallel mechanism with CLSCs follows a similar procedure to that of the planar mechanisms in Chapter 3, with many resultant screws having identical definitions. Therefore, in order to avoid repetition of definitions and processes, the following section refers to equations and methods proposed in Section 3.3.

5.3.1 Mechanism Twist

The input twist of the spatial mechanisms is identical to that defined for the planar mechanisms in Eqn. (3.12). For a serial spatial chain, the twists are determined through the same method as described in Section 3.3.1. The SCARA-Tau mechanism
5.3 Screws for Singularity and Transmission Analysis

incorporates a single serial chain in a RSS or RSU arrangement. Through inspection, it is evident that this chain cannot resist any forces or moments and therefore, a mobile platform attached to this chain has a full 6-DOF.

As with the Delta, H4 and Orthoglide mechanisms, through interactions between chains the CLSCs remain planar. As detailed in Section 3.3.1, the equivalent twists of these chains with CLSCs are determined by substituting the CLSCs by their generalised kinematic pair.

5.3.2 Mechanism Wrenches

The actuation and constraint wrenches form the foundation of the singularity and transmission analysis. Therefore, they must be determined consistently for all variants. The serial chain analysis process is identical to the method described in Section 3.3.2. Applying this method identifies the RSS chains as producing no constraints and an actuation wrench coaxial to its distal link, as illustrated in Fig. 5.3.

Closed-Loop Sub-Chain Wrenches

To determine the wrenches of chains with CLSCs, a similar procedure to that developed for the planar mechanisms is implemented. However, for the analysed spatial

\[ \hat{A}_i \]

\[ \hat{B}_i \]

\[ \hat{C}_i \]

\[ \hat{S}_i \]

\[ \hat{S}_m \]

Figure 5.3: Serial RSS chain with its actuation wrench illustrated.
mechanisms, these chains may contain more than one CLSC per chain, hence, additional steps are required to determine the equivalent wrenches.

**Constraint Wrenches** The constraint wrenches of the chain with two distal links, $R(SS)_2$, shown in Fig. 5.4 are described by the screws,

\[
\begin{align*}
\hat{s}_{C_{i,1}} &= \begin{bmatrix} \hat{s}_1 \\ b_{i,1} \times \hat{s}_1 \end{bmatrix} \text{ and } \hat{s}_{C_{i,1}}' &= \begin{bmatrix} \hat{s}_2 \\ b_{i,2} \times \hat{s}_2 \end{bmatrix},
\end{align*}
\]

(5.12)

where

\[
\hat{s}_1 = \frac{c_{i,1} - b_{i,1}}{\|c_{i,1} - b_{i,1}\|} \text{ and } \hat{s}_2 = \frac{c_{i,2} - b_{i,2}}{\|c_{i,2} - b_{i,2}\|}
\]

for the $i$th chain, as seen in Fig. 5.4(a).

The equivalent constraint produced by the two wrenches in Eqn. 5.12 is an infinite pitch screw normal to the plane defined by the two distal links,

\[
\hat{s}_{C_{i,1}'} = \begin{bmatrix} 0_{(3\times1)} \\ \hat{n}_1 \end{bmatrix},
\]

(5.13)

where, * signifies an equivalent wrench and $\hat{n}_1$ is the unit vector normal to the CLSC’s plane, as illustrated in Fig. 5.4(b). This constraint restricts the rotation of the mobile.

**Figure 5.4:** The $R(SS)_2$ chain with its (a) constraint generating distal link vectors and wrenches and (b) its equivalent constraint wrench.
platform about the $\hat{n}_1$ direction.

The chain with the cluster of three distal links, $\mathbf{R}(SS)_3$, shown in Fig. 5.5 produces two linearly independent infinite pitch equivalent constraint wrenches that restrict two platform rotations. One of the wrenches is normal to the vertical parallelogram, as defined in Eqn. (5.13), and the other reduces to a wrench that is parallel with the vertical parallelogram and orthogonal to the direction of the third distal link $\hat{s}_3$, as illustrated in Fig. 5.5 and is defined as,

$$
\hat{\mathbf{s}}_{Cl,2^*} = \begin{bmatrix} 0_{(3\times1)} \\ \hat{n}_2 \end{bmatrix},
$$

(5.14)

where $\hat{n}_2 = \hat{n}_1 \times \hat{s}_3$.

**Actuation Wrenches**  Any chain with a CLSC that is not oriented parallel to the actuated joint’s axis requires the definition of two different actuation wrenches, one for the ITS and one for the OTS, as required for the planar mechanisms in Chapters 3 and 4. The process of determining the ITS actuation wrench is shown in Fig. 5.6 and described below,

1. Project the $i$th chain onto the normal plane of the actuated joints axis. For the mechanisms analysed herein, this is the $xy$-plane as shown in Fig. 5.6(a).

![Figure 5.5](image_url) **Figure 5.5:** The $\mathbf{R}(SS)_3$ chain with its constraint generating limb vectors and equivalent linearly independent constraints.
5.3 Screws for Singularity and Transmission Analysis

Figure 5.6: Determining the ITS actuation wrench of the cluster of three distal links. (a) The three distal links projected onto the actuated revolute joints normal plane for the obtuse trapezium variant; (b) The projection of the ITS actuation wrench on the actuated revolute joints normal plane for the parallelogram variant; (c) The projection of the ITS actuation wrench on the actuated revolute joints normal plane for the obtuse trapezium variant; (d) The cluster of three distal links projected onto the vertical parallelogram’s plane with the ITS projection shown; (e) The resultant ITS actuation wrench for the cluster of three distal links for the obtuse trapezium variant.
2. Determine the intersection point of the distal link projections.

- If the distal link projections are parallel then the intersection point is located at infinity and the \(xy\)-projection of the actuation wrench is parallel to these links and passes through the point of analysis \(X\), as shown in Fig. 5.6(b).

- If the intersection point is finite, then the \(xy\)-projection of the actuation wrench passes through the intersection point and the projection of the point of analysis \(X\), as illustrated in Fig. 5.6(c).

3. Project the \(i\)th chain including the mobile platform onto a plane parallel to the vertical parallelogram. The projection of the actuation wrench in this plane is parallel to the distal links of the vertical parallelogram and intersects the projection of \(X\), as shown in Fig. 5.6(d).

4. The final ITS actuation wrench is along the line that satisfies the two projections, as shown in Fig. 5.6(e).

The OTS actuation wrenches are determined using the above procedure by replacing the point of analysis \(X\) with the intersection point of the \(xy\)-projection of the distal-links of chains one and three. For the case examined herein, this point is along the axis \(C_{1,1}C_{1,2}C_{3,1}\), as illustrated in Fig. 5.7. The position of the point along this axis is not important due to the rotational constraints applied on the mobile platform, therefore, mobile platform joint \(C_{1,1}\) can be utilised in this case.

The other chain with a CLSC, seen in Fig. 5.1, has its parallelogram CLSC oriented parallel to the actuated joint’s axis. This results in the ITS and OTS actuation wrenches being the same wrench, which is coplanar with the parallelogram and parallel to its distal links. Again, the vertical position of the wrench is not important
and can be selected as joint $C_{1,1}$.

The three constraint wrenches and three actuation wrenches presented above can be utilised in the singularity and motion/force transmission analysis of the 3-DOF SCARA-Tau parallel mechanism.

5.4 Singular Location Detection Verification

Before examining the motion/force transmission results in detail, an analysis of the singular locations detected using the screw theory based method from Section 3.4, with the proposed constraint and actuation wrenches, is performed. The results are verified against the singular locations detected by the zero points of the numerically calculated input-output Jacobian’s conditioning index. The analysis utilises the parameter set in Table 5.1. The wrench definitions and methodology are further verified through the analysis of an additional SCARA-Tau variant, for which the $xy$-projection of the three link cluster in chain two forms an obtuse trapezium arrangement, as seen in Fig. 3.12(b).
To visualise the singular locations within the workspace, a two dimensional quasi-random set of $10^6$ points is generated with bounds of $0 < x < (l_{a1,1} + l_{b1,1})$ and $-l_{b1,1} < z < (l_{b1,1} + h_{1,2})$. At each location, the ITS, OTS and conditioning index are evaluated to determine the singular locations. To improve computational speed, parallel processing was implemented with a pool of eight workers through MATLAB’s parallel computing toolbox. Collisions between the distal links and the fixed base column can result in a thin section of unreachable workspace close to the base column. As the size of this section is dependent on the radius of both the base column and the distal links, as well as the link length, these collisions are not examined herein.

Fig. 5.8(a) shows the distribution plot where $\min(\text{ITS,OTS}) \to 0$ and Fig. 5.8(b) visualises the distribution of the locations where the conditioning index $1/\kappa \to 0$. 

### Table 5.1: Modified SCARA-Tau Parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l_{a1,1}$</td>
<td>0.9000</td>
</tr>
<tr>
<td>$l_{a1,2}$</td>
<td>0.9000</td>
</tr>
<tr>
<td>$l_{a2,1}$</td>
<td>0.9000</td>
</tr>
<tr>
<td>$l_{a2,2}$</td>
<td>0.7250</td>
</tr>
<tr>
<td>$l_{a3,1}$</td>
<td>0.9000</td>
</tr>
<tr>
<td>$h_{1,1}$</td>
<td>0.2000</td>
</tr>
<tr>
<td>$h_{1,2}$</td>
<td>0.2000</td>
</tr>
<tr>
<td>$h_{2,1}$</td>
<td>-0.0600</td>
</tr>
<tr>
<td>$h_{2,2}$</td>
<td>-0.0600</td>
</tr>
<tr>
<td>$h_{3,1}$</td>
<td>0.9250</td>
</tr>
<tr>
<td>$h_{b1,1}$</td>
<td>0.5000</td>
</tr>
<tr>
<td>$h_{b1,2}$</td>
<td>0.5000</td>
</tr>
<tr>
<td>$h_{b2,1}$</td>
<td>0.5000</td>
</tr>
<tr>
<td>$h_{b2,2}$</td>
<td>0.5000</td>
</tr>
<tr>
<td>$h_{b3,1}$</td>
<td>1.0000</td>
</tr>
<tr>
<td>$c_{1,1}$</td>
<td>$[0 \ 0.1305 \ -0.1000]^T$</td>
</tr>
<tr>
<td>$c_{1,2}$</td>
<td>$[0 \ 0.1305 \ 0.1000]^T$</td>
</tr>
<tr>
<td>$c_{2,1}$</td>
<td>$[0 \ -0.1305 \ -0.1000]^T$</td>
</tr>
<tr>
<td>$c_{2,2}$</td>
<td>$[0 \ -0.1305 \ 0.1000]^T$</td>
</tr>
<tr>
<td>$c_{2,3}$</td>
<td>$[-0.0731 \ 0.0285 \ -0.1600]^T$</td>
</tr>
<tr>
<td>$c_{1,2}$</td>
<td>$[0 \ 0.1305 \ 0.1600]^T$</td>
</tr>
</tbody>
</table>
It should be noted that the actual values of the indices are not important, only the locations of the minimums, representing the singularities, are important.

The ranges of the minimums have been selected to aid in visualisation of locations where the two distributions approach zero. For Fig. 5.8(a) black $0 \leq \min(ITS,OTS) \leq 0.015$ and blue $0.015 < \min(ITS,OTS) < 0.100$ and Fig. 5.8(b) black $0 \leq 1/\kappa \leq 0.007$ and blue $0.007 < 1/\kappa < 0.06$. A selection of data points are shown in each figure displaying the $x$ and $z$ location and the respective index value. The selected points are a sample of the minimums along the black singularity bands. Comparing the two figures, the detected singular locations are identical for the input-output Jacobian analysis based method and the screw theory based method, verifying the validity of the proposed constraint and actuation wrenches.

To further demonstrate this correlation, Fig. 5.9 shows a sectional view of the plots.
Figure 5.9: The profile of the workspace of the mechanism with the modified SCARA-Tau parameters from Table 5.1 (a) \( \min(\text{ITS,OTS}) \rightarrow 0 \) and (b) \( 1/\kappa \rightarrow 0 \). Selected minimum points are listed on each plot.

from Fig. 5.8. The plots are looking at the \( z \)-axis section \(-0.13 \leq z \leq 0.11 \) facing the negative \( x \)-direction. The vertical axis is the respective value of the measure and the black region is the black arc in Figs. 5.8. Another set of data points along the respective detected singularity bands are shown in each figure. Through comparison of the two figures, the singular locations detected are identical for the input-output Jacobian based method and the screw theory based method, this further validates the proposed constraint and actuation wrenches. Additionally, the shape drawn by tracing the minimums of each plot also correlate when scaled appropriately, as seen in the plots.

There are two ITS bands seen towards the workspace front in Figs. 5.8. As discussed in Section 3.7.1, the multiple ITS bands occur as a result of utilising the parallelogram CLSC to constrain the mobile platform’s yaw, in chain two of the SCARA-Tau. The first ITS band occurs when the \( xy \)-projection of chain two is in the
configuration shown in Fig. 3.15(b), while the other occurs on the front workspace boundary, with the chain configuration illustrated in Fig. 3.15(d). There is also a third singularity band in this region, which results in a simultaneous ITS, CTS and ICCS, due to the collapse of the yaw constraining CLSC. The configuration of chain two at this singularity band is illustrated in Fig. 3.15(c). However, as discussed in Section 3.7.1 and visualised in Fig. 3.16, to reach this configuration the point of analysis X first passes through this $x$-coordinate in a different configuration and then moves backwards from the first ITS band towards the fixed base column, along the green path in Fig. 3.16. This means that, at this $x$-coordinate the mechanism is along the green path and thus in a non-singular configuration. This illustrates why the simultaneous ITS, CTS and ICCS band is not seen in Figs. 5.8(a) and (b).

To further verify the proposed wrench definitions, an identical analysis is now performed on an obtuse trapezium variant. The difference between the variants is the $xy$-projection of the three link cluster in chain two now forms the obtuse trapezium arrangement seen in Fig. 3.12(b). This is achieved by modifying the parameter $l_{a2,3} = 0.800\text{m}$. The analysis results are displayed in Fig. 5.10. As with the previous analysis, these results show identical singularity locations for the screw theory based and numerical input-output Jacobian based detection methods. Unlike the parallelogram variant, for this mechanism variant there only exists a single ITS band at the workspace front, located on the boundary.

The above analysis clearly verifies the definitions of the ITS and OTS actuation wrenches and the screw theory based singularity detection method through its correlation with the minimums of the conditioning index of the numerically determined input-output Jacobian matrix.
5.5 Motion/Force Transmission Analysis

In order to exemplify the method and results, this section provides a performance analysis of the SCARA-Tau mechanism with respect to the ITI, OTI, CTI and ICCI. The parameters of the original SCARA-Tau physical prototype are utilised in the analysis. These parameters are identical to those listed in Table 5.1, except for the distal link lengths, which are now \( l_{b3,1} = 1.285\,\text{m} \) and for all other distal links \( l_{b_1,j} = 1.100\,\text{m} \). The analysis is set up following the same procedure as stated in Section 5.4.

The ITI distribution plots are shown in Fig. 5.11. To aid intuitive interpretation of the results, the index values are mapped to colours with dark blue signifying a singular location and red representing locations of optimal motion/force transmission. The colour mapping displayed in Fig. 5.11(a) applies to all distribution plots in the
remainder of this thesis. Examination of the individual ITI distribution plots indicates that each chain independently limits the mechanism’s reach in certain directions. Chains one and two, in Figs. 5.11(b) and (c) respectively, limit the mechanism’s reach in the positive $z$-direction. Additionally, chain two also limits the reach in the positive and negative $x$-directions. Figure 5.11(d) shows that chain three limits
the reach in the negative z-direction as well as a small section in the negative x-direction. The minimum ITI, shown in Fig. 5.11(e), is calculated from the minimum ITI value of all three chains, $\min(\text{ITI}_1, \text{ITI}_2, \text{ITI}_3)$. The distribution plot shows the existence of ITS along the top, bottom and left workspace boundaries. Three close, but separate, bands of ITS exist towards the front of the workspace, however, for the reasons discussed in Section 5.4, only the first ITS encountered is analysed.

The areas of high motion/force transmission for chains one and two emanate from regions in the lower central workspace, while chain three’s maximum ITI is located towards the upper front of the workspace. The distribution plots, can be utilised to visualise how modifications to a chain’s structural parameters affect a chain’s, and in turn, the mechanism’s reachable workspace and ITI motion/force transmission abilities.

Figures 5.12(a) to (c) show the distribution of the OTI throughout the reachable workspace for chains one, two and three, respectively. These plots demonstrate the actuation wrenches effectiveness in resisting a potential motion along their respective output twists. The comparatively higher average OTI value for chain two is due to the configuration of the mechanism. When viewed from the fixed base column and facing the mobile platform, the SCARA-Tau prototype in Fig. 5.1 is assembled with chain one and three on the left hand side and chain two on the right. Therefore, the axis of chain two’s output twist is commonly directed towards the right side of the mechanism. Hence, it is more optimally aligned with the chain’s OTS actuation wrench, resulting in a higher average OTI for chain two. If the mechanism is reconfigured so that chains two and three are on the same side, chain one’s average OTI distribution is now comparatively higher than the other two chains.

Figure 5.12(d) shows the overall minimum OTI for all chains, calculated by
Figure 5.12: Distribution of the OTI results throughout the workspace of the SCARA-Tau mechanism for chain (a) one, (b) two and (c) three. The overall minimum OTI, $\min(\text{OTI}_1, \text{OTI}_2, \text{OTI}_3)$, for the mechanism is shown in (d).

The resultant value of the minimum OTI is smoothly distributed throughout the workspace, except at the top extreme where the mechanism enters another working mode, near $x = 1.00$ and $z = 1.18$. This leads to the rapid change in the OTI value seen on the workspace boundary. The issue can be resolved by enforcing additional restrictions on the analysis, such as limiting the mechanism to a single working mode. Furthermore, examination of the overall ITI distribution at this location shows the mechanism is in an ITS, hence, this issue does not affect the
5.5 Motion/Force Transmission Analysis

analysis when examining the overall minimum ITI and OTI performance of the mechanism. Therefore, it can be stated that the SCARA-Tau with the current parameter set does not experience any OTS within its usable workspace.

The distribution of the CTI throughout the workspace is shown in Figs. 5.13(a) to (c) for the equivalent constraints produced by the vertical CLSC in chain one, the vertical CLSC in chain two and the yaw constraining CLSC in chain two, respectively. The CTI results illustrate an equivalent constraint wrench’s ability to oppose the motion not constrained by all other wrenches. Away from a CTS, the constraints

![Figure 5.13: Distribution of the CTI results throughout the workspace of the SCARA-Tau mechanism for the equivalent constraints produced by (a) the vertical CLSC in chain one, (b) the vertical CLSC in chain two and (c) the yaw constraining CLSC in chain two. The overall minimum CTI, min(CTI₁, CTI₂₁, CTI₂₂), for the mechanism is shown in (d).]
restrict the three rotations of the mobile platform. Therefore, in a CTS one or more of these rotations become unconstrained.

From the overall minimum CTI, \( \min(\text{CTI}_{1,1}, \text{CTI}_{2,1}, \text{CTI}_{2,2}) \), shown in Fig. 5.13(d), it is evident that the central workspace is free from constraint singularities, however, they are present at locations near the top of the workspace boundary. The constraints involved in each CTS location can be determined by pairing the dark blue bands in the distribution plots. The dark blue section at the top left of the workspace plot in Figs. 5.13(a) and (b) is due to a linear dependence between the constraints produced by the two vertical parallelograms. This occurs when the planes of the parallelograms becomes parallel, as seen in the three views illustrated in Fig. 5.14 at workspace location \([x \ y \ z] = [0.598 \ 0.000 \ 1.053] \text{m}\). The gradual reduction in CTI value towards the front of the workspace, evident in Figs. 5.13(a) and (b), is also due to the interactions between the two vertical parallelograms. When close to the front boundary, the two planes of the vertical CLSCs are nearly parallel to each other, resulting in a substantial reduction in the mechanism’s ability to effectively resist

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.14.png}
\caption{The SCARA-Tau mechanism in a constraint singularity involving the two vertical parallelograms at workspace location \([x \ y \ z] = [0.598 \ 0.000 \ 1.053] \text{m}\). (a) Side view, (b) front view and (c) top view. The mechanism is illustrated following the convention established in the description of Fig. 3.13, which is utilised for all the mechanism configuration illustrations throughout this thesis.}
\end{figure}
5.5 Motion/Force Transmission Analysis

A moment or force about the calculated output twist. The CTI distribution plot in Fig. 5.13(c) indicates that the yaw constraining CLSC effectively constrains the mobile platform’s yaw rotation in the central and lower portions of the workspace. As the mechanism moves upwards, the CLSC begins inclining, until becoming vertical at the top of the workspace. At this location the equivalent constraint produced by this CLSC is coplanar with the constraints produced by the two vertical CLSC resulting in a CTS, as indicated by the dark blue point at the top of the plot.

Figure 5.15 displays the ICCI distributions throughout the mechanism’s workspace for the three CLSC. It should be noted that for a general four-bar CLSC, two ICCI results are generated, one for each wrench in Eqn. (5.12). However, in the case of a parallelogram, the two ICCI results for each CLSC are identical, thus, produce the same distribution plot. Hence, a single distribution plot is shown for each CLSC. The distributions for the two vertical parallelograms are seen in Figs. 5.15(a) and (b), respectively. Due to the identical distal link lengths, proximal joint heights, and mobile platform joint heights, the two ICCI distributions are indistinguishable. The ICCI performance of the two vertical parallelograms is high throughout the mechanism’s lower and middle workspace. They reduce in constraint efficiency toward the top of the workspace as their respective CLSCs near collapse. The mechanism’s selected parameters also result in these ICCI distributions matching that of the CTI for the yaw constraining CLSC in Fig. 5.13(c). Different distributions could be obtained by modifying the initial vertical angle of the yaw constraining CLSC or the mobile platform joint positions of the vertical CLSCs.

The ICCI distribution for the yaw constraining CLSC is shown in Fig. 5.15(c). In contrast to the vertical CLSC plots, the distribution for this sub-chain is related to both the z and x-position of the mobile platform. The CLSC is close to collapse at the workspace front and towards the fixed base column. The minimum ICCI in
5.5 Motion/Force Transmission Analysis

Figure 5.15: Distribution of the ICCI results throughout the workspace of the SCARA-Tau mechanism for (a) the vertical CLSC in chain one, (b) the vertical CLSC in chain two and (c) the yaw constraining CLSC in chain two. The overall minimum ICCI, min(ICCI$_{1,1}$, ICCI$_{2,1}$, ICCI$_{2,2}$), for the mechanism is shown in (d).

Fig. 5.15(d) visualises the mechanism’s closeness to an ICCS. This analysis reveals that the mechanism’s CLSCs provide optimal constraints in the central and lower portions of the workspace, while the effectiveness of the constraints are reduced near the workspace front, top and close to the fixed base column.

Comparing Figs. 5.13(d) and 5.15(d) highlights the importance of examining both the CTI and ICCI for spatial mechanisms that contain these CLSCs. The CTI plot in Fig. 5.13(d) shows that the mobile platform is well constrained through the central band of the workspace. However, examination of the ICCI, reveals that near the
front of the workspace and fixed base column the yaw constraining CLSC is close to an ICCS. Therefore, the mechanism cannot resist certain externally applied forces or moments. This has been verified on the physical SCARA-Tau prototype, shown in Fig. 5.1(a), by positioning the prototype close to the fixed base column and applying forces on the mobile platform to generate a moment in the platform’s z-direction. The test was repeated in the central workspace away from all singularities and towards the front of the workspace. Comparing the three locations from a qualitative perspective, it was clearly evident that the mechanism could not effectively resist the applied moment when positioned close to the fixed base and near the front of the workspace, however, when positioned in the central workspace it could. These observations correlate with the ICCI results obtained.

In order to develop a complete characterisation of the SCARA-Tau, the overall minimum of all indices is calculated as the \( \min(\text{ITI}_i, \text{OTI}_i, \text{CTI}_{i,k}, \text{ICCI}_{q,i,k}) \) for the \( q \)th component of the \( k \)th equivalent constraint in the \( i \)th chain. As discussed in Section 4.4.1, the ICCI’s relation to the transmission angle results in a lower acceptable bound on its value of 0.64. Going beyond this can lead to poor operational characteristics of the constraint produced by the CLSC. Therefore, two plots can be generated, the first by taking the ICCI’s minimum with all other indices and the second applying a lower acceptable bound on the ICCI. These distributions are shown in Fig. 5.16(a) and (b) respectively.

Examining Fig. 5.16(a), the central workspace is singularity free, with ITS occurring around the workspace boundary and CTS at a small section in the top left. The even colouration of the workspace indicates stable motion/force transmission abilities. Through examination of all the individual index distribution plots, the main limitation on the motion/force transmission effectiveness of the SCARA-Tau is due to chains one and three, evident in Figs. 5.12(a) and (c). This highlights the ease
5.5 Motion/Force Transmission Analysis

The overall minimum for all motion/force transmission measures throughout the workspace of the SCARA-Tau mechanism is shown in (a). The overall minimum with a lower bound on the ICCI of 0.64 is shown in (b).

of identifying a mechanism’s performance limiting chains through inspection of the index distribution plots. These plots relate to the output transmission characteristics of the mechanism, and as discussed earlier in this section, the reduced OTI of chains one and three is a result of the actuator and link scheme utilised and is an innate limitation with the SCARA-Tau axis-symmetric parallel mechanism. This results in an average overall minimum index value of 0.4338, with large portions of the workspace maintaining values above 0.500. The performance specifications for the physical SCARA-Tau prototype are shown in Table 5.2, hence, even with an average overall minimum motion/force transmission index of 0.4338, it is evident that the SCARA-Tau is capable of very high performance.

Figure 5.16(b) is generated by reassigning any value of ICCI < 0.64 as zero, with

**Table 5.2:** The properties of the physical SCARA-Tau prototype [26].

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Repeatability</td>
<td>4μm</td>
</tr>
<tr>
<td>Absolute accuracy</td>
<td>15μm</td>
</tr>
<tr>
<td>Lowest resonance frequency</td>
<td>30Hz</td>
</tr>
<tr>
<td>Path accuracy at 1 m/s</td>
<td>100μm</td>
</tr>
<tr>
<td>Linear acceleration</td>
<td>5g</td>
</tr>
<tr>
<td>Maximum speed</td>
<td>5m/s</td>
</tr>
</tbody>
</table>
the bound selected as described in Section 4.4.1. This crops the reachable workspace, leaving a reduced usable workspace with an acceptable transmission performance of the CLSCs. Examining the cropped workspace demonstrates the effect CLSCs can have on the motion/force transmission ability and usable workspace size of a mechanism. This highlights the importance of analysing the ICCI in addition to the ITI, OTI and CTI in mechanisms that utilise CLSCs of the type studied herein.

5.6 Limitations of the Presented Method

Up to this point, the methodology completely characterises the singularities and motion/force transmission abilities of the exemplified variants. However, two restriction have been implemented that do not always hold true in other variants from the same family of mechanisms. The first is with reference to the CLSCs utilised. All CLSCs analysed so far have been planar, meaning their links remain in a single plane, or could be simplified to planar, due to constraints within the chain. Planar CLSCs are present in mechanisms such as the Delta [13], H4 [15], SCARA-Tau [21] and Orthoglide [18]. However, certain axis-symmetric parallel mechanism variants can experience non-planar CLSCs, which increase the complexity of their analysis. The second involves the layout of the mobile platform joints. The analysed mobile platform joint arrangements have been found to inadvertently assist the unique definition of the OTS actuation wrenches. However, other platform joint configurations can require in depth and specialised knowledge of a mechanism’s motions and constraints in order to determine the equivalent actuation wrenches, in turn making this method less accessible. These two limitations are further detailed in the following sections.
5.6 Limitations of the Presented Method

5.6.1 Non-Planar Closed-Loop Sub-Chains

If a CLSC is planar, the equivalent zero or infinite pitch wrenches of the chain can be determined through the inspection method using observations one to three from Section 2.3.1. However, if a CLSC is non-planar, its equivalent wrenches become too complex to determine through inspection. This is due to the equivalent wrenches of such a closed-loop possessing non-zero finite pitches. This case is examined below for the non-planar (SS)$_2$ closed-loop, illustrated in Fig. 5.17. Joints A, B, C and D do not lie in the same plane and are all spherical joints. The links AB and CD are therefore both SS links. In order to simplify the resulting equations, a local coordinate system is attached to the centre of spherical joint A, with the $z$-axis directed along distal link AB and at this instant the $y$-axis is along AD, with the $x$-axis as per the right hand rule. Link AD is initially fixed and link BC is the output.

The wrenches along the distal links, defined relative to the local coordinate system, equal

$$\hat{\mathbf{w}}_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T \quad \text{and} \quad \hat{\mathbf{w}}_2 = \begin{bmatrix} a & b & c & c & 0 \end{bmatrix}^T$$

where all lowercase italic letters are generic scalars. The reciprocal twist system to

![Figure 5.17: Non-planar (SS)$_2$ CLSC with its distal link wrenches shown.](image)
the above wrenches is determined as

\[
\begin{align*}
\hat{s}_{t1} &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T, \\
\hat{s}_{t2} &= \begin{bmatrix} a & b & c & 0 & 0 \end{bmatrix}^T, \\
\hat{s}_{t3} &= \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \end{bmatrix}^T, \\
\hat{s}_{t4} &= \begin{bmatrix} 0 & 0 & 0 & -b & a & 0 \end{bmatrix}^T.
\end{align*}
\] (5.16)

The resultant twists are illustrated in Fig. 5.18(a) and are described respectively as, a rotation about the \(y\)-axis, a rotation about the axis \([a \ b \ c]^T\), a rotation about the \(z\)-axis and a translation along the axis \([-b \ a \ 0]^T\). Therefore, the non-planar \((\text{SS})_2\) closed-loop is equivalent to a complex 4-DOF joint with three rotations and one translation [115].

In axis-symmetric parallel mechanisms, the actuator’s input twist is commonly oriented in either the \(y\)-direction or \(xz\)-direction of the local coordinate system. The latter case can be further simplified by examining the instant when the input twist is

\[\begin{align*}
\hat{s}_{t1} &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T, \\
\hat{s}_{t2} &= \begin{bmatrix} a & b & c & 0 & 0 \end{bmatrix}^T, \\
\hat{s}_{t3} &= \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \end{bmatrix}^T, \\
\hat{s}_{t4} &= \begin{bmatrix} 0 & 0 & 0 & -b & a & 0 \end{bmatrix}^T.
\end{align*}\]

**Figure 5.18:** (a) Twists associated with the \((\text{SS})_2\) non-planar CLSC and (b) a visualisation of why the inspection method does not work for this case.
5.6 Limitations of the Presented Method

parallel to the \( x \)-direction. Therefore, the two input twists are defined as

\[
\begin{align*}
\hat{s}_{t5y} &= \begin{bmatrix} 0 & 1 & 0 & d & f \end{bmatrix}^T \\
\hat{s}_{t5x} &= \begin{bmatrix} 1 & 0 & 0 & 0 & -e \end{bmatrix}^T,
\end{align*}
\]

respectively.

The incorporation of an input twist into the twist system in Eqn. (5.16) adds a DOF to the chain, in turn unfixing link AD. Two twist systems are possible by using either \( \hat{s}_{t5y} \) or \( \hat{s}_{t5x} \) as the input twist. Calculating the reciprocal wrench system for each twist system results in

\[
\begin{align*}
\hat{s}_{w5y} &= \begin{bmatrix} -af & -bf & ad & -cf & 0 & af \end{bmatrix}^T \\
\hat{s}_{w5x} &= \begin{bmatrix} -ae & -be & -c & -ce & 0 & ae \end{bmatrix}^T,
\end{align*}
\]

respectively. From the two results in Eqns. (5.18), it is evident that the pitch of the wrenches are non-zero and finite, as identified by Eqn. (2.7) producing a non-zero result.

The reason why the inspection method does not work in this case can be explained geometrically. Figure 5.18(b) shows the four twists of the CLSC and the two input twist cases. For a zero pitch wrench to be reciprocal to all twists, it must intersect all twists of zero pitch and be perpendicular to all twists of infinite pitch, as defined by observations one and three in Section 2.3.1. Using the input twist parallel to the \( y \)-axis, Eqn. (5.17a), the complete chain is analysed by first sketching a plane, labelled \( \Omega_1 \), which includes \( \hat{s}_{t2} \) and is perpendicular to \( \hat{s}_{t4} \). Therefore, any line in the \( \Omega_1 \) plane is perpendicular to \( \hat{s}_{t4} \) satisfying the requirement of observation three. A line directed between the origin and the point where \( \hat{s}_{t5y} \) passes through \( \Omega_1 \) is perpendicular to \( \hat{s}_{t4} \) and intersects all other twists except \( \hat{s}_{t3} \). This line is therefore not reciprocal to
§t3. No other line is possible that fulfils the requirements, hence a pure force wrench cannot exist.

From observation three, an infinite pitch wrench is reciprocal to all twists when it is perpendicular to all zero pitch twists. Applying this to Fig. 5.18(b), it becomes evident that this is not possible in the non-planar (SS)2 closed-loop. Therefore, as determined in Eqn. (5.18a) the constraint must possess a non-zero finite pitch. The same conclusion is reached when using the other possible input twist from Eqn. (5.17b).

5.6.2 The Mobile Platform Joints

As stated, the configuration of the mobile platform joints in the previous analysis assisted the unique definition of the OTS actuation wrenches. The mobile platform joints of chain one and three were placed along the same vertical axis, hence, when projected onto a plane normal to the common axis of the actuated joints, a unique interaction point between the chains can be identified. This allows the simple and unique definition of the point of application of the OTS actuation wrenches. However, if restrictions are not placed on the mobile platform joint arrangement and chain composition, except for in this case the existence of the two vertical parallelograms, then determining the equivalent OTS actuation wrench of a CLSC can become more complex due to the interactions between the chains. Certain mobile platform joint arrangements require knowledge about a mechanism’s motions and constraints at an expert level in order to define the equivalent actuation wrenches. This reduces the intuitiveness and accessibility of the method.

Therefore, an alternative approach is required that is more accessible, intuitive, detects all singularities and provides a measure of the motion/force transmission of these mechanisms with potentially non-planar CLSCs and different mobile platform joint arrangements.
5.7 Chapter Conclusion

This chapter extended the application of the motion/force transmission indices onto spatial mechanisms with CLSCs. The position and kinematic equations describing the SCARA-Tau parallel mechanism were derived, along with the required twists and wrenches. The detected singular locations, using the screw theory based method with the proposed twists and wrenches, were verified against the singular locations detected by the zero points of the numeric input-output Jacobian’s conditioning index. Thereafter, the ITI, OTI, CTI and ICCI were applied to the SCARA-Tau parallel mechanism. It was found that the inclusion of the ICCI was essential to completely characterise its motion/force transmission abilities, as a result of the equivalent CLSC wrenches utilised.

The presented method was found to encounter limitations when applied to certain mechanisms, namely mechanisms with non-planar CLSCs or modified mobile platform joint arrangements. Therefore, the following chapter introduces a general method applicable to all axis-symmetric parallel mechanisms, which is intuitive, detects all singularities, has a physical meaning and provides a measure of a mechanism’s motion/force transmission performance.
6.1 Overview

This chapter addresses the limitations of the proposed methodology that were highlighted at the conclusion of the previous chapter. A general approach to the singularity and motion/force transmission analysis of axis-symmetric parallel mechanisms, including variants with a different configuration of the mobile platform joints, non-parallelogram CLSCs and potentially non-planar CLSCs, is presented in this chapter. In order to verify the general method, both this method and the method from the previous chapter are applied to a SCARA-Tau mechanism with the parallelogram CLSCs. The correlation between the detected singularities is verified. Then the inclusion of
the ICCI in the developed analysis is verified and exemplified through examination of the transmission performance of a planar 2-DOF axis-symmetric parallel mechanism.

The developed analysis method is then exemplified using a spatial 3-DOF axis-symmetric parallel mechanism based on the SCARA-Tau design, with a modified distal link grouping and mobile platform joint arrangement. This mechanism includes a potentially non-planar CLSC, along with a complex equivalent OTS actuation wrench. The developed methodology is shown to be intuitive to implement, provide results with a physical meaning, detect all singularities and completely characterise the mechanisms’ motion/force transmission performance.

6.2 The General Methodology

The following methodology can be applied to any mechanism that implements the link schemes analysed throughout this thesis. Fig. 6.1 illustrates a variety such mechanisms. Figs. 6.1(a) to (d) use the same actuation scheme as analysed in this thesis. Fig. 6.1(a) shows a 4-DOF axis-symmetric parallel mechanism with an articulated mobile platform, enabling infinite rotation of the mobile platform and control of its three translations. I analysed this mechanism with Gosselin, Isaksson and Laliberté [116]. The mechanisms in Figs. 6.1(b) and (c) are based on patents [8, 67] and exhibit no parasitic yaw rotation during radial and vertical translations. I analysed these mechanisms for singularities and workspace conditioning using the analytical derivative of their inverse kinematic equations [27]. Fig. 6.1(d) illustrates a SCARA-Tau variant with a 2/2/2 distal link clustering, I analysed this style of mechanism with Isaksson and Brogårdh utilising screw theory to examine the yaw constraining ability of different variants [117]. Figs. 6.1(e) to (g) utilise actuated linear guide rails with the same distal link configurations as the mechanisms analysed herein. The
6.2 The General Methodology

Figure 6.1: Other mechanisms implementing the link scheme utilised in this thesis. (a) 4-DOF axis-symmetric parallel mechanism utilising redundant actuation to achieve infinite mobile platform rotation and vertical translation; (b) and (c) 3-DOF axis-symmetric parallel mechanisms with a tangentially constant platform; (d) SCARA-Tau variant with a 2/2/2 distal link clustering and parallelogram CLSCs; (e) 4-DOF axis-symmetric parallel mechanism that makes use of the redundant actuation to operate a gripper and can achieve infinite gripper rotation; (f) 4-DOF linearly actuated parallel mechanism utilising redundant actuation to achieve infinite mobile platform rotation and vertical translation; (g) 3-DOF Gantry-Tau mechanism based on the SCARA-Tau design actuated along linear guide rails; (h) 4-DOF H4 mechanism with an articulated mobile platform, revolute actuated proximal links and parallelogram CLSCs [15]; (i) 4-DOF Delta mechanism with parallelogram CLSCs and revolute actuated proximal links [11].
6.2 The General Methodology

The mechanism in Fig. 6.1(e) has 4-DOF and is kinematically redundant. The kinematic redundancy is utilised to operate a gripping mechanism. I analysed the performance of this mechanism with Isaksson and Gosselin [118]. Fig. 6.1(f) is a variation of the mechanism in Fig. 6.1(a), utilising the same distal link and articulated mobile platform design to achieve vertical translation and infinite rotation of the mobile platform [116]. The mechanism illustrated in Fig. 6.1(g) is the Gantry-Tau and is based on the SCARA-Tau design, however, it utilises linear actuators instead of the actuated coaxial proximal links [119]. The final two figures, Fig. 6.1(h) and (i), are the H4 [15] and Delta [11] mechanisms, respectively. Revolute actuated proximal links and planar parallelogram CLSCs are utilised in both designs. The mechanisms described above are examples of the variety of mechanisms that can be analysed using the following method. It is important to note that because the described spatial mechanisms all contain planar parallelogram CLSCs, the previous method can also be utilised for their singularity and motion/force transmission analysis. However, the method presented in this chapter simplifies this analysis, especially for cases where the CLSCs are non-planar and non-parallelogram or where the mobile platform joint placements add complexity to the previous analysis, in terms of defining the equivalent constraint and actuation wrenches. The new method is exemplified on such a mechanism later in this chapter.

The previous method, required calculation of the equivalent constraint and actuation wrenches in order to determine the OTI, ITI and CTI, along with the need to define a point of application for the equivalent OTS actuation wrenches. The new method simplifies the approach by removing the requirement of determining the equivalent OTS actuation wrenches, and instead utilising the zero pitch wrenches directed along each of the distal links, as seen in Figs. 6.2(a) and (b) for a planar
6.2 The General Methodology

Figure 6.2: Wrenches directed along the distal links used in the OPI calculations for (a) the planar and (b) spatial mechanisms.

and spatial variant respectively. Furthermore, the line of action of each wrench constantly intersects the mobile platform joint of its distal link. Therefore, any ambiguity in the selection of these wrenches and their points of application is eliminated. This simplifies the analysis process, particularly for mechanisms with non-planar or non-parallelogram CLSCs, or where the mobile platform joint placements would result in difficulties in defining the equivalent constraint and actuation wrenches.

As discussed in the previous chapters, determining the equivalent wrenches also removes important information about the performance of a CLSC, hence the ICCI was proposed to monitor their performance. As a result of not utilising the equivalent wrenches, the new method retains all information about a CLSC’s performance. Therefore, the new method measures a mechanism’s total output transmission performance, with the OTI, CTI and ICCI combined into a single index, termed the
output performance index (OPI). The OPI’s inclusion of these three indices is exemplified in Section 6.3. The term OPI was selected to distinguish this measure from the previously proposed TI [108] and GTI [110] indices.

The presented method requires no assumptions or simplifications for the output transmission analysis, however, the input transmission analysis still requires the definition of equivalent ITS actuation wrenches for the chains with CLSCs. The possibility of non-planar CLSCs means that some knowledge about the constraints within the mechanism is still required. Due to the variety of possible actuation and constraint schemes these ITS wrenches must be determined on a case-by-case basis. A systematic procedure for determining the ITS actuation wrench for these mechanisms with two vertical CLSCs is detailed below.

### 6.2.1 ITS Actuation Wrench

The process for determining the ITS actuation wrenches for the chains with CLSCs requires a similar procedure to that implemented in Chapters 3 to 5. As mentioned above, the ITS actuation wrenches need to be determined on a case-by-case basis. The spatial mechanisms examined in this chapter contain a vertical planar parallelogram CLSC in chains one and two, as present in the SCARA-Tau. From the previous analysis, it is known that these CLSCs constrain the mobile platform’s pitch and roll. Therefore, the pitch and roll of the output link of the yaw constraining CLSC, which is rigidly attached to the mobile platform, is also constrained. Therefore, the twists of the non-planar CLSC, in Eqn. (5.16), can be simplified to a zero pitch twist in the z-direction about the IC of the yaw constraining CLSC’s xy-projection and an infinite pitch twist parallel to the common perpendicular between the CLSC’s distal links. Hence, the ITS actuation wrench can now be determined through inspection
and must be orthogonal to the infinite pitch twist and intersect the zero pitch twist.

The process outlined below is with reference to the 3-DOF axis-symmetric parallel mechanisms that contain two planar vertical parallelogram CLSCs and two other distal links, where at least one of these distal links is in a separate chain to the vertical parallelograms. This arrangement is seen in the SCARA-Tau in Fig. 5.1(a) and its variant with a 2/2/2 distal link clustering in Fig. 6.1(d). This process must be applied to any chain in which the CLSC is non-planar or its plane is not parallel to the chain’s input twist.

1. Project the chain onto the plane, $\Omega_1$, normal to the actuated joints axis, $\hat{s}_I$. For the mechanisms analysed herein this is the $xy$-plane, as shown in Fig. 6.3(a).

2. Determine the intersection point of the distal link $\Omega_1$-projections, $IC_{\Omega_1}$.
   - If $IC_{\Omega_1}$ is located at infinity, the projection of the distal links are parallel. The $\Omega_1$-projection of the ITS actuation wrench’s direction vector, $\hat{s}_{A_{\Omega_1}}$, is parallel to the distal links and intersects the projection of the point of analysis X in $\Omega_1$, $X_{\Omega_1}$, as shown in Fig. 6.3(b).
   - If $IC_{\Omega_1}$ is finite, then $\hat{s}_{A_{\Omega_1}}$ passes through $IC_{\Omega_1}$ and $X_{\Omega_1}$, as illustrated in Fig. 6.3(c).

3. Generate the plane $\Omega_2$, which is perpendicular to the $\Omega_1$-plane and includes the vector $\hat{s}_{A_{\Omega_1}}$. The normal to this plane is $\hat{n}_{\Omega_2}$, as illustrated in Fig. 6.3(d).

4. Calculate the common perpendicular between the two distal links, $\hat{n}_D$. Then determine the common perpendicular between $\hat{n}_D$ and $\hat{n}_{\Omega_2}$, resulting in a vector, $\hat{s}_{A_{\Omega_2}}$, which is coplanar with $\Omega_2$ and normal to $\hat{n}_D$, as seen in Fig. 6.3(d).
6.2 The General Methodology

Figure 6.3: Determining the ITS actuation wrench for spatial mechanism variants with a potentially non-planar CLSC. (a) Projection of the CLSC on the $\Omega_1$-plane which is normal to $\hat{s}_I$. (b) $\Omega_1$-projection of the ITS actuation wrench direction vector $\hat{s}_{A\Omega_1}$ for a parallelogram projection and (c) for a general projection. (d) ITS actuation wrench direction vector perpendicular to $\hat{n}_D$ and laying in the $\Omega_2$-plane.

5. The ITS actuation wrench is the zero pitch screw in the $\hat{s}_{A\Omega_2}$ direction. Due to the constraints on the output link’s roll and pitch, produced by the two vertical parallelograms, the point of application of this wrench can be any point in the $\Omega_2$ plane. Hence, the intersection between the proximal link and the $\Omega_2$-plane is utilised herein, $\mathbf{p}_{\Omega_2}$. The wrench is therefore defined as

\[
\hat{s}_{A_{IT}} = \begin{bmatrix} \hat{s}_{A\Omega_2} \\ \mathbf{p}_{\Omega_2} \times \hat{s}_{A\Omega_2} \end{bmatrix}.
\] (6.1)

For any chains with a single planar parallelogram CLSC, where its plane is parallel to the chain’s input twist, the resultant ITS actuation wrench is coplanar and parallel
to the CLSC’s distal links. The vertical position of the wrench is not important and can be selected as any point in the CLSC’s plane, for the same reason that was stated above.

6.2.2 The Analysis Procedure

To determine the input transmission characteristics, the above ITS actuation wrenches and the input twists are applied to the ITI defined in Eqn. (4.4). The OPI analysis utilises the wrenches directed along the distal links. Therefore, the studied planar mechanisms with a CLSC have three OPI wrenches and three permanent planar constraint wrenches, whereas the studied spatial variants with CLSCs have six OPI wrenches, as illustrated in Figs. 6.2(a) and (b), respectively.

The output twist of each distal link OPI wrench is then calculated using the process outlined in Section 3.4.4. Next, the power coefficient is applied to each wrench and its respective output twist, as with the OTI defined in Eqn. (4.5), producing a measure of the mechanism’s output transmission characteristics. As a consequence of not simplifying the CLSC into their equivalent actuation and constraint wrenches, the overall minimum of the OPI of each distal link contains the same information as the OTI, CTI and ICCI. This is exemplified in the following section, along with a method to individually identify the OTS and CTS locations through examination of the similarities between the OPI plots for each distal link.

6.3 Singular Location Detection Verification

In this section, the detected singular locations utilising the OTI and CTI, are compared against those detected by the OPI. The modified SCARA-Tau mechanism is
6.3 Singular Location Detection Verification

utilised in this analysis, with the parameter set defined in Table 5.1. The ITS actuation wrenches are the same for the SCARA-Tau using both methods, hence, are not examined in this section. The analysis set up is identical to that defined in Chapter 5, with $10^6$ analysis points and eight workers performing parallel computing.

Figures 6.4(a) and (b) show the overall minimum OTI and CTI distribution plots, respectively. The minimum of these two distributions, $\min(\text{OTI, CTI})$, is shown in Fig. 6.4(c), while Fig. 6.4(d) displays the overall minimum OPI distribution plot. The detected singular locations are indicated by the dark blue areas of the plots. These locations have been extracted from the $\min(\text{OTI, CTI})$ and OPI plots and are illustrated in Figs. 6.4(e) and (f) repetitively. The colour map for the indices is shown in (g). Index values other than $\approx 0$ are not important in the present verification, however, are included in the plots to demonstrate the mechanism’s performance over the entire workspace.

It is evident from Figs. 6.4(e) and (f) that the detected singular locations are identical for both analysis methods. Furthermore, it can be seen that the general method’s OPI combines the OTI and CTI result into a single distribution. The presence of the ICCI in the OPI is less obvious, however, the gradual reduction in the OPI’s value towards the workspace front and rear, compared with the sudden drop in the $\min(\text{OTI, CTI})$ value indicates its presence. The inclusion of the ICCI in the OPI calculation is more clearly verified in Section 6.3.1.

Through comparison of Figs. 6.4(a) and (b) against Fig. 6.4(d), the respective OTS and CTS bands can be identified. Thus, the OTS band curves towards the right of the plot, while the CTS band curves towards the left.

The OTS and CTS bands can also be identified from the general method alone, by studying the OPI distribution plot for each individual distal link. These plots are shown in Figs. 6.5(a) to (f) for distal links (1,1), (1,2), (2,1), (2,2), (2,3) and
6.3 Singular Location Detection Verification

Figure 6.4: The (a) OTI, (b) CTI, (c) min(OTI, CTI) and (d) OPI distribution plots of the modified SCARA-Tau with the parameters listed in Table 5.1. The singularity bands from plots (c) and (d), identified by the dark blue sections, have been extracted and are shown in (e) and (f) repetitively. The colour map for the indices is shown in (g).
6.3 Singular Location Detection Verification

Figure 6.5: The OPI distribution plots (a) to (f) for the modified SCARA-Tau using the proposed general method with the parameters listed in Table 5.1, for distal links (1,1), (1,2), (2,1), (2,2), (2,3) and (3,1), respectively. Chain three cannot be involved in constraint singularities, due to not applying any constraints on the mobile platform when its actuator is unlocked. Therefore, the dark blue area in Fig. 6.5(f) must represent the OTS locations. The mechanism’s configuration at point $\begin{bmatrix} x & y & z \end{bmatrix} = \begin{bmatrix} 0.569 & 0.000 & 0.240 \end{bmatrix}$ m along this band is illustrated in Figs. 6.6(a) to (c). Examining the front and top views in Figs. 6.6(a) and (b), the cause of the OTS is not obvious, however, it becomes apparent in Fig. 6.6(c), where the wrench along each distal link, overlaid as orange arrows, are in parallel planes and thus are unable to resist a force applied normal to these planes, indicated by the green arrow. For the remaining plots in Figs. 6.5(a) to (e), any other dark blue areas not along this OTS band must therefore be CTS locations. The CTS
6.3 Singular Location Detection Verification

Figure 6.6: An OTS configuration of the modified SCARA-Tau mechanism at workspace location $[x, y, z] = [0.569, 0.000, 0.240]$ m. (a) Front view, (b) top view and (c) side view. The direction vector of each distal link wrench is overlaid in (c) to demonstrate that they are all in parallel planes and visualise the cause of the singular configuration. The unconstrained DOF is indicated by the green arrow. The mechanism is illustrated following the convention established in the description of Fig. 3.13, which is utilised for all the mechanism configuration illustrations throughout this thesis.

band starting at the top of the workspace and curving towards the left of the plot ending near $x = 0.750$ m, represents a CTS involving the two vertical parallelograms. This is evident from the lack of this CTS band in Figs. 6.5(e) and (f), which are for the two links not part of the vertical parallelograms. The mechanism’s configuration at point $[x, y, z] = [0.706, 0.000, 0.320]$ m along this CTS band is illustrated in Figs. 6.7(a) to (c). From the figures it is evident that the planes of the two vertical parallelograms are parallel and therefore results in an unconstrained motion of the mobile platform.

The dark blue band near the front of the workspace, seen in Figs. 6.5(c) to (e), signifies a CTS configuration where the three distal links of chain two become coplanar. This results in a loss of the yaw constraint produced by the CLSC.

The above analysis verifies the detected singular locations using the proposed
6.3 Singular Location Detection Verification

Figure 6.7: A CTS configuration of the modified SCARA-Tau mechanism involving the two vertical parallelograms at workspace location $[x\ y\ z] = [0.706\ 0.000\ 0.320]$m. (a) Front view, (b) top view and (c) side view.

OPI against the previously presented OTI and CTI. The OTS and CTS locations throughout the workspace were shown correlate exactly between both methods.

6.3.1 ICCI Verification

To further verify the inclusion of the ICCI in the OPI measure, the transmission performance of a planar 2-DOF axis-symmetric parallel mechanism with a CLSC is examined. The parameters of this mechanism, listed in Table 6.1, have been selected to provide a clear exemplification of the ICCI’s existence in the OPI.

The geometry and parameters of this mechanism are shown in Fig. 6.8(a), joints $C_{1,1}$ and $C_{2,2}$ are coaxial, with the point of analysis $X$ located at this point. The length of the distal link $L_{b_{2,1}}$ has been shortened to induce a collapse of the CLSC within the workspace.

Through inspection, the equivalent ITS and OTS actuation wrenches are directed along the distal links $L_{b_{1,1}}$ and $L_{b_{2,2}}$ for chains one and two respectively, as shown
6.3 Singular Location Detection Verification

Table 6.1: Mechanism parameters for testing ICCI inclusion in OPI.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proximal</td>
<td></td>
</tr>
<tr>
<td>$l_{a1,1}$</td>
<td>0.900</td>
</tr>
<tr>
<td>$l_{a2,1}$</td>
<td>0.900</td>
</tr>
<tr>
<td>$l_{a2,2}$</td>
<td>0.800</td>
</tr>
<tr>
<td>Distal</td>
<td></td>
</tr>
<tr>
<td>$l_{b1,1}$</td>
<td>1.100</td>
</tr>
<tr>
<td>$l_{b2,1}$</td>
<td>1.080</td>
</tr>
<tr>
<td>$l_{b2,2}$</td>
<td>1.100</td>
</tr>
<tr>
<td>Mobile Platform</td>
<td></td>
</tr>
<tr>
<td>$c_{1,1}'$</td>
<td>$[x \ y \ z]^T$</td>
</tr>
<tr>
<td>$c_{2,1}'$</td>
<td>$[0 \ 0 \ 0]^T$</td>
</tr>
<tr>
<td>$c_{2,2}'$</td>
<td>$[0 \ -0.100 \ 0]^T$</td>
</tr>
</tbody>
</table>

in Fig. 6.8(b). The CLSC produces an equivalent infinite pitch wrench normal to its plane, $\hat{s}_{C2,1}$*, which constrains the mobile platform’s yaw. The wrenches utilised in the OPI analysis are coaxial with each distal link, as illustrated in Fig. 6.8(c). The remaining three wrenches, common between both methods, are the permanent planar set of constraints that restrict the mobile platform’s roll, pitch and translation in the $z$-direction.

Using the wrenches defined above, the ITI, OTI, ICCI and OPI are calculated, with the results shown in Figs. 6.9(a) to (d), respectively. Examining Figs. 6.9(a) to

Figure 6.8: Planar mechanism for verification of the ICCI within the OPI measure, utilising the parameters in Table 6.1. (a) The configuration and parameters of the mechanism, (b) the equivalent actuation and constraint wrenches with $\hat{s}_{C2,1}$* directed normal to the CLSC and (c) the OPI wrenches.
6.3 Singular Location Detection Verification

(c), reinforces the requirement to analyse the ICCI when implementing the previous transmission analysis method using the equivalent wrenches.

It should be noted that the ICCI distributions are not required to be identical to individual components of the OPI distributions in Fig. 6.9(d). The reason for this is the ICCI independently monitors the transmission performance of the CLSC itself, whereas, the OPI considers the interactions of all chains together. Therefore, an exact match between the distribution of the ICCI and components of the OPI is not important, what is important is that a reduction in the OPI is observed as the CLSC approaches its internal singularity. Comparing Figs. 6.9(c) and (d), it is evident that the minimum OPI and the ICCI gradually reduce to zero toward the left of the distribution plots, both reaching zero at $x = 0.686\text{m}$. This indicates that the OPI does include information about the transmission performance of the CLSC.
Therefore, from the results obtained in the last two sections, it can be concluded that the OPI can measure the closeness to OTS, CTS and ICCS, hence, the OPI is a complete output performance measure.

6.4 Motion/Force Transmission Analysis of a SCARA-Tau 2/2/2 Variant

The SCARA-Tau 2/2/2 variant link assembly is shown in Fig. 6.10, with its OPI wrenches overlaid along each distal link. The term 2/2/2 is with reference to the clustering of the distal links, where for this variant there are two distal links per chain. In the original SCARA-Tau the distal links are arranged in a 3/2/1 clustering, that is, there are three distal links in one chain, two in another and one in the final chain. As with the previously analysed spatial variants, the 2/2/2 link configuration contains two vertical parallelograms in chains one and two, constraining the pitch and roll of the mobile platform. The mobile platform’s yaw angle is now constrained by

\[
\begin{align*}
\hat{A}_{1,1} & \quad \hat{A}_{1,2} \\
\hat{A}_{2,1} & \quad \hat{A}_{2,2} \\
\hat{A}_{3,1} & \quad \hat{A}_{3,2} \\
\end{align*}
\]

Figure 6.10: The 2/2/2 variant of the SCARA-Tau mechanism with the wrenches used in the OPI calculations overlaid. Each wrench is coaxial with its respective distal link.
the CLSC in chain three. This CLSC does not always remain planar and could result in equivalent wrenches with finite pitch, as described in Section 5.6.1. Furthermore, if a non-parallelogram CLSC is utilised in chain three, the mobile platform’s joint arrangement significantly increases the complexity of defining an equivalent OTS actuation wrench, due to the selection of its point of intersection.

The new method is exemplified on a SCARA-Tau 2/2/2 variant with an obtuse trapezium CLSC in chain three. The obtuse trapezium is with reference to the shape of the CLSC’s $xy$-projection. The utilised parameter set is given in Table 6.2 and is illustrated in Figs. 6.11(a) to (c), from an elevated view, top view and side view, respectively.

Table 6.2: Parameters for the SCARA-Tau 2/2/2 mechanism with an obtuse trapezium CLSC in chain three.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proximal</td>
<td></td>
</tr>
<tr>
<td>$l_{a1,1}$</td>
<td>0.900</td>
</tr>
<tr>
<td>$l_{a1,2}$</td>
<td>0.900</td>
</tr>
<tr>
<td>$l_{a2,1}$</td>
<td>0.900</td>
</tr>
<tr>
<td>$l_{a2,2}$</td>
<td>0.900</td>
</tr>
<tr>
<td>$l_{a3,1}$</td>
<td>0.800</td>
</tr>
<tr>
<td>$l_{a3,2}$</td>
<td>0.900</td>
</tr>
<tr>
<td>Distal</td>
<td></td>
</tr>
<tr>
<td>$h_{1,1}$</td>
<td>0</td>
</tr>
<tr>
<td>$h_{1,2}$</td>
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</tr>
<tr>
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</tr>
<tr>
<td>$h_{2,2}$</td>
<td>0.200</td>
</tr>
<tr>
<td>$h_{3,1}$</td>
<td>0.925</td>
</tr>
<tr>
<td>$h_{3,2}$</td>
<td>0.925</td>
</tr>
<tr>
<td>Mobile Platform</td>
<td></td>
</tr>
<tr>
<td>$e_{1,1}'$</td>
<td>$[x \ y \ z]^T$</td>
</tr>
<tr>
<td>$e_{1,2}'$</td>
<td>$[0 \ 0.1305 \ -0.100]^T$</td>
</tr>
<tr>
<td>$e_{2,1}'$</td>
<td>$[0 \ -0.1305 \ -0.100]^T$</td>
</tr>
<tr>
<td>$e_{2,2}'$</td>
<td>$[0 \ -0.1305 \ 0.100]^T$</td>
</tr>
<tr>
<td>$e_{3,1}'$</td>
<td>$[-0.0731 \ -0.0285 \ 0.160]^T$</td>
</tr>
<tr>
<td>$e_{3,2}'$</td>
<td>$[0 \ 0.1305 \ 0.160]$</td>
</tr>
</tbody>
</table>
6.4 Motion/Force Transmission Analysis of a SCARA-Tau 2/2/2 Variant

Figure 6.11: The SCARA-Tau 2/2/2 mechanism with an obtuse trapezium CLSC, illustrating an (a) elevated view, (b) top view and (c) side view.

The ITI analysis results are shown in Figs. 6.12(a) to (d) for chain one, two, three and the overall minimum, respectively. The mechanism experiences ITS around the workspace boundary, with the region of optimal transmission located toward the front central workspace. Chains one and two limit the mechanism’s forward and top reach, as seen in Figs. 6.12(a) to (b), while chain three limits its back and lower reach, evident in Fig. 6.12(c). The configurations of maximum ITI for chains one and two, with the vertical parallelograms, occur when the distal links of the parallelograms are horizontal and perpendicular to their respective distal link axis. The maximum ITI configuration for chain three is illustrated in Figs. 6.13(a) and (c), showing front, top and side views of the mechanism’s configuration. In this configuration, the mechanism’s CLSC in chain three is planar, while being coplanar with the chains proximal link. Its ITS actuation wrench, calculated using the method described in Section 6.2.1, is illustrated as an orange arrow.

The output transmission characteristics of this mechanism are now analysed using the OPI. The results are shown in Figs. 6.14(a) to (g) for the distal links (1,1), (1,2), (2,1), (2,2), (3,1), (3,2) and the overall minimum, respectively. Figures 6.14(a) to
6.4 Motion/Force Transmission Analysis of a SCARA-Tau 2/2/2 Variant

Figure 6.12: The ITI distribution plots for the SCARA-Tau 2/2/2 mechanism with an obtuse trapezium CLSC utilising the parameters in Table 6.2, for chain (a) one, (b) two and (c) three. The overall minimum ITI is shown in (d).

Figure 6.13: The SCARA-Tau 2/2/2 mechanism with an obtuse trapezium CLSC in the maximum ITI configuration for chain three at location $[x, y, z] = [1.563, 0.000, 0.765]$ m. (a) Front view, (b) top view and (c) side view. The ITS actuation wrench for chain three is shown in orange.
Figure 6.14: The OPI distribution plots (a) to (f) are for the SCARA-Tau 2/2/2 mechanism with an obtuse trapezium CLSC utilising the parameters in Table 6.2, for distal links (1,1), (1,2), (2,1), (2,2), (3,1) and (3,2), respectively. The overall minimum is shown in (g).
6.4 Motion/Force Transmission Analysis of a SCARA-Tau 2/2/2 Variant

(d) reveal the interactions between the distal links forming the two vertical parallelograms. The dark blue area along the front edge of the workspace is a result of the planes of the two vertical parallelograms nearly becoming parallel, meaning the mechanism is very close to a CTS. This results in a substantially reduced rotational constraint on the mobile platform. The dark blue section near the workspace top, between $x = 0.480$ and $x = 0.888$, are also CTS locations involving the two vertical parallelograms. The mechanism’s configuration at point, $[x \ y \ z] = [0.765 \ 0.000 \ 1.143]$m, along this CTS is illustrated in Fig. 6.15 from the front, top and side viewpoints, with the uncontrollable output twist shown in green. The linear dependence amongst the four distal links of the two vertical parallelograms becomes apparent in Figs. 6.15(b) and (c) due to the planes formed by the two vertical parallelograms becoming parallel. This results in an uncontrollable DOF of the mobile platform, described by a 0.1230 pitch output twist with the Plücker coordinates

$$\hat{\xi}_O = \begin{bmatrix} 0.0863 & 0.9958 & 0.0315 & -1.2223 & 0.2054 & 0.7614 \end{bmatrix}^T, \quad (6.2)$$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure615.png}
\caption{The SCARA-Tau 2/2/2 mechanism with an obtuse trapezium CLSC in a CTS involving the two vertical parallelograms at workspace location $[x \ y \ z] = [0.765 \ 0.000 \ 1.143]$m. (a) Front view, (b) top view and (c) side view. The unconstrained DOF is a 0.1230 pitch twist and is illustrated in green.}
\end{figure}
illustrated as the green arrow in Fig. 6.15. An equivalent configuration for the modified SCARA-Tau 3/2/1 was shown in Fig. 6.7.

The reason for the high OPI value near the bottom of Fig. 6.14(b), at location \( \begin{bmatrix} x & y & z \end{bmatrix} = [0.959, 0.000, -0.515] \) m is not obvious at first. However, through examination of the output twist gained when the wrench \( \mathbf{\hat{s}}_{A_{1,2}} \) is removed, it becomes clear that the wrench and its output twist are well positioned for efficient motion/force transmission between them. The Plücker coordinates of the wrench and its output twist are given by

\[
\mathbf{\hat{s}}_{A_{1,2}} = \begin{bmatrix} 0.5485 & -0.6217 & -0.5591 & -0.3272 & 0.3318 & -0.6899 \end{bmatrix}^T \quad \text{and} \quad \mathbf{\hat{s}}_{O_{1,2}} = \begin{bmatrix} 0.6225 & 0.7677 & -0.1519 & 0.4255 & -0.2622 & 0.6987 \end{bmatrix}^T,
\]

respectively. Therefore, the numerator of the OPI, which is equal to the absolute value of the reciprocal product between the two screws, is

\[
\left| \mathbf{\hat{s}}_{A_{1,2}} \circ \mathbf{\hat{s}}_{O_{1,2}} \right| = 0.1616.
\]

Then from the method described in Section 4.2.1, the denominator of the OPI, which equals the potential maximum value of the numerator, is

\[
\left| \mathbf{\hat{s}}_{A_{1,2}} \circ \mathbf{\hat{s}}_{O_{1,2}} \right|_{\text{max}} = 0.2003,
\]

resulting in an OPI of

\[
\frac{\left| \mathbf{\hat{s}}_{A_{1,2}} \circ \mathbf{\hat{s}}_{O_{1,2}} \right|}{\left| \mathbf{\hat{s}}_{A_{1,2}} \circ \mathbf{\hat{s}}_{O_{1,2}} \right|_{\text{max}}} = 0.8068
\]

for this location. Therefore, at this location, in terms of motion/force transmission,
the wrench $\hat{\mathbf{A}}_{1,2}$ is well positioned to effectively constrain the motion of the mobile platform not constrained by the other five distal links, described by the output twist $\hat{\mathbf{O}}_{1,2}$. The examination of a distal link’s individual OPI distribution plot provides valuable insight into the link’s specific effects on a mechanism’s overall performance. However, at each workspace location a mechanism’s maximum motion/force transmission performance is limited by its worst performing link, therefore, the performance of a mechanism cannot be judged by the performance of a single link alone, and must also consider the performance effects due to the other distal links.

The overall minimum OPI, shown in Fig. 6.14(g), highlights the mechanism’s poor output transmission characteristics near the top, front, and bottom workspace boundaries. The central and rear of the workspace is OTS and CTS free with the minimum OPI results above 0.40 in these regions.

Figure 6.16 shows the mechanism’s overall minimum of the ITI and OPI, calculated by $\min(ITI_i, OPI_{i,j})$. The central workspace is seen to be singularity free, with singular locations around the workspace boundary and a CTS band seen at the top of the workspace. Comparing the minimum ITI and minimum OPI distribution plots

![Figure 6.16: The overall minimum ITI and OPI, \(\min(ITI_i, OPI_{i,j})\), for the SCARA-Tau 2/2/2 with the obtuse trapezium CLSC in chain three, utilising the parameters in Table 6.2.](image)
with the $\min(\text{ITI}_i, \text{OPI}_{i,j})$ distribution, it is evident that the output transmission characteristics predominantly reduce the overall performance of the mechanism in terms of motion/force transmission.

All the mechanisms studied in this thesis are seen to experience ITS at, or very close to, their workspace boundary, thus, by slightly cropping the workspace the ITS can be avoided. With this in mind, the independent analysis of the OPI can provide sufficient information to validate most designs. The benefit of this approach is that it avoids the process of deriving the equivalent ITS actuation wrenches of the CLSCs and simply utilises the wrenches directed along each of the distal links. Additionally, the implementation of the OPI is straightforward and detects all singularities that can occur within the cropped workspace, as well as delivers intuitive and meaningful results.

6.5 Chapter Conclusion

The method presented in this chapter was developed to address the limitations of the methodology proposed in the previous chapters when applied to spatial mechanisms. A general approach to the singularity and motion/force transmission analysis of axisymmetric parallel mechanisms was presented. A systematic method was detailed for the analysis of a mechanism’s input and output transmission characteristics. If a mechanism under analysis contains only planar CLSCs, the ITI analysis requires no assumptions, however, if the CLSCs are non-planar then the ITI methodology requires some known observations about the mechanism’s constraints. A systematic procedure for the calculation of the ITS actuation wrenches for axis-symmetric parallel mechanisms with two vertical parallelograms and a non-planar CLSC was detailed. A new index, called the OPI, was developed for measuring the output transmission
characteristics of a mechanism. The method is intuitive to implement, detects OTS, CTS and ICCS locations and provides meaningful results.

The singular locations detected by the new method were verified against the results obtained using the previously presented screw theory based method, along with an examination of the singular configurations of the mechanisms. The inclusion of the ICCI in the OPI was demonstrated utilising a planar 2-DOF axis-symmetric parallel mechanism with a CLSC. The new method was exemplified on a SCARA-Tau 2/2/2 variant with an obtuse trapezium CLSC in chain three, which is not always planar. Its mobile platform arrangement also introduces complexities in determining the equivalent OTS actuation wrench utilised in the previous chapters. It was shown that the new analysis process is intuitive and systematic, and that its results can be logically interpreted to determined the cause and consequence of a singularity, along with an intuitive visual representation of the motion/force transmission abilities of each chain and the overall mechanism as a whole.
Conclusions and Future Work

This thesis presented several results regarding the kinematic performance analysis of axis-symmetric parallel mechanisms with closed-loop sub-chains (CLSCs) using screw theory based methods.

A succinct introduction to the field of parallel mechanisms, including their benefits, drawbacks and the axis-symmetric topology was first presented, in order to assimilate the reader and highlight a clear motivation for the research. To standardise the nomenclature utilised throughout this thesis, key terminology associated with parallel mechanisms were defined.

Then a detailed review of the origins to the current state of the art of axis-symmetric parallel mechanisms was provided, revealing clear trends and gaps in the research. One such trend is the utilisation of CLSCs in the mechanisms’ designs to provide constraints on the mobile platform’s output motion. Other parallel mechanisms also include CLSCs, such as the world’s best-selling parallel mechanism, the
Delta, along with the H4 and Orthoglide. The CLSCs in these mechanisms are all four-bar planar parallelograms. However, many of the proposed axis-symmetric designs also possess non-parallelogram and potentially non-planar CLSCs. The singularities and performance effects of non-parallelogram and non-planar CLSCs had not previously been examined. Furthermore, some researchers presented results on the singularity analysis of axis-symmetric parallel mechanisms with CLSCs, however, multiple methods were required to detect all singularity types and resulted in limited physical insight into the cause or consequence of the singularity. These methods relied heavily on a researcher’s expert knowledge and understanding of the studied mechanisms to interpret the results. Therefore, an intuitive and systematic analysis tool was required that delivered logical results with a physical meaning to determine the cause and consequence of a singularity. This led to the adoption of screw theory techniques in this thesis. Screw theory is a mathematical tool utilised for the analysis of kinematics, statics and dynamics of interconnected bodies. It produces results with definitive physical meaning, expressed clearly and intuitively using geometrical concepts and common algebraic calculations. The origins and mathematical foundation of screw theory were then presented, providing the reader with the required basis to follow this thesis.

The principles of screw theory were then applied to the singularity analysis of a family of planar 2-DOF axis-symmetric parallel mechanisms with CLSCs. The mechanisms were parametrised and their general inverse kinematic solutions were defined in order to provide a foundation for the derivation of the twists and wrenches associated with their motions and forces, respectively. In the screw theory singularity analysis of a mechanism with CLSCs, the literature states that each chain with a CLSCs must be substituted by its equivalent serial chain through determining their equivalent twists and wrenches. A systematic method of generating the twists and
wrenches for use in the singularity analysis of these mechanisms with CLSCs was detailed. The mechanisms’ design, coupled with the use of CLSCs, resulted in the necessity to develop modified wrench definitions compared to the conventional equivalent serial chain wrenches proposed. The singularity types experienced by parallel mechanisms were then detailed. The selected singularity measures were based on the reciprocal product between specific mechanism twists and wrenches. The result of this product being the instantaneous work between the two screws. These singularities can be grouped into three types, one occurring when the actuators can no longer produce motion of the mobile platform’s point of analysis in a certain direction, termed the input transmission singularity (ITS), one where a previously controllable mobile platform motion becomes uncontrollable, termed an output transmission singularity (OTS) and ones where a once constrained motion of the mobile platform is now unconstrained, termed a constraint transmission singularity (CTS).

It was shown that any chain with a CLSC that is not oriented parallel to the actuated joint’s axis actually required the definition of two different actuation wrenches, one for use in the ITS detection and one for use in the OTS detection. For the mechanisms analysed herein, both wrenches must intersect the instantaneous centre of the CLSC, then the ITS actuation wrench intersects the mobile platform’s point of analysis and the OTS actuation wrench intersects the interaction point between the two chains. It was determined that the chain with a CLSC completely controls the mobile platform’s motion and thus, the mobile platform can be thought of as an extension of this chain. Hence, the point of interaction between the two chains was shown to be the serial chain’s mobile platform joint.

The detected singular locations, through the screw theory based method with the new wrench definitions, were found to exactly match the zero points of the numerical input-output Jacobian’s conditioning index throughout the workspace. The Jacobian
is formed by the numerical derivatives of the inverse kinematic equations, hence, its resultant analysis can only detect the ITS and OTS singularity types. However, the new wrench definitions only involve the ITS and OTS, hence, through the above comparison the validity of the new wrench definitions were confirmed. The presented screw theory based method, with the new wrench definitions, was shown to produce results with intuitive physical meaning regarding the causes and consequences of the singularities for these planar mechanisms with various CLSCS. Furthermore, the analysis discovered two ITS locations and a combined ITS and CTS location near the front of the workspace for the parallelogram and equal crossed CLSC variants. The configurations of the chain with the parallelogram CLSC were examined in these three locations and a clear explanation of their sources was determined. The novel attribute of these singular configurations is that one of the ITS locations occurs before the absolute workspace boundary, which is determined by the other ITS. It was shown that to reach the combined ITS and CTS location, the point of analysis on the mobile platform had to move forward through an ITS then reverse its motion direction, as was illustrated in Fig. 3.16. This singularity only occurs in mechanisms with a parallelogram CLSC constraining the mobile platform’s yaw orientation and is dependent on the selected choice of the point of analysis on the mobile platform.

The presented screw theory based method determines the exact workspace location, mechanism configuration, cause and consequence of a singularity, which is of great importance, however, the performance of a mechanism is not only affected at these singular locations, but also in the regions surrounding the singularities. This motivated the next stage in the research, which involved the modification and application of a singularity closeness measure.

Such a measure was implemented using a set of recently proposed indices that examine the motion/force transmission performance of a mechanism. These indices
were developed through the application of the power coefficient to the ITS, OTS and CTS measures, which produces the input transmission index (ITI), the output transmission index (OTI) and the constraint transmission index (CTI), respectively. The principles of screw theory enabled the generation of singularity closeness measures that are applicable to purely translational, purely rotational and combined motion parallel mechanisms. Furthermore, the indices are finite, dimensionless and frame invariant, with values ranging from zero to unity. These indices were applied to the above mentioned planar 2-DOF parallel mechanisms with various CLSCs. It was shown that the CTI remained at unity throughout the workspace, only instantaneously equalling zero where linear dependence occurs within the CLSC. The ITI measure was found to require modification of its denominator when applied to any chain with a CLSC that is not oriented parallel to the actuated joint’s axis. This involved the definition of a new method of determining the potential maximum of the reciprocal product between the ITS actuation wrench and its input twist. This new method for calculating the ITI of chains with these CLSCs was detailed, tested and verified, forming a contribution of this research.

It was demonstrated that the set of performance indices could not completely characterise the motion/force transmission abilities of mechanisms with CLSCs. Hence, a new performance index, termed the intra-chain constraint index (ICCI), was proposed based on the principles of screw theory and the power coefficient. The ICCI was shown provide important information about the effectiveness of the constraint produced by a CLSC, for which the other indices do not. The development of this index marks one of the major contributions of this thesis.

The application of the modified ITI and the OTI along with the ICCI enabled, for the first time, the complete motion/force transmission analysis of eight planar 2-DOF axis-symmetric parallel mechanism with various CLSCs, for two different parameter
sets. The analysis of these planar mechanisms provided an intuitive understanding of
the motion/force transmission effects in the presence of CLSCs and highlighted the need for the developed ICCI.

The motion/force transmission indices were then introduced to spatial mechanisms with CLSCs and exemplified on the SCARA-Tau axis-symmetric parallel mechanism. The kinematic equations describing the SCARA-Tau parallel mechanism were defined and a systematic procedure for determining the ITS and OTS actuation wrenches of this family of spatial mechanisms was proposed and detailed. The detected singular locations, using these ITS and OTS actuation wrenches, were verified against the singular locations detected by the zero points of the numeric input-output Jacobian’s conditioning index. Thereafter, the ITI, OTI, CTI and ICCI were applied to the SCARA-Tau parallel mechanism. As a result of utilising the equivalent CLSC wrenches, it was exemplified that the addition of the ICCI was essential to completely characterise the mechanism’s motion/force transmission abilities, through the inclusion of the transmission effects of the CLSCs themselves.

The application of this method to other spatial variants was found to significantly increase in complexity when applied to certain mechanism variants, namely ones with non-planar CLSCs or modified mobile platform joint arrangements. These variants can require in depth and specialised knowledge of a mechanism’s motions and constraints in order to determine the equivalent actuation wrenches, in turn making this method less accessible. This motivated the development of a general method applicable to all axis-symmetric parallel mechanisms, which is intuitive, detects all singularities, has a physical meaning and provides a complete measure of a mechanism’s motion/force transmission performance.

The developed general method is simple to implement for the output transmission analysis of these mechanisms with the presented link configurations. The developed
output transmission measure was termed the output performance index (OPI). For the OPI analysis, the wrenches along the six distal links of the spatial mechanisms, or the three for the planar mechanisms plus their three permanent planar constraints, are utilised. The output twist of each wrench is then calculated and applied with its respective wrench to the power coefficient. When only planar CLSCs are utilised, the ITI analysis follows the previously defined procedure. However, if any CLSCs are non-planar then the ITS calculation requires some knowledge about the mechanism’s constraints and as such, must be determined on a case-by-case basis. For the studied axis-symmetric parallel mechanisms with two vertical parallelograms and a non-planar CLSC, a systematic procedure was detailed for the calculation of its ITS actuation wrench. All detected singular locations were shown to correlate with those detected by the other screw theory based method. Furthermore, it was demonstrated that the OTI, CTI and ICCI were all now integrated into the OPI. A method was also detailed to identify the OTS and CTS locations directly from examining the individual OPI distribution plots of each wrench. The complete method was exemplified on a SCARA-Tau 2/2/2 variant with an obtuse trapezium CLSC in chain three, where this CLSC is not always planar and possesses a different mobile platform joint arrangement to the original SCARA-Tau, which also introduces complexities into the previous analysis methodology.

The new analysis methodology was shown to be intuitive and systematic. Its results are meaningful and determined the cause and consequence of a singularity, along with providing a clear visual representation of the motion/force transmission abilities of each chain and the overall mechanism. This enables performance changes due to modification of mechanism parameters and configurations to be clearly understood and quantitatively measured.

This research contributes to the field of kinematic performance analysis of parallel
mechanisms by extending and adapting the use of current indices, along with the generation of new indices, onto a family of planar and spatial parallel mechanisms with CLSCs. This culminated in the development of a formal procedure for the systematic singularity and motion/force transmission analysis of such mechanisms. The systematic approach enables a complete and intuitive understanding of the motion/force transmission performance and singularities of mechanisms with these CLSCs.

7.1 Directions for Future Work

This thesis applied and extended screw theory based methods for the kinematic performance analysis of axis-symmetric parallel mechanisms with CLSCs. Some natural progressions of this work include:

- An experimental investigation to quantify and compare the measured kinematic performance of a mechanism against the calculated motion/force transmission results. The experiment should establish the ability of a set of physical parallel mechanisms to generate and resist forces and moments about their calculated output twists throughout their workspace. This would enable quantification of a set of minimum acceptable ICCI, ITI and OPI values.

- The application of the ITI and OPI indices as part of a multi-objective optimisation procedure would enable the generation of an optimal set of mechanism parameters in terms of its motion/force transmission ability. The optimisation should also consider other factors such as reducing collisions between links, generating a workspace of usable size and shape, decreasing the parasitic yaw angle of the mobile platform and the minimising the total moving mass of the mechanism. Considering the application dependent nature of many of the listed
objectives, in order to generate meaningful results, the optimisation should be performed with respect to a specific application.

- An extension of the indices to also consider joint and link flexibilities, friction, gravity and the mass distribution within a parallel mechanism. This would enable a more comprehensive understanding of a mechanism’s overall motion/force transmission abilities to be generated.

- The utilisation of the indices on other parallel mechanisms with CLSCs, along with an investigation into their application on parallel mechanisms that include multiple actuators per chain, entirely passive chains or are overconstrained.
References


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