Partial State Estimation of Time-Delay Systems

by

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PARTIAL state reconstruction of dynamic systems is extremely important in practical applications, when the information on a few functions of internal states of a system are needed, instead of the full set of the states. This problem, which is referred to as Functional Observer (FO) design, is crucial for different purposes, such as monitoring, fault detection and isolation, and output feedback control of dynamic systems. In addition, time lag, is present in several applications including biological systems, communication systems, chemical processes, robotics, neural networks, etc. It is well-known that time delays, even in very small amount, can deeply affect the stability and the performance of dynamic systems in negative or sometimes in positive ways. Moreover, the analysis of time-delay systems are far more complicated than ordinary delay-free systems, because of their extremely different behaviour.

This thesis investigates the problem of functional observer design for time-delay systems. To this aim, three categories of achievements are presented. Firstly, a novel delay-dependent stability criterion is proposed for Linear Time Invariant (LTI) systems with interval time-varying delays. A novel inequality is proposed to effectively estimate the quadratic upper-bound of a class of double-integral terms that commonly appear in the stability analysis of interval time-delay systems, when using the Lyapunov Krasovskii approach. The stability criteria are obtained by applying the latter inequality, and employing other advanced techniques in analyzing the new Lyapunov Krasovskii functional.
Secondly, a practical algorithm is proposed to design minimum possible order (minimal order) functional observers for ordinary LTI systems. The new algorithm employs the concept of functional observability to update the desired functions, by augmenting auxiliary functions in the minimum required number to the original function that should be estimated. In addition, a novel approach is proposed to solve the observer equations that come up in the design procedure, which can numerically outperform some of the existing approaches.

The third class of contributions is devoted to the main topic, which is FO design for retarded systems. First, the novel problem of partial state estimation of LTI systems with multiple mixed time-varying state and input delays, is investigated. It is assumed that the state delays are known with either unknown or small derivatives, and the input delays are assumed to be unknown and thus arbitrary. The exponential convergence of the estimation error is assured in the proposed observer design algorithm. The observer parameters are obtained from a delay-dependent stability criteria, expressed in terms of a linear matrix inequality. Next, the problem of FO design for LTI systems with known interval time-varying state delays is studied. The delay-derivative is assumed to be bounded with arbitrarily large upper-limit, which is addressed for the first time. A new less conservative methodology is developed, using the Lyapunov Krasovskii approach to guarantee the asymptotic stability of the filter.

Finally, the important problem of functional observer design for unknown time-varying delay systems is investigated. A novel sliding-mode observer structure possessing an auxiliary delay is proposed to this aim. The observer parameters are obtained via a delay-dependent practical framework that takes into account the available information on the delay zone.
List of Publications

Accepted Journal Papers:


Under Review Journal Papers:


Conference Papers:


Scientific Contributions

This dissertation proposes a part of the author’s research contributions during his PhD studies. This research has resulted in over 17 published or under-review papers, including more than 11 peer-reviewed Journal articles, and 6 high quality international IEEE conference papers, which are detailed in the List of Publication’s section. To be more specific, Chapter 3 is reported in the paper number (8) of the list. Moreover, Chapter 4 is published in Papers (1) and (16); and Chapters 5-7 are given in Papers (5), (7), and (10), respectively.

Due to space limitations, the author could not put all of the achieved contributions. However, enthusiastic readers are highly recommended to refer to Papers (3) and (14) for information on our contributions on designing minimal-order unknown-input functional observer for LTI systems. Moreover, Paper (15) describes an application of functional observers to pre-compensated systems that was considered during my PhD. In addition, Papers (2), (4), (13) and (17) contain additional valuable material and contributions on functional observer design for time-delay systems. Furthermore, Papers (6) and (9) are the results of further studies on the stability analysis of interval time-varying delay systems, respectively performed on generalized neural networks and neutral systems with mixed time-varying delays and nonlinear disturbances, which are more general than the system investigated in Chapter 3. Finally, Paper (11) is an interdisciplinary work collaborated with the Bharathiar University, on the application of time-delay systems in the robust control of vehicle suspension systems.
Acronyms

**FO**: functional observer;

**UIFO**: unknown-input functional observer;

**LTI**: linear time-invariant;

**LTV**: linear time-varying;

**ODE**: ordinary differential equation;

**FDE**: functional differential equation;

**DDE**: delay differential equation;

**LKF**: Lyapunov Krasovskii functional;

**DD**: delay-dependent;

**LMI**: linear matrix inequality;

**MAUBD**: maximum allowable upper-bound of delay;

**GA**: genetic algorithm;
I would like to use this opportunity to express my profound gratitude to my Ph.D supervisors Dr Hamid Abdi and Professor Saeid Nahavandi, without whom this research was not possible. Their friendly support, encouragements, and helpful advices have enlightened my way throughout this journey. Dr Abdi held regular and fruitful meetings with me throughout my studies at Deakin University, and beside being a great mentor, he has been a terrific friend.

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Chapter 1

Introduction

1.1 Introduction to Time-Delay Systems

Time delay can be present in several practical systems, such as network control systems [1, 2], process control [3, 4], neural networks [5–7], and biological systems [3]. Time lag can create several problems and challenges in the stability analysis, controller and observer design of linear or nonlinear systems. Mathematical modelling, analysis and synthesis of time-delay systems are different from ordinary systems. The main difference between an ordinary systems and a time-delay system comes from the fact that ordinary systems are modelled as Ordinary Differential Equations (ODEs), while time-delay systems are modelled as Functional Differential Equations (FDEs) (see e.g., [3, 8, 9] and the references therein). This area of research has been the focus of attention for several years (see e.g., [4, 8, 10–12]). However, the topic is still subject to plenty of open problems related to the effects of time-delay on different systems behaviour.

To gain a better feeling for the after-effects of delays on a system, a tangible example is provided. It can be simply realized that stabilizing a pencil on a finger is a very difficult
task. Only a small proportion of people are able to accomplish this task, after rigorously practising. The reason for this inconvenience partially lies in the disability of our internal control systems to provide suitable control actions to stabilize the pencil (Fig. 1.1). This issue is due to the time-delay between our observation and our control actions. The dynamic equation of motion of the pen can be written as

\[ ml\ddot{\theta}(t) - \gamma l\dot{\theta}(t) - mg\sin(\theta(t)) = f(\theta(t - \tau), \dot{\theta}(t - \tau)), \]  

(1.1)

where \( m \) is the mass, \( l \) is the length, and \( \theta \) is the deviation from the vertical position of the pencil. In addition, \( g \) is the gravitational acceleration, \( \gamma \geq 0 \) is the damping coefficient, and \( \tau \) is the input time delay due to reaction of the human operator. The closed-loop dynamics of the pencil shows that the control action is a function of both magnitude and the rate of the deviation parameter \( \theta \), which suffers from a small reaction delay. This delay is sufficient for destabilizing the closed-loop control system.

Time delay systems have plenty of applications in teleoperation, robotic systems, network control systems, industrial processes such as chemical reactors, combustion engines, population models, biological models, financial systems. In such systems, it is essential to consider time delays to have an accurate model of the system, and being able to analyse and synthesise it properly. Due to space limitations, this text does not further illustrate the applications and only focuses on the mathematical challenges that usually arise in investigating retarded systems.

Time delays can be present in the states, inputs, outputs, or a combination of these points of a block diagram of a dynamic control system (Fig. 1.2). Each case brings about new challenges and problems that invoke the necessity of a vast amount of research. If the delay is an inherent property of the system, it will appear in all or a number of the states of the system, and is thus called a state-delay (see e.g. [8,9,14,15], and the references therein).
1.1 Introduction to Time-Delay Systems

If it is present in the actuators of the system, then it is called an input-delay [16–21]. Finally, if time-delay appears as the sensors’ latency, then it is called an output or measurement delay [20, 22–24].

The model of the system is also potentially considered as another challenge. The majority of the research in this area has been devoted to LTI systems (see e.g. [9, 25–28], and the references therein). Recently, nonlinear systems with specific structures are being investigated [29–34]. However, since analyzing nonlinear time-delay systems is rather complicated and needs advanced mathematical manipulations and knowledge, this area is still not well-developed and is subject to further development by mathematicians in control theory. According to this author’s understandings, nonlinear systems can be more conveniently be analyzed and synthesized in the time domain. The best tool for this goal has
been the Lyapunov theory. This renown and extremely handful approach, is extended to
time-delay systems that are indeed infinite-dimensional, by two Russian mathematicians
named Krasovskii and Razuminikhin. The theorems that they developed are called Lyapunov Krasovskii and Lyapunov Razuminikhin theorems, respectively [3, 8, 9, 35].

Moreover, the delay can be time-varying, instead of being constant, or even with random values [21, 28, 36–40]. This factor has created several challenges, and dealing with this problem still needs sophisticated theories and mathematics. Another challenge related to the delay itself is its uncertainty. If the time-delay is unknown or not deterministic, which is very common in a wide range of real applications, the problem becomes even more complicated, and has been barely addressed in the literature (see e.g. [41–44], and the references therein). This particular problem is indeed different from the uncertainty of the dynamic parameters of the system, and cannot be handled by ordinary identification techniques and adaptive control methods applied to linear or nonlinear systems, which are modelled as ODEs.
The investigation of any time-delay system can always be classified in one of the following categories: 1) stability analysis 2) controller design 3) observer design 4) output feedback control, or observer based controller design [24, 45–49]. Moreover, in the stability analysis, and the controller or observer design for a hereditary system, one can perform a delay-dependent, or a delay-independent analysis. In the delay-dependent analysis and synthesis [37, 50–53], the admissible values of the upper-bound, the lower bound, and the rate of the delay are crucial, and should be properly estimated using particular techniques, such as those employed in Lyapunov Krasovskii approaches. The upper-bound value of the delay value is usually obtained from solving a Linear Matrix Inequality (LMI), obtained from the Lyapunov Krasovskii analysis, and commonly the obtained result is conservative. In other words, there always exists the opportunity to increase the upper-bound of the delay, obtained from the latter approach [37, 39, 40]. On the other hand, the delay-independent analysis and synthesis of time-delay systems can be relatively more conservative. This approach is solely applicable for the case that the admissible delay zone can be arbitrarily large and even infinite. Apart from the conservative results, the delay-independent controllers may lead to high gain and unimplementable control signals. Hence, this type of controllers has recently lost their importance in the research community.

In addition, there can be a single or multiple discrete delays present in the states, inputs, or the outputs of the system. In the multiple-delay case, the delays values can be independent of each other or they might be multiplicities of the minimum non-zero delay in the system. In the latter case, the delays are called commensurate delays [9, 52]. Commensurate delays are usually considered, when a distributed delay is approximated as a summation of discrete delay terms [52].

Overall, from the above descriptions, it can be concluded that there are plenty of problems about time-delay systems that are caused due to the delay effects. Any control problem that has been solved for ordinary un-delayed systems can be widely extended to time-delay
systems, and it is hard to see an ending for this area of research. To further clarify the above discussions, a number of interesting works in this area are summarized in the following.

LTI systems have been investigated in several works, and from different points of view. Since these systems are relatively simpler to be analyzed in the Laplace domain, particularly in the frequency domain, the main proportion of the accomplished studies on time-delay systems have been devoted to this class of systems. Fu et al. [54], consider the delay-dependent robust stability analysis and synthesis of LTI systems with single discrete constant state delay. Integral quadratic constraint approach is employed in the stability analysis and the controller design. The designed controller is static, and the control gain is attained from solving an LMI problem. In [37], reachable sets for linear time-delay systems with time-varying delays are defined and an elliptic bound for that is derived using the Lyapunov Razuminikin approach. Moreover, both delay-dependent and delay-independent results are presented. It is however assumed that the time delay must be lower bounded by zero, and the initial function should be zero that are some restrictions.

Lin and Fang [55, 56], consider controller design for LTI systems with pure input delay (i.e., there is no non-delayed input signal). The delay is single and known. It is shown that if the open loop system has positive poles, the stability of the system would always be delay-dependent. It is an important point, which emphasizes that an unstable nominal LTI system that is delay-free cannot be stabilized by any control action, if the action is delayed with an arbitrarily large value. This characteristic can also be justified by realizing that an unstable system can diverge in a finite time, if the control input does not properly act on it. Both delay-dependent and delay-independent controllers are designed in [56].

Fridman [39] deals with the stability analysis of linear continuous time sampled-data control systems. The sampling period is assumed to be time-varying, and there is an upper-bound on it. The important point is that a sampled-data input with time-varying sampling period can be modelled as a continuous time-varying input-delay. In addition, the stability
of LTI systems with single bounded time-varying state delay, with arbitrary derivative, is analyzed in [28].

Network control systems that are a class of sampled-data systems are studied in [57]. These systems can be generally modelled as continuous linear or nonlinear systems with discretized input, which can be modelled as time-varying delayed input. The step-size of the discretization induces the time delay, and it is assumed to be bounded with known lower and upper bounds. LMI conditions that guarantee the asymptotic and the exponential stability of the closed loop system under asynchronous sampling, which is equivalent to unknown time-varying delay, are established in [57], using the Lyapunov Razumikhin approach.

1.2 An Introduction to Functional Observers

Functional observers (FOs) are the generalized class of Luenberger observers that aim to estimate a number of functions of the states of the system, instead of the full set of the states (see e.g. [58–62] and the references therein). This feature induces more complexities in the observer design procedure, but can help in reducing the order of the observer, as well as lessening the observability/detectability requirements to less restrictive conditions [59, 63–65]. Under a special situation, when the functional distribution matrix is fixed at the identity matrix, the problem of ordinary full-order observer is enclosed. The notion of a functional observer was first defined in Luenberger’s pioneering work on observers for linear multi-variable systems [66]. Although Luenberger type full-order observer has been well studied for several years, FO is still an interesting topic for research (see e.g. [63, 67–69]).

Functional observers have been studied for linear time-invariant (LTI) [58,62,70], linear time-varying (LTV) [71–73], or even nonlinear systems [60,62,74]. It is cost effective to
use functional observers whenever the number of the states of a system are considerably larger than the number of the outputs and we do not need to observe all of them. FOs are very useful in output feedback control, system monitoring and fault detection and isolation of dynamic systems, such as interconnected systems and power systems (see e.g. [75–85]). For instance, in the observer-based control of interconnected systems or descriptor systems [86, 87], it has been shown that functional observers are more cost effective and applicable than full-order or reduced order Luenburger observers. More importantly, in the static output-feedback controller design problem, the control signal $u(t) = Kx(t)$ should be calculated, instead of the whole set of the states of the system. When the control gain $K$ is in the space spanned by the output distribution matrix $C$, i.e. the control signal is a function of the measurement outputs, then there is ideally no need for an observer for the system, and the observability requirement is thus not necessary to be satisfied. However, if the system is not detectable, and for control aims the designer needs more information on the states of the system, rather than only the output measurements, then functional observers can be utilized as an effective solution (see e.g. [68, 79, 88]). Moreover, in fault detection and isolation of a large scale electromechanical system, mostly the states related to the current, velocity, and acceleration are desired. As a result, this area of research have received an increasing attention in the recent years.

One of the critical problems in the majority of the functional observer design approaches is solving a set of generalized Sylvester equations [62, 89]. Aldeen and Trinh [90] design a functional observer for linear time invariant systems. Here, it is required that the number of the estimated states, $p$, to be larger than a specified value ($p > \frac{m(n-r)}{r}$, where $n$ is the number of states, $m$ is the number of inputs, and $r$ is the number of outputs). In [58] a functional observer is designed for an LTI system, and the necessary and sufficient conditions for solving the generalized Sylvester equations is proposed. This paper is important, since the necessary and sufficient conditions for the existence of a minimum-order FO for
LTI systems are obtained, while all of the previous papers in this regards had only proposed sufficient conditions.

It is important to remark that when the minimum-order FO does not exist, it might be possible to increase the order of the observer in an appropriate way (see [62, 69], and also Chapter 4). If the order of the observer is increased in the minimum required number, then it is called minimal-order or minimal functional observer. The very first attempt to design a minimal FO in a general way is proposed in [70]. Nevertheless, the approach does not guarantee the stability of the observer. Fernando and Trinh [59] define the concept of functional observability/detectability for LTI systems. The necessary and sufficient conditions for functional observability/detectability is proposed in Jennings et al. [64]. The concept is crucial for investigating the existence of an asymptotically stable (or asymptotic) functional observer for an LTI system.

Unknown-Input Observers (UIOs) have been the focus of research for several years [91–94]. This is due to the wide range of applications that already exist for this theory, like fault detection and observer-based control of electromechanical systems that are subjected to measurement noise, uncertainties, and disturbances [84, 95, 96]. In this line, the objective could be disturbance-decoupled observer design [97, 98], and/or the estimation of the disturbance signals [84, 99]. Analogous to UIOs, unknown-input functional observers [62, 100, 101] are far more useful than functional observers in practical applications. This application is particularly highlighted in Chapter 5, where input-delay terms are treated as unknown-inputs. The author has had more contributions in this area [65, 68, 69], which are not included in this thesis due to space limitations.

Nevertheless, FO design for time-delay systems has been fairly overlooked, such that a great proportion of the few existing works in this area only consider single constant delay in the states [74, 101–107]. Darouach [103] designs a functional observer for LTI systems with bounded time varying but known delay. The problem of functional observer design
for linear time delay systems with slow-varying state delay and unknown-input is recently investigated [105, 108]. Ezzine et al. [87] study the design of minimum-order functional observers for LTI descriptor systems with constant state delays and unknown-inputs in both the state and measurement equations.

Plenty of open problems exist on this topic, including delay-dependent FO design for systems with multiple state and input delays, considering interval time-delay in the states, and more importantly considering unknown state delays in designing delay-dependent functional observers. The latter topics have been in-depth investigated in Chapters 5, 6, and 7, respectively.

1.3 The Structure of the Thesis

The dissertation is structured as follows. In Chapter 2, some basic concepts, lemmas and definitions related to time-delay systems and their stability in the time domain, as well as the Lyapunov theory for hereditary systems, are presented. Next, in Chapter 3, one of our contributions on the stability analysis of interval time-varying delay systems using the Lyapunov Krasovskii approach is illustrated and justified.

Chapter 4 investigates the problem of minimum possible order functional observer design for ordinary LTI systems, and proposes a new effective methodology to design the observer. The findings of this chapter are crucial for the next Chapters 5-7.

Thereafter, Chapter 5 demonstrates a methodology to design delay-dependent exponentially stable functional observers for LTI systems with multiple time-varying state and input delays. The state delays are mixed, bounded, and two scenarios for their rates are considered: 1- slow-varying and 2- unknown (or arbitrary) rates. Moreover, the input delays are assumed to be unknown in the design procedure.

Next, the new problem of delay-dependent FO design for LTI systems with interval
time-varying delays of bounded rates not limited to be less than unity, is addressed in Chapter 6. Inspired by the theoretical findings of Chapter 3, an advanced Lyapunov Krasovskii functional and contemporary estimation techniques are employed to obtain less conservative asymptotic stability criteria for the observer.

In Chapters 5 and 6, the state delays are assumed to be available to the designer, which can be violated in several applications. This critical issue is addressed in Chapter 7, wherein a new sliding-mode FO structure is demonstrated that contains an auxiliary time-varying delay. A practical auxiliary delay-dependent algorithm is proposed to design the observer parameters, using the Lyapunov Krasovskii methodology. Finally, the thesis is concluded in Chapter 8, and some future directions are given.

It is remarked that due to highly mathematical nature of the thesis, several variables are used in each chapter. Hence, to avoid any possible conflict and further complexities, the variables in each chapter are considered independent of those of the other chapters. In addition, the notations used in each chapter are explicitly illustrated in the preliminaries section of that chapter. This point hopefully makes it more convenient for the readers to follow the story of each individual chapter.
Chapter 2

Principles of Time-Delay Systems

2.1 Modelling of Time-Delay Systems

There are three basic methods in modelling time-delay systems:

1. Modelling over rings of operators [9, 109, 110];

2. Modelling as differential equations on an infinite dimensional abstract linear space [9, 111];

3. Modelling using functional differential equations [3, 8];

The first two methods are mathematically involved and are only applicable to LTI systems with constant delays. These methods do not have applicability to LTV or nonlinear systems, and need explicit knowledge in abstract linear algebra (advanced algebra) and ring theory. Hence, they have not become popular methods for modelling time-delay systems. As a result, the first two methods in mathematical modelling of time delay systems are not discussed in this dissertation. On the other hand, most of the recent existing works in
the literature has adopted the third approach for the mathematical modelling of the system [52]. The mathematical model of a retarded (say delayed) dynamic system, described as functional differential equation, is also called *retarded functional differential equation* (RFDE) [8]. To be more specific, the definitions of a functional differential equation (FDE), and a delay differential equation (DDE) are separately described in the sequel.

**Definition 2.1.1** ([8, 13]). A functional differential equation is an equation for an unknown function (e.g., \( x(\cdot) \)) with different arguments, wherein the derivative of the unknown function is a function of itself, and possibly its derivatives with different arguments.

The following differential equations are samples of functional differential equations.

- \( \dot{x}(t) = x(t - h) + x(t) \),
- \( \dot{x}(t) = t^2 x(t) - \dot{x}(t - 1) \),
- \( \ddot{x}(t) = -\dot{x}(t) + \sin(x(t)) + x(t - 5) + x^2(t - 3) \),
- \( \dot{x}(t) = x(t) - x(t/2) \).

**Definition 2.1.2** ([8, 13]). A retarded functional differential equation (RFDE), or a delay differential equation (DDE), is a functional differential equation, wherein the highest derivative of the unknown function (\( x(\cdot) \) for example) only appears with one argument, and this argument is not less than the other arguments of the unknown function, or its lower order derivatives in the equation.

In the example (2.1), the first and the third FDEs are also RFDEs. The fourth FDE is not an RFDE when \( t < 0 \), and the second FDE is actually a *neutral* system that will be explained later. The general form of a RFDE is as below,
2.1 Modelling of Time-Delay Systems

\[ \dot{x}(t) = f(t, x_t) \]
\[ x_{t_0}(\theta) = \phi(\theta), \quad \theta \in [t_0 - h, t_0] \]

(2.1)

where \( x_t(\theta) = x(t + \theta) \quad \forall \theta \in [t - h, t] \), and \( t_0 \) is the initial time. Moreover, the function \( f : \Omega \subset \mathbb{R} \times C([-h, 0], \mathbb{R}^n) \rightarrow \mathbb{R}^n \) is continuous, and (locally) Lipschitz with respect to the second variable, and \( f(t, 0) = 0 \) for all \( t \). In addition, \( C([-U, \mathbb{R}^n]) \) is the space of continuous functions mapping from \( U \) into \( \mathbb{R}^n \) with the topology of uniform convergence.

It is worthwhile to mention that in Equation (2.1), the state is not a single point \( x(t_1) \) at time \( t_1 \in \mathbb{R} \). Indeed it is the function \( x_t(\theta) \), wherein \( \theta \in [t_1 - h, t_1] \). This is the significant difference between an ordinary differential equation (ODE) and a functional differential equation. This feature results in the infinite-dimensional characteristic of an FDE or equivalently an RFDE. Analogously, the initial condition is not in a single point, but is a function on the interval \( [t_0 - h, t_0] \), which is prescribed as \( \phi : [t_0 - h, t_0] \rightarrow \mathbb{R}^n \).

The initial function can be a continuous map \( (\phi \in C([-t_0 - h, t_0], \mathbb{R}^n)) \), or it can have some bounded discontinuous points. An RFDE has a forward solution (for \( t \geq t_0 \)), and a backward solution (for \( t \leq t_0 \)). Generally, a solution to the RFDE (2.1) is defined as follows.

**Definition 2.1.3 ([8]).** A function \( x \) is a solution of (2.1) on \( [\sigma - h, \sigma + a] \), if given \( \sigma \in \mathbb{R} \) and \( a > 0 \) we have \( x \in C([-\sigma + h, \sigma + a], \mathbb{R}^n) \) for all \( t, x_t \in \Omega \), and \( x(t) \) satisfies RFDE (2.1) for all \( t \in [\sigma - h, \sigma + a] \). Moreover, for given \( \sigma \in \mathbb{R} \) and \( \phi \in C([-h, 0], \mathbb{R}^n) \), \( x(\sigma, \phi, f) \) is said to be a solution of (2.1) with the initial condition \( \phi \) at \( \sigma \), or simply a solution through \( (\sigma, \phi) \), if there exist a constant \( a > 0 \), such that \( x(\sigma, \phi, f) \) is a solution of (2.1) on \( t \in [\sigma - h, \sigma + a] \) and \( x_{\sigma}(\phi, f) = \phi \).

Furthermore, suppose that \( U \) is an open set, and \( f \in C(U, \mathbb{R}^n) \). A backward continuation of the solution through \( (\sigma, \phi) \) is a function \( x \in C([\sigma - h - \alpha, \sigma], \mathbb{R}^n), \alpha > 0 \), such that \( x_{\sigma} = \phi \), and \( \forall \sigma_1 \in [\sigma - \alpha, \sigma], (\sigma_1, x_{\sigma_1}) \in \Omega \), and \( x \) is a solution of (2.1) on \( [\sigma_1 - h, \sigma] \).
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through \((\sigma_1, x_{\sigma_1})\).

Similar to ordinary differential equations if \(f(\cdot, \cdot) \in \mathcal{C}(\Omega, \mathbb{R}^n)\), then there exists a solution of (2.1) through \((\sigma, \phi) \in \Omega\). Moreover, if \(f(\cdot, \cdot)\) is Lipschitz in its second argument in each compact set in \(\Omega\) (locally Lipschitz), then the solution is unique. However, unlike ODEs, the backward continuation of an RFDE is not necessarily unique. It means that two RFDEs might have different initial functions but their solutions might coincide after some time \(t_1 \geq t_0\). The uniqueness of the backward continuation depends on the behaviour of \(f(\cdot, \cdot)\) with respect to the initial function \(\phi(\cdot)\). Indeed, the backward solution of (2.1) is unique if an additional condition called atomicity of \(f\) is also satisfied (see e.g. [8, 52]).

An RFDE (2.1) is said to be autonomous or time-invariant if \(f(t, \phi) = g(\phi)\). It is linear if \(f(t, \phi) = L(t)\phi + h(t)\), where the operator \(L(\cdot)\) is linear. Moreover, it is linear time-invariant if \(f(t, \phi) = L\phi\).

A neutral functional differential equation (NFDE) is also a delay differential equation, which involves the highest derivative of the unknown function for both time \(t\) and past time(s) \(t - h, \ h > 0\). An NFDE in general is represented as,

\[
\dot{x}(t) = f(x_t, t, \dot{x}_t) \tag{2.2}
\]

A sample neutral system is shown in Example 2.1.

A retarded LTI system in general can be demonstrated as follows,

\[
\dot{x}(t) = \sum_{i=0}^{k_i} (A_i x(t - h_i) + \sum_{j=1}^{k_j} B_j u(t - h_j)) \\
+ \sum_{i=1}^{k_i} \int_{t-\tau_i}^{t} G_i(\theta)x(\theta) d\theta + \sum_{s=1}^{k_s} \int_{t-\tau_s}^{t} H_s(\theta)u(\theta) d\theta \\
y(t) = \sum_{i=1}^{k_i} C_i x(t - h_i) + \sum_{j=1}^{k_j} \int_{t-\tau_j}^{t} N_j(\theta)x(\theta) d\theta \tag{2.3}
\]

where \(x \in \mathbb{R}^n\) is the state, \(u \in \mathbb{R}^m\) is the input, and \(y \in \mathbb{R}^p\) is the output signals of the system. Moreover, \(A_i, B_j, \text{ and } C_i\) are constant matrices with appropriate dimensions. In
addition, \( x(t-h_i) \) and \( u(t-h_j) \) are the state and the input discrete delay terms, respectively. Furthermore, \( \int_{t-\tau_l}^{t} G_l(\theta)x(\theta)d\theta \) and \( \int_{t-\tau_s}^{t} H_s(\theta)u(\theta)d\theta \) respectively represent the state and the input distributed delay terms.

**Remark 2.1.1.** The definitions of controllability and observability of time-delay systems (RFDE systems) are quite different from ODE systems. The basic definitions and the criteria to examine them are illustrated in [111–119] and the references therein. It is clear from these references that there are several definitions related to the controllability and the observability of a time-delay system, but still the controllability and observability are dual concepts in time-delay systems. Among the definitions of controllability (observability) of time-delay systems, the following can be pointed out (a dual of each definition exists for observability): 1) initial controllability; 2) \( \mathbb{R}^n \) controllability; 3) asymptotic controllability; 4) final controllability; 5) infinite time controllability; 6) spectral controllability; 7) essential controllability; 8) hyper controllability; 9) controllability over the field of rational functions \( \mathbb{R}(d) \); 10) controllability over the ring of polynomials \( \mathbb{R}[d] \); 11) weak controllability; and 12) strong controllability [52, 111–114, 118].

Some of the above definitions are equivalent, while a number of them are stronger than the others. However, according to [120, 121], the best definition for controllability or observability, for the aim of designing a suitable controller or observer for the system, is the spectral controllability (observability), which is stronger than the weak controllability (observability), and weaker than the strong controllability (observability). Similar to ODE systems, the stabilizability and the detectability concepts can be defined as milder conditions of controllability and observability, respectively.
2.2 Preliminaries

In the sequel a number of concepts, lemmas, and theorems that are crucial in this area of research are briefly explained. First, some important notations are given.

**Notations:** Throughout the section, $X \succ 0$ and $X \succeq 0$ respectively state that the symmetric matrix $X$ is positive definite (PD) and positive semi-definite (PSD); $\mathbb{R}^n$ is the $n$-dimensional vector space; $\mathbb{R}^+$ is the set of non-negative real numbers; $\mathbb{R}^{n \times m}$ and $\mathbb{S}^n$ respectively are the spaces of $n \times m$ real matrices and $n \times n$ real symmetric matrices. Moreover, $C_n(\Omega)$ is the space of continuous functions, mapping from $\Omega$ to $\mathbb{R}^n$ with the topology of uniform convergence. Further, $*$ in a symmetric matrix stands for the transpose of the associated off-diagonal block element, superscript “$T$” denotes the transposition operator, and $\otimes$ is the Kronecker product operator, which carries the following properties for any real matrices $A = [a_{ij}] \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{l \times d}$, $C \in \mathbb{R}^{n \times r}$, and $D \in \mathbb{R}^{d \times f}$:

(i) $A \otimes B = [a_{ij}B] \in \mathbb{R}^{ml \times nd}$,
(ii) $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$,
(iii) $(AC) \otimes (BD) = (A \otimes B)(C \otimes D)$.

Next, some important inequalities are introduced.

**Lemma 2.2.1** ([44]). *Young’s inequality:*

\[
xy \leq \frac{\gamma}{2}x^2 + \frac{1}{2\gamma}y^2, \quad \forall \gamma > 0
\]  \hspace{1cm} (2.4)

**Lemma 2.2.2** ([44]). *Cauchy-Schwarz inequality:*

\[
\int_0^1 u(x)g(x)dx \leq \sqrt{\int_0^1 u(x)^2dx} \sqrt{\int_0^1 g(x)^2dx}
\]  \hspace{1cm} (2.5)
Lemma 2.2.3 (Schur complement lemma [4, 122]). Given matrices $A$, $B$, and $C$ of appropriate dimensions, the following statements are equivalent:

I. $\begin{bmatrix} A & B \\ * & C \end{bmatrix} \succ 0$ \hspace{1cm} (2.6)

II. $C \succ 0$, and $A - BC^{-1}B^T \succ 0$.

Lemma 2.2.4 (Jensen’s inequality [4, 123]). For any scalars $a, b, c, d \in \mathbb{R}$, $\eta_1 \triangleq b - a$, $\eta_2 \triangleq b - c$, and any constant real symmetric matrix $M \succ 0$ of appropriate dimension, if $a < c$, $a < d$, $\eta_1 > \eta_2$, the following inequalities are satisfied upon the existence of the integrals:

\[ \int_a^d x^T(\omega)Mx(\omega)d\omega \geq \frac{1}{d-a} \left( \int_a^d x^T(\omega)d\omega \right) M \left( \int_a^d x(\omega)d\omega \right) \tag{2.7} \]

\[ \int_a^c \int_s^b x^T(\omega)Mx(\omega)d\omega ds \geq \frac{2}{\zeta_2(\eta_1, \eta_2)} \left( \int_a^c \int_s^b x^T(\omega)d\omega ds \right) M \left( \int_a^c \int_s^b x(\omega)d\omega ds \right) \tag{2.8} \]

\[ \int_a^c \int_s^b \dot{x}^T(\omega)M\dot{x}(\omega)d\omega ds \geq \frac{2}{\zeta_2(\eta_1, \eta_2)} \left( \eta_3 x^T(b) - \int_a^c x^T(\omega)d\omega \right) \]

\[ \times M \left( \eta_3 x(b) - \int_a^c x(\omega)d\omega \right) \tag{2.9} \]

\[ \int_a^b \int_s^0 \dot{x}^T(t+u)M\dot{x}(t+u)dudsds_1 \geq \frac{6}{\zeta_3(b,a)} \chi^TM\chi \tag{2.10} \]

where $\eta_3 \triangleq c - a$, $\zeta_i(a, b) \triangleq a^i - b^i$, $i = 1, 2, \cdots$, and

$\chi \triangleq 0.5\zeta_2(b,a)x(t) - \int_a^b \int_s^0 x(t+u)duds$.

Lemma 2.2.5 (Wirtinger’s inequality [40]). Given a constant matrix $M \succ 0 \in \mathbb{S}^n$, the
following inequality is satisfied for any continuous function \( x : [a, b] \rightarrow \mathbb{R}^n \):

\[
\int_a^b x^T(u) M x(u) du \geq \frac{2}{b-a} \left[ \int_a^b x(u) du \right]^T \Theta(a, b) \left[ \int_a^b x(u) du \right]
\]

(2.11)

where \( \Theta(a, b) \triangleq \begin{bmatrix} 2 & -\frac{3}{b-a} \\ * & \frac{6}{(b-a)^2} \end{bmatrix} \otimes M \).

Remark 2.2.1. Using the relation \( \int_a^b x^T(\omega) d\omega - \frac{2}{b-a} \int_a^b x^T(\omega) d\omega du = -\int_a^b x^T(\omega) d\omega + \frac{2}{b-a} \int_a^b x^T(\omega) d\omega du \), Inequality (2.11) can be rewritten as

\[
\int_a^b x^T(u) M x(u) du \geq \frac{2}{b-a} \left[ \int_a^b x(u) du \right]^T \Theta(a, b) \left[ \int_a^b x(u) du \right]
\]

(2.12)

Lemma 2.2.6 (Weighted integral inequality [124]). For any given matrix \( X \in \mathbb{S}^m \), a scalar \( \alpha > 0 \), and a vector function \( \rho(\cdot) \in C_m([a, b]) \), the inequality below is satisfied

\[
\int_a^b e^{\alpha(s-b)} \rho^T(s) X \rho(s) ds \geq \zeta^T \left[ \begin{array}{ccc} \frac{\alpha}{\gamma_0} + \frac{\alpha}{\rho_0} & -\alpha^2 \frac{\gamma_0}{\rho_0 \gamma_1} \\ * & \frac{\rho_0}{\rho_0 \gamma_1} & \frac{\alpha^3}{\rho_0 \gamma_1} \end{array} \right] \otimes X \zeta \]

(2.13)

where \( \zeta^T \triangleq \left[ \int_a^b \rho^T(s) ds \quad \int_a^s \int_a^b \rho^T(s_1) ds_1 ds \right] \), \( \gamma_0 \triangleq e^{\alpha(b-a)} - 1 \), \( \gamma_1 \triangleq e^{\alpha(b-a)} - \alpha(b-a) - 1 \), and \( \rho_0 \triangleq \frac{\gamma_0}{\gamma_1} \left( \gamma_0^2 - \alpha^2 (b-a)^2 e^{\alpha(b-a)} \right) \).

Lemma 2.2.7. (Reciprocally convex combination technique [10]) If functions \( f_1, f_2, \cdots, f_N : \mathbb{R}^n \rightarrow \mathbb{R} \) are positive in an open subset \( \mathcal{D} \) of \( \mathbb{R}^n \), then the reciprocally convex combination
of \( f_i \)'s over \( D \) satisfies

\[
\min_{\{\alpha_i | \alpha_i > 0, \sum \alpha_i = 1\}} \sum_i \frac{1}{\alpha_i} f_i(t) = \sum_i f_i(t) + \max \sum_{i \neq j} g_{i,j}(t)
\]

subject to

\[
\begin{cases}
ush{g_{i,j}(t)}: \mathbb{R}^n \to \mathbb{R}, \quad \begin{bmatrix} f_i(t) & g_{i,j}(t) \\ \ast & f_j(t) \end{bmatrix} \succeq 0
\end{cases}
\]

**Remark 2.2.2.** Lee et al. [125] very recently have extended the reciprocally convex combination lemma (Lemma 2.2.7) for quadratic functions to an \( l \)th order case. Accordingly, for any positive scalars \( \alpha \) and \( \beta \), such that \( \alpha + \beta = 1 \), the following inequality always holds:

\[
\frac{1}{\alpha^l} x^T(t) X x(t) + \frac{1}{\beta^l} y^T(t) Y y(t) \geq \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}^T \begin{bmatrix} X & \sum_{i=1}^l H_i \\ \ast & Y \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix},
\]

subject to:

\[
\begin{bmatrix}
\binom{l}{i} X H_i \\
\ast & \binom{l}{i} Y
\end{bmatrix} \succ 0,
\]

where \( \binom{l}{i} = \frac{l!}{i!(l-i)!} \).

**Lemma 2.2.8.** Let \( G: [a, b] \to \mathbb{R}^{n \times n} \) be a convex function of its argument. Then, \( G(t) \prec 0 \), \( \forall t \in [a, b] \), if and only if \( G(a) \prec 0 \) and \( G(b) \prec 0 \).
Proof. The proof follows from the definition of a convex function [126], which implies that if $G$ is a convex function of $t \in [a, b]$, then $G(t) \preceq \alpha \in \mathbb{R}$ for all $t \in [a, b]$ if and only if $G(a) \preceq \alpha$ and $G(b) \preceq \alpha$.

Now, a few crucial definitions and theorems are introduced that shed some effective light into the contributions in the forthcoming chapters. Consider a function $f : \mathbb{R}^+ \to \mathbb{R}^n$. The $L_p$ norm for $p \in [1, \infty]$, is defined as

$$
\|f\|_p = \begin{cases} 
(f_0^\infty |f(t)|^p dt)^{1/p}, & p \in [1, \infty) \\
\sup_{t \geq 0}|f(t)|, & p = \infty
\end{cases}
$$

(2.14)

Definition 2.2.1 ([127]). A continuous function $f : [0, a) \to \mathbb{R}^+$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $f(0) = 0$. In addition, it is of class $\mathcal{K}_\infty$ if $a = \infty$ and $f(t) \to \infty$ when $t \to \infty$.

Definition 2.2.2 ([127]). A continuous function $\beta : [0, a) \times \mathbb{R}^+ \to \mathbb{R}^+$ is of class $\mathcal{KL}$ if for each fixed $t$, the mapping $\beta(s, t)$ as a function of $t$ is of class $\mathcal{K}$, and for each fixed $s$, $\beta(s, t)$ is decreasing as a function of $t$, and $\lim_{t \to \infty} \beta(s, t) = 0$. Moreover, $\beta(\cdot, \cdot)$ is said to be of class $\mathcal{KL}_\infty$ if in addition the mapping $\beta(s, t)$ is of class $\mathcal{K}_\infty$ for fixed $t$, as a function of $s$.

Consider the following non-autonomous general system

$$
\dot{x} = f(x, t)
$$

(2.15)

where $f : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ is locally Lipschitz ([127]) in $x$, and piecewise continuous in $t$.

Definition 2.2.3 (Stability [44]). The trivial solution (zero solution) of the system (2.15) is
2.2 Preliminaries

- **Uniformly stable** if there exist a class $K$ function $\gamma(\cdot)$ and a positive constant $c$, independent of $t_0$, such that

  \[ |x(t)| \leq \gamma(|x(t_0)|) \quad \forall t \geq t_0 \geq 0, \forall |x(t_0)| < c; \quad (2.16) \]

- **Uniformly asymptotically stable** if there exists a class $K\mathcal{L}$ function $\beta(\cdot, \cdot)$, and a positive constant $c$, independent of $t_0$, such that

  \[ |x(t)| \leq \beta(|x(t_0)|, t - t_0) \quad \forall t \geq t_0 \geq 0, \forall |x(t_0)| < c; \quad (2.17) \]

- **Exponentially stable** if (2.17) is satisfied with $\beta(s, t) = kse^{-\alpha t}$, $k > 0, \alpha > 0$;

- **Globally uniformly stable** if (2.16) is satisfied for any initial state $x(t_0)$, with $\gamma(\cdot) \in K\infty$;

- **Globally uniformly asymptotically stable** if (2.17) is satisfied with $\beta(\cdot, \cdot) \in K\mathcal{L}_\infty$, and for any initial state $x(t_0)$;

- **Globally exponentially stable** if (2.17) is satisfied for any initial state $x(t_0)$ with $\beta(s, t) = kse^{-\alpha t}$, $k > 0, \alpha > 0$.

Hereafter, one of the most important theorems in the control theory is presented that is the main tool in the analysis and synthesis of dynamical systems in the time domain.

**Theorem 2.2.1** (Lyapunov theorem [127]). Let $x = 0$ be an equilibrium point of (2.15) and let $D = \{x \in \mathbb{R}^n| |x| < r\}$. Further, assume that $V : D \times \mathbb{R}^n \rightarrow \mathbb{R}^+$ is a continuously differentiable function such that $\forall t \geq 0$ and $\forall x \in D$,

\[ \gamma_1(|x|) \leq V(x, t) \leq \gamma_2(|x|) \quad (2.18) \]
\[
\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x, t) \leq -\gamma_3(|x|) \tag{2.19}
\]

Then the equilibrium point \(x = 0\) is

- Uniformly stable if the functions \(\gamma_1\) and \(\gamma_2\) are of class \(K\), and \(\gamma_3(\cdot) \geq 0\) on \([0, r)\);
- Uniformly asymptotically stable if \(\gamma_1, \gamma_2,\) and \(\gamma_3\) are all class \(K\) functions on \([0, r)\);
- Exponentially stable if \(\gamma_i(\rho) = k_i \rho^\alpha\) on \([0, r)\), \(k_i > 0, \alpha > 0, i = 1, 2, 3;\)
- Globally uniformly stable if \(D = \mathbb{R}^n, \gamma_1, \gamma_2\) are functions of class \(K_\infty\), and \(\gamma_3 \geq 0\) on \(\mathbb{R}^+;\)
- Globally uniformly asymptotically stable if \(D = \mathbb{R}^n, \gamma_1\) and \(\gamma_2\) are class \(K_\infty\) functions, and \(\gamma_3\) is of class \(K\) on \(\mathbb{R}^+;\)
- Globally exponentially stable if \(D = \mathbb{R}^n\) and \(\gamma_i(\rho) = k_i \rho^\alpha\) on \(\mathbb{R}^+\), \(k_i > 0, \alpha > 0, i = 1, 2, 3.\)

We now introduce two important extensions of the Lyapunov second theorem, Theorem 2.2.1. Consider the initial value problem for the following retarded functional differential equation (RFDE),

\[
\dot{x}(t) = f(t, x_t), \quad t \geq 0 \tag{2.20}
\]

where \(x_t(\theta) = x(t + \theta),\) and \(x_{t0} = \phi(\theta)\) for \(-h \leq \theta \leq 0.\) Moreover, \(\phi(\theta) \in C_n([-h, 0]),\) the mapping \(f : \mathbb{R} \times Q_H \to \mathbb{R}^n\) is continuous and Lipschitz in \(\phi \in Q_H,\) and \(f(t, 0) = 0.\) To add, \(Q_H = \{ \phi \in C_n([-h, 0]) | \rho(\phi, 0) \leq H \}\) is a sphere in \(C_n([-h, 0]),\) where \(\rho(\phi, \psi) = \sum_{j=1}^\infty 2^{-j} \| \phi - \psi \|_j (1 + \| \phi - \psi \|_j)^{-1}\) is a metric in \(C_n([-h, 0]),\) and \(\| \phi \|_j = \max |\phi|, \quad t \in [-j, 0].\)
Theorem 2.2.2 (Lyapunov Razumikhin theorem [3, 8, 9]). Consider the system (2.20). Assume that there exists a continuous positive definite function \( V(t, x) \), \( V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \), whose derivative, computed along the trajectories of (2.20), is non-positive for any solution \( x(t) \) satisfying the inequality

\[
V(s, x(s)) \leq V(t, x(t)), \quad s \leq t, \quad t \geq t_0
\]  

(2.21)

then the trivial solution of (2.20) is stable. In addition, if \( V(t, x) \) is upper-bounded and \( \dot{V}(t) \) is negative definite along any trajectory \( x(t) \) of (2.20), in a way that

\[
V(s, x(s)) \leq w(V(t, x(t))), \quad s \leq t, \quad t \geq t_0
\]  

(2.22)

where \( w \) is a continuous function and \( w(u) > u, \forall u > 0 \), then the trivial solution of (2.20) is uniformly asymptotically stable.

Now, let \( V : \mathbb{R} \times \mathcal{Q}_H \to \mathbb{R} \) to be a continuous functional such that \( V(t, 0) = 0 \). The functional \( V(t, \phi) \) is positive definite if there exists a continuous, non-decreasing function \( \omega_1 \), such that

\[
V(t, \phi) \geq \omega_1(|\phi(0)|), \quad \phi \in \mathcal{Q}_H, \quad t \in \mathbb{R}
\]  

(2.23)

Moreover, the functional \( V(t, \phi) \) is upper-bounded if there exists a continuous, non-decreasing function \( \omega_2 \), such that

\[
V(t, \phi) \leq \omega_2(|\phi(\theta)|)
\]  

(2.24)

In the Lyapunov Krasovskii analysis, the initial function \( \phi(\theta) \) is replaced by \( x_t(\theta) \) in the functional \( V(t, \phi) \), and the resulting functional \( V(t, x_t(\theta)) \) is denoted by \( V(t) \) in abbreviation. The time derivative of \( V(t, x_t(\theta)) \), which is denoted by \( \dot{V}(t) \) is a right derivative
2.3 Summary

defined as below

\[ \dot{V}(t) \triangleq \lim_{\Delta t \to 0^+} \sup (V(t + \Delta t) - V(t))/\Delta t \]  \hspace{1cm} (2.25)

**Theorem 2.2.3** (Lyapunov Krasovskii theorem [3, 8, 9]). *If there exists a continuous positive definite functional \( V(t, \phi) \) for the system (2.20), such that \( \dot{V}(t) \leq 0 \), then the trivial solution of (2.20) is stable. Moreover, if there exists a continuous functional \( V(t, \phi) \), such that

\[ \omega_1(||\phi(0)||) \leq V(t, \phi) \leq \omega_2(||\phi(\theta)||), \quad \dot{V}(t) \leq -\omega_3(|x(t)|) \]  \hspace{1cm} (2.26)

where \( \omega_i(\cdot), \ i = \{1, 2, 3\} \) are continuous, non-negative, and non-decreasing functions. Then, the zero solution of (2.20) is uniformly asymptotically stable. Conversely, if the zero solution of (2.20) is uniformly asymptotically stable, then there is a continuous functional \( V(t, \phi) \) that satisfies (2.26), and is locally Lipschitz in \( \phi \).*

2.3 Summary

Overall, this chapter has given a broad overview on the topic of time-delay systems and their importance in the stability analysis, controller design, and state estimation of practical dynamic systems. Various classes of time-delay systems, together with the problems and challenges that encounter in the analysis of those systems have been briefly highlighted. In addition, critical and key definitions, lemmas, and theorems that are useful in deriving the results and algorithms of the forthcoming chapters have been explained in Section 2.2.
Chapter 3

New Stability Criteria for
Interval-Time-Varying Delay Systems

3.1 Introduction

The stability analysis of retarded systems, which are generally modelled as functional
differential equations, has been very attractive for many researchers in recent years (see
e.g. [5, 128, 129]). This is also the key point in the controller and observer design for this
class of systems. In the recent literature, the most popular approach to this aim has been
using the Lyapunov Krasovskii methodology, which results in criteria expressed in terms of
LMIs that can be readily and efficiently examined using numerical approaches. However,
the stability conditions that are obtained via employing this method, are generally strictly
conservative. Hence, finding less restrictive stability conditions for time-delay systems us-
ing new Lyapunov Krasovskii functionals (LKF) and/or estimating tighter upper-bounds
of the LKF derivatives, has become a crucial topic, which is the focus of diverse researches
(see e.g., [5, 12, 128–131]).
One important class of hereditary systems is systems with *interval time-varying delays*, wherein the information of both the lower and upper bounds of the time-varying delay is employed for the aims of the stability analysis, observer, and controller design for the system. The additional information on the lower-bound of delay can be helpful in achieving a less conservative delay upper-bound using delay-dependent stability analysis techniques [132, 133]. Among the effective approaches that have recently been introduced in the stability analysis of interval time-varying delay systems, some try to modify the LKF, and others attempt to reduce from the conservatism that is inherently involved in deriving the stability conditions in terms of LMIs. The schemes in the first group include delay partitioning and delay decomposition approaches [130,131,133–136], utilizing triple-integral terms [131–133, 137–139], and employing slack variables or free-weighting matrices [130,134]. A number of these schemes can result in the excessive number of decision variables that can induce computational difficulties. In the meanwhile, some of those approaches might not be necessarily effective enough in expanding the stability regions.

On the other hand, approaches like using convex combination technique [130,131,138], reciprocallly convex combination approach [133, 136, 137, 140–142], and Wirtinger-based integral inequalities [136,139], can effectively reduce the conservatism, and at the same time minimally add to the number of decision variables. It is well-known that in constructing the stability conditions of a time-delay system using Lyapunov Krasovskii schemes, some nonlinear cross integral terms are generated that their upper-bounds should be estimated as quadratic terms. Jensen’s inequalities [123] are prevalently applied to this aim in the recent literature (see e.g. [130,132,133,140,143] and the references therein). However, this type of inequalities can be severely conservative. Hence, Seuret and Gouaisbaut [40] introduced a Wirtinger-based integral inequality to obtain a tighter upper-bound for single integral quadratic terms. Kwon et al. [144] applied this inequality to obtain a new less
3.1 Introduction

conservative stability criteria for LTI systems with time-varying state delays. Nevertheless, the inequality proposed in [40] is only applicable for single integral terms of the form $\int_a^b x^T(s)Mx(s)ds$. Recently, Park et al. [12] extended the Wirtinger inequality from single-integral terms to a class of double-integral inequalities of the form $\int_a^b \int_s^b x^T(u)Mx(u)duds$. However, the latter is not directly applicable to general quadratic terms of the form $\int_a^b \int_s^c x^T(u)Mx(u)duds$, which commonly appear in the stability analysis of retarded systems with interval-time-varying delays. In addition, none of the above technical contributions obtains the analytical measure of improvement to the conventional Jensen’s inequalities.

In this chapter, as the first contribution, we have obtained the generalized form of Wirtinger-based double-integral inequalities, and have calculated the analytical amount of improvement that it brings with respect to the associated Jensen’s inequality. As the second contribution, employing the new tighter upper-bound for cross integral terms, new delay-dependent sufficient conditions in terms of LMIs are established to justify the stability of LTI systems with interval time-varying delays. Both the known and unknown upper-bounds of the delay derivative are dealt with. Besides applying Wirtinger-based integral inequalities, the established stability criteria can be less conservative and numerically more efficient than the state of the art existing criteria on this problem due to:

A. Using triple integral terms in the LKF;

B. Employing the reciprocally convex combination approach and its extension for the aim of splitting both single and double quadratic integral terms;

C. Obtaining a condition for the positive-definiteness of the LKF instead of solely assuming positive-definite parameters;

D. Utilizing an effective combination of splitting an integral term and using essential
free-weighting matrices;

E. Avoiding ineffective terms in the LKF that can increase conservativeness, as well as delay-partitioning approach.

Two descriptive numerical examples, and simulation results, evaluate the effectiveness of the new established criteria in comparison with some existing ones in the current literature. The effectiveness of the Wirtinger-based integral inequalities, as well as Items B and C listed above, are particularly discussed.

The rest of the chapter is structured as follows. First the problem is demonstrated in Section 3.2. Next, a new double-integral inequality is introduced in Section 3.3.1, and several stability criteria for the system are derived in Section 3.3.2. Illustrative numerical examples are given in Section 3.4, and the chapter is summed up in Section 3.5.

### 3.2 Problem Statement and Preliminaries

**Notations:** Throughout the chapter, similar notations of Section 2.2 are employed. Moreover, \( \text{sym}(X) = X + X^T \), and \( I_n \) and \( 0 \) denote the \( n \times n \) identity matrix and the zero matrix of appropriate dimensions, respectively. Further, we define \( \zeta_i(a, b) \triangleq a^i - b^i, \ i = \{1, 2, \cdots\} \), and \( \bar{\zeta}(a, b) \triangleq 3\zeta_2(a, b)\zeta_4(a, b) - 2\zeta_3^2(a, b) \) for any scalars \( a \) and \( b \).

Our objective is to derive a new strictly less conservative set of sufficient conditions for the delay-dependent asymptotic stability of a retarded LTI system with the below dynamics,

\[
\begin{align*}
\dot{x}(t) &= F_0x(t) + F_1x(t - h(t)) \\
x(\theta) &= \phi(\theta), \ \forall \theta \in [-h_2, 0],
\end{align*}
\] (3.1)

where \( x(\cdot) \in \mathbb{R}^n \) is the state vector of the system, \( F_0, F_1 \in \mathbb{R}^{n \times n} \) are constant matrices that characterize the system, and \( \phi(\cdot) \in C_n([-h_2, 0]) \) is the initial function of the system.
Moreover, the continuous time-varying delay satisfies the below conditions

\[ 0 < h_1 \leq h(t) \leq h_2, \quad \dot{h}(t) \leq \mu, \]  

(3.2)

where \( \mu, h_1, \) and \( h_2 \) are given constants. In addition, let us define \( h_{12} \triangleq h_2 - h_1. \)

### 3.3 New Stability Analysis Method

#### 3.3.1 A New Double-Integral Inequality

First, a new Wirtinger-based double-integral inequality inspired by [12] is derived, which gives a tighter upper-bound for the left-hand-side of (2.8).

**Lemma 3.3.1.** Given real scalars \( a, b, c \), such that \( a < c \) and \( \eta_1 > \eta_2 \), and any constant real symmetric PD matrix \( M \), the following inequality holds if the integrations exist,

\[
\int_a^c \int_s^b x^T(\omega)Mx(\omega)d\omega ds \geq \frac{2\zeta_4(\eta_1, \eta_2)}{\zeta(\eta_1, \eta_2)} \left( \int_a^c \int_s^b x^T(\omega)d\omega ds \right) M \left( \int_a^c \int_s^b x(\omega)d\omega ds \right) \\
+ \frac{4\zeta_4(\eta_1, \eta_2)}{\zeta(\eta_1, \eta_2)} \Upsilon^T \begin{bmatrix} M & -3\frac{\zeta_3(\eta_1, \eta_2)}{\zeta_4(\eta_1, \eta_2)} M \ 
-3\frac{\zeta_3(\eta_1, \eta_2)}{\zeta_4(\eta_1, \eta_2)} M & 9\frac{\zeta_2(\eta_1, \eta_2)}{\zeta_4(\eta_1, \eta_2)} M \end{bmatrix} \Upsilon,
\]  

(3.3)

where \( \Upsilon^T \triangleq \left[ \int_a^c \int_s^b x^T(\omega)d\omega ds \int_a^c \int_s^b x^T(\omega)d\omega ds_1 \int_s^b x^T(\omega)d\omega ds_2 \right] \), and \( \eta_i, i = \{1, 2, 3\} \) are defined in Lemma 2.2.4.

**Proof.** Applying Lemma 2.2.5 to the cross term \( \int_s^b x^T(\omega)Mx(\omega)d\omega \) results in the following inequality,

\[
\int_s^b x^T(\omega)Mx(\omega)d\omega \geq \bar{v}^T(s)\Omega(s)\bar{v}(s),
\]  

(3.4)
3.3 New Stability Analysis Method

where \( \bar{v}^T(s) = \begin{bmatrix} \int_s^b x^T(\omega)d\omega & \int_s^b \int_s^b x^T(\omega)d\omega du \end{bmatrix} \), and \( \Omega(s) = \begin{bmatrix} \frac{4}{b-s} M & -\frac{6}{(b-s)^2} M \\ -\frac{6}{(b-s)^2} M & \frac{12}{(b-s)^3} M \end{bmatrix} \).

Now, applying the Schur compliment lemma \([122]\) to (3.4) gives,

\[
\begin{bmatrix} \int_s^b x^T(\omega)Mx(\omega)d\omega & \bar{v}^T(s) \\ \bar{v}(s) & \bar{\Omega}(s) \end{bmatrix} \succeq 0, \tag{3.5}
\]

where \( \bar{\Omega} \triangleq \Omega^{-1}(s) \), which considering Property (ii) of the Kronecker products, it can be written as,

\[
\bar{\Omega} = \begin{bmatrix} \frac{b-s}{2} & \frac{(b-s)^2}{2} \\ \frac{(b-s)^2}{3} & \frac{1}{2} \end{bmatrix} \otimes M^{-1} \succ 0, \tag{3.6}
\]

Thereafter, integrating (3.5) from \( a \) to \( c \), results

\[
\begin{bmatrix} \int_a^c \int_s^b x^T(\omega)Mx(\omega)d\omega ds & \int_a^c \bar{v}^T(s)ds \\ \int_a^c \bar{v}(s)ds & \int_a^c \bar{\Omega}(s)ds \end{bmatrix} \succeq 0, \tag{3.7}
\]

where

\[
\int_a^c \bar{v}^T(s)ds = \begin{bmatrix} \int_a^c \int_s^b x^T(\omega)d\omega ds & \int_a^c \int_s^b \int_s^b x^T(\omega)d\omega ds \int_s^b x^T(\omega)d\omega ds_1 ds_1 \end{bmatrix} \bar{\Upsilon},
\]

and

\[
\int_a^c \bar{\Omega}(s)ds = \begin{bmatrix} -0.5(b-s)^2 & -\frac{1}{6}(b-s)^3 \\ -\frac{1}{12}(b-s)^4 & \frac{1}{6} (\eta_1^2 - \eta_2^2) \end{bmatrix} \otimes M^{-1}
\]

\[
= \begin{bmatrix} 0.5(\eta_1)^2 & \frac{1}{6} (\eta_1^3 - \eta_2^3) \\ \frac{1}{12} (\eta_1^4 - \eta_2^4) \end{bmatrix} \otimes M^{-1}. \tag{3.8}
\]

Applying the Schur complement lemma on (3.7), one can obtain

\[
\int_a^c \int_s^b x^T(\omega)Mx(\omega)d\omega ds \geq \bar{\Upsilon} \left( \int_a^c \bar{\Omega}(s)ds \right)^{-1} \bar{\Upsilon}. \tag{3.9}
\]
3.3 New Stability Analysis Method

In addition, from Property (ii) of the Kronecker product and (3.8), we have

\[
\left( \int_a^c \bar{\Omega}(s) ds \right)^{-1} = \begin{bmatrix}
0.5\zeta_2(\eta_1, \eta_2) & \frac{1}{6}\zeta_3(\eta_1, \eta_2) \\
\frac{1}{6}\zeta_3(\eta_1, \eta_2) & \frac{1}{12}\zeta_4(\eta_1, \eta_2)
\end{bmatrix}^{-1} \otimes M
\]

After some basic calculations, the following is obtained,

\[
\begin{bmatrix}
0.5\zeta_2(\eta_1, \eta_2) & \frac{1}{6}\zeta_3(\eta_1, \eta_2) \\
\frac{1}{6}\zeta_3(\eta_1, \eta_2) & \frac{1}{12}\zeta_4(\eta_1, \eta_2)
\end{bmatrix}^{-1} = \frac{2}{\zeta(\eta_1, \eta_2)} \begin{bmatrix}
3\zeta_4(\eta_1, \eta_2) & -6\zeta_3(\eta_1, \eta_2) \\
-6\zeta_3(\eta_1, \eta_2) & 18\zeta_2(\eta_1, \eta_2)
\end{bmatrix}. \quad (3.10)
\]

Finally, after substituting (3.10) into (3.9), and after some manipulations, Inequality (3.3) is achieved, which completes the proof of the lemma.

\[\square\]

Corollary 3.3.1. Given an arbitrary constant matrix \(M \succ 0 \in \mathbb{S}^n\), and for any continuous differentiable function \(x : [a, \max\{c, b\}] \rightarrow \mathbb{R}^n\), assuming \(\eta_1 > \eta_2\), the following inequality is satisfied

\[
\int_a^c \int_s^b \dot{x}^T(u)M\dot{x}(u)duds \geq \frac{6\zeta_4(\eta_1, \eta_2)}{\zeta(\eta_1, \eta_2)} \tilde{\Upsilon}^T \begin{bmatrix}
1 & -\frac{2\zeta_3(\eta_1, \eta_2)}{\zeta_4(\eta_1, \eta_2)}
\end{bmatrix} \otimes M \tilde{\Upsilon}, \quad (3.11)
\]

where

\[
\tilde{\Upsilon}^T \triangleq \left[ \eta_3x^T(b) - \int_a^c x^T(u)du, \frac{\zeta_2(\eta_1, \eta_2)}{2}x^T(b) - \int_a^c \int_s^b x^T(u)duds \right].
\]

**Proof.** The proof can be directly obtained from Lemma 3.3.1, and using the fact that Inequality (3.3) is equivalent to the following inequality,
3.3 New Stability Analysis Method

\[ \int_a^c \int_s^b x^T(u)Mx(u)du ds \geq \frac{6\zeta_4(\eta_1, \eta_2)}{\zeta(\eta_1, \eta_2)} \Upsilon^T \begin{bmatrix} 1 & -\frac{2\zeta_3(\eta_1, \eta_2)}{\zeta_4(\eta_1, \eta_2)} \\ -\frac{2\zeta_3(\eta_1, \eta_2)}{\zeta_4(\eta_1, \eta_2)} & \frac{6\zeta_2(\eta_1, \eta_2)}{\zeta_4(\eta_1, \eta_2)} \end{bmatrix} \otimes M \Upsilon. \quad (3.12) \]

Remark 3.3.1. If the upper limits of the integrals on the left-hand-side of (3.3) are equal (i.e. \( b = c \)), then \( \eta_2 = 0 \), and Lemma 3.3.1 simply results Corollary 1 of [12]. Hence, the main result of the paper [12] is a special case of Lemma 3.3.1.

Next, it is proved that the new proposed integral inequality is strictly less conservative than the associated Jensen’s inequality (2.8).

Proposition 3.3.1. Given a constant real symmetric and PD matrix \( M \), the satisfaction of inequality (3.3) strictly concludes inequality (2.8). Similarly, the satisfaction of inequality (3.11) strictly implies (2.9).

Proof. The proof of the second part can be directly deduced from the validity of the first part. Therefore, only the first statement is verified. To show that the upper-bound given in Lemma 3.3.1 is strictly tighter than the upper-bound value given in (2.8), let us rewrite (3.3) as follows:

\[ \int_a^c \int_s^b x^T(\omega)Mx(\omega)d\omega ds \geq \Upsilon^T \Xi(\eta_1, \eta_2) \otimes M \Upsilon \\
+ \frac{2}{\zeta_3(\eta_1, \eta_2)} \left( \int_a^c \int_s^b x^T(\omega)d\omega ds \right) M \left( \int_a^c \int_s^b x(\omega)d\omega ds \right), \quad (3.13) \]

where \( \Xi(\eta_1, \eta_2) \triangleq \begin{bmatrix} 6\zeta_4(\eta_1, \eta_2) - \frac{2}{\zeta_2(\eta_1, \eta_2)} & -12\zeta_3(\eta_1, \eta_2) \\ -12\zeta_3(\eta_1, \eta_2) & 36\zeta_2(\eta_1, \eta_2) \end{bmatrix} \). Let us further define \( \Upsilon^T \triangleq \begin{bmatrix} \psi_1^T \\ \psi_2^T \end{bmatrix} \). Moreover, \( \frac{6\zeta_4(\eta_1, \eta_2)}{\zeta(\eta_1, \eta_2)} - \frac{2}{\zeta_2(\eta_1, \eta_2)} = \frac{4\zeta_2^2(\eta_1, \eta_2)}{\zeta(\eta_1, \eta_2)\zeta_2(\eta_1, \eta_2)} > 0 \). In addition, calculating the eigenvalues of \( \Xi(\eta_1, \eta_2) \), it is found that this matrix has a zero eigenvalue and a positive
3.3 New Stability Analysis Method

one, which is the sum of the diagonal terms, i.e. \( \lambda = \frac{4\zeta_2^2(\eta_1,\eta_2) + 36\zeta_2^2(\eta_1,\eta_2)}{\zeta_2(\eta_1,\eta_2)\zeta(\eta_1,\eta_2)} \). Hence, it can be concluded that \( \Xi(\eta_1,\eta_2) \succeq 0 \). Since \( \Xi(\eta_1,\eta_2) \) is symmetric positive semi-definite, its singular value decomposition can be written as

\[
\Xi(\eta_1,\eta_2) = U \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} U,
\]

(3.14)

where \( U \triangleq \begin{bmatrix} v_1 & v_2 \end{bmatrix} \), and \( v_1^T = \begin{bmatrix} a \\ b \end{bmatrix} \) and \( v_2^T = \begin{bmatrix} b \\ -a \end{bmatrix} \) are the eigenvectors of \( \Xi(\eta_1,\eta_2) \), with \( a, b \) as some scalars.

Therefore, considering properties (i) and (iii) of Kronecker products, the following equations can be achieved,

\[
\Upsilon^T \Xi(\eta_1,\eta_2) \otimes M \Upsilon = \Upsilon^T U \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix} U \otimes M \Upsilon
\]

(3.15)

\[
= \lambda \Upsilon^T v_1 v_1^T \otimes M \Upsilon.
\]

Since \( v_1 \) is a normal vector and \( v_1 v_1^T \) is of rank 1, one can deduce that \( v_1 v_1^T \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \).

In conclusion,

\[
\lambda \Upsilon^T v_1 v_1^T \otimes M \Upsilon \equiv \lambda \Upsilon^T \begin{bmatrix} 0 & 0 \\ 0 & M \end{bmatrix} \Upsilon = \lambda \psi_2^T M \psi_2
\]

(3.16)

Considering (3.13), the relations (3.15) and (3.16) result in the following inequality,

\[
f_a^c \int_s^b x^T(\omega) M x(\omega) d\omega ds \geq \frac{2}{\zeta_2(\eta_1,\eta_2)} \psi_1^T M \psi_1 + \lambda \psi_2^T M \psi_2
\]

(3.17)

This concludes the first statement of the theorem. \( \Box \)
3.3 New Stability Analysis Method

Remark 3.3.2. Inequality (3.17) gives an analytical amount of improvement to the Jensen’s inequality, when using the new integral inequality. This value, which is equal to
\[ \lambda(\int_a^c \int_s^b \int_s^b x_T(\omega) d\omega ds_1 ds) M(\int_a^c \int_s^b \int_s^b x(\omega) d\omega ds_1 ds), \]
depends on the values of \( \eta_1, \eta_2 \), the norm of matrix \( M \), as well as the norm of the vector \( x(\cdot) \).

Remark 3.3.3. The analytical amount of improvement to the Jensen’s inequality, using Lemma (3.3.1), when \( c = b \), is thus
\[ \frac{4\eta_1^2 + 36}{\eta_1^2}(\int_a^b \int_s^b \int_s^b x_T(\omega) d\omega ds_1 ds) M(\int_a^b \int_s^b \int_s^b x(\omega) d\omega ds_1 ds), \]
which is not mentioned in [12].

3.3.2 New Stability Criteria

Now, the second main result of the chapter is summarized in the following theorem.

Theorem 3.3.1. Given scalars \( 0 < h_1 < h_2 \) and \( \mu \), if there exist constant parameters
\( P \in \mathbb{S}^{3n}, \) \( Q_i \in \mathbb{S}^n, i = \{1, 2\}, Q_3 \succ 0 \in \mathbb{S}^n, R_j \in \mathbb{S}^n, j = \{1, 2, 3, 4\}, S_k \in \mathbb{S}^n, k = \{1, 2\}, \) and any matrices \( M_i \in \mathbb{R}^{n \times n}, X_i \in \mathbb{R}^{n \times n}, i = \{1, 2, 3\}, \) and \( T_{ij} \in \mathbb{R}^{n \times n}, i, j = \{1, 2\}, \) such that the following LMIs are satisfied, then for any delay satisfying (3.2), the system (3.1) is globally asymptotically stable.

\[ \Psi \prec 0, \quad (3.18) \]
\[ \Pi \succ 0, \quad (3.19) \]
\[ \begin{bmatrix}
  2R_4 & -3R_4 & T_{11} & T_{12} \\
  * & 2R_4 & T_{21} & T_{22} \\
  * & * & 2R_4 & -3R_4 \\
  * & * & * & 2R_4
\end{bmatrix} \succeq 0, \quad (3.20) \]

where \( \Psi = [\Psi_{i,j}]_{12 \times 12} \) and \( \Pi = [\Pi_{i,j}]_{7 \times 7} \) are symmetric matrices that are articulated in the sequel. The non-zero elements of the matrix \( \Psi \) are:
\begin{align*}
\Psi_{1,1} &= \text{sym} (X_1 F_0) + \text{sym} (P_3) + Q_1 + h_1^2 R_1 - 4 R_3 - 6 h_1^2 S_1 + h_{12}^2 R_2 - 12 a_{11} S_2, \\
\Psi_{1,2} &= F_0^T X_2^T + X_1 F_1 + P_2, \quad \Psi_{1,3} = -P_3 - 2 R_3, \quad \Psi_{1,4} = -P_2, \quad \Psi_{1,5} = P_6 + \frac{6}{h_1} R_3 - 12 (2 h_{12}^2 \zeta_3 - 3 h_{12}^2 \zeta_2^2) S_2, \\
\Psi_{1,6} &= 12 S_1, \quad \Psi_{1,7} = -12 (\zeta_2 \zeta_3 - h_{12} \zeta_4) S_2 + P_5^T, \\
\Psi_{1,8} &= -12 (2 h_{12}^2 \zeta_3 - 3 \zeta_2^2) S_2, \quad \Psi_{1,9} = P_2, \quad \Psi_{1,10} = P_1 - X_1 + F_0^T X_2^T, \\
\Psi_{2,2} &= -(1 - \mu) Q_3 - 8 R_4 - \text{sym} (M_2) + 2 \text{sym} (T_{11} + T_{12} + T_{21} + T_{22}) + \text{sym} (X_2 F_1), \\
\Psi_{2,3} &= M_2 - M_1^T - 2 R_4 - 2 T_{11} - 2 T_{21}, \quad \Psi_{2,4} = -2 R_4 - 2 T_{11}^T - 2 T_{21}^T, \quad \Psi_{2,5} = P_3, \\
\Psi_{2,7} &= P_4, \quad \Psi_{2,9} = -M_2 + M_3^T, \quad \Psi_{2,10} = F_1^T X_3^T - X_2, \quad \Psi_{3,3} = \text{sym} (M_1) - 4 R_4 - 4 R_3 + Q_3 + Q_2 - Q_1, \\
\Psi_{3,4} &= 2 T_{11}^T, \quad \Psi_{3,5} = -P_6 + \frac{6}{h_1} R_3, \quad \Psi_{3,7} = -P_5^T, \\
\Psi_{3,9} &= -M_1 + M_3^T, \quad \Psi_{3,11} = 6 R_4 + 2 T_{21}^T, \quad \Psi_{4,4} = -Q_2 - 4 R_4, \quad \Psi_{4,5} = -P_3, \\
\Psi_{4,7} &= -P_4, \quad \Psi_{4,11} = 6 R_4, \quad \Psi_{4,12} = 2 T_{12}, \quad \Psi_{5,5} = -4 R_1 - \frac{12}{h_1} R_3 - 12 S_1 - 72 h_{12}^2 \zeta_2 S_2, \\
\Psi_{5,6} &= \frac{6}{h_1} R_1 + \frac{24}{h_1^2} S_1, \quad \Psi_{5,7} = 24 h_{12} \zeta_3 S_2, \quad \Psi_{5,8} = -72 h_{12} \zeta_2 S_2, \\
\Psi_{5,9} &= P_5^T, \quad \Psi_{5,10} = P_5^T, \quad \Psi_{6,6} = -\frac{12}{h_1^2} R_1 - \frac{72}{h_1^3} S_1, \quad \Psi_{7,7} = -4 R_2 - 12 \zeta_4 S_2, \\
\Psi_{7,8} &= \frac{6}{h_{12}} R_2 + 24 \zeta_3 S_2, \quad \Psi_{7,9} = P_4, \quad \Psi_{7,10} = P_2^T, \quad \Psi_{8,8} = -\frac{12}{h_{12}} R_2 - 72 \zeta_2 S_2, \\
\Psi_{9,9} &= -\text{sym} (M_3), \quad \Psi_{10,10} = h_1^2 R_3 + h_{12}^2 R_4 + h_1^4 S_1 + \zeta_2 \tilde{\zeta} S_2 - \text{sym} (X_3), \\
\Psi_{11,11} &= \Psi_{12,12} = -12 R_4, \quad \Psi_{11,12} = 2 T_{22},
\end{align*}

where \( a_{11} \triangleq h_{12}^2 \zeta_4 - 2 h_{12} \zeta_3 \zeta_2 + 1.5 \zeta_2^3 \), \( \zeta_i = \zeta_i (h_2, h_1) \), \( i \in \{2, 3, 4\} \), and \( \tilde{\zeta} = \tilde{\zeta} (h_1, h_2) \).

Similarly, all of the non-zero elements of \( \Pi \) are described in the following:

\begin{align*}
\Pi_{1,1} &= P_1 + 3 h_1 R_3 + h_{12} b_{11} R_4 + 3 h_1^3 S_1 + 3 \frac{h_2 \tilde{\zeta}}{h_4} S_2, \quad \Pi_{1,2} = P_2 + h_{12} b_{12} R_4, \\
\Pi_{1,3} &= P_3, \quad \Pi_{1,4} = b_{11} h_{12} R_4 - 6 \frac{\zeta_4}{\zeta_3} S_2, \quad \Pi_{1,5} = -6 h_1 S_1 - \frac{6}{h_1} R_3, \quad \Pi_{1,22} = P_4 + \frac{4}{h_{12}} Q_2 + 6 h_{12} \frac{\zeta_4}{\zeta_3} R_4, \\
\Pi_{2,3} &= P_5 + \frac{6}{h_{12}} Q_2, \quad \Pi_{2,4} = -\frac{6}{h_{12}} Q_2 - 12 \frac{\zeta_4}{\zeta_3} R_4, \quad \Pi_{3,3} = P_6 + \frac{12}{h_{12}} Q_2 + \frac{4}{h_1} Q_1 + \frac{6}{h_{12}} R_3, \\
\Pi_{3,4} &= -\frac{12}{h_{12}} Q_2, \quad \Pi_{3,5} = -\frac{6}{h_1} Q_1 - \frac{12}{h_1} R_3, \quad \Pi_{4,4} = \frac{12}{h_{12}} Q_2 + 6 \frac{h_{12} \zeta_4}{\zeta_3} R_2 + \frac{12 \zeta_3}{\zeta_4} S_2 + 36 \frac{h_{12} \zeta_4}{\zeta_3} R_4, \\
\Pi_{4,7} &= -12 \frac{h_{12} \zeta_4}{\zeta_3} R_2, \quad \Pi_{5,5} = \frac{12}{h_1} Q_1 + \frac{6}{h_1} R_1 + \frac{36}{h_1} R_3 + \frac{12}{h_1} S_1, \\
\Pi_{5,6} &= -\frac{12}{h_1} R_1, \quad \Pi_{6,6} = \frac{36}{h_1} R_1, \quad \Pi_{7,7} = 36 h_{12} \frac{\zeta_4}{\zeta_3} R_2, \\
\end{align*}

where \( b_{11} \triangleq \frac{6}{\zeta} (h_{12} \zeta_4 - 2 h_{12} \zeta_3 \zeta_2 + 1.5 \zeta_2^3) \), \( b_{12} \triangleq \frac{6}{\zeta} (\zeta_2 \zeta_3 - h_{12} \zeta_4) \), and \( b_{14} \triangleq \frac{6}{\zeta} (2 h_{12} \zeta_3 - 3 \zeta_2^2) \).

**Proof.** Let us first define the following variables, wherein the arguments are ignored for
simplicity: $z_1 \triangleq x(t)$, $z_2 \triangleq x(t - h(t))$, $z_3 \triangleq x(t - h_1)$, $z_4 \triangleq x(t - h_2)$, $z_5 \triangleq \int_{t-h_1}^{t} x(s) \, ds$, $z_6 \triangleq \int_{-h_1}^{0} \int_{s}^{0} x(t + u) \, du \, ds$, $z_7 \triangleq \int_{-h_1}^{-h_1} x(t + u) \, du$, $z_8 \triangleq \int_{-h_2}^{-h_1} \int_{s}^{-h_1} x(t + u) \, du \, ds$, $z_9 \triangleq \int_{-h(t)}^{-h(t)} \dot{x}(t + s) \, ds$, $z_{10} \triangleq \dot{x}(t)$, $z_{11} \triangleq \frac{1}{h_2 - h(t)} \int_{-h_2}^{-h(t)} x(t + u) \, du$, and $z_{12} \triangleq \frac{1}{h(t) - h_1} \int_{-h(t)}^{-h_1} x(t + u) \, du$.

Now, consider the following LKF candidate as,

$$V(t, x(t), x_t, \dot{x}_t) = \sum_{i=1}^{4} V_i(t), \quad (3.21)$$

where

$$V_1(t, x_t) = \Gamma^T(t) \Gamma(t)$$

$$V_2(t, x_t) = \int_{t-h_1}^{0} x^T(s) Q_1 x(s) \, ds + \int_{t-h_1}^{t-h_2} x^T(s) Q_2 x(s) \, ds + \int_{t-h_2}^{t-h(t)} x^T(s) Q_3 x(s) \, ds,$$

$$V_3(x_t) = h_1 \int_{-h_1}^{0} \int_{s}^{t} x^T(s) R_1 x(s) \, ds \, ds_1 + h_1 \int_{0}^{0} \int_{t}^{t} \dot{x}^T(s) R_2 \dot{x}(s) \, ds \, ds_1$$

$$+ h_2 \int_{-h_2}^{-h_1} \int_{s}^{t} x^T(s) R_2 x(s) \, ds \, ds_1 + h_1 \int_{-h_1}^{-h_1} \int_{s}^{t} \dot{x}^T(s) R_3 \dot{x}(s) \, ds \, ds_1,$$

$$V_4(\dot{x}_t) = 2h_1 \int_{-h_1}^{0} \int_{s}^{0} \int_{s}^{t} \dot{x}^T(s) S_1 \dot{x}(s) \, ds \, ds_2 \, ds_1$$

$$+ 2\zeta \int_{-h_2}^{-h_1} \int_{s}^{0} \int_{s}^{t} \dot{x}^T(s) S_2 \dot{x}(s) \, ds \, ds_2 \, ds_1.$$

with

$$\Gamma(t) = \begin{bmatrix} x^T(t), & \int_{t-h_1}^{t-h_2} x^T(s) \, ds, & \int_{t-h_1}^{t} x^T(s) \, ds \end{bmatrix}^T,$$

$$P \triangleq \begin{bmatrix} P_1 & P_2 & P_3 \\ * & P_4 & P_5 \\ * & * & P_6 \end{bmatrix}.$$

In addition, the following relations are true,
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\[
\int_{-h_2}^{-h_1} \dot{x}^T(t + u) R_4 \dot{x}(t + u) du = \int_{-h_2}^{-h(t)} \dot{x}^T(t + u) R_4 \dot{x}(t + u) du
\]

\[
+ \int_{-h(t)}^{-h_1} \dot{x}^T(t + u) R_4 \dot{x}(t + u) du,
\]

\[
\int_{-h_2}^{-h(t)} \dot{x}(t + u) du = x(t - h(t)) - x(t - h_2),
\]

\[(3.22)\]

\[
\int_{-h_2}^{-h_1} x(t + u) duds = \int_{-h_2}^{-h_1} x(t + u) duds + h_{12} \int_{-h_1}^{0} x(t + u) du.
\]

\[(3.24)\]

Differentiating the LKF (3.21) along the solution of (3.1), one obtains

\[
\dot{V}_1(t) = \text{sym} \left( \begin{bmatrix} z_1 \\ z_7 \\ z_5 \end{bmatrix}^T \begin{bmatrix} P_1 & P_2 & P_3 \\ * & P_4 & P_5 \\ * & * & P_6 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 - z_4 + z_9 \\ z_1 - z_3 \end{bmatrix} \right),
\]

\[(3.25)\]

\[
\dot{V}_2(t) \leq z_1^T Q_1 z_1 + z_3^T (Q_2 + Q_3 - Q_1) z_3 - z_4^T Q_2 z_4 - (1 - \mu) z_2^T Q_3 z_2,
\]

\[(3.26)\]

\[
\dot{V}_3(t) = h_1 z_1^T R_1 z_1 + h_1^2 \dot{x}^T(t) R_3 \dot{x}(t) + h_2 z_3^T R_2 z_3 + h_2^2 \dot{x}^T(t) R_4 \dot{x}(t)
\]

\[
- h_1 \int_{-h_1}^{0} \dot{x}^T(t + u) R_3 \dot{x}(t + u) du - h_1 \int_{-h_2}^{0} \dot{x}^T(t + u) R_1 \dot{x}(t + u) du
\]

\[
- h_{12} \int_{-h_2}^{-h_1} \dot{x}^T(t + u) R_2 \dot{x}(t + u) du - h_{12} \int_{-h_2}^{-h_1} \dot{x}^T(t + u) R_4 \dot{x}(t + u) du,
\]

\[(3.27)\]

\[
\dot{V}_4(t) = h_1^2 \dot{x}^T(t) S_1 \dot{x}(t) + \zeta_2 \dot{x}^T(t) S_2 \dot{x}(t) - 2h_1 \int_{-h_1}^{0} \dot{x}^T(t + u) S_1 \dot{x}(t + u) duds
\]

\[
- 2\zeta \int_{-h_2}^{-h_1} \int_{s}^{0} \dot{x}^T(t + u) S_2 \dot{x}(t + u) duds.
\]

\[(3.28)\]
Henceforth, applying Lemmas 2.2.5, 2.2.7, and 3.3.1, as well as Corollary 3.3.1, together with taking (3.22)-(3.24) into account, and assuming \( \zeta_i = \zeta_i(h_2, h_1), \ i = \{2, 3, 4\} \), and \( \tilde{\zeta} = \tilde{\zeta}(h_2, h_1) \), (3.27) and (3.28) are upper-bounded as follows provided that the inequality (3.19) is feasible,

\[
\begin{align*}
\dot{V}_3(t) & \leq h_1^2 z_1^T R_1 z_1 + h_1^2 z_{10}^T R_3 z_{10} + h_{12}^2 z_1^T R_2 z_1 + h_{12}^2 z_{10}^T R_4 z_{10} \\
& - 2 \begin{bmatrix} z_5 \\ z_6 \end{bmatrix}^T \begin{bmatrix} 2 & \frac{-3}{h_1} \\ \ast & \frac{6}{h_1^2} \end{bmatrix} \otimes R_1 \begin{bmatrix} z_5 \\ z_6 \end{bmatrix} \\
& - 2 \begin{bmatrix} z_1 - z_3 \\ h_1 z_1 - z_5 \end{bmatrix}^T \begin{bmatrix} 2 & \frac{-3}{h_1} \\ \ast & \frac{6}{h_1^2} \end{bmatrix} \otimes R_3 \begin{bmatrix} z_1 - z_3 \\ h_1 z_1 - z_5 \end{bmatrix} \\
& - 2 \begin{bmatrix} z_7 \\ z_8 \end{bmatrix}^T \begin{bmatrix} 2 & \frac{-3}{h_{12}} \\ \ast & \frac{6}{h_{12}^2} \end{bmatrix} \otimes R_2 \begin{bmatrix} z_7 \\ z_8 \end{bmatrix} \\
& - 2 \begin{bmatrix} z_2 - z_4 \\ z_2 - z_{11} \\ z_3 - z_2 \\ -z_2 + z_{12} \end{bmatrix}^T \begin{bmatrix} 2R_4 & -3R_4 & T_{11} & T_{12} \\ \ast & 2R_4 & T_{21} & T_{22} \\ \ast & \ast & 2R_4 & -3R_4 \\ \ast & \ast & \ast & 2R_4 \end{bmatrix} \begin{bmatrix} z_2 - z_4 \\ z_2 - z_{11} \\ z_3 - z_2 \\ -z_2 + z_{12} \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\dot{V}_4(t) & \leq h_{11}^2 z_{11}^T S_1 z_{11} + \zeta_2 \tilde{\zeta} z_{11}^T S_2 z_{11} \\
& - 12 \begin{bmatrix} h_1 z_1 - z_5 \\ \frac{1}{2} \tilde{z}_1 - z_6 \end{bmatrix}^T \begin{bmatrix} 1 & \frac{-2}{h_1} \\ \ast & \frac{6}{h_1^2} \end{bmatrix} \otimes S_1 \begin{bmatrix} h_1 z_1 - z_5 \\ \frac{1}{2} \tilde{z}_1 - z_6 \end{bmatrix} \\
& - 12 \begin{bmatrix} \tilde{\zeta}_{21} z_1 - z_7 \\ \frac{\tilde{\zeta}}{2} z_1 - z_8 - \tilde{\zeta}_{12} z_5 \end{bmatrix}^T \begin{bmatrix} \zeta_4 & -2\zeta_3 \\ \ast & 6\zeta_2 \end{bmatrix} \otimes S_2 \begin{bmatrix} \tilde{\zeta}_{21} z_1 - z_7 \\ \frac{\tilde{\zeta}}{2} z_1 - z_8 - \tilde{\zeta}_{12} z_5 \end{bmatrix}
\end{align*}
\]
In addition, for any matrices $M_i$ and $X_i$, $i = \{1, 2, 3\}$, with appropriate dimensions, the following zero equations hold:

\[
sym \left( [z_3^T M_1 + z_2^T M_2 + z_9^T M_3] [z_3 - z_2 - z_9] \right) = 0 \tag{3.32}
\]

\[
sym \left( [z_1^T X_1 + z_2^T X_2 + z_{10}^T X_3] [F_0 z_1 + F_1 z_2 - z_{10}] \right) = 0 \tag{3.33}
\]

Adding Inequalities (3.25), (3.26), and (3.29)-(3.33), and reformulating the result, finally yields the following inequality:

\[
\dot{V}(t) \leq Z^T \Psi Z \tag{3.34}
\]

where $Z = \left[ z_1^T \ z_2^T \cdots \ z_{12}^T \right]^T$. To apply the Lyapunov Krasovskii theorem it is necessary that the LKF (3.21) to be positive definite. To satisfy this requirement by applying Lemmas 2.2.4, 2.2.5, and 3.3.1, as well as Corollary 3.3.1 to the cross product integral terms of $V(t) = V(t, x(t), x_t, \dot{x}_t)$ we obtain

\[

V(t) \geq \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}^T \begin{bmatrix} P_1 & P_2 & P_3 \\ * & P_4 & P_5 \\ * & * & P_6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \frac{2}{h_1} \begin{bmatrix} v_3 \\ v_5 \end{bmatrix}^T \begin{bmatrix} 2 & -\frac{3}{h_1} \\ * & \frac{6}{h_1^2} \end{bmatrix} \otimes Q_1 \begin{bmatrix} v_3 \\ v_5 \end{bmatrix} + \frac{2}{h_{12}} \begin{bmatrix} v_2 \\ v_4 - h_{12}v_3 \end{bmatrix}^T \begin{bmatrix} 2 & -\frac{3}{h_{12}} \\ * & \frac{6}{h_{12}} \end{bmatrix} \otimes Q_2 \begin{bmatrix} v_2 \\ v_4 - h_{12}v_3 \end{bmatrix} + \frac{6}{h_1} \begin{bmatrix} h_1v_1 - v_3 \\ 0.5h_1^2v_1 - v_5 \end{bmatrix}^T \begin{bmatrix} 1 & -\frac{2}{h_1} \\ * & \frac{6}{h_1} \end{bmatrix} \otimes R_3 \begin{bmatrix} h_1v_1 - v_3 \\ 0.5h_1^2v_1 - v_5 \end{bmatrix}
\]
where \( v_1 \triangleq x(t), v_2 \triangleq \int_{t-h_2}^{t-h_1} x(u)du, v_3 \triangleq \int_{t-h_1}^{t} x(s)ds, v_4 \triangleq \int_{-h_1}^{-h_2} \int_{s}^{0} x(t + u)duds, \\
v_5 \triangleq \int_{-h_1}^{0} \int_{s}^{0} x(t + u)duds, v_6 \triangleq \int_{-h_1}^{0} \int_{s}^{0} \int_{s_1}^{0} x(t + u)duds_1ds, \) and \( v_7 \triangleq \int_{-h_1}^{0} \int_{s}^{0} \int_{s_1}^{0} \int_{s_1}^{0} x(t + u)duds_1ds. \) Accordingly, it can be seen that

\[
V(t) \geq \beta^T \Pi \beta,
\tag{3.36}
\]

where \( \beta = \begin{bmatrix} v_1^T & v_2^T & \cdots & v_7^T \end{bmatrix}^T. \) Hence, if the LMIs (3.18) and (3.19) are satisfied, then according to the Lyapunov Krasovskii theorem the system (3.1) is globally asymptotically stable, which concludes the proof of the theorem.

**Remark 3.3.4.** Free-weighting matrices can be useful to push the solution of LMIs (3.18) and (3.19) from a marginally feasible condition into a feasible region, when a numerical LMI programming toolbox is used. However, one should bear in mind that the excessive usage of free-weighting matrices can significantly increase the number of decision variables, thus can cause computational difficulties. On the other hand, employing Wirtinger-based integral inequalities, and considering positive-definiteness of the LKF as a condition of stability do not add to the number of decision variables.
Remark 3.3.5. Zero equation (3.32) is indeed essential to reflect the effects of the structural system’s parameters $F_i, i = \{0, 1\}$. Moreover, considering Zero equation (3.33) is also necessary to use the expression $z_2 - z_4 + z_9$ as a substitute to $z_3 - z_4$ in (3.25), wherein more off-diagonal terms in the second row (and column) of the matrix $\Psi$ are created. Otherwise, the $(9, 9)$ diagonal term was missing, which made the LMI (3.18) infeasible. Our experience shows that this special combination of employing splitting the integrals and zero equations is effective in reducing from the conservativeness of the derived stability conditions.

Remark 3.3.6. Our observations show that the process of splitting the single-integral term $\int_{-h_1}^{-h_2} \dot{x}^T(t + u)R_4\dot{x}(t + u)du$ according to Relation (3.22) and applying the reciprocally convex optimization approach is very effective in reducing the conservatism of the proposed criterion of Theorem 3.3.1. This operation indeed results in creating more negative-definite terms in the $(2, 2)$ element of the matrix $\Psi$, and thus may not be ignored.

To emphasize the effectiveness of Lemmas 2.2.5 and 3.3.1, and Corollary 3.3.1 in the numerical examples the following stability criterion is established with the aim of Jensen’s inequalities, instead of Wirtinger-based inequalities.

**Corollary 3.3.2.** Given the scalars $0 < h_1 < h_2$ and $\mu$, if there exist constant positive definite matrices $P \in S^{3n}, Q_i \in S^n, i = \{1, 2, 3\}, R_j \in S^n, j = \{1, 2, 3, 4\}, S_k \in S^n, k = \{1, 2\}$, and any matrices $M_i \in \mathbb{R}^{n \times n}, X_i \in \mathbb{R}^{n \times n}, i = \{1, 2, 3\},$ and $T_1 \in \mathbb{R}^{n \times n}$, such that the following LMIs are satisfied, then for any delay fulfilling (3.2), the system (3.1) is globally asymptotically stable.

$$\bar{\Psi} < 0,$$

$$\begin{bmatrix} R_4 & T_1 \\ * & R_4 \end{bmatrix} \succeq 0,$$
where the matrix $\bar{\Psi} = [\bar{\Psi}_{i,j}]_{8 \times 8}$ has the following non-zero components:

\[
\bar{\Psi}_{1,1} = \text{sym}(P_3) + Q_1 + h_1^2 R_1 - R_3 - 4h_1^2 S_1 + h_1^2 R_2 - 4h_1^2 \tilde{\zeta}_2 S_2 + \text{sym}(X_1 F_0),
\]
\[
\bar{\Psi}_{1,2} = P_2 + X_1 F_1 + F_0^T X_2^T, \quad \bar{\Psi}_{1,3} = -P_3 + R_3, \quad \bar{\Psi}_{1,4} = -P_2, \quad \bar{\Psi}_{1,5} = P_6 + 4h_1 S_1,
\]
\[
\bar{\Psi}_{1,6} = 4h_1^2 \tilde{\zeta}_2 S_2 + P_5^T, \quad \bar{\Psi}_{1,7} = P_2, \quad \bar{\Psi}_{1,8} = P_1 - X_1 + F_0^T X_3^T, \quad \bar{\Psi}_{2,2} = -(1 - \mu) Q_3 + T_1 - R_4 - \text{sym}(M_2) + \text{sym}(X_2 F_1), \quad \bar{\Psi}_{2,3} = M_2 - M_1^T - T_1, \quad \bar{\Psi}_{2,4} = R_4, \quad \bar{\Psi}_{2,5} = P_3,
\]
\[
\bar{\Psi}_{2,6} = P_4, \quad \bar{\Psi}_{2,7} = -M_2 - M_3^T, \quad \bar{\Psi}_{2,8} = -X_2 + F_1^T X_3^T, \quad \bar{\Psi}_{3,3} = \text{sym}(M_1) - R_3 + Q_3 + Q_2 - Q_1, \quad \bar{\Psi}_{3,4} = T_1^T, \quad \bar{\Psi}_{3,5} = -P_6, \quad \bar{\Psi}_{3,6} = -P_5^T, \quad \bar{\Psi}_{3,7} = -M_1 + M_3^T,
\]
\[
\bar{\Psi}_{4,4} = -R_4 - Q_2, \quad \bar{\Psi}_{4,5} = -P_5, \quad \bar{\Psi}_{4,6} = -P_4, \quad \bar{\Psi}_{5,5} = -R_1 - 4S_1, \quad \bar{\Psi}_{5,7} = P_5^T, \quad \bar{\Psi}_{5,8} = P_3^T, \quad \bar{\Psi}_{6,6} = -R_2 - 4\tilde{\zeta}_2 S_2, \quad \bar{\Psi}_{6,7} = P_4, \quad \bar{\Psi}_{6,8} = P_2^T, \quad \bar{\Psi}_{7,7} = -\text{sym}(M_3), \quad \bar{\Psi}_{8,8} = -\text{sym}(X_3) + h_1^2 R_3 + h_1^2 R_4 + h_1^4 S_1 + \zeta_2 \tilde{\zeta} S_2,
\]

where $\zeta_2$ and $\tilde{\zeta}$ are defined in Theorem 3.3.1.

**Proof.** The corollary can be proved by following the same line of the proof of Theorem 3.3.1, with the only difference in calculating the upper-bounds of $\dot{V}_3(t)$ and $\dot{V}_4(t)$ given in (3.29) and (3.30), respectively. Let us define $\bar{z}_i \triangleq z_i, \quad i = \{1, 2, \cdots, 5\}, \quad \bar{z}_6 \triangleq z_7, \quad \bar{z}_7 \triangleq z_9, \quad \bar{z}_8 \triangleq z_{10}$, where $z_i, \quad i = \{1, 2, \cdots, 10\}$ are as defined in the proof of Theorem 3.3.1. Applying the inequalities of Lemma 2.2.4 to the single and double integral terms of (3.27) and (3.28), together with the reciprocally convex approach, we realize that upon the feasibility of the inequality (3.38), the following inequalities are satisfied,

\[
\dot{V}_3(t) \leq h_1^2 \bar{z}_1^T R_1 \bar{z}_1 + h_1^2 \bar{z}_8^T R_3 \bar{z}_8 + h_1^2 \bar{z}_1^T R_2 \bar{z}_1 + h_1^2 \bar{z}_8^T R_4 \bar{z}_8 - (\bar{z}_1 - \bar{z}_3)^T R_3 (\bar{z}_1 - \bar{z}_3) - \bar{z}_6^T R_1 \bar{z}_6
\]
\[
\dot{V}_4(t) \leq h_1^4 \bar{z}_8^T S_1 \bar{z}_8 + \zeta_2 \tilde{\zeta} \bar{z}_8^T S_2 \bar{z}_8
\]
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\[ -4 \left( h_1 \bar{z}_1 - \bar{z}_5 \right)^T S_1 \left( h_1 \bar{z}_1 - \bar{z}_5 \right) \]

\[ -4 \frac{\tilde{\zeta}}{\zeta} \left( h_12 \bar{z}_1 - \bar{z}_6 \right)^T S_2 \left( h_12 \bar{z}_1 - \bar{z}_6 \right)^T. \]

It follows that replacing (3.29) and (3.30) respectively by Inequalities (3.39) and (3.40) in attaining the matrix inequality associated with the negative-definiteness of the LKF derivative, the LMI (3.37) is achieved as a substitute to (3.18), which completes the proof of the corollary.

Inspired by Remark 2.2.2, we construct another stability criterion by further developing Theorem 3.3.1 through splitting the double-integral term \[ \int_{-h_1}^{0} \int_{s}^{0} \dot{x}^T(t+u)S_2 \dot{x}(t+u) duds, \] before calculating its lower-bound. The following theorem summarizes the third main contribution of the chapter.

**Theorem 3.3.2.** Given scalars \( 0 < h_1 < h_2 \) and \( \mu \), if there exist constant parameters \( P \in S^{3n} \), \( Q_i \in S^n \), \( i = \{1, 2\} \), \( Q_3 \succ 0 \in S^n \), \( R_j \in S^n \), \( j = \{1, 2, 3, 4\} \), \( S_k \in S^n \), \( k = \{1, 2\} \), and any matrices \( M_i \in \mathbb{R}^{n \times n} \), \( X_i \in \mathbb{R}^{n \times n} \), \( i = \{1, 2, 3\} \), \( T_{ij} \in \mathbb{R}^{n \times n} \), \( H_{ij} \), and \( \bar{H}_{ij} \), \( i,j = \{1, 2\} \), such that the following LMIs are satisfied, then for any delay satisfying (3.2), the system (3.1) is globally asymptotically stable.

\[ \tilde{\Psi}_i \prec 0 \quad i = \{1, 2\}, \quad (3.41) \]

\[ \tilde{\Pi} \succ 0, \quad (3.42) \]

\[
\begin{bmatrix}
2R_4 & -3R_4 & T_{11} & T_{12} \\
* & 2R_4 & T_{21} & T_{22} \\
* & * & 2\bar{R}_i & -3\bar{R}_i \\
* & * & * & 2\bar{R}_i
\end{bmatrix} \succeq 0 \quad i = \{1, 2\}, \quad (3.43)
\]
\[ \begin{bmatrix} 2S_2 & -4S_2 & H_{11} & H_{12} \\ * & 12S_2 & H_{21} & H_{22} \\ * & * & 2S_2 & -4S_2 \\ * & * & * & 12S_2 \end{bmatrix} \succeq 0, \]

\[ \begin{bmatrix} S_2 & -2S_2 & \bar{H}_{11} & \bar{H}_{12} \\ * & 6S_2 & \bar{H}_{21} & \bar{H}_{22} \\ * & * & S_2 & -2S_2 \\ * & * & * & 6S_2 \end{bmatrix} \succeq 0, \]

where \( \bar{R}_i = \bar{R}(h(t))|_{h(t)=h_i} \), and \( \bar{R}(h(t)) = R_4 + 2h_{12}(h_2 - h(t))S_2 \). Moreover, \( \tilde{\Psi}_i = \tilde{\Psi}(h(t))|_{h(t)=h_i} \), wherein \( \tilde{\Psi}(h(t)) = [\tilde{\Psi}_{i,j}(h(t))]_{16 \times 16} \) with the following non-zero elements:

\[ \tilde{\Psi}_{1,1} = \text{sym}(X_1F_0) + \text{sym}(P_3) + Q_1 + h_2^2R_1 - 4(R_3 + 2h_{12}^2S_2) - 6h_1^2S_1 + h_{12}^2R_2, \]

\[ \tilde{\Psi}_{1,2} = F_0^TX_2^T + X_1F_1 + P_2, \quad \tilde{\Psi}_{1,3} = -P_3 - 2(R_3 + 2h_{12}^2S_2), \quad \tilde{\Psi}_{1,4} = -P_2, \]

\[ \tilde{\Psi}_{1,5} = P_6 + \frac{6}{h_1}(R_3 + 2h_{12}^2S_2), \quad \tilde{\Psi}_{1,6} = 12S_1, \quad \tilde{\Psi}_{1,7} = \tilde{\Psi}_{1,8} = P_5^T, \quad \tilde{\Psi}_{1,10} = P_1 - X_1 + F_0^TX_3^T, \quad \tilde{\Psi}_{1,11} = P_2, \quad \tilde{\Psi}_{2,2} = -(1 - \mu)Q_3 - 4R_4 - 4\bar{R}(h(t)) - \text{sym}(M_2) + 2\text{sym}(T_1 + T_2 - T_{21} + T_{22}) + \text{sym}(X_2F_1) - 6(h_2 - h(t))^2S_2, \quad \tilde{\Psi}_{2,3} = M_2 - M_1^T - 2\bar{R}(h(t)) - 2T_{11} - 2T_{21} - 12(h_2 - h(t))(h(t) - h_1)(H_{11} + \bar{H}_{11} + H_{12} + \bar{H}_{12} + 0.5(H_{21} + \bar{H}_{21}) + 0.25(H_{22} + \bar{H}_{22})) \]

\[ \tilde{\Psi}_{2,4} = -2R_4 - 2T_{11} - 2T_{12}^T, \quad \tilde{\Psi}_{2,5} = P_5, \quad \tilde{\Psi}_{2,7} = P_4, \]

\[ \tilde{\Psi}_{2,8} = P_4 + 12(h_2 - h(t))\{H_{11} + \bar{H}_{11} + 0.5(H_{22} + \bar{H}_{22})\}, \quad \tilde{\Psi}_{2,10} = F_1^TX_3^T - X_2, \]

\[ \tilde{\Psi}_{2,11} = -M_2 + M_3^T, \quad \tilde{\Psi}_{2,12} = 6R_4 - 2(T_{11}^T + T_{22}^T), \quad \tilde{\Psi}_{2,13} = -2(T_{12} + T_{22}) + 6\bar{R}(h(t)), \]

\[ \tilde{\Psi}_{2,14} = 12(h_2 - h(t))S_2, \quad \tilde{\Psi}_{2,15} = 12(h_2 - h(t))\{H_{12} + \bar{H}_{12} + 0.5(H_{22} + \bar{H}_{22})\}, \]

\[ \tilde{\Psi}_{3,3} = \text{sym}(M_1) - 4\bar{R}(h(t)) - 4(R_3 + 2h_{12}^2S_2) + Q_3 + Q_2 - Q_1 - 6(h(t) - h_1)^2S_2, \]

\[ \tilde{\Psi}_{3,4} = 2T_{11}^T, \quad \tilde{\Psi}_{3,5} = -P_6 + \frac{6}{h_1}(R_3 + 2h_{12}^2S_2), \quad \tilde{\Psi}_{3,7} = -P_5^T + 12(h(t) - h_1)\{H_{11}^T + \bar{H}_{11}^T + 0.5(H_{12}^T + \bar{H}_{12})\}, \quad \tilde{\Psi}_{3,8} = -P_5^T, \quad \tilde{\Psi}_{3,11} = -M_1 + M_3^T, \quad \tilde{\Psi}_{3,12} = 2T_{21}^T, \]

\[ \tilde{\Psi}_{3,13} = 6\bar{R}(h(t)), \quad \tilde{\Psi}_{3,15} = 12(h(t) - h_1)S_2, \quad \tilde{\Psi}_{4,4} = -Q_2 - 4R_4, \quad \tilde{\Psi}_{4,5} = -P_5, \]
3.3 New Stability Analysis Method

\[ \tilde{\Psi}_{4,7} = \tilde{\Psi}_{4,8} = -P_4, \quad \tilde{\Psi}_{4,12} = 6R_4, \quad \tilde{\Psi}_{4,13} = 2T_{12}, \quad \tilde{\Psi}_{5,5} = -4R_1 - \frac{12}{h^2_1}(R_3 + 2h_1^3S_2) - 12S_1, \quad \tilde{\Psi}_{5,6} = \frac{6}{h^1_1}R_1 + \frac{24}{h^1_1}S_1, \quad \tilde{\Psi}_{5,10} = P^T_3, \quad \tilde{\Psi}_{5,11} = P^T_5, \quad \tilde{\Psi}_{6,6} = -\frac{12}{h^2_1}R_1 - \frac{72}{h^1_1}S_1, \quad \tilde{\Psi}_{7,7} = -4R_2 - 12S_2, \quad \tilde{\Psi}_{7,8} = -4R_2 - 12(H_{11} + \tilde{H}_{11}), \quad \tilde{\Psi}_{7,9} = \frac{6}{h^1_1}R_2, \]
\[ \tilde{\Psi}_{7,10} = \tilde{\Psi}_{8,10} = P^T_2, \quad \tilde{\Psi}_{7,11} = \tilde{\Psi}_{8,11} = P_4, \quad \tilde{\Psi}_{7,14} = \tilde{\Psi}_{8,15} = 24S_2, \quad \tilde{\Psi}_{7,15} = -12(H_{12} + \tilde{H}_{12}), \quad \tilde{\Psi}_{8,8} = -4R_2 - 12S_2, \quad \tilde{\Psi}_{8,9} = \frac{6}{h^1_2}R_2, \quad \tilde{\Psi}_{8,14} = -12(H^T_{21} + H^T_{21}), \quad \tilde{\Psi}_{9,9} = -\frac{12}{h^2_1}R_2, \quad \tilde{\Psi}_{10,10} = h^2_1R_3 + h^2_1R_4 + h^2_1S_1 + \gamma_2h^2_2S_2 - \text{sym}(X_3), \quad \tilde{\Psi}_{11,11} = -\text{sym}(M_3), \quad \tilde{\Psi}_{12,12} = -12R_4, \quad \tilde{\Psi}_{12,13} = 2T_{22}, \quad \tilde{\Psi}_{13,13} = -12R(h(t)), \quad \tilde{\Psi}_{14,14} = -72S_2, \quad \tilde{\Psi}_{14,15} = 12(H_{22} + \tilde{H}_{22}), \quad \tilde{\Psi}_{15,15} = -72S_2. \]

Moreover, all of the non-zero elements of \( \tilde{\Pi} = [\tilde{\Pi}_{i,j}]_{7 \times 7} \) are the same as those of the matrix \( \Pi \) defined in Theorem 3.3.1, except with minor differences in the following terms:
\[ \tilde{\Pi}_{1,1} = P_1 + 3h_1R_3 + h_12b_{11}R_4 + 3h_1^3S_1 + 3\frac{\gamma_2h^2_2}{\zeta_3}S_2, \quad \tilde{\Pi}_{1,4} = b_{14}h_12R_4 - 6\frac{\gamma_2h^2_2}{\zeta_3}S_2, \quad \tilde{\Pi}_{4,4} = \frac{12}{h^1_2}Q_2 + 6\frac{\gamma_1\zeta_4}{\zeta_3}R_2 + \frac{12h^2_1}{\zeta_3}S_2 + 36\frac{\gamma_2}{\zeta}R_4. \]

**Proof.** The LKF candidate \( \tilde{V}(t, x(t), x_t, \dot{x}_t) \) is defined similar to (3.21) with the exceptional difference in the coefficient of the \( S_2 \) integral term, i.e.
\[ \tilde{V}_4(\dot{x}_t) = \cdots + 2h^2_1 \int_{-h_2}^{-h_1} \int_{-h_2}^{0} \int_{t+s_2}^{t} \dot{x}^T(s)S_2\dot{x}(s)dsdsds_1. \]

Moreover, let us define \( \tilde{z}_i \triangleq z_i \) for \( i = \{1, \cdots, 6\} \), \( \tilde{z}_7 \triangleq \int_{-h_2}^{-h_1} x(t + u)du \), \( \tilde{z}_8 \triangleq \int_{-h_2}^{-h_1} x(t + u)duds \), \( \tilde{z}_{10} \triangleq \int_{-h_2}^{-h_1} \dot{x}(t + u)du \), \( \tilde{z}_{12} \triangleq \frac{1}{h^2_1-h(t)}\tilde{z}_7, \tilde{z}_{13} \triangleq \frac{1}{h^2_1-h(t)}\tilde{z}_8, \tilde{z}_{14} \triangleq \frac{1}{h^2_1-h(t)}\int_{-h_2}^{-h_1} \int_{-h_2}^{-h_1} x(t + u)duds \), and \( \tilde{z}_{15} \triangleq \frac{1}{h^2_1-h(t)}\int_{-h_2}^{-h_1} \int_{-h_2}^{-h_1} x(t + u)duds. \) Furthermore, the following identity is employed in estimating the upper-bound of the derivative of \( \tilde{V}(t) = \tilde{V}(t, x(t), x_t, \dot{x}_t) \):
\[
\begin{align*}
\int_{-h_2}^{-h_1} \int_{s}^{0} \dot{x}^T(t + u)S_2 \dot{x}(t + u) \, du &= \int_{-h_2}^{-h(t)} \int_{s}^{-h(t)} \dot{x}^T(t + u)S_2 \dot{x}(t + u) \, du \\
&+ \int_{-h_1}^{-h(t)} \int_{s}^{-h_1} \dot{x}^T(t + u)S_2 \dot{x}(t + u) \, du \\
&+ (h_2 - h(t)) \int_{-h(t)}^{-h_1} \dot{x}^T(t + u)S_2 \dot{x}(t + u) \, du \\
&+ h_{12} \int_{-h_1}^{0} \dot{x}^T(t + u)S_2 \dot{x}(t + u) \, du.
\end{align*}
\] (3.46)

Thereafter, differentiating \( \bar{V}(t) \) along the solution of (3.1), taking the relations (3.22)-(3.24) and (3.46) into account, and applying Lemmas 2.2.5, 2.2.7, and 3.3.1, as well as Corollary 3.3.1 and Remark 2.2.2, the following can be attained subject to the fulfilment of the inequalities (3.43)-(3.45):

\[
\begin{align*}
\dot{\bar{V}}_1(t) &= \text{sym} \left( \begin{bmatrix} \tilde{z}_1 & * & * \\ \tilde{z}_7 + \tilde{z}_8 & P_1 & P_2 \\ \tilde{z}_5 & * & P_3 \end{bmatrix} \begin{bmatrix} P_1 & P_2 & P_3 \\ * & P_4 & P_5 \\ * & * & P_6 \end{bmatrix} \begin{bmatrix} \tilde{z}_{10} \\ \tilde{z}_2 - \tilde{z}_4 + \tilde{z}_{11} \\ \tilde{z}_1 - \tilde{z}_3 \end{bmatrix} \right), \\
\dot{\bar{V}}_2(t) &\leq \tilde{z}_1^T Q_1 \tilde{z}_1 + \tilde{z}_3^T (Q_2 + Q_3 - Q_1) \tilde{z}_3 - \tilde{z}_4^T Q_2 \tilde{z}_4 - (1 - \mu) \tilde{z}_2^T Q_3 \tilde{z}_2, \\
\dot{\bar{V}}_3(t) + \dot{\bar{V}}_4(t) &\leq h_1^2 \tilde{z}_1^T R_1 \tilde{z}_1 + h_2^2 \tilde{z}_3^T R_2 \tilde{z}_3 + h_3^2 \tilde{z}_{10}^T R_3 \tilde{z}_{10} + h_4^2 \tilde{z}_{10}^T R_4 \tilde{z}_{10} \\
&- 2 \begin{bmatrix} \tilde{z}_5 \\ \tilde{z}_6 \end{bmatrix}^T \begin{bmatrix} 2 & -\frac{3}{h_1} \\ * & \frac{6}{h_1^2} \end{bmatrix} \otimes R_1 \begin{bmatrix} \tilde{z}_5 \\ \tilde{z}_6 \end{bmatrix} \\
&- 2 \begin{bmatrix} \tilde{z}_1 - \tilde{z}_3 \\ h_1 \tilde{z}_1 - \tilde{z}_5 \end{bmatrix}^T \begin{bmatrix} 2 & -\frac{3}{h_1} \\ * & \frac{6}{h_1^2} \end{bmatrix} \otimes \left( R_3 + 2h_4^2 S_2 \right) \begin{bmatrix} \tilde{z}_1 - \tilde{z}_3 \\ h_1 \tilde{z}_1 - \tilde{z}_5 \end{bmatrix} \\
&- 2 \begin{bmatrix} \tilde{z}_7 + \tilde{z}_8 \\ \tilde{z}_9 \end{bmatrix}^T \begin{bmatrix} 2 & -\frac{3}{h_1} \\ * & \frac{6}{h_1^2} \end{bmatrix} \otimes R_2 \begin{bmatrix} \tilde{z}_7 + \tilde{z}_8 \\ \tilde{z}_9 \end{bmatrix} \\
&+ h_{12} \int_{-h_1}^{0} \dot{x}^T(t + u)S_2 \dot{x}(t + u) \, du.
\end{align*}
\]
3.3 New Stability Analysis Method

\[ \dot{\tilde{V}}(t) \leq \tilde{Z}^T \tilde{\Psi}(h(t)) \tilde{Z}, \]  \hspace{1cm} (3.50)\]

where \( \tilde{Z} = \left[ \begin{array}{c} \tilde{z}_1^T \\ \tilde{z}_2^T \\ \vdots \\ \tilde{z}_{15}^T \end{array} \right]^T \). Moreover, it is clear that the inequality (3.36) is modified to

\[ \tilde{V}(t) \geq \beta^T \tilde{\Pi} \beta. \]  \hspace{1cm} (3.51)\]
Hence, considering Inequalities (3.50) and (3.51), and according to Lemma 2.2.8, the statement of the theorem is concluded from the Lyapunov Krasovskii theorem.

**Remark 3.3.7.** On the numerical complexity and costs point of view the criterion of Theorem 3.3.2 is considerably more complicated than that of Theorem 3.3.1, thus implies more computational burden. Beside having more decision variables involved, the main reason for this additional cost is the extra number of LMIs and the larger sizes of the LMI matrices.

The extension of Theorem 3.3.2 to the case of unknown derivative of delay can be directly achieved by assuming \( Q_3 \equiv 0 \).

**Corollary 3.3.3.** Given positive scalars \( h_1 < h_2 \), if there exist constant matrices \( P \in \mathbb{S}^{3n} \), \( Q_i \in \mathbb{S}^n, i = \{1, 2\} \), \( R_i \in \mathbb{S}^n, i = \{1, 2, 3, 4\} \), \( S_i \in \mathbb{S}^n, i = \{1, 2\} \), and free-weighting matrices \( M_i \in \mathbb{R}^{n \times n}, X_i \in \mathbb{R}^{n \times n}, i = \{1, 2, 3\} \), \( T_{ij} \in \mathbb{R}^{n \times n}, H_{ij} \) and \( \bar{H}_{ij} \), \( i, j = \{1, 2\} \), such that the LMIs below together with the LMIs (3.42)-(3.45) are satisfied, then for any delay satisfying (3.2) with arbitrary \( \mu \), the system (3.1) is globally asymptotically stable.

\[
\Psi_i < 0 \quad i = \{1, 2\},
\]

where \( \Psi_i = \Psi_i|_{Q_3=0} \).

To emphasize the effectiveness of the positive-definiteness LMI (3.42) in the numerical examples the following Corollary, which is directly deduced from Theorem 3.3.2, is given.

**Corollary 3.3.4.** Given the scalars \( 0 < h_1 < h_2 \) and \( \mu \), if there exist constant positive definite matrices \( P \in \mathbb{S}^{3n} \), \( Q_i \in \mathbb{S}^n, i = \{1, 2, 3\} \), \( R_i \in \mathbb{S}^n, i = \{1, 2, 3, 4\} \), \( S_i \in \mathbb{S}^n, i = \{1, 2\} \), and free-weighting matrices \( M_i \in \mathbb{R}^{n \times n}, X_i \in \mathbb{R}^{n \times n}, i = \{1, 2, 3\} \), \( T_{ij} \in \mathbb{R}^{n \times n}, H_{ij} \) and \( \bar{H}_{ij} \), \( i, j = \{1, 2\} \), such that the LMIs (3.41), and (3.43)-(3.45) are realized, then for any delay satisfying (3.2), the system (3.1) is globally asymptotically stable.
3.4 Benchmarking Numerical Examples of the Stability Analysis of the System

In this section two numerical examples are given to evaluate and compare the effectiveness and the less conservatism of the proposed delay-dependent stability conditions.

3.4.1 Example 1

Consider the system (3.1) with the parameters given below

\[ F_0 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad F_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}. \]

Based on the above parameter values maximum allowable upper bounds of delay (MAUBD) for different values of \( h_1 \) and \( \mu \), for which the system remains stable were calculated from Theorems 3.3.1 and 3.3.2, and Corollaries 3.3.2 and 3.3.4. The results are tabulated in Table 3.1 and are benchmarked against some already reported data in the recent relevant papers. The values that are put blank in the table and also in the next benchmarking tables are either not reported by the referenced works, or their associated criteria are infeasible. Moreover, the “Non” expression stands for the inapplicability of the associated criterion in calculating an upper-bound of delay for given \( h_1 \) and \( \mu \) parameters. To evaluate the validity of the calculated MAUBDs the stability of the system is shown in Fig. 3.1 via simulations in the Simulink environment for two conditions of the delays with the largest calculated admissible \( h_2 \).

The table demonstrates the superiority of the criteria proposed in Theorems 3.3.1 and 3.3.2 in achieving less conservative upper-bounds of the delay values. In addition, comparing the results obtained from Corollary 3.3.2 and Theorem 3.3.1 emphasizes the significant
value that applying the Wirtinger-based integral inequalities described in Lemmas 2.2.5 and 3.3.1 bring, in comparison with the conventional Jensen's inequalities, in estimating the upper-bounds of the cross integral terms. Surprisingly, the stability condition of Corollary 3.3.2, which has been obtained by employing the same LKF as those of Theorems 3.3.1 and 3.3.2 and Corollary 3.3.4, is the least effective method among all of the benchmarked criteria for reducing the inherent conservatism of the Lyapunov Krasovskii stability analysis approach.

Further investigation of the proposed stability criteria of the chapter in Table 3.1 demonstrates that for small values of the lower-bound of delay (here $h_1 \leq 1s$) Theorem 3.3.1 is slightly more effective in estimating a higher MAUBD. On the other hand, by increasing $h_1$ Theorem 3.3.2 significantly outperforms the other established criteria, and Corollary 3.3.4 also exhibits a better performance with regard to Theorem 3.3.1. This point is more highlighted in Fig. 3.2, which is depicted based on the data reported in the table. In addition, by comparing the results of Theorem 3.3.2 and Corollary 3.3.4 in Fig. 3.2 it can be seen that for higher values of $h_1$ the positive-definiteness condition of the LKF can be very helpful in relieving the conservatism of the stability criteria.

Moreover, under the unknown delay-derivative condition MAUBD was calculated from Corollary 3.3.3 for different values of $h_1$. Table 3.2 illustrates the obtained results and compares them with those of the papers [132, 135, 136, 141]. The table highlights the effectiveness of the proposed new criteria, particularly for higher values of $h_1$.

### 3.4.2 Example 2

A system with the following parameters is taken into consideration:

$$F_0 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$$
3.4 Benchmarking Numerical Examples of the Stability Analysis of the System

The stability criteria of the chapter were tested on this system, and the obtained MAUBDs are benchmarked in Table 3.3 against the results of the papers [132, 138, 145, 146]. The stability of the system for two sets of conditions of the delay values with largest calculated admissible $h_2$ is justified in Fig. 3.3.

The table confirms that the stability criteria in Theorem 3.3.2 and Corollary 3.3.4 can be less restrictive, particularly for larger lower-bounds of delay. To add, it is clear that Corollary 3.3.2 that differs from the other constructed criteria only in employing Jensen’s

| Table 3.1: MAUBDs for different values of $h_1$ and $\mu$ in Example 1 |
|-----------------|---------|---------|---------|---------|---------|
| $\mu$          | Method  | 0.3     | 0.8     | 1       | 2       | 3       | 4       | 5       |
| 0.3            | Theorem 2 in [133] | 2.4322  | 2.4305  | 2.4232  | —       | —       | —       | —       |
|                 | Theorem 2 in [132] | —       | —       | —       | 3.0129  | 3.3408  | 4.1690  | 5.0275  |
|                 | Theorem 3.1 in [140] | —       | —       | —       | 3.03    | 3.43    | 4.23    | —       |
|                 | Corollary 3.3.2   | 2.3826  | 2.3877  | 2.3153  | Non     | Non     | Non     | Non     |
|                 | Theorem 3.3.1     | 2.9437  | 3.0734  | 3.1059  | 3.2779  | 3.4895  | 4.2938  | 5.1372  |
|                 | Corollary 3.3.4   | 2.9421  | 3.0734  | 3.1095  | 3.5825  | 4.2574  | 4.9039  | 5.5445  |
|                 | Theorem 3.3.2     | 2.9430  | 3.0734  | 3.1123  | 4.1166  | 5.6941  | 7.3855  | 9.1169  |
| 0.5            | Theorem 2 in [132] | —       | —       | —       | 2.5663  | 3.3408  | 4.1690  | 5.0275  |
|                 | Theorem 3.1 in [140] | —       | —       | —       | 2.71    | 3.43    | 4.23    | —       |
|                 | Theorem 1 (N=4) in [135] | —       | —       | 2.5898  | 3.1469  | 3.7895  | 4.4928  | 5.2377  |
|                 | Corollary 3.3.2   | 1.6927  | 1.5511  | 1.4130  | Non     | Non     | Non     | Non     |
|                 | Theorem 3.3.1     | 2.4245  | 2.7200  | 2.8538  | 2.7488  | 3.4895  | 4.2938  | 5.1372  |
|                 | Corollary 3.3.4   | 2.4242  | 2.4712  | 2.4717  | 2.8768  | 3.7306  | 4.6270  | 5.5442  |
|                 | Theorem 3.3.2     | 2.4245  | 2.4712  | 2.4717  | 3.3477  | 4.7947  | 6.3236  | 7.8880  |
| 0.9            | Theorem 2 in [132] | —       | —       | —       | 2.5663  | 3.3408  | 4.1690  | 5.0275  |
|                 | Theorem 3.1 in [140] | —       | —       | —       | 2.71    | 3.43    | 4.23    | —       |
|                 | Theorem 1 (N=4) in [135] | —       | —       | 2.5898  | 3.1469  | 3.7895  | 4.4928  | 5.2377  |
|                 | Theorem 1 (N=4) in [135] | —       | —       | 3.8859  | 4.0448  | 4.2117  | 4.3868  | —       |
|                 | Corollary 3.3.3   | 1.8201  | 2.0394  | 2.1343  | 2.8719  | 3.7306  | 4.6270  | 5.5441  |
|                 | Theorem 3.3.2     | 1.8212  | 2.0394  | 2.1344  | 3.1972  | 4.5555  | 5.9543  | 7.3668  |

| Table 3.2: MAUBDs for different values of $h_1$ and unknown $\mu$ in Example 1 |
|-----------------|---------|---------|---------|---------|---------|
| $\mu$          | Method  | 0.3     | 0.6     | 1.5     | 3       | 4       | 5       |
| 2              | Corollary 2 in [141] | 2.08    | 2.15    | 2.47    | —       | —       | —       |
|                 | Corollary 7 in [132] | —       | —       | —       | 3.3408  | 4.1690  | 5.0275  |
|                 | Corollary 2 ($\lambda = 3$) in [136] | —       | —       | —       | 3.499   | 4.2966  | 5.14    |
|                 | Theorem 1 (N=4) in [135] | —       | —       | —       | 3.7895  | 4.4928  | 5.2377  |
|                 | Corollary 3.3.3   | 1.8216  | 1.9588  | 2.5637  | 4.5555  | 5.9543  | 7.3668  |
3.4 Benchmarking Numerical Examples of the Stability Analysis of the System

inequalities as a substitute to Wirtinger-based integral inequalities gives the worst estimation of the upper-bounds of delay, even compared to other methods. Moreover, analogous to Example 1 it can be seen from the table and more clearly from Fig. 3.4 that for small values of \( h_1 \) Theorem 3.3.1 delivers a slightly tighter estimation of the MAUBD for the system, while Theorem 3.3.2 and Corollary 3.3.4 are superior to this aim for larger values of \( h_1 \). The effectiveness of the LMI (3.42) associated with the positive-definiteness of the LKF is also highlighted by comparing the results of Theorem 3.3.2 and Corollary 3.3.4.

**Remark 3.4.1.** Based on the results of the preceding examples it can be remarked that among the methods proposed in the chapter, employing Theorem 3.3.1 is more efficient in application (e.g. controller design for the system) whenever the lower-bound of delay is relatively small (in these examples \( h_1 \leq 1s \)). This choice is particularly supported due to the fact that it can significantly save for the computational costs as earlier stated in Remark 3.3.7.

The next chapter sheds some light to the topic of functional observers and provides a
new recursive, systematic, and reliable algorithm to design minimum possible order FOs for ordinary LTI systems, whenever it is applicable to do so.

3.5 Conclusions

New delay-dependent stability criteria for LTI systems with interval time-varying delay have been established in terms of LMI conditions. Both cases of upper-bounded and unknown delay-derivatives have been considered. The new stability conditions are obtained

Figure 3.2: The comparison of three proposed criteria of the chapter under different conditions of the delay’s derivative and lower-bound in Example 1
3.5 Conclusions

Table 3.3: MAUBDs for different values of \( h_1 \) and \( \mu \) in Example 2

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>Method</th>
<th>( 0.3 )</th>
<th>( 0.8 )</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
<th>( 4 )</th>
<th>( 5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>Theorem 2 in [132]</td>
<td>2.22</td>
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<td>2.4798</td>
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<td></td>
<td>[146] (N=2)</td>
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<td>2.4335</td>
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<td>3.5074</td>
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<td>2.4275</td>
<td>2.4334</td>
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<td>6.0000</td>
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Figure 3.3: The convergence of the states of the system in Example 2 under largest MAUBDs

based on using a new Wirtinger-based double-integral inequality that has been derived and analyzed in Section 3.3.1. The inequality generalizes the recently obtained inequality in [12]. Using the latter bounding techniques, along with employing the reciprocally convex optimization methods, a condition on positiveness of the LKF, and a few necessary slack variables, the proposed criteria can improve the conservatism problem of the other existing criteria. The effectiveness of the proposed criteria have been highlighted via two
Figure 3.4: The comparison of three proposed criteria of the chapter under different conditions of the delay’s derivative and lower-bound in Example 2

illustrative benchmarking numerical examples. It is believed that the combination of the proposed analysis method with the delay-partitioning approach can further reduce from the conservatism of the stability conditions.
Chapter 4

Minimal Order Functional Observers for LTI Systems

4.1 Introduction

Recently, one of the important research directions in the area of FO design has been devoted to finding the least possible order for a functional observer \([59, 64, 147–149]\), and designing an observer with that order. In \([59, 64]\) the concept of *functional observability* is introduced and the necessary and sufficient conditions for functional observability or functional detectability of a system are derived. Fernando and Trinh \([61]\) propose a scheme for designing functional observers for linear systems even if the necessary and sufficient conditions of Darouach \([58]\) are not satisfied, and even the system is not observable/detectable, provided that it is functional observable/detectable. Although the proposed method might result in the design of a reduced order observer, the resulted observer is not necessarily *minimal* (of minimum possible order).

In this chapter, to address the minimality requirement of the functional observer, the
definitions of functional observability are revisited. As the first contribution, we elaborate on a practical recursive algorithm to increase the order of the FO in a way that the necessary and sufficient conditions of the existence of a functional observer are satisfied. This is accomplished by appending the minimum required number of auxiliary functions to the original function that is desired to be observed.

In addition, an important problem that is addressed in most of the significant previous works in this field is solving a number of coupled matrix equations called interconnected generalized (or constrained) Sylvester equations [150]. This set of equations might have infinite number of solutions and each solution is a set of observer parameters for the system that should satisfy the observer equations. However, to the best of the author’s knowledge, there are three state of the art methodologies to solve the constrained Sylvester equations existing in the literature: 1) transformation based approach [58, 62, 90, 147], 2) parametric approach [149, 150], and 3) direct approach [148].

The first method, which has been the most popular one, is based on a number of matrix transformations that break the unknown matrices into smaller sub-matrices and increase the number of observer equations [58, 62]. This approach can also be classified to three different categories that are more illustrated in Section 4.2.1. The transformation-based approaches have been applied to single and multi-functional observers, as well as unknown-input functional observers [80, 104, 151–153]. It can be remarked that the majority of the recent contributions in this field of research have used one of the schemes of this approach in their design algorithms. This can be due to the simplicity of converting the methodologies of this class to numerical algorithms. Moreover, they can be applied to general MIMO linear systems, as well as a class of nonlinear systems. However, it is shown in this chapter that the schemes of this category have different numerical properties and might have some numerical issues in particular situations. Motivated by this observation, we have shed some effective light into this problem by proposing a new transformation-based design algorithm
that improves the performance of the observer, as the second main novelty of the chapter. Our observations show that the proposed algorithm can be more reliable and effective than the other existing methods. The necessary and sufficient conditions are derived for the existence of an asymptotic functional observer for the system using the novel design approach. The equivalence between the obtained new conditions and the renown conditions proposed by Darouach [58] are also verified. Numerical examples and simulation results show the effectiveness of our design algorithms, as well as some of the issues related to the other conventional design schemes.

The parametric approach, as the second framework of solving the observer equations, propose a rather rigorous algorithm to solve the constrained observer equations. The method requires the system to be observable, and the eigenvalues of the observer must be distinct, and they should be chosen via a trial and error procedure.

On the other hand, the direct approach proposed by Rotella and Zambettakis [148] gives a straightforward procedure to find the minimal order single functional observer for the system. This approach is the only available FO design method that has been applied to linear time-varying systems [73]. Moreover, this is the only algorithm that does not need to solve the Sylvester equations to find the observer parameters. These advantages make this approach, which is also one of the most recent ones, quite useful. However, as it will be more discussed in this chapter, the original direct method is not applicable to design multi-functional observers in general. Hence, as the third contribution of this chapter, an extension to the direct approach is proposed that makes it practical for designing minimal multi-functional observers. This extension directly owes to the first contribution of the chapter.
4.2 Problem Statement and Preliminaries

Notations: Throughout the chapter the following simplified notations are used. The expression \([A_1; A_2]\) is equivalent to \([A_1^T, A_2^T]^T\); \(\text{rows}(X)\) indicates the number of rows of the matrix \(X\); \(I_k\) and \(I_{k \times l}\) respectively denote the \(k \times k\) and \(k \times l\) identity matrices; and \(g_{k \times l}\) denotes a \(k \times l\) matrix with elements all equal to number “\(g\)”. Furthermore, \(X^\dagger\) represents the Moore-Penrose pseudo-inverse or the generalized inverse of the matrix \(X\), and \(X^\perp\) denotes a right orthogonal matrix of \(X\), wherein \(XX^\perp = 0\). In addition, denote \(\mathcal{R}(M)\) and \(\mathcal{N}(M)\) as the row space and the null space of the matrix \(M\), respectively. To add, \([[S]]\) denotes a matrix of row bases vectors for the subspace \(S\). Furthermore, let us define \(H_2 \triangleq [[\mathcal{R}([C; CA; L])]]\), \(H_3 \triangleq \left[\mathcal{R}\left(\begin{bmatrix} H_2 \\ LA \end{bmatrix}\right)\right]\), \(a \triangleq \text{rank}(H_2)\), \(b \triangleq \text{rank}(H_3)\).

Consider the following LTI system,

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) \\
z(t) &= Lx(t)
\end{align*}
\tag{4.1}
\]

where \(A \in \mathbb{R}^{n \times n}\), \(B \in \mathbb{R}^{n \times m}\), \(C \in \mathbb{R}^{p \times n}\) and \(L \in \mathbb{R}^{l \times n}\) are constant and known matrices. The main aim of the observer is to reconstruct a functional \(z_0(t) = L_0x(t)\) with \(z_0 \in \mathbb{R}^{l_0}\), where \(l_0 \leq l\). The matrix \(L = [L_0; R_1; R_2]\) is defined in a way that a stable functional observer can be designed for the system. If \(l > l_0\), then \(l - l_0\) auxiliary functions are appended to the vector \(z_0(\cdot)\) to form the new vector \(z(\cdot)\) in an attempt to design the minimum-order functional observer for the new system. Clearly, if any auxiliary functions are appended to the system’s nominal vector \(z_0(t)\), the designed observer cannot be of minimum-order, while it can be of minimal order. One of our goals is thus to find appropriate matrices \(R_1\) and \(R_2\) with the minimum possible number of rows.

The following assumptions are considered in the chapter,
A. The matrices \( C \) and \( L_0 \) are of full row rank. Moreover, \( \text{rank}([C; L_0]) = p + l_0 \).

B. The number of functions to be observed are not larger than the difference between the number of states and outputs of the system \( (l \leq n - p) \), and also \( n > p \).

C. The triple \( \Sigma = (A, C, L_0) \) is functional observable, or at least functional detectable.

Assumptions A and B do not fail the generality of the chapter. For example, if Assumption B is not satisfied, the least possible order for the functional observer is \( n - p \), which is already solved using the reduced order Luenberger observer \([66, 154]\). Furthermore, if Assumption C is violated, then it is not possible to design a functional observer for the triplet \( \Sigma [59, 155] \). On the other hand, if Assumption C is satisfied, then the rows of \( R_1 \) and \( R_2 \) can be determined (this is later described in Algorithm 1).

The following theorem shows a simple method for examining the functional observability/detectability of the system \( \Sigma \),

**Theorem 4.2.1** ([59, 64]). The system \((A, C, L_0)\) is functional detectable if and only if

\[
\text{rank} \begin{bmatrix}
  sI_n - A \\
  C \\
  L_0
\end{bmatrix} = \text{rank} \begin{bmatrix}
  sI_n - A \\
  C
\end{bmatrix}, \quad \forall s \in \mathbb{C}^+.
\]  

Moreover, it is functional observable if and only if (4.2) holds for all \( s \in \mathbb{C} \). Equivalently,
Σ is functional observable if and only if

\[
\begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{n-1}
\end{bmatrix}
\begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{n-1}
\end{bmatrix} =
\begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{n-1}
\end{bmatrix}.
\]  

(4.3)

Remark 4.2.1. Condition (4.2) can be directly testified in MATLAB, using the invariant zeros of the Rosenbrock’s matrix. If the matrices of both sides of (4.2) possess similar invariant zeros, then they have equal ranks.

The \( l \)’th order functional observer structure, which is the minimal-order observer for the system \( Σ \) is given as follows.

\[
\dot{\omega}(t) = F\omega(t) + Gu(t) + Hy(t) \\
\hat{z}(t) = \omega(t) + Vy(t),
\]  

(4.4)

where \( \omega(\cdot) \in \mathbb{R}^l \) is the observer’s state vector, and \( \hat{z}(\cdot) \) is the estimation of \( z(\cdot) \). Moreover, \( F \in \mathbb{R}^{l \times l}, \ G \in \mathbb{R}^{l \times m}, \ H \in \mathbb{R}^{l \times p} \) and \( V \in \mathbb{R}^{l \times p} \) are the parameters that should be designed. The block diagram of the observer is depicted in Fig. 4.1.

Let us define the estimation error signal \( e(t) \triangleq \hat{z}(t) - z(t) \). Then, according to [58] the necessary and sufficient conditions of the existence of a stable functional observer (4.4) for the system (4.1) are Conditions I and II given in the sequel:
4.2 Problem Statement and Preliminaries

Condition I:

\[
\text{rank } \begin{bmatrix}
    LA \\
    CA \\
    C \\
    L
\end{bmatrix} = \text{rank } \begin{bmatrix}
    CA \\
    C \\
    L
\end{bmatrix}
\] \hspace{1cm} (4.5)

Condition II:

\[
\text{rank } \begin{bmatrix}
    sL - LA \\
    CA \\
    C
\end{bmatrix} = \text{rank } \begin{bmatrix}
    CA \\
    C \\
    L
\end{bmatrix} \quad \forall s \in \mathbb{C}^+ \hspace{1cm} (4.6)
\]

Literally, Matrices \( R_1 \) and \( R_2 \) in \( L \) are added to satisfy Conditions I and II, respectively. It is noted that Condition I is equivalent to \( \mathcal{R}(H_2) = \mathcal{R}(H_3) \), or \( a = b \). If Conditions I and II are
fulfilled, then the following necessary and sufficient conditions of the asymptotic stability of the observer (4.4), such that \( \lim_{t \to \infty} e(t) = 0 \) can be achieved via the appropriate design of the observer parameters [58]:

(a) The observer matrix \( F \) is Hurwitz.

(b) There exists a matrix \( T \in \mathbb{R}^{l \times n} \), such that the set of equations

\[
FT - TA + HC = 0 \quad (4.7a)
\]

\[
G = TB \quad (4.7b)
\]

\[
L - T - VC = 0 \quad (4.7c)
\]

are satisfied using appropriate matrices \( F, H, G \) and \( V \).

### 4.2.1 Conventional Observer Design Methods

Although the set of matrix equations (4.7) might have infinite number of solutions if they are solvable, finding an analytical solution of them is stringent. This is due to the difficulties that are raised from the interconnection of the Sylvester equations (4.7a) and (4.7c). The Sylvester equation (4.7a) confers that the matrix \( \bar{H} \triangleq TA - FT \in \mathbb{R}^{l \times n} \) must lie in the row space spanned by the matrix \( C \). It means that although \( \bar{H} \) can potentially be of rank \( l \), it must be at most of rank \( p \), i.e, the rank of Matrix \( C \). However, \( l \) might be larger than \( p \), and the main challenge in solving (4.7a) is thus to find matrices \( T \) and \( F \), such that \( \bar{H} \) lies in the row space of \( C \), while simultaneously \( F \) is Hurwitz. Moreover, Matrix \( T \) must also satisfy (4.7c), which is thus tangled to (4.7a) that increases the complexity of the problem.

From the main three approaches proposed to solve this problem and briefly mentioned in the introduction, the transformation-based approach [62, 90, 149] is likely to be more
4.2 Problem Statement and Preliminaries

effective and easier in design and practice. This is the reason that this methodology has been the commonly used one in solving (4.7). Nevertheless, this approach can also be classified into three different frameworks. Two frameworks are briefly studied here and the third one, which is the most recent method, is briefly described in Remark 4.3.7.

Let us define a square matrix $\bar{C} \in \mathbb{R}^{n \times n}$, which is always of full rank (due to Assumption A), as

$$
\bar{C} \triangleq \begin{bmatrix} C^\top & C^\perp \end{bmatrix}
$$

(4.8)

Next, $T_1 \in \mathbb{R}^{l \times p}$, $T_2 \in \mathbb{R}^{l \times (n-p)}$, $A_{11} \in \mathbb{R}^{p \times p}$, $A_{12} \in \mathbb{R}^{p \times (n-p)}$, $A_{21} \in \mathbb{R}^{(n-p) \times p}$, $A_{22} \in \mathbb{R}^{(n-p) \times (n-p)}$, $L_1 \in \mathbb{R}^{l \times p}$, and $L_2 \in \mathbb{R}^{l \times (n-p)}$ are defined as follows,

$$
\begin{bmatrix} T_1 & T_2 \end{bmatrix} \triangleq T\bar{C}
$$

$$
\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \triangleq \bar{C}^{-1}A\bar{C}
$$

$$
\begin{bmatrix} L_1 & L_2 \end{bmatrix} \triangleq L\bar{C}
$$

Finally, the interconnected Sylvester equations (4.7a) and (4.7c) are post-multiplied by $\bar{C}$ that results

$$
V = L_1 - T_1
$$

(4.9)

$$
T_2 = L_2
$$

(4.10)

$$
H = T_1 A_{11} + T_2 A_{21} - F T_1
$$

(4.11)

$$
F T_2 - T_1 A_{12} - T_2 A_{22} = 0
$$

(4.12)
Now, from (4.10) and (4.12) we have

\[
\begin{bmatrix}
F & -T_1
\end{bmatrix}
\Omega_1 = L_2 A_{22}
\]  
(4.13)

where \( \Omega_1 \triangleq [L_2; A_{12}] \in \mathbb{R}^{(l+p) \times (n-p)} \). According to [58, 156], the relation (4.13) has a solution for \( F \) and \( T_1 \) if and only if

\[
L_2 A_{22} (I_{n-p} - \Omega_1 \Omega_1^\dagger) = 0
\]  
(4.14)

This is equivalent to

\[
rank \left( \begin{bmatrix}
\Omega_1 \\
L_2 A_{22}
\end{bmatrix} \right) = rank (\Omega_1)
\]  
(4.15)

It can be proved using the same procedure as in [58, 62, 105] that condition (4.15) is equivalent to Condition I described in (4.5). Hence, if Condition I is satisfied, then from (4.13)

\[
\begin{bmatrix}
F & -T_1
\end{bmatrix} = U_1 + Z_1 U_2,
\]  
(4.16)

where \( U_1 \triangleq L_2 A_{22} \Omega_1^\dagger \in \mathbb{R}^{l \times (p+l)} \), \( U_2 \triangleq I_{p+l} - \Omega_1 \Omega_1^\dagger \in \mathbb{R}^{(p+l) \times (p+l)} \), and \( Z_1 \in \mathbb{R}^{l \times (p+l)} \) is an arbitrary design matrix, which will be used in obtaining convenient matrices \( F \) and \( T_1 \).

By appropriately partitioning \( U_1 \) and \( U_2 \) as \( U_1 = \begin{bmatrix} U_{11} & U_{12} \end{bmatrix} \) and \( U_2 = \begin{bmatrix} U_{21} & U_{22} \end{bmatrix} \), the following are obtained from (4.16):

\[
T_1 = -U_{12} - Z_1 U_{22},
\]  
(4.17)

\[
F = U_{11} + Z_1 U_{21}.
\]  
(4.18)
To define $Z_1$ in (4.18), in a way that the matrix $F$ is Hurwitz, it is necessary and sufficient that the pair $(U_{11}, U_{21})$ is observable/detectable. It can be attained after some algebraic manipulations as described in [58, 62, 105] that this is equivalent to Condition II. Hence, if Condition II is satisfied, the matrix $Z_1$ can be computed, in a way that the matrix $F$ is Hurwitz with the most possible predefined eigenvalues. The eigenvalues are defined in a way to achieve the desired observer’s performance. Accordingly, $T_1, H, V,$ and $G$ are calculated from (4.17), (4.11), (4.9), and (4.7b), respectively.

Besides, Darouach [58] proposes another transformation based methodology to solve the interconnected equations (4.7). The transformation matrix $\bar{L} = [\bar{L}^\dagger, I_n - \bar{L}^\dagger \bar{L}]$ is employed to this aim. It is obtained after post-multiplying (4.7a) by $\bar{L}$ that

$$F = TAL^\dagger - KCL^\dagger,$$

$$T \bar{A} = \bar{K} \bar{C},$$  \hspace{1cm} (4.19) \hspace{1cm} (4.20)

where $\bar{A} \triangleq A(I_n - L^\dagger L)$, $\bar{C} \triangleq C(I_n - L^\dagger L)$, and $K \triangleq H - FV$. Substituting $T$ from (4.7c) into (4.20), gives

$$\begin{bmatrix} V & K \end{bmatrix} \bar{\Sigma} = L \bar{A},$$ \hspace{1cm} (4.21)

where $\bar{\Sigma} \triangleq \begin{bmatrix} C\bar{A} \\ \bar{C} \end{bmatrix}$. If Condition I is satisfied, then rank $\begin{bmatrix} L\bar{A} \\ \bar{\Sigma} \end{bmatrix} = \text{rank}\bar{\Sigma}$. Consequently the following holds

$$\begin{bmatrix} V & K \end{bmatrix} = L\bar{A}\bar{\Sigma}^\dagger + Z_2(I_{2p} - \bar{\Sigma}\bar{\Sigma}^\dagger)$$ \hspace{1cm} (4.22)

Furthermore, after some algebraic manipulations it is obtained that
4.2 Problem Statement and Preliminaries

\[
F = N_1 - Z_2 N_2, \quad (4.23)
\]

where \( N_1 \triangleq LAL^\dagger - L\bar{A}\Sigma^\dagger \begin{bmatrix} CAL^\dagger \\ CL^\dagger \end{bmatrix}, \ N_2 \triangleq (I_{2p} - \Sigma\Sigma^\dagger) \begin{bmatrix} CAL^\dagger \\ CL^\dagger \end{bmatrix}, \) and \( Z_2 \in \mathbb{R}^{l \times 2p} \) is the observer design matrix with the similar role of \( Z_1 \) in the previous framework. It is further confirmed in [58] that the detectability of the pair \((N_1, N_2)\) is equivalent to Condition II, which is the necessary and sufficient condition for the existence of a matrix \( Z_2 \), such that the matrix \( F \) becomes Hurwitz. After calculating an appropriate matrix \( Z_2 \) through a pole-placement approach, the matrix \( F \) and the other observer parameters, can also be obtained.

**Remark 4.2.2.** Although the conventional approaches described above solve the same problem, they use different approaches, possess distinct numerical properties, and give different results for similar observer features. The observer equations usually have infinite number of solutions even for a particular convergence rate. Each method finds a set of these solutions. However, there are also some sources of numerical error generation, such as multiple pseudo-inverses, inverses, and null space calculations that are performed during the design procedure. In addition, the pole-placement technique that is employed in the final step is a numerical method that can induce some errors. Hence, lacking enough degrees of freedom provided by the design parameter, can result in insufficient performance of the observer. By the way, both of these approaches suffer from some numerical issues in particular situations. This is illustrated via some numerical examples in Section 4.4. To exemplify, the first approach usually faces numerical issues whenever the matrix \( C \) is not in the canonical form \( C = [I_p, 0] \) (see Examples 2 and 4 in Section 4.4). The second conventional method on the other hand can confront numerical deficiencies whenever \( \bar{\Sigma} \) in (4.21) is close to a singular configuration (see Example 3 in Section 4.4). These phenomena have instigated a desire to increase the numerical flexibility of the observer design parameter.
(matrices $Z_1$ and $Z_2$ in the first and second conventional algorithms, respectively).

### 4.3 Minimal Order Functional Observer Design Algorithm

Firstly, a recursive algorithm to obtain a minimal matrix $L$ is proposed using the concept of functional observability [59]. Secondly, the new functional observer design approach is illustrated, and compared to the conventional schemes. Thirdly, a recursive algorithm for designing a functional observer using the new methodology is demonstrated. Finally, employing the proposed algorithm in the first stage, an extension to the direct approach is advised.

**Algorithm 1 (a procedure for finding a minimal matrix $L$):**

1. Examine whether Condition (4.2) of Theorem 4.2.1 is satisfied. If so, then set $i = 0$, $L_i = L_0$, and proceed to the next step. Otherwise, there is no functional observer for the system and the algorithm stops.

2. Testify Condition I. If it is satisfied set $L_\beta = L_0$, and jump to Step 6. Otherwise proceed to the next step. $L_\beta$ is the (modified) matrix $L$ that satisfies Conditon I.

3. Define $H^i_2 \triangleq [\mathcal{R}([C; CA; L_i])], H^i_3 \triangleq [\mathcal{R}([H^i_2; L_i A])], a_i = \text{rank}(H^i_2)$, and $b_i = \text{rank}(H^i_3)$.

4. Calculate $\Pi_i \triangleq [\mathcal{R}(\Theta_i)]\Phi_i H^i_3$, where $\Phi_i$ and $\Theta_i$ are defined as follows,

\[
\begin{bmatrix}
\Phi_i & \Psi_i
\end{bmatrix} \triangleq \begin{bmatrix}
\mathcal{N} \left( \begin{bmatrix}
H^i_2 A \\
H^i_3
\end{bmatrix} \right) \end{bmatrix}
\] (4.24)

\[
\begin{bmatrix}
\Theta_i & \Gamma_i
\end{bmatrix} \triangleq \begin{bmatrix}
\mathcal{N} \left( \begin{bmatrix}
\Phi_i H^i_3 \\
H^i_2
\end{bmatrix} \right) \end{bmatrix}
\] (4.25)
Next, for \( j = \{1, \cdots, \text{rows}(\Pi_i)\} \) define \( L_i^j = [L_i^{j-1}; q_j] \), and \( L_i^1 = [L_i; q_1] \), where \( q_j \) is the \( j'th \) row of \( \Pi_i \). If \( L_i^j \) satisfies Condition I, then select \( L_\beta = L_i^j \), and continue to Step 6. Otherwise proceed to the next step.

5. Construct \( \Lambda_i \triangleq \begin{bmatrix} R(H_i^3) \cap R \left( \begin{bmatrix} H_2^3 \\ \Pi_i \end{bmatrix} \right)^\perp \end{bmatrix} \). Next, define \( L_{i+1} = [L_i; \Pi_i; \Lambda_i] \), set \( i = i + 1 \), and go back to Step 3.

6. Examine Condition II. If it is satisfied, choose \( L = L_\beta \), and the algorithm stops. Otherwise, calculate \( \begin{bmatrix} \Xi_1 & \Delta_1 \end{bmatrix} \triangleq \begin{bmatrix} \mathcal{N} \left( \begin{bmatrix} C \Lambda^2 \\ H_2^3 \end{bmatrix} \right) \end{bmatrix} \), in which the number of columns of \( \Xi_1 \) is equal to the number of the rows of \( C \). Moreover, define \( \Pi = \mathcal{R}(\Xi_1C\Lambda) \). Now, for \( j = \{1, \cdots, \text{rows}(\Pi)\} \), let \( L_\beta^j = [L_\beta^{j-1}; q_j] \), and \( L_\beta^1 = [L_\beta; q_1] \), where \( q_j \) is the \( j'th \) row of the matrix \( \Pi \). If \( L_\beta^j \) satisfies Condition II, then set \( L = L_\beta^j \), and the algorithm is terminated.

As an illustration of Algorithm 1, it is worthwhile to remark that Steps 4 and 5 are given to find the minimum number of rows to append to \( L_0 \), such that Condition I is satisfied. In addition, the definitions (4.24) and (4.25) simply indicate \( \Phi_iH_3^3A + \Psi_iH_3^3 = 0 \) and \( \Theta_i\Phi_iH_3^3 + \Gamma_iH_2^\beta = 0 \), respectively. To satisfy Condition I, it is necessary that each appended row to \( L_0 \), increase the right-hand side of (4.5), while keeping its left-hand side unaltered. Hence, each selected row must be in the product space spanned by \( H_3^3 \) and \( \Phi_i \), while orthogonal to \( \Theta_i \). This is exactly what \( \Pi_i \) defined in Step 4 demonstrates. In other words, each row selected from \( \Pi_i \) in Step 4 increases \( a_i \) by one, while does not alter \( b_i \). On the other hand, each row selected from \( \Lambda_i \) in Step 5 increases both \( a_i \) and \( b_i \) by one [59].

Analogously, Step 6 of the algorithm is aimed at finding the minimum number of rows to be appended to \( L_\beta \) to satisfy Condition II, whereas Condition I is not violated. It can be shown that each of these row vectors (named \( q \) for example), should lie in the row space of
4.3 Minimal Order Functional Observer Design Algorithm

$CA$ [59], and to simultaneously satisfy Condition I, $qA$ also must be in the range space of $H^2_2$. This results in the last step of the algorithm.

**Remark 4.3.1.** The concept of functional observability is a certificate for the existence of a functional observer for the system (4.1). Indeed, in Step 1 of the algorithm, it is assured that there exists a functional observer for the system. Thereafter, it is the minimum possible number of rows are sought to be appended to $L$, such that Conditions I and II are fulfilled.

**Remark 4.3.2.** In Step 6 of the algorithm, Condition II can be explored for all the variables $s$ in the complex plane. Then, the asymptotic stability of the observer (4.4) can be assured with an arbitrary convergence rate. Hence, this step is flexible and the minimum required order of the observer can be increased for improving the performance of the observer.

**Remark 4.3.3.** Algorithm 1 is independent of the method chosen to solve the interconnected equations (4.7). After finding a workable matrix $L$ that satisfies Conditions I and II, any effective method can be used to solve (4.7), including the conventional methods summarized in Section 4.2.1.

**Remark 4.3.4.** Although Algorithm 1 might look algebraically complicated at the first glance, it can be simply implemented in any matrix programming software like MATLAB.

### 4.3.1 A New Algorithm to Solve the Observer Equations

If Conditions I and II are satisfied, the design parameters $F, H, V, G$, and $T$ can always be sought, such that equations (4.7) are fulfilled. Unlike the conventional approaches explained in the previous section that commence with Equation (4.7a), our design procedure starts with Equation (4.7c), which is rearranged as

$$\begin{bmatrix} T & V \end{bmatrix} \begin{bmatrix} I_n \\ C \end{bmatrix} = L$$

(4.26)
4.3 Minimal Order Functional Observer Design Algorithm

Denote \( M \triangleq [I_n; C] \in \mathbb{R}^{(n+p) \times n} \). Since \( n + p > n \), and the matrix \( M \) is of full column rank, from (4.26) we have

\[
\begin{bmatrix}
T & V
\end{bmatrix} = LM^\dagger + Z(I_{n+p} - MM^\dagger),
\] (4.27)

where \( Z \in \mathbb{R}^{l \times (n+p)} \) is an arbitrary matrix of appropriate dimension, which is our first design parameter. This matrix plays the same role as the matrix \( T \) in the sequel. In addition, let us partition the matrices \( M \dagger \), and \( I_{n+p} - MM^\dagger \) as

\[
\begin{bmatrix}
M_1 & M_2
\end{bmatrix} \triangleq M^\dagger,
\]
\[
\begin{bmatrix}
E_1 & E_2
\end{bmatrix} \triangleq I_{n+p} - MM^\dagger,
\]

where \( M_1, M_2, E_1, \) and \( E_2 \) are \( n \times n \), \( n \times p \), \((n + p) \times n \), and \((n + p) \times p \) constant matrices, respectively. Hence, it is obtained from (4.27) that

\[
T = LM_1 + ZE_1, \quad (4.28)
\]
\[
V = LM_2 + ZE_2. \quad (4.29)
\]

If the left hand side of (4.7a) is post-multiplied by Matrix \( \tilde{C} \), the following are achieved

\[
H = -FTC^\dagger + TAC^\dagger, \quad (4.30)
\]
\[
FTC^\perp = TAC^\perp. \quad (4.31)
\]

Substituting \( T \) from (4.28) into (4.31) yields

\[
FLM_1C^\perp + FZE_1C^\perp = LM_1AC^\perp + ZE_1AC^\perp \quad (4.32)
\]
It can be shown after some algebraic manipulations that the matrix $E_1$ is orthogonal to $C^\perp$, i.e., $E_1C^\perp = 0$. To verify this, we show that $E_1$ lies in the range space of the matrix $C$. Recalling the computation of Matrix $E_1$ we have

$$\begin{bmatrix} E_1 & E_2 \end{bmatrix} = \begin{bmatrix} I_{n+p} - \begin{bmatrix} I_n \\ C \end{bmatrix} \left( \begin{bmatrix} I_n \\ C \end{bmatrix} \right)^\dagger \end{bmatrix}$$ (4.33)

There are a variety of approaches in computing of the generalized inverse of a matrix [156, 157]. One well-known method in calculating the pseudo-inverse of a matrix $X \in \mathbb{R}^{k \times s}$, $k > s$, $k, s \in \mathbb{N}$, is the following relation

$$X^\dagger = \left( X^T X \right)^{-1} X^T$$ (4.34)

Using (4.34), we have $X^\dagger X = I_t$ if and only if $X$ is of full column rank. However, considering that $k > s$, the multiplication $XX^\dagger$ is not the identity matrix. Taking (4.34) into account, (4.33) is reformulated as

$$\begin{bmatrix} E_1 & E_2 \end{bmatrix} = I_{n+p} - \begin{bmatrix} I_n \\ C \end{bmatrix} \left( \begin{bmatrix} I_n + C^T C \end{bmatrix} \right)^{-1} \begin{bmatrix} I_n & C^T \end{bmatrix}$$ (4.35)

that is equivalent to

$$\begin{bmatrix} E_1 & E_2 \end{bmatrix} = I_{n+p} - \begin{bmatrix} (I_n + C^T C)^{-1} & (I_n + C^T C)^{-1} C^T \\ C (I_n + C^T C)^{-1} & C (I_n + C^T C)^{-1} C^T \end{bmatrix}$$ (4.36)

Hence, Matrix $E_1$ can be written as follows
\[ E_1 = \begin{bmatrix} I_n - (I_n + C^T C)^{-1} \\ -C (I_n + C^T C)^{-1} \end{bmatrix}. \]  

(4.37)

Now, remarking that

\[(I_n + C^T C)^{-1} = I_n - (I_n + C^T C)^{-1} C^T C,\]  

(4.38)

Equation (4.37) can be written as

\[ E_1 = \begin{bmatrix} (I_n + C^T C)^{-1} C^T C \\ C - C (I_n + C^T C)^{-1} C^T C \end{bmatrix}. \]  

(4.39)

Accordingly, \( E_1 \) is in the space spanned by the row vectors of the matrix \( C \). Hence, it is clear that since \( CC^\perp = 0 \), the multiplicative term \( E_1 C^\perp \) is also equal to zero. Thus, (4.32) can be written as

\[ FLM_1 C^\perp = LM_1 AC^\perp + ZE_1 AC^\perp. \]  

(4.40)

Thereafter, to find convenient matrices \( F \) and \( Z \), (4.40) is reformulated as

\[
\begin{bmatrix}
F \\
-Z
\end{bmatrix}
\Omega = \Phi,
\]  

(4.41)

where \( \Omega \triangleq \begin{bmatrix} LM_1 C^\perp \\ E_1 AC^\perp \end{bmatrix} \in \mathbb{R}^{(n+p+l)\times(n-p)} \) and \( \Phi \triangleq LM_1 AC^\perp \in \mathbb{R}^{l\times(n-p)}. \)

It is well-known that (4.41) has a solution if and only if the following important rank condition is satisfied [62, 156].

**Condition III:**

\[
rank \left( \begin{bmatrix} \Omega \\ \Phi \end{bmatrix} \right) = rank(\Omega)
\]  

(4.42)
4.3 Minimal Order Functional Observer Design Algorithm

Hence, if Condition III is realized, Matrices $F$ and $Z$ can be obtained as follows,

\[
\begin{bmatrix}
F & -Z
\end{bmatrix} = \Phi \Omega^\dagger + \tilde{Z} (I_{n+p+l} - \Omega \Omega^\dagger) \tag{4.43}
\]

where $\tilde{Z} \in \mathbb{R}^{l \times (n+p+l)}$ is an arbitrary matrix that is to be defined shortly. Let us denote $N_1 \triangleq \Phi \Omega^\dagger \in \mathbb{R}^{l \times (n+p+l)}$, and $N_2 \triangleq (I_{n+p+l} - \Omega \Omega^\dagger) \in \mathbb{R}^{(n+p+l) \times (n+p+l)}$. Moreover, partition $N_1$ and $N_2$ as $N_1 = \begin{bmatrix} N_{11} & N_{12} \end{bmatrix}$, and $N_2 = \begin{bmatrix} N_{21} & N_{22} \end{bmatrix}$, where $N_{11}, N_{12}, N_{21},$ and $N_{22}$ are of dimensions $l \times l$, $l \times (n+p)$, $(n+p+l) \times l$, and $(n+p+l) \times (n+p)$, respectively. As a result one has

\[
F = N_{11} + \tilde{Z} N_{21}, \tag{4.44}
\]

\[
Z = -N_{21} - \tilde{Z} N_{22}. \tag{4.45}
\]

If the pair $(N_{11}, N_{21})$ is observable or detectable, Matrix $\tilde{Z}$ can be defined such that the observer parameter $F$ is stable. This itself satisfies Condition (a) that was mentioned in Section 4.2. Hence, the second necessary condition for the existence of an asymptotic observer with structure (4.4) for the system is the following rank condition:

**Condition IV:**

\[
\text{rank} \begin{bmatrix}
I_l & -N_{11} \\
N_{21}
\end{bmatrix} = l \quad \forall s \in \mathbb{C}^+ \tag{4.46}
\]

It is clear that if Condition (4.46) is satisfied for all complex values of $s \in \mathbb{C}$, then the pair $(N_{11}, N_{21})$ is observable, and arbitrary eigenvalues can be assigned for $F$. Otherwise, solely the observable poles can be arbitrarily placed in the the complex plane.

**Remark 4.3.5.** If Matrix $L$ that is the augmented version of $L_0$, $R_1$, and $R_2$ is defined by employing Algorithm 1, then Conditions III and IV are automatically satisfied. Hence,
it is sufficient to use the conventional rank conditions (4.5) and (4.6), as a substitute of the obtained new Conditions (4.42) and (4.46). To justify this, it is sufficient to show the equivalence between Conditions (4.42) and (4.5), as well as the correspondence between Conditions (4.46) and (4.6).

As a summary of the above illustrations, the following theorem and lemmas are given.

**Theorem 4.3.1.** Conditions III and IV are the necessary and sufficient conditions for the existence of an asymptotic functional observer with structure (4.4) for the system (4.1).

**Lemma 4.3.1.** The rank condition (4.42) is equivalent to Condition I described in (4.5).

**Proof.** To prove the lemma, first it is remarked that

\[
\begin{bmatrix}
\Omega \\
\Phi
\end{bmatrix}
= \begin{bmatrix}
LM_1 C^\perp \\
E_1 AC^\perp \\
LM_1 AC^\perp
\end{bmatrix}
\]  

(4.47)

According to its definition and (4.38), \(M_1\) can be written as

\[
M_1 = (I_n + C^T C)^{-1} \\
= I_n - (I_n + C^T C)^{-1} C^T C
\]

(4.48)

Accordingly,

\[
LM_1 C^\perp = L(I_n - (I_n + C^T C)^{-1} C^T C)C^\perp \\
= LC^\perp
\]

(4.49)

In addition, since the matrix \((I_n + C^T C)^{-1}\) is of full rank, we have

\[
rank \left( LM_1 AC^\perp \right) = rank \left( L(I_n + C^T C)^{-1} AC^\perp \right) \\
= rank \left( LAC^\perp \right)
\]

(4.50)
Moreover, according to the definition of $\mathcal{E}_1$ in (4.37) and the useful relation (4.38), the following can be obtained

$$\text{rank} \left( \mathcal{E}_1 AC^\perp \right) = \text{rank} \left( \begin{bmatrix} C^T C AC^\perp \\ -C AC^\perp \end{bmatrix} \right) = \text{rank} \left( C AC^\perp \right)$$

As a result, the rank condition (4.47) can be written as

$$\text{rank} \left( \begin{bmatrix} \Omega \\ \Phi \end{bmatrix} \right) = \text{rank} \left( \begin{bmatrix} LC^\perp \\ CAC^\perp \\ LAC^\perp \end{bmatrix} \right)$$

In a similar way, the right hand side of (4.42) is equivalent to

$$\text{rank} \left( \Omega \right) = \text{rank} \left( \begin{bmatrix} LC^\perp \\ CAC^\perp \end{bmatrix} \right)$$

On the other side, the left hand side of (4.5) is manipulated as

$$\text{rank} \left( \begin{bmatrix} L \\ C \\ CA \\ LA \end{bmatrix} \right) = \text{rank} \left( \begin{bmatrix} LA \\ CA \\ C \\ L \end{bmatrix} \begin{bmatrix} C^\dagger \\ C^\perp \end{bmatrix} \right)$$
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\[ = \text{rank} \begin{pmatrix} \begin{bmatrix} LAC^\dagger & LAC^\perp \\ CAC^\dagger & CAC^\perp \\ I_p & 0_p \\ LC^\dagger & LC^\perp \end{bmatrix} \end{pmatrix} \]

\[ = p + \text{rank} \begin{pmatrix} \begin{bmatrix} LC^\perp \\ CAC^\perp \\ LAC^\perp \end{bmatrix} \end{pmatrix} \quad \text{(4.54)} \]

Likewise, the right hand side of (4.5) is obtained as

\[ \text{rank} \begin{pmatrix} \begin{bmatrix} L \\ C \\ CA \end{bmatrix} \end{pmatrix} = p + \text{rank} \begin{pmatrix} \begin{bmatrix} LC^\perp \\ CAC^\perp \end{bmatrix} \end{pmatrix} \quad \text{(4.55)} \]

Hence, from (4.52), (4.53), (4.54), and (4.55), it is clear that Condition (4.42) can always be concluded from (4.5), and the proof is complete.

Now, showing the equivalence between Conditions (4.46) and (4.6) is straightforward.

Lemma 4.3.2. The rank condition (4.46) is equivalent to Condition II, described in (4.6).

Proof. The lemma is proved using contradiction. Let us assume that the statement of the lemma is false. Since Conditions I and II are necessary and sufficient for the existence of an asymptotic functional observer, and Conditions I and III are equivalent (Lemma 4.3.1), it can be concluded that Conditions III and IV are not necessary and sufficient for the existence of an asymptotic functional observer. This is a contradiction according to Theorem 4.3.1, and concludes the statement of the lemma.

To summarize the aforementioned design methodology, a recursive algorithm is given that covers both of the functional observability and functional detectability scenarios of the
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triple $\Sigma$.

**Algorithm 2:**

1. Fix $L = L_0$

2. Compute the matrices $M, M_1, M_2, E_1, E_2, \bar{C}, \Omega,$ and $\Phi$ from their corresponding definitions.

3. Examine if Condition III is satisfied. If so, then continue to the next step. Otherwise, append a row vector to the matrix $L$ as articulated in Algorithm 1 and move back to Step 2.

4. Testify Condition IV. If it holds, then continue to the next step. Otherwise, append a row vector to the matrix $L$, selected according to Algorithm 1. Next, move back to Step 2.

5. Examine the observability or the detectability of the pair $(N_{11}, N_{21})$. If it is observable, then proceed to the next step. If it is detectable and the triplet $\Sigma$ is functional detectable, or if it is not essential to assign all of the poles of the dynamic observer, then jump to Step 7. Otherwise, if the pair $(N_{11}, N_{21})$ is detectable, but the triplet $\Sigma$ is functional observable, and in addition we need to assign all of the observer’s eigenvalues, then increase the order of the observer by one. To do this, append another row vector to the latest updated matrix $L$ as instructed in Algorithm 1, and move back to Step 2.

6. Solve for $\tilde{Z}$ in (4.44) by employing a pole-placement algorithm, such that the matrix $F$ is stable with desired eigenvalues.

7. Solve for $\tilde{Z}$ in (4.44) using a pole-placement approach, such that the observable eigenvalues of $N_{11}$ are assigned to the desired values.
8. Calculate the remaining observer parameters $Z, T, V, H,$ and $G$ from (4.45), (4.28), (4.29), (4.30), and (4.7b), respectively.

**Remark 4.3.6.** To compare the design Algorithm 2 and the conventional approaches summarised in Section 4.2.1, it is noted that each of those schemes use a different approach to solve the observer design problem. It is clear that by using the new design method (Algorithm 2) the dimension of the observer design parameter is increased (that is $l \times (n + p + l)$, instead of $l \times (p + l)$ and $l \times 2p$ in the first and second conventional schemes, respectively). In addition, in (4.41) the matrix $\Omega$ has the size of $(n + p + l) \times (n - p)$. In other words, it cannot have more columns than rows. Additionally, let us recall the situations that can result in numerical issues, where some of them summarised in Remark 2 as near-singular configurations. These configurations are partly created due to the size of the free matrix parameters that should satisfy the resulted observer equations. By increasing the number of the free elements in those matrices, more accurate mathematical operations that are approximate in nature can follow, due to dealing with numerically better-behaved matrices. As such, by employing the new methodology such near-singular configurations are less likely to appear, because the increased size of the observer parameter (as our free parameter) helps the mathematical operations to be more reliable. This helpful point is more justified and exemplified in Section 4.4.

**Remark 4.3.7.** Recently Fernando and Trinh [147] have also proposed a methodology to design a minimal multi-functional observer for LTI systems only if the system $\Sigma$ is functional observable, even when Conditions I and II are not satisfied. To the best of the author’s knowledge, this is the only published paper in the Literature that address this specific problem. The proposed approach designs a minimum possible order functional observer for the system (4.1), using the reduced order Luenberger observer design technique, and system decomposition based on some matrix transformations proposed in [158]. Although
it solves a similar problem as ours, the methodology of [147] involves complex algebraic and numeric processes. Moreover, it does not cover the case, when the system $\Sigma$ is functional detectable but not functional observable. In addition, it is not clear that if it is possible to extend the developed method to unknown-input functional observer design as well as the other classes of dynamic systems, per se linear-time-varying, nonlinear, and time-delay systems.

### 4.3.2 Extension of the Direct Approach

The direct approach was firstly proposed for single functional observers [148]. In this section, this algorithm is revisited to address the multiple-functional observer design problem. Following that, the problems that might arise using this method are illustrated, together with our proposed solutions. Let $1 \leq \kappa \leq n - 1$ be the functional observability index of the triple $(A, C, L_0)$. That is $\kappa$ is the least possible integer that

$$\text{rank}(\Sigma_{\kappa}) = \text{rank} \left( \begin{bmatrix} \Sigma_{\kappa} \\ L_0 A^\kappa \end{bmatrix} \right),$$

(4.56)

where

$$\Sigma_{\kappa} = \begin{bmatrix} C \\ L_0 \\ CA \\ \vdots \\ CA^{\kappa-1} \\ L_0 A^{\kappa-1} \\ CA^{\kappa} \end{bmatrix}. $$
Then, from (4.56) it is concluded that

$$L_0A^\kappa = \sum_{i=0}^{\kappa} \Gamma_i CA^i + \sum_{j=0}^{\kappa-1} \Lambda_j L_0 A^j \quad (4.57)$$

where $\Gamma_i \in \mathbb{R}^{l_0 \times p}$, $i = \{0, \cdots, \kappa\}$, and $\Lambda_j \in \mathbb{R}^{l_0 \times l_0}$, $j = \{0, \cdots, \kappa - 1\}$ are constant matrices that can be found using least squares. By $q$ times differentiating $z_0(t)$ with respect to time $t$, one gets

$$z_0^{(\kappa)}(t) = L_0 A^\kappa x(t) + \sum_{i=0}^{\kappa-1} L_0 A^i Bu^{(\kappa-1-i)}(t) \quad (4.58)$$

Now, substituting from (4.57) into (4.58), and using (4.1) it is obtained after some algebraic manipulations that

$$z_0^{(\kappa)}(t) - \Gamma_\kappa y^{(\kappa)}(t) = \sum_{i=0}^{\kappa-1} \Gamma_i y^{(i)}(t) + \Lambda_i z_0^{(i)}(t) + \Phi_i u^{(i)}(t) \quad (4.59)$$

where for $i = \{0, \cdots, \kappa - 2\}$

$$\Phi_i \triangleq \begin{bmatrix} L_0 A^{\kappa-1-i} - \sum_{j=i+1}^{\kappa} \Gamma_j CA^{j-i-1} - \sum_{j=i+1}^{\kappa-1} \Lambda_j L_0 A^{j-i-1} \end{bmatrix} B,$$

and $\Phi_{\kappa-1} \triangleq (L_0 - \Gamma_\kappa C)B$. The dynamic equation (4.59) can be written in the observable canonical form. Let us define $\bar{x}_1(t) = z_0(t) - \Gamma_\kappa y(t)$. Then (4.59) can be reformulated as
4.3 Minimal Order Functional Observer Design Algorithm

\[
\dot{\bar{x}}(t) = \begin{bmatrix}
\Lambda_{\kappa-1} & I_{l_0} & * & 0_{l_0} \\
\vdots & * & \ddots & \vdots \\
\Lambda_1 & 0_{l_0} & \cdots & I_{l_0} \\
\Lambda_0 & 0_{l_0} & \cdots & 0_{l_0}
\end{bmatrix} \bar{x}(t) + \begin{bmatrix}
\Phi_{\kappa-1} \\
\vdots \\
\Phi_1 \\
\Phi_0
\end{bmatrix} u(t) + \begin{bmatrix}
\Gamma_{\kappa-1} + \Lambda_{\kappa-1} \Gamma_{\kappa} \\
\vdots \\
\Gamma_1 + \Lambda_1 \Gamma_{\kappa} \\
\Gamma_0 + \Lambda_0 \Gamma_{\kappa}
\end{bmatrix} y(t),
\]

(4.60)

where \( \bar{x}(t) = \begin{bmatrix} \bar{x}_1^T & \cdots & \bar{x}_{\kappa-1}^T \end{bmatrix}^T(t) \), and "*" represents zero matrices.

Comparing (4.60) with the observer structure (4.4), it is concluded that by defining

\[
F = \begin{bmatrix}
\Lambda_{\kappa-1} & I_{l_0} & * & 0_{l_0} \\
\vdots & * & \ddots & \vdots \\
\Lambda_1 & 0_{l_0} & \cdots & I_{l_0} \\
\Lambda_0 & 0_{l_0} & \cdots & 0_{l_0}
\end{bmatrix}, \quad G = \begin{bmatrix}
\Phi_{\kappa-1} \\
\vdots \\
\Phi_1 \\
\Phi_0
\end{bmatrix}, \quad H = \begin{bmatrix}
\Gamma_{\kappa-1} + \Lambda_{\kappa-1} \Gamma_{\kappa} \\
\vdots \\
\Gamma_1 + \Lambda_1 \Gamma_{\kappa} \\
\Gamma_0 + \Lambda_0 \Gamma_{\kappa}
\end{bmatrix}, \quad P = 
\begin{bmatrix}
I_{l_0} & 0_{l_0} & \cdots & 0_{l_0}
\end{bmatrix}, \quad V = \Gamma_{\kappa},
\]

the observer (4.4) is asymptotically stable if and only if Matrix \( F \) is Hurwitz.

For single functional observers (case \( l_0 = 1 \)), the proposed method results in the minimal order functional observer. However, for multiple functional observers, i.e., \( l_0 > 1 \), the resulting observer parameters (if they can be computed) do not necessarily represent the minimum possible order observer. In this scenario, the order of the observer is \( \kappa \times l_0 \), which might be even larger than \( n \). In other words, the order of the observer might be even larger than the ordinary Luenberger observer.

Moreover, the parameters \( \Lambda_i, i = \{0, \cdots, \kappa-1\} \), which are computed from (4.57) using a least squares approach, must be designed in a way that the eigenvalues of the matrix \( F \) can be fixed at arbitrary locations in the left half of the complex plane. Since the matrix \( F \) is not symmetric, nor diagonal, the parametric calculation of its determinant is quite difficult even for a simple case \( l_0 = 2 \), and \( \kappa \geq 2 \). Hence, calculating the matrices \( \Lambda_i \) from (4.57), such that the resulting matrix \( F \) becomes stable with predefined eigenvalues is numerically
complicated, and practically impossible in some situations. This point is further clarified in Section 4.4 via a numerical example (see Example 3).

The above problems make the direct approach impractical for multi-functional observers, when used in its initial format. If Condition (4.56) is satisfied for $\kappa = 1$, then it is possible to obtain an asymptotic functional observer of order $l_0$, which is the minimum-order observer with arbitrary convergence rate. However, if any of Conditions I or II is not satisfied, but the triple $(A, C, L_0)$ is functional observable, then Condition (4.56) is attained for a $\kappa \geq 2$, whence the original direct approach is generally impractical. To resolve this issue, the order of the observer is increased via applying Algorithm 1. In conclusion, the direct approach can be used in one step (i.e., $\kappa = 1$), which makes this approach handful and practical one. Evidently, Condition II must be satisfied for any point in the complex plane.

In addition, the output of Algorithm 1 is a matrix $L$ that together with the pair $(A, C)$ satisfy the following rank condition

$$\text{rank}(\Sigma) = \text{rank}\left(\begin{bmatrix} \Sigma \\ LA \end{bmatrix}\right), \quad (4.61)$$

where $\Sigma = [L; C; CA]$. In consequence, we obtain

$$LA = \Gamma_0 C + \Gamma_1 CA + \Lambda_0 L. \quad (4.62)$$

This is equivalent to

$$X \Sigma = LA, \quad (4.63)$$

where $X \triangleq \begin{bmatrix} \Lambda_0 & \Gamma_0 & \Gamma_1 \end{bmatrix}$. Hence, since Condition (4.61) is satisfied, (4.63) gives

$$X = LA \Sigma^\dagger + \tilde{Z}(I_{2p+l} - \Sigma \Sigma^\dagger), \quad (4.64)$$
where $\bar{Z} \in \mathbb{R}^{l \times (2p+l)}$ is an arbitrary matrix that is our design parameter. Let us define
\[
\begin{bmatrix}
  M_1 & M_2 & M_3
\end{bmatrix} \triangleq L A \Sigma^\dagger, \quad \text{and} \quad
\begin{bmatrix}
  S_1 & S_2 & S_3
\end{bmatrix} \triangleq I_{2p+l} - \Sigma \Sigma^\dagger,
\]
where $M_1, M_2, M_3, S_1, S_2,$ and $S_3$ are $l \times l, l \times p, l \times p, (2p+l) \times l, (2p+l) \times p,$ and $(2p+l) \times p$ matrices, respectively.

Now define $\bar{x}(t) = z(t) - \Gamma_1 y(t)$. Similar to the previously explained procedure, it is obtained that
\[
\dot{\bar{x}}(t) = \Lambda_0 \bar{x}(t) + (\Gamma_0 + \Lambda_0 \Gamma_1) y(t) + \Phi_0 u(t).
\] (4.65)

Hence, comparing (4.60) and (4.65) it is concluded that defining $F = \Lambda_0$, $H = \Gamma_0 + \Lambda_0 \Gamma_1$, $G = \Phi_0$, $P = I_l$, and $V = \Gamma_1$, one achieves an $l$’th order asymptotic functional observer, if and only if the matrix $\Lambda_0$ is Hurwitz.

On the other hand, from (4.64) we have
\[
\Lambda_0 = M_1 + \bar{Z} S_1.
\] (4.66)

Accordingly, using a pole placement technique the eigenvalues of $\Lambda_0$ can be arbitrarily assigned via properly designing the matrix $\bar{Z}$. Moreover, it is obtained from (4.64), and the definition of $X$ that
\[
\Gamma_0 = M_2 + \bar{Z} S_2
\] (4.67)
\[
\Gamma_1 = M_3 + \bar{Z} S_3
\] (4.68)

**Remark 4.3.8.** The proposed direct FO design scheme does not need to solve the interconnected Sylvester equations (4.7), and at the same time returns the minimal-order observer with an arbitrary asymptotic convergence rate. However, it is only applicable to functional observable, and functional detectable systems that are not functional observable are excluded. Nevertheless, it can still be less conservative than the observability requirement of
4.4 Numerical Examples

The following input signal was arbitrarily selected in all of the upcoming examples, which is more illustrated in Fig. 4.2. Since observer design is conceptually independent of the controller design, the controllability and the stability of the studied systems are not of our concern in this section.

\[ u(t) = 2 + 10e^{-0.4t}\cos(2t), \quad t \geq 0 \]  

(4.69)

![Figure 4.2: The arbitrary input of the system u(t)](image)

4.4.1 Example 1, Non-observable System, Minimum-order FO not Realized

Consider the system (4.1) with the following parameters,
4.4 Numerical Examples

\[ A = \begin{bmatrix}
1 & 0 & 0 & -6 & 0 & 1 & 0 & 2 & -2 \\
0 & -1 & 29 & 136 & -64 & -64 & -123 & -91 & 112 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
2 & 1 & 0 & -2 & 3 & 2 & 0 & 1 & -2 \\
-3 & 1 & 0 & 1 & 0 & -1 & 1 & 1 & 0 \\
-1 & 0 & 1 & 3 & 0 & -1 & 1 & 1 & 0 \\
0 & 0 & 2 & 1 & 0 & 4 & -5 & 1 & -1 \\
1 & 1 & 1 & 0 & 1 & -2 & -3 & 0 & 0 \\
-1 & 1 & 1 & 0 & 0 & -1 & 2 & 4 & -3 \\
\end{bmatrix}, \]

\[ B = [1; -3; -5; 5; -2; 3; 5; 8; 2; 1], \quad C = \begin{bmatrix} I_2 & 0_{2 \times 8} \end{bmatrix} \text{ and} \]

\[ L_0 = \begin{bmatrix}
0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 \\
-0.1767 & -0.0256 & -0.2836 & -0.2836 & -0.2886 \\
1 & 1 & 1 & 1 & 0 \\
-1 & 1 & 1 & 0 & 0 \\
-0.2886 & -0.3662 & -0.3662 & 1 & 0 \\
\end{bmatrix}. \]

It can be checked out that the pair \((A, C)\) is not observable. However, examining Condition (4.3), it is found that the triple \((A, C, L_0)\) is functional observable. In addition, it can be shown that Condition I is satisfied with \(L = L_0\), while Condition II is not realized. Hence, Algorithm 1 was applied to add an auxiliary row to \(L_0\) to satisfy Condition II. The auxiliary row was obtained as

\[ q = \begin{bmatrix} 1 & 3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \]
4.4 Numerical Examples

Hence, \( L = [L_0; q] \) satisfies both Conditions I and II.

Now, the results of applying the new and the conventional algorithms to solve the observer problem are reported. By applying Algorithm 2, the following observer parameters were attained for assigning the eigenvalues of the matrix \( F \) to arbitrary values \((−3, −6, −7, −8)\):

\[
\tilde{Z} = 10^3 \begin{bmatrix} 0_{4 \times 3}, \begin{bmatrix} -2.9692 & 2.9692 & 2.9692 \\ -1.2379 & 1.2379 & 1.2379 \\ -0.3192 & 0.3192 & 0.3192 \\ -0.0315 & 0.0315 & 0.0315 \end{bmatrix}, 0_{4 \times 8}, \begin{bmatrix} -2.9692 & -2.9692 \\ -1.2379 & -1.2379 \\ -0.3192 & -0.3192 \\ -0.0315 & -0.0315 \end{bmatrix} \end{bmatrix}, \begin{bmatrix} -2.9692 & 2.9692 \\ -1.2379 & 1.2379 \\ -0.3192 & 0.3192 \\ -0.0315 & 0.0315 \end{bmatrix} \]

\[
T = \begin{bmatrix} -2969 & -2969 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ -1238 & -1238 & 0 & 0 & -1 & -1 & 1 & 1 & 0 \\ -319.3 & -319.3 & -0.3 & -0.3 & -0.3 & -0.4 & -0.4 & 1 & 0 \\ -31 & -31 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 2969 & 2969 \\ 1238 & 1238 \\ 319.1 & 319.3 \\ 32 & 34 \end{bmatrix},
\]

\[
F = \begin{bmatrix} 3 & 0 & 0 & -2970 \\ 0.1 & 1.6 & 1.9 & -1237.6 \\ -0.6 & 0 & 0.9 & -319 \\ 0.4 & -0.5 & -0.4 & -29.5 \end{bmatrix}, \quad G = 10^3 \begin{bmatrix} 5.954 \\ 2.488 \\ 0.6355 \\ 0.062 \end{bmatrix},
\]

\[
H = 10^4 \begin{bmatrix} -8.6132 & -9.207 \\ -3.6708 & -3.9184 \\ -1.188 & -1.2519 \\ -0.0584 & -0.0646 \end{bmatrix}, \quad V = 10^3 \begin{bmatrix} -2.9692 & -2.9692 \\ -1.2379 & -1.2379 \\ -0.3192 & -0.3192 \\ -0.0315 & -0.0315 \end{bmatrix}
\]

The obtained parameters satisfy Equations (4.7), which confirms the validity of the solution.

Similarly, using the first conventional design scheme in obtaining the unknown observer parameters, yielded the same results as the previous algorithm except for \( Z_1 = \)
10^3 \begin{bmatrix}
-2.9693 & 2.9693 & 2.9693 \\
-1.2379 & 1.2379 & 1.2379 \\
-0.3192 & 0.3192 & 0.3192 \\
-0.0305 & 0.0305 & 0.0305
\end{bmatrix}

Likewise, the constrained Sylvester equations are fulfilled in this case.

On the contrary, the second conventional algorithm does not satisfy the Sylvester equation (4.7a). The following parameters were obtained after applying this method:

\begin{bmatrix}
-7.3704 & -1.2073 & -4.5181 & 3.157 \\
-1.2334 & -0.4398 & -0.6454 & 0.1751 \\
0.1446 & 0.0405 & 0.0808 & -0.037 \\
-0.2118 & -0.0027 & -0.1447 & 0.1382
\end{bmatrix}

\begin{bmatrix}
-605.9603 & 376.4509 & 1 & 1 & 1 \\
-20.8024 & 160.3568 & 0 & 0 & -1 \\
6.0902 & -14.3731 & -0.2836 & -0.2836 & -0.2886 \\
-29.8792 & -3.9329 & 1 & 1 & 0
\end{bmatrix}

\begin{bmatrix}
1 & 1 & 1 & 1 & 0 \\
-1 & 1 & 1 & 0 & 0 \\
-0.2886 & -0.3662 & -0.3662 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}

\begin{bmatrix}
-12.3376 & 3.2237 & 83.0857 & -25.6485 \\
-2.7117 & 2.1675 & 17.2046 & 86.8105 \\
-0.3233 & -0.1119 & -0.8202 & -5.6845 \\
-0.0284 & -0.4295 & 1.8177 & -13.0097
\end{bmatrix}

\begin{bmatrix}
605.9603 & -376.4509 \\
20.8024 & -160.3568 \\
-6.2669 & 14.3475 \\
30.8792 & 6.9329
\end{bmatrix}
Substituting the above parameters in (4.7a) yields:

\[ H = \begin{bmatrix} -8720.7 & 5144.8 \\ 974.7 & 1520.7 \\ -368.8 & 87.5 \\ -440.8 & 10.9 \end{bmatrix}, \quad \text{and} \quad G = \begin{bmatrix} -1719.3 \\ -489.9 \\ 46.2 \\ -18.1 \end{bmatrix}. \]

This particular example emphasize that in some situations the second conventional approach does not work, even when Conditions I and II are satisfied. To magnify these findings, simulation results obtained from the Simulink environment are illustrated in Figs. 4.3 and 4.4. As it is shown in Fig. 4.3 the performance of our new methodology and the first conventional method are similar, which was expected due to their similarly obtained observer parameters. However, the observer working based on the parameters obtained from the second conventional algorithm is not convergent that is clear from Fig. 4.4 that reports 0.1 seconds of the simulation time.
4.4 Numerical Examples

Figure 4.3: The estimation errors of the functions of Example 1 obtained from using different design algorithms.
4.4.2 Example 2, Non-functional observable system, Minimum-order FO not Realized

In this example, the following system’s distribution matrices are considered,

\[
A = \begin{bmatrix}
-5 & -2 & 5 & 0 & 1 \\
2 & -6 & 1 & 0 & -3 \\
0 & -2 & -8 & 0 & 0 \\
-6 & 5 & 7 & -5 & -5 \\
-2 & 0 & -4 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
1 \\
-3 \\
-5 \\
5 \\
-2
\end{bmatrix}, \quad C = \begin{bmatrix}
0 & 1 \\
1 & 1 \\
0_{2 \times 3}
\end{bmatrix},
\]

and \(L_0 = \begin{bmatrix}
0_{2 \times 2} & I_2 & 0_{2 \times 1}
\end{bmatrix}\).

Here the system \(\Sigma\) is not functional observable, but it is functional detectable. This can be shown via testing Condition (4.2) of Theorem 4.2.1. Considering \(L = L_0\), Condition I is satisfied, but Condition II is practically not satisfied. This is because the system is
4.4 Numerical Examples

nearly undetectable. To resolve this issue, Algorithm 1 was used to increase the order of the observer. As a result, a row vector \( q = \begin{bmatrix} -3 & -8 & 6 & 0 & -2 \end{bmatrix} \) was obtained as an auxiliary row to be appended to \( L_0 \). Hence, \( L = [L_0; q] \) satisfies both Conditions I and II, or analogously Conditions III and IV. Calculating the invariant zeros of the matrix pencil

\[
S = \begin{bmatrix}
sI_n - A & 0_{n \times d} \\
C & 0_{(p+l) \times d}
\end{bmatrix},
\]

with \( d \) as an arbitrary integer, it is obtained that the invariant zero of the Rosenbrock’s matrix of the system is equal to “\(-5\)”. Hence, \( \lambda = -5 \) is the undetectable eigenvalue of the system, and should be included in the desired observer’s eigenvalues.

Applying Algorithm 2 for obtaining the observer parameters by assigning the observer’s pole locations to \( \{-10.2648, -5, -9\} \), wherein \( \{-10.2648, -9\} \) were selected arbitrarily, the following parameters were achieved:

\[
\tilde{Z} = \begin{bmatrix}
-8.6776 & 0 & 0.8397 & 0.7873 & -0.2974 \\
0.06 & 0 & -1.0066 & 0.9953 & 1.0028 \\
1.2371 & 0 & -5.2751 & 5.0432 & 5.1978
\end{bmatrix}, \quad 0_3 = \begin{bmatrix}
1.0847 & -0.7873 \\
-0.0075 & -0.9953 \\
-0.1546 & -5.0432
\end{bmatrix},
\]

\[
T = \begin{bmatrix}
-0.1875 & -0.0625 & 1 & 0 & 0 \\
-1 & -2 & 0 & 1 & 0 \\
-2.7648 & -7.7648 & 6 & 0 & -2
\end{bmatrix}, \quad F = \begin{bmatrix}
-9 & 0 & 0 \\
0 & -5 & 0 \\
0 & 0 & -10.2648
\end{bmatrix},
\]

\[
V = \begin{bmatrix}
-0.125 & 0.1875 \\
1 & 1 \\
0 & -0.2352
\end{bmatrix}, \quad H = \begin{bmatrix}
-0.9375 & -0.875 \\
19 & -10 \\
-13.5 & -26.0857
\end{bmatrix}, \text{ and } G = \begin{bmatrix}
-5 \\
10 \\
-5.4704
\end{bmatrix}.
\]
Evidently, these parameters satisfy the equations (4.7). Moreover, using the second conventional algorithm resulted in the same observer parameters satisfying (4.7), except for the matrix $Z_2$.

On the contrary, applying the first conventional scheme yields in the following observer parameters that do not satisfy conditions (4.7a) and (4.7c):

\[
Z_1 = \begin{bmatrix}
-8.7429 & 0 & 0.7259 & -1.0929 & 0.9134 \\
0.0302 & 0 & -0.5028 & 0.0038 & 0.4972 \\
1.2854 & 0 & -6.6353 & 0.1607 & 6.3943
\end{bmatrix}, \quad T = \begin{bmatrix}
0.125 & -0.1875 & 1 & 0 & 0 \\
-1 & -1 & 0 & 1 & 0 \\
-5 & -2.7648 & 6 & 0 & -2
\end{bmatrix},
\]

\[
G = \begin{bmatrix}
-4.3125 \\
7 \\
-22.7056
\end{bmatrix}, \quad \text{and the other parameters were obtained as the previous algorithms.}
\]

Substituting the obtained parameters in (4.7a) and (4.7c) gives

\[
E_1 = \begin{bmatrix}
1 & -1 & 1.4375 & 0 & 0.6875 \\
2 & -1 & 1 & 0 & -3 \\
-1.7679 & 25.7944 & -6.176 & 0 & -17.2352
\end{bmatrix},
\]

and $E_2 = \begin{bmatrix}
-0.3125 & 0.125 \\
0 & -1 \\
2.2352 & -5
\end{bmatrix} \mathbf{0}_3$, respectively.

Simulation results are demonstrated in Fig. 4.6. As this figure highlights, the observers obtained from the new algorithm and the second conventional scheme have similar tracking performances, which was indeed expected due to their similar design parameters. However, the observer designed via the first conventional algorithm is not asymptotically stable.
4.4 Numerical Examples

4.4.3 Example 3, Extension of the Direct Approach

This example is devoted to clarify our contribution on extending the direct approach, explained in Section 4.3.2. Consider the system (4.1) with the parameters described as follows,
4.4 Numerical Examples

\[
A = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\
0 & 2 & 0 & 0 & -3 & 0 & -4 & -1 & 0 & 1 \\
0 & 2 & 0 & 0 & -3 & 0 & -4 & -1 & 0 & 1 \\
0 & 2 & 0 & 0 & -3 & 0 & -4 & -1 & 0 & 1 \\
0 & 2 & 0 & 0 & -3 & 0 & -4 & -1 & 0 & 1 \\
\end{bmatrix},
\]

\[
L_0 = \begin{bmatrix}
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 \\
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
2 & 0 \\
0 & 1 \\
\end{bmatrix}^{0_{2 \times 8}}, \text{ and } B = [1; 0; 0; 5; -2; 0; 5; 8; 2; 0].
\]

Testifying Condition (4.56), it is found that the functional observability index is \( \kappa = 2 \), which indicates that the observer’s order, when designed using the original direct approach is \( \kappa \times l = 4 \). However, this can not necessarily conclude that the minimal order is also equal to four. Nevertheless, using Algorithm 1, it is observed that Condition I is not satisfied. Hence, applying Steps 2-5 of the algorithm, it was found that

\[
L_\beta = \begin{bmatrix}
0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0.0248 & 0.0598 & -0.0657 & -0.0657 & 0.1332 & 0.1332 \\
1 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 \\
-0.0811 & -0.0811 & -0.9536 & 0 \\
\end{bmatrix}. 
\]
Next, examining Condition II, it is clear that $L = L_β$ does not fulfil this rank condition. Hence, applying Step 6 of Algorithm 1 returned the row vector

$$q = \begin{bmatrix} 2 & 6 & 2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

such that $L = [L_β; q]$ satisfies both Conditions I and II. Therefore, employing the new matrix $L$, one can apply the direct approach to design a minimal-order FO for the system. By the way, in this example the minimal order is equal to four, that is $κ \times l$.

After applying the algorithm proposed in Section 4.3.2 to find the observer parameters, in a way that the eigenvalues of the observer be assigned at arbitrary locations $\{-3, -6, -7, -8\}$, the following were attained:

$$Z = 10^3 \begin{bmatrix} 0_{4 \times 3} & \begin{bmatrix} -1.4846 \\ -0.619 \\ 0.5228 \\ -0.0329 \end{bmatrix} \\ \begin{bmatrix} 1.4846 & 2.9692 \\ 0.619 & 1.238 \\ -0.5228 & -1.0456 \\ 0.0329 & 0.0658 \end{bmatrix} \end{bmatrix},$$

$$F = \begin{bmatrix} 3 & 0 & 0 & -1485 \\ -0.4 & 1.2 & -2.6 & -618.9 \\ 0.1 & 0 & 1.2 & 522.8 \\ 1 & -0.9 & 1 & -29.5 \end{bmatrix}, \quad H = 10^4 \begin{bmatrix} -4.3066 & -9.2070 \\ -1.8354 & -3.9184 \\ 1.6178 & 3.4446 \\ -0.0584 & -0.1292 \end{bmatrix},$$

and $G = 10^3 \begin{bmatrix} -2.951 \\ -1.223 \\ 1.0421 \\ -0.052 \end{bmatrix}$.

Simulation results obtained from the Simulink environment are reported in Fig. 4.6. It is clear that all of the estimated functions have asymptotically converged to their true
values. Moreover, the fourth function $z_4$, shows the fastest convergence rate, while the first function $z_1$, has the slowest convergence rate, which was indeed expected due to the desired observer’s eigenvalues.

![Figure 4.6: The estimation errors of the desired functions, obtained from using the extended direct approach (Example 3)](image)

Although using the original format of the direct approach also results in a fourth order observer in the present example, the design procedure is partly difficult and impossible in particular situations. To justify this point it is tried to apply the original direct order-increase algorithm proposed in Section 4.3.2. Since $LA^2 = \Gamma_0 C + \Gamma_1 CA + \Gamma_2 CA^2 + \Lambda_0 L + \Lambda_1 LA$, we have

$X\Sigma_2 = LA^2$  \hspace{1cm} (4.70)

where $\Sigma_2 = [L; LA; C; CA; CA^2]$. Hence,
4.4 Numerical Examples

\[ X = LA^2\Sigma_2^\dagger + \bar{Z}(I_{10} - \Sigma_2\Sigma_2^\dagger), \]

(4.71)

where \( \bar{Z} \in \mathbb{R}^{2 \times 10} \) is an arbitrary design matrix. Moreover, let us define

\[
\begin{bmatrix}
M_1 & M_2 & M_3 & M_4 & M_5
\end{bmatrix} \triangleq LA^2\Sigma_2^\dagger,
\]

and

\[
\begin{bmatrix}
S_1 & S_2 & S_3 & S_4 & S_5
\end{bmatrix} \triangleq I_{10} - \Sigma_2\Sigma_2^\dagger,
\]

such that \( M_1, M_2, M_3, M_4, \) and \( M_5 \) have the same sizes of \( \Lambda_0, \Lambda_1, \Gamma_0, \Gamma_1, \) and \( \Gamma_2, \) respectively. Similar correspondence holds for \( S_i \)s \( i = \{1, \ldots, 5\}. \) Next, the relation (4.71) gives

\[
\begin{align*}
\Lambda_0 &= M_1 + \bar{Z}S_1 \\
\Lambda_1 &= M_2 + \bar{Z}S_2
\end{align*}
\]

(4.72)

Now, from (4.60) we have

\[
\dot{x}(t) = \begin{bmatrix}
\Lambda_0 & I_2 \\
\Lambda_1 & 0_2
\end{bmatrix} x(t) + \begin{bmatrix}
\Phi_1 \\
\Phi_0
\end{bmatrix} u(t) + \begin{bmatrix}
\Gamma_1 + \Lambda_1\Gamma_2 \\
\Gamma_0 + \Lambda_0\Gamma_2
\end{bmatrix} y(t)
\]

(4.73)

Hence, \( \bar{Z} \) must be properly designed, such that \( F = \begin{bmatrix}
\Lambda_0 & I_2 \\
\Lambda_1 & 0_2
\end{bmatrix} \) becomes an stable matrix with preselected eigenvalues \( \{-3, -6, -7, -8\}. \) However, the analytic computation of the eigenvalues of the matrix \( F \) is a difficult task, even in this example that \( \kappa = 2 \) and \( l_0 = 2, \) which is the simplest scenario of using the original direct approach. One idea for solving this problem is to first consider \( \Lambda_0 \) and \( \Lambda_1 \) as unknown 2 by 2 matrices. Next, we should solve a system of algebraic equations with 4 equations, and 8 unknowns, obtained from the analytic calculation of the eigenvalues of \( F. \) Eventually, the matrix \( \bar{Z} \) can be obtained from \( \bar{Z} \begin{bmatrix}
S_1 & S_2
\end{bmatrix} = \begin{bmatrix}
\Lambda_0 - M_1 & \Lambda_1 - M_2
\end{bmatrix}. \) However, the latter equation has no solution in this particular example, because \( \begin{bmatrix}
S_1 & S_2
\end{bmatrix} \) is not a full column rank matrix.
As a result, the original direct approach does not work in this example.

4.5 Conclusions

The problem of designing minimal multi-functional observers for LTI systems has been addressed. A new design algorithm has been proposed that enables finding a new and more reliable way to solve the observer equations. Our observer scheme has only assumed the functional observability/detectability of the system, which is necessary and sufficient for the existence of an asymptotic functional observer for the system. In the new observer design scheme, more numerical degrees of freedom are provided for the design parameter that results in better performance and simplicity, as well as higher reliability with regard to the other existing transformation-based approaches. The necessary and sufficient conditions have been obtained for the existence of an asymptotic observer when using the new methodology, and the equivalence of these conditions to renown Conditions I and II has been verified. In addition, an extension of the direct approach is advised to make it workable for designing FOs for multiple-functions. Three illustrative numerical examples and simulation results clarified the usefulness and superiority of the proposed design methods, as well as some drawbacks of the conventional observer design schemes.
Chapter 5

Functional Observer Design for LTI Systems with Multiple Time-Varying State and Input Delays

5.1 Introduction

Due to its importance and wide range of applications, the problem of ordinary filter design for hereditary systems has been the focus of attention by the control community for many years, and even very advanced problems including observer design for stochastic time-delay systems have been studied in the literature (see e.g. [159–165]). Nevertheless, the problem of delay-dependent functional observer design for time-delay systems has been fairly overlooked. There are only a few papers in this area [101, 103, 105–107]. From those contributions it is clear that FO design for time-delay systems involves significantly more complexities and challenges, and one cannot simply and directly extend a methodology for the full-order observer design to its corresponding problem in functional observers. These
challenges are partly related to the constrained equations that should be satisfied along with the matrix inequalities associated with the stability conditions.

The key paper in this area is the first one proposed by Darouach in 2001 [103], which studies delay-dependent stability of LTI systems with a single slow-varying state delay. Next, in [101] the same author using a similar approach, extends the subject to unknown-input functional observer design problem. Teh and Trinh [105] consider the simultaneous estimation of the states and the unknown inputs for LTI systems with a single slow-varying state delay. Ha et al. [106], consider constant equal delays in the input and output channels of the observer structure (not in the nominal system). Nam et al. [107] study the problem of [101], but assumes a single constant state delay in the system. However, they consider $H_\infty$ observer design to relax the structural constraints that are created due to designing unknown-input functional observers.

In addition, the majority of the above papers use simple LKFs and conservative techniques in establishing their observers stability criteria. This point essentially induces heavy restrictions on the application of these approaches, since they may result in small stability regions in terms of the upper-bound of the delay and its derivative. This drawback is particularly exaggerated, when studying systems with multiple mixed state delays. Furthermore, only a few contributions address the exponential convergence of the observer, which is essential for regulating the performance of the FO [106, 107].

Other papers that study functional observer design for time-delay systems mainly consider delay-free observer structures (see e.g. [68, 108, 166]). It is well-known that ignoring the knowledge of state delays into the observer structure and treating the delayed terms as unknown-inputs, can induce significantly conservative constraints on the structure of the system, which is not desirable.

In this chapter, we study the novel fundamental problem of delay-dependent functional observer design for LTI systems with mixed and known time-varying state delays
and multiple *unknown* time-varying input-delays. The observer is of minimum-order, and the time-varying input delays are arbitrary with no constraint on their upper-bounds, but the state delays are upper-bounded. It is well-known that in many practical applications including network control systems, teleoperation over internet, and sampled-data control systems [167–169], the actuators delays (input delays) are time-varying and have uncertain nature, which emphasize the necessity of making the observer robust against these uncertainties. Moreover, two scenarios are considered for the rates of the state-delays: I) derivatives less than one, and II) unknown delay derivatives. *Delay-dependent* (DD) sufficient conditions in terms of LMIs are constructed that guarantee the global exponential stability of the observer in both scenarios. The Lyapunov Krasovskii approach is employed in this regard. Using more effective techniques like the *descriptor transformation* and a contemporary weighted integral inequality, our results can be less conservative than the majority of the existing relevant papers in this field. Moreover, an intelligent method based on the *genetic algorithm* (GA) is proposed to adjust a weighting matrix in the obtained LMI criteria. The proposed observer design algorithm can also be directly applied to design *unknown-input* functional observer for LTI systems with multiple time-varying state delays.

The rest of the chapter is organised as follows. The problem is introduced and some preliminaries are given in Section 5.2. The main results are presented in Section 5.3, explaining the derivation of the observer equations, stability analysis, and the design algorithms. An illustrative numerical example is given in Section 5.4, and the chapter is concluded in Section 5.5.
5.2 Problem Statement and Preliminaries

Notations: Throughout the chapter, $|f(\cdot)|$ denotes the absolute value of $f(\cdot)$; $\otimes$ is the Kronecker product; $\max\{c_1, \cdots, c_n\}$ is the largest number of the set $\{c_1, \cdots, c_n\}$; $\text{diag}(X_1, \cdots, X_n)$ is a block diagonal matrix with $X_1, \cdots, X_n$ as its diagonal blocks; $[X; Y] \triangleq [X^T, Y^T]^T$, where $X$ and $Y$ could be scalars, vectors, or matrices with appropriate dimensions; $\text{sym}(X) = X + X^T$; $\star$ in a symmetric matrix stands for the symmetric element, and $\rho(X)$ represents the rank of the matrix $X$. Additionally, $\mathbb{R}^{n \times m}$ and $\mathbb{S}^n$ respectively denote the space of $n \times m$ real matrices, and the space of $n \times n$ symmetric matrices; and $\mathbb{C}^+$ is the set of complex numbers with non-negative real parts. Moreover, $C_n([a, b])$ is the space of continuous functions mapping from the set $[a, b]$ to $\mathbb{R}^n$, with the topology of uniform convergence, and $x_t(\theta) \triangleq x(t + \theta)$, $\forall \theta \in [-h, 0]$. Furthermore, $X^+$ and $X^\perp$ respectively represent the generalized-inverse and a right-orthogonal matrices of $X$, such that $XX^\perp = 0$, where $0$ is the zero matrix with appropriate dimensions. In addition, $I_n$ denotes the $n \times n$ identity matrix, and $0_{n \times m}$ is the $n \times m$ zero matrix. Finally, $X \succ 0$ ($\prec 0$) and $X \succeq 0$ ($\preceq 0$) report that the matrix $X$ is positive definite (negative definite) and positive semi-definite (negative semi-definite), respectively.

The following LTI system with two known time-varying state delays and two unknown time-varying input delays are considered,

\begin{align*}
\dot{x}(t) &= A_1 x(t) + \sum_{i=2}^{3} A_i x(t - h_i(t)) \\
&
+ B_1 u_1(t) + \sum_{i=2}^{3} B_i u_i(t - \tau_i(t)) \\
y(t) &= C x(t) \\
z(t) &= L x(t) \\
x(\theta) &= \phi_x(\theta) \ \forall \theta \in [-h_u, 0], \\
u(\theta) &= \phi_u(\theta) \ \forall \theta \in [-\tau_u, 0],
\end{align*}

(5.1)
5.2 Problem Statement and Preliminaries

where \( x(\cdot) \in \mathbb{R}^n \) is the state vector, \( u_i(\cdot) \in \mathbb{R}^{m_i}, \ i = \{1, 2, 3\} \) is the input vector, \( y(\cdot) \in \mathbb{R}^p \) is the output vector, and \( z(\cdot) \in \mathbb{R}^l \) is the vector of functions to be estimated. Moreover, \( \phi_x(\cdot) \in C_n([-h_u, 0]) \) and \( \phi_u(\cdot) \in C_n([-\tau_u, 0]), h_u = max\{h_{iu}\}, \tau_u = sup\{\tau_i(t)\} \) for \( i = \{2, 3\} \), are the initial state and the initial input functions, respectively. Furthermore, \( A_i \in \mathbb{R}^{n \times n}, B_i \in \mathbb{R}^{n \times m}, i = \{1, 2, 3\}, C \in \mathbb{R}^{p \times n}, \) and \( L \in \mathbb{R}^{l \times n} \) are constant known matrices that describe the dynamics of the system. In addition, the state and input delays satisfy the following conditions for \( i = \{2, 3\} \),

\[
0 \leq h_i(t) \leq h_{iu}, \quad \dot{h}_i(t) \leq \mu_i < 1, \quad \text{arbitrary} \ \tau_u. \tag{5.2}
\]

Furthermore, throughout the chapter, without loss of generality, it is assumed that the matrices \( C \) and \( L \) are of full row rank, and \( B_i, i = \{1, 2, 3\} \) are full column rank, and in addition \( p < n \).

Remark 5.2.1. Two mixed state delays and two input delays are taken into account. However, this is mainly for the sake of simplicity with notations, and essentially does not reduce from the generality of the results. The obtained results can be directly extended to systems with a larger number of delays in the states and/or the inputs.

The aim of this chapter is to design a delay-dependent exponentially stable functional observer of minimum-order (order equal to the number of functions) for the system (5.1), assuming known state delays and unknown (arbitrary) input-delays. State delays can be constant or bounded time-varying, and two scenarios are considered for the time-derivative of the state-delays: I) slow-varying delays, wherein the absolute value of each delay derivative is essentially less than one, and II) fast-varying delays, with arbitrary (or unknown) delays rates.

Throughout this chapter, the following dynamic observer structure is considered for the system,
\[ \dot{\omega}(t) = F_1 \omega(t) + \sum_{i=2}^{3} F_i \omega(t - h_i(t)) + G_1 u_1(t) + H_1 y(t) \]
\[ + \sum_{i=2}^{3} H_i y(t - h_i(t)) + G_1 u_1(t) \]
\[ \dot{\hat{z}}(t) = \omega(t) + V y(t) \]
\[ \omega(\theta) = 0 \quad \forall \theta \in [-h_u, 0], \]

(5.3)

where \( \omega(\cdot) \in \mathbb{R}^l \) is the observer’s state vector, and \( \hat{z}(\cdot) \in \mathbb{R}^l \) is the estimated functional. In addition, \( F_i \in \mathbb{R}^{l \times l}, \quad H_i \in \mathbb{R}^{l \times p}, \quad i = \{1, 2, 3\}, \quad G_1 \in \mathbb{R}^{l \times m_1}, \) and \( V \in \mathbb{R}^{l \times p} \) are the observer parameters that should be designed. Let us define the following error parameters,

\[ e(t) \triangleq \omega(t) - T x(t), \]  
(5.4)

\[ e(t) \triangleq \hat{z}(t) - z(t), \]  
(5.5)

where \( T \in \mathbb{R}^{l \times n} \) is a constant matrix that will be used in the design framework.

Definition 5.2.1. The observer (5.3) is globally \( \alpha \)-exponentially stable for a scalar \( \alpha > 0 \), if for any initial condition \( \phi(\cdot) \in C_n([-h_u, 0]) \), the estimation error satisfies

\[ |e(t, \phi(\theta))|_c \leq e^{-\alpha t} |e(0, \phi(\theta))|, \quad \forall t \geq 0, \quad \forall \theta \in [-h_u, 0] \]

where \( |e(t)|_c = \sup_{-h_u \leq \theta \leq 0} |e(t + \theta)|. \)

Hence, delay-dependent sufficient conditions should be found together with a set of appropriate observer parameters that can guarantee the global \( \alpha \)-exponential stability of the observer, upon each specific assumption on the rates of the state-delays.
5.3 Detailed Observer Design Procedure

5.3.1 Deriving the Observer Equations

First, the following theorem summarizes the necessary and sufficient conditions for the existence of an \( l \)'th order functional observer with the desired features.

**Theorem 5.3.1.** There exists a globally exponentially stable and proper minimum-order functional observer with structure (5.3) for the system (5.1), if and only if there exists a matrix \( T \) together with a set of observer parameters \( F_i, H_i, G_1, V, i = \{1, 2, 3\} \) such that the sequel conditions are realized

\[
\begin{align*}
(A) \quad & \text{The following error dynamics is globally exponentially stable for delays satisfying (5.2)} \\
& \dot{\epsilon}(t) = F_1 \epsilon(t) + \sum_{i=2}^{3} F_i \epsilon(t - h_i(t)) \\
& \epsilon(\theta) = -T \phi_+(\theta), \quad \forall \theta \in [-h_u, 0]
\end{align*}
\]

\[
\begin{align*}
(B) \quad & \text{The following constrained matrix equations are satisfied} \\
& T + VC - L = 0, \quad (5.7a) \\
& F_i T - T A_i + H_i C = 0, \quad i = \{1, 2, 3\} \quad (5.7b) \\
& TB_j = 0, \quad j = \{2, 3\} \quad (5.7c) \\
& G_1 = TB_1. \quad (5.7d)
\end{align*}
\]

**Proof.** The proof of both necessity and the sufficiency of the theorem are straightforward, and only the sufficiency part of the theorem is proved. Differentiating (5.6), substituting
from (5.1) and (5.3), and after some basic algebraic manipulations, it follows that

\[ \dot{\epsilon}(t) = F_1 \epsilon(t) + \sum_{i=2}^{3} F_i \epsilon(t - h_i) + (F_1 T - TA_1 + H_1 C)x(t) + (G_1 - TB_1)u_1(t) - \sum_{i=2}^{3} TB_j u_j(t - \tau_j(t)) + \sum_{i=2}^{3} (F_i T - TA_i + H_i C)x(t - h_i(t)) \] (5.8)

Hence, if Condition (A) together with the observer equations (5.7b)-(5.7d) are satisfied, then the error signal \( \epsilon(t) \) is globally exponentially stable. Moreover, from (5.5), it is obtained that 

\[ e(t) = \epsilon(t) + (T + VC - L)x(t). \] 

Hence, if in addition (5.7a) is satisfied, then the error signal \( e(t) \) is globally exponentially stable, and the sufficiency of the theorem is verified.

Hereafter, the main aim of the chapter is to find sufficient conditions, whence appropriate observer parameters can be sought, such that the conditions of Theorem 5.3.1 are satisfied.

**Remark 5.3.1.** As can be observed from the observer structure and conditions (5.7c), the input delay terms are treated as unknown-inputs. In consequence, the results of the chapter can also be regarded as a methodology to design minimum-order unknown-input functional observers for LTI systems with multiple time-varying state delays. To our knowledge, this chapter proposes one of the first works that address this crucial problem. Moreover, if the online values of the input-delays are available, then the input-delay terms can be employed in the observer structure, and even less conservative results can be achieved by replacing the restrictive conditions (5.7c), by non-conservative ones \( G_j = TB_j, j = \{2, 3\} \).

The interconnected observer equations (5.7) must be solved for the parameters \( F_i, i = \{1, 2, 3\} \), in a way that Condition (A) of Theorem 5.3.1 is fulfilled. Similar to Chapter 4,
first let us define the matrix \( M \triangleq [I_n; C] \in \mathbb{R}^{(n+p) \times n} \), which is of full column rank. Therefore, it can always be obtained from (5.7a) that

\[
T = LM_1 + Z\mathcal{E}_1, \tag{5.9}
\]

\[
V = LM_2 + Z\mathcal{E}_2, \tag{5.10}
\]

where \( Z \in \mathbb{R}^{l \times (n+p)} \) is a parameter matrix,

\[
\begin{bmatrix}
M_1 & M_2
\end{bmatrix} \triangleq M^\dagger,
\]

and \( \begin{bmatrix} \mathcal{E}_1 & \mathcal{E}_2 \end{bmatrix} \triangleq I_{n+p} - MM^\dagger \), and \( M_1, M_2, \mathcal{E}_1, \) and \( \mathcal{E}_2 \) have appropriate dimensions.

Later on, equations (5.7b) are post-multiplied by the transformation matrix \( \bar{U} \triangleq \begin{bmatrix} C^\dagger & C^\perp \end{bmatrix} \), and for \( i = \{1, 2, 3\} \) it simply gives the following

\[
H_i = -F_iTC^\dagger + T A_i C^\dagger, \tag{5.11}
\]

\[
F_iTC^\perp = T A_i C^\perp. \tag{5.12}
\]

After substituting \( T \) from (5.9) into (5.12) and (5.7c), and considering the relation \( \mathcal{E}_1 C^\perp = 0 \) (see Section 4.3.1), it can be obtained for \( i = \{1, 2, 3\} \) and \( j = \{2, 3\} \) that

\[
F_iLM_1 C^\perp = LM_1 A_i C^\perp + Z\mathcal{E}_1 A_i C^\perp, \tag{5.13}
\]

\[
(LM_1 + Z\mathcal{E}_1)B_j = 0. \tag{5.14}
\]

Reformulating equations (5.13) and (5.14), we have

\[
\begin{bmatrix}
F_1 & F_2 & F_3 & -Z
\end{bmatrix} \Omega = \Phi, \tag{5.15}
\]
where

$$\Phi \triangleq \begin{bmatrix} LM_1 A_1 C \perp & LM_1 A_2 C \perp & LM_1 A_3 C \perp & LM_1 B_2 & LM_1 B_3 \end{bmatrix},$$

and

$$\Omega \triangleq \begin{bmatrix} LM_1 C \perp & 0 & 0 & 0 & 0 \\ 0 & LM_1 C \perp & 0 & 0 & 0 \\ 0 & 0 & LM_1 C \perp & 0 & 0 \\ \mathcal{E}_1 A_1 C \perp & \mathcal{E}_1 A_2 C \perp & \mathcal{E}_1 A_3 C \perp & \mathcal{E}_1 B_2 & \mathcal{E}_1 B_3 \end{bmatrix}.$$  

According to [63, 170], (5.15) has a solution for $F_i, \ Z_i, \ i = \{1, 2, 3\}$, if and only if $\rho([\Omega; \Phi]) = \rho(\Omega)$. Moreover, it is shown in Chapter 4 that the relations $LM_1 C \perp = LC \perp$, $\rho(\mathcal{E}_1 A_i C \perp) = \rho(\mathcal{E} A_i C \perp)$, and $\rho(LM_1 A_i C \perp) = \rho(LA_i C \perp)$, always hold. Hence, here comes a necessary condition for the existence of a minimum-order FO of the form (5.3) for the system (5.1)

**Condition I**

$$\rho \left( \begin{bmatrix} LA_1 C \perp & LA_2 C \perp & LA_3 C \perp & LB_2 & LB_3 \\ CA_1 C \perp & CA_2 C \perp & CA_3 C \perp & CB_2 & CB_3 \\ LM_1 C \perp & 0 & 0 & 0 & 0 \\ 0 & LM_1 C \perp & 0 & 0 & 0 \\ 0 & 0 & LM_1 C \perp & 0 & 0 \\ \mathcal{E}_1 A_1 C \perp & \mathcal{E}_1 A_2 C \perp & \mathcal{E}_1 A_3 C \perp & \mathcal{E}_1 B_2 & \mathcal{E}_1 B_3 \end{bmatrix} \right) =$$

$$\rho \left( \begin{bmatrix} CA_1 C \perp & CA_2 C \perp & CA_3 C \perp & CB_2 & CB_3 \\ LM_1 C \perp & 0 & 0 & 0 & 0 \\ 0 & LM_1 C \perp & 0 & 0 & 0 \\ 0 & 0 & LM_1 C \perp & 0 & 0 \end{bmatrix} \right).$$

(5.16)
5.3 Detailed Observer Design Procedure

Upon the fulfilment of Condition I, (5.15) gives

\[
\begin{bmatrix}
F_1 & F_2 & F_3 & -Z
\end{bmatrix} = N_1 + JN_2,
\]

(5.17)

where \( N_1 \triangleq \Phi \Omega^\dagger \in \mathbb{R}^{l \times (3l+n+p)} \), \( N_2 \triangleq (I_{3l+n+p} - \Omega \Omega^\dagger) \in \mathbb{R}^{(3l+n+p) \times (3l+n+p)} \), and \( J \in \mathbb{R}^{l \times (3l+n+p)} \) is a free design parameter. Partitioning \( N_1 \) and \( N_2 \) appropriately as \( N_1 =: \begin{bmatrix} N_{11} & N_{12} & N_{13} & N_{14} \end{bmatrix} \), and \( N_2 =: \begin{bmatrix} N_{21} & N_{22} & N_{23} & N_{24} \end{bmatrix} \), one obtains

\[
F_i = N_{1i} + JN_{2i}, \quad i = \{1, 2, 3\},
\]

(5.18)

\[-Z = N_{14} + JN_{24}.\]

(5.19)

Hereafter, the free parameter \( J \) is designed in a way that Condition (A) of Theorem 5.3.1 is strictly satisfied.

5.3.2 Stability Analysis

Since the state delays can accept zero values, firstly system (5.1) with all zero state-delays is investigated, \textit{i.e.} \( h_i(.) \equiv 0 \), \( i = \{2, 3\} \). In this particular case, Condition (A) deals with analyzing an intrinsically simpler dynamic equation, \textit{i.e.}, \( \dot{\epsilon}(t) = \bar{F}\epsilon(t) \), wherein \( \bar{F} \triangleq \sum_{i=1}^{3} F_i \).

The latter dynamic is an ordinary differential equation rather than a rigorous functional differential equation. Let us further define \( \bar{N}_1 \triangleq N_{11} + N_{12} + N_{13} \) and \( \bar{N}_2 \triangleq N_{21} + N_{22} + N_{23} \).

Hence, considering (5.18) the auxiliary error dynamics (5.6) can be written as

\[
\dot{\epsilon}(t) = (\bar{N}_1 + J\bar{N}_2)\epsilon(t)
\]

(5.20)

Therefore, under the assumption of all zero state-delays, if the parameter \( J \) can be found in a way that the matrix \( \bar{N}_1 + J\bar{N}_2 \) is strictly Hurwitz, then the auxiliary error signal \( \epsilon(t) \),
and consequently \(e(t)\) asymptotically converge to zero. To this aim, the pair \((\bar{N}_1, \bar{N}_2)\) must be observable, or detectable [62, 63], which can be mathematically expressed as

**Condition II**

\[
\rho \left( \begin{bmatrix} sI_l - \bar{N}_1 \\ \bar{N}_2 \end{bmatrix} \right) = l, \quad \forall s \in \mathbb{C}^+.
\] (5.21)

The following theorem summarizes the above illustrations.

**Theorem 5.3.2.** Under the condition of all zero state-delays, there exists a minimum-order asymptotic multi-functional observer with structure (5.3) for the system (5.1), if and only if Conditions I and II are satisfied.

**Remark 5.3.2.** A methodology to design the observer parameters considering zero state-delays is fully investigated in [68]. In addition, Conditions I and II apparently should be satisfied for designing a stable functional observer for the system in its general form.

Now, sufficient conditions for the fulfilment of Condition (A) in Theorem 5.3.1 are established under slow and fast-varying delays scenarios.

**Theorem 5.3.3.** Given \(h_{iu} > 0, \mu_i < 1, i = \{2, 3\}, \text{ and } \alpha > 0\), the delayed error dynamics (5.6) is globally \(\alpha\)-exponentially stable for any delays satisfying (5.2), if there exist matrices \(P_1 \succ 0 \in \mathbb{S}^l, P_2 \text{ and } P_3 \in \mathbb{R}^{l \times l}, R_i \succ 0 \in \mathbb{S}^l, \text{ and } S_i \succ 0 \in \mathbb{S}^l, i = \{2, 3\}\) that the following matrix-inequality is feasible,
where

\[
E \triangleq \begin{bmatrix} I_l & 0 \\ 0 & 0 \end{bmatrix}, \quad P \triangleq \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix},
\]

\[
\Pi^{s}_{1,1} = \text{sym} \left( P^T \begin{bmatrix} 0 & I_l \\ F_1 & -I_l \end{bmatrix} \right) + 2\alpha EP + \sum_{i=2}^{3} \begin{bmatrix} S_i - \eta_i R_i & 0 \\ 0 & h_{iu} R_i \end{bmatrix},
\]

\[
\Pi^{s}_{1,5} = \left( \frac{\alpha}{\gamma_{02}} + \frac{\alpha}{\rho_{02}} - h_{2u} \alpha^2 \frac{\gamma_{02}}{\rho_{02} \gamma_{12}} \right) R_2, \quad \Pi^{s}_{1,6} = \left( \frac{\alpha}{\gamma_{03}} + \frac{\alpha}{\rho_{03}} - h_{3u} \alpha^2 \frac{\gamma_{03}}{\rho_{03} \gamma_{13}} \right) R_3,
\]

\[
\Pi^{s}_{1,7} = \left( -\alpha^2 \frac{\gamma_{02}}{\rho_{02} \gamma_{12}} + h_{2u} \alpha^3 \frac{\gamma_{02}}{\rho_{02} \gamma_{12}} \right) R_2, \quad \Pi^{s}_{1,8} = \left( -\alpha^2 \frac{\gamma_{03}}{\rho_{03} \gamma_{13}} + h_{3u} \alpha^3 \frac{\gamma_{03}}{\rho_{03} \gamma_{13}} \right) R_3,
\]

\[
\Pi^{s}_{6,6} = - \left( \frac{\alpha}{\gamma_{03}} + \frac{\alpha}{\rho_{03}} \right) R_3, \quad \Pi^{s}_{6,8} = -\alpha^3 \frac{\gamma_{02}}{\rho_{02} \gamma_{12}} R_2, \quad \text{and} \quad \Pi^{s}_{8,8} = -\alpha^3 \frac{\gamma_{03}}{\rho_{03} \gamma_{13}} R_3. \quad \text{In addition, for} \ i = \{2, 3\}, \gamma_{0i} \triangleq e^{\alpha h_{iu}} - 1, \gamma_{1i} \triangleq e^{\alpha h_{iu}} - \alpha h_{iu} - 1, \rho_{0i} \triangleq \frac{\gamma_{0i}}{\gamma_{1i}} \left( \gamma_{0i} - \alpha^2 h_{iu} e^{\alpha h_{iu}} \right), \text{and} \eta_i \triangleq \frac{\alpha}{\gamma_{0i}} + \frac{\alpha}{\rho_{0i}} - 2h_{iu} \alpha^2 \frac{\gamma_{0i}}{\rho_{0i} \gamma_{1i}} + h_{iu} \alpha^3 \frac{\gamma_{0i}}{\rho_{0i} \gamma_{1i}}.
\]

**Proof.** Consider the following Lyapunov-Krasovskii functional candidate facilitated with the descriptor transformation [4].

\[
V(t, \varepsilon_i) = V_1(t) + V_2(t, \varepsilon_i) + V_3(t, \varepsilon_i)
\]

\[
(5.23)
\]
5.3 Detailed Observer Design Procedure

where

\[ V_1(t) \triangleq \bar{\epsilon}^T(t) EP\bar{\epsilon}(t), \]
\[ V_2(t) \triangleq \sum_{i=2}^{3} \int_{-h_i}^{0} \int_{t+\theta}^{t} e^{2\alpha(s-t)} \hat{\epsilon}^T(s) R_i\hat{\epsilon}(s) ds d\theta, \]
\[ V_3(t) \triangleq \sum_{i=2}^{3} \int_{t-h_i(t)}^{t} e^{2\alpha(s-t)} \hat{\epsilon}^T(s) S_i\epsilon(s) ds. \]

In addition, \( \bar{\epsilon}(t) \triangleq [\epsilon(t); \hat{\epsilon}(t)] \). According to the definition of \( V(t, \bar{\epsilon}_t) \), positive constants \( \alpha_1 \) and \( \alpha_2 \) can be found for any time \( t \geq 0 \) such that \( \alpha_1|\epsilon(t)| \leq V(t, \bar{\epsilon}_t) \leq \alpha_2|\bar{\epsilon}_t| \). Now, differentiating (5.23) along the solution of (5.6) gives

\[
\frac{d}{dt} V(t, \epsilon_t) |_{(5.6)} = \frac{d}{dt} V_1(t) |_{(5.6)} + \frac{d}{dt} V_2(\epsilon_t) |_{(5.6)} + \frac{d}{dt} V_3(\epsilon_t) |_{(5.6)}, \quad (5.24)
\]

\[
\frac{d}{dt} V_1(t) |_{(5.6)} = -2 \sum_{i=2}^{3} \hat{\epsilon}^T(t) P^T \left[ \begin{array}{c} 0 \\ F_i \end{array} \right] \int_{t-h_i(t)}^{t} \epsilon(s) ds \\
+ \bar{\epsilon}^T(t) \text{sym} \left( P^T \left[ \begin{array}{cc} 0 & I_l \\ \sum_{i=1}^{3} F_i & -I_l \end{array} \right] \right) \bar{\epsilon}(t), \quad (5.25)
\]

\[
\frac{d}{dt} V_2(t, \epsilon_t) |_{(5.6)} = -2\alpha V_2(t, \epsilon_t) + \sum_{i=2}^{3} \left( \int_{-h_i}^{0} \hat{\epsilon}^T(t) R_i\hat{\epsilon}(t) d\theta \\
- \int_{-h_i}^{0} e^{2\alpha\theta} \hat{\epsilon}^T(t + \theta) R_i\hat{\epsilon}(t + \theta) d\theta \right), \quad (5.26)
\]

\[
\frac{d}{dt} V_3(t, \epsilon_t) |_{(5.6)} = -2\alpha V_3(t, \epsilon_t) + \sum_{i=2}^{3} (\epsilon^T(t) S_i\epsilon(t) \\
- (1 - \frac{d}{dt} h_i(t)) e^{-2\alpha h_i(t)} \epsilon^T(t - h_i(t)) S_i\epsilon(t - h_i(t))) \right). \quad (5.27)
\]
Moreover, for \( i = \{2, 3\} \) we always have

\[
-(1 - \frac{d}{dt} h_i(t)) e^{-2\alpha h_i(t)} e^T(t - h_i(t)) S_i \epsilon(t - h_i(t)) \leq
-(1 - \mu_i) e^{-\alpha h_i(t)} e^T(t - h_i(t)) S_i \epsilon(t - h_i(t)) \tag{5.28}
\]

In addition, Lemma 2.2.6 is employed to obtain the following inequality for \( i = \{2, 3\} \)

\[
- \int_{-h_{iu}}^{0} e^{\alpha \theta} e^T(t + \theta) R_i \dot{\epsilon}(t + \theta) d\theta \leq -\xi_i^T \left[ \alpha \gamma_{i0} + \frac{\alpha}{\rho_{i0}} \begin{bmatrix} -\alpha^2 \gamma_{i0} \\ 0 \\ \rho_{i0} \gamma_{i1} \end{bmatrix} \right] \otimes R_i \xi_i, \tag{5.29}
\]

where \( \xi_i \triangleq \begin{bmatrix} \epsilon(t) - \epsilon(t - h_{iu}) \\ h_{iu} \epsilon(t) - \int_{-h_{iu}}^{0} \epsilon(t + s) ds \end{bmatrix} \).

Now, in light of (5.25)-(5.29), and bearing in mind that \( \int_{t-h_i(t)}^{t} \dot{\epsilon}(s) ds = \epsilon(t) - \epsilon(t - h_i(t)) \), the following inequality can be written

\[
\frac{d}{dt} V(t, \epsilon_t) \big|_{(5.6)} \leq \zeta^T(t) \Pi^\epsilon \zeta(t) - 2\alpha V(t, \bar{\epsilon}_t) \tag{5.30}
\]

where

\[
\zeta(t) \triangleq \begin{bmatrix} \bar{\epsilon}(t); \epsilon(t - h_2(t)); \epsilon(t - h_3(t)); \epsilon(t - h_{2u}); \epsilon(t - h_{3u}); \int_{t-h_{2u}}^{t} \epsilon(s) ds; \int_{t-h_{3u}}^{t} \epsilon(s) ds \end{bmatrix},
\]

and \( \Pi^\epsilon \) is the whole matrix on the left-hand side of the inequality (5.22). Hence, if the matrix inequality (5.22) is fulfilled, then \( \frac{d}{dt} V(t, \epsilon_t) \big|_{(5.6)} + 2\alpha V(t, \bar{\epsilon}_t) \leq 0 \). As a result, differentiating \( v(t) \triangleq e^{2\alpha t} V(t, \bar{\epsilon}_t) \) along the solution of (5.6) results \( \dot{v}(t) < 0 \). Finally, integrating \( \dot{v}(t) \) from 0 to \( t \), and substituting from the definition of \( v(t) \), simply gives

\( V(t, \bar{\epsilon}_t) < e^{-2\alpha t} V(0, \bar{\phi}) \), where \( \bar{\phi}(\theta) \triangleq \bar{\epsilon}_0(\theta), \forall \theta \in [-h_u, 0] \). This point also indicates that \( \epsilon(t), \dot{\epsilon}(t), \) and thus \( \bar{\epsilon}(t) \) are \( \alpha \)-exponentially stable in the large. The proof is now complete.
Corollary 5.3.1. Given \( h_{iu} > 0, i = \{2, 3\}, \alpha > 0 \), the delayed error dynamics (5.6) is globally \( \alpha \)-exponentially stable for any delays satisfying (5.2) and unknown \( \mu_i s \), if there exist matrices \( P_1 \succ 0 \in \mathbb{S}^l, P_2 \) and \( P_3 \in \mathbb{R}^{l \times l}, R_i \succ 0 \in \mathbb{S}^l, i = \{2, 3\} \) that satisfy the following matrix-inequality,

\[
\begin{bmatrix}
\Pi_{1,1}^f & \Pi_{1,3}^f & \Pi_{1,4}^f & \Pi_{1,5}^f & \Pi_{1,6}^f \\
* & \Pi_{3,3}^f & 0 & \Pi_{3,5}^f & 0 \\
* & * & \Pi_{4,4}^f & 0 & \Pi_{4,6}^f \\
* & * & * & \Pi_{5,5}^f & 0 \\
* & * & * & * & \Pi_{6,6}^f
\end{bmatrix} \prec 0,
\]

where \( \Pi_{1,1}^f = \Pi_{1,1}^s, \Pi_{1,3}^f = \Pi_{1,5}^s, \Pi_{1,4}^f = \Pi_{1,6}^s, \Pi_{3,3}^f = \Pi_{3,3}^s, \Pi_{3,5}^f = \Pi_{3,5}^s, \Pi_{4,6}^f = \Pi_{4,6}^s, \Pi_{5,5}^f = \Pi_{5,5}^s - e^{-2\alpha h_{iu}} S_2, \Pi_{4,4}^f = \Pi_{6,6}^s - e^{-2\alpha h_{iu}} S_3, \Pi_{5,5}^f = \Pi_{7,7}^s, \) and \( \Pi_{6,6}^f = \Pi_{8,8}^s \).

Proof. The proof of the theorem is alike the proof of Theorem 5.3.3, with the critical difference in the definition of the functional \( V_3(t, \epsilon_t) \) as

\[
V_3(t, \epsilon_t) \triangleq \sum_{i=2}^{3} \int_{t-h_{iu}}^{t} e^{2\alpha(s-t)} \epsilon^T(s) S_i \epsilon(s) ds.
\]

Hence, following the same line of the proof of Theorem 5.3.3 in using the Lyapunov Krasovskii theorem results in the statement of the corollary.

5.3.3 FO Design for Slow-Varying Delays Scenario

Sufficient conditions for the \( \alpha \)-exponential stability of the auxiliary error dynamics (5.6) have been established in Theorem 5.3.3 in terms of an LMI. In addition, an illustrative framework to obtain the observer parameters is presented in Section 5.3.1. Substituting for each \( F_i \) from (5.18) into (5.22), \( i = \{1, 2, 3\} \), it is found that the observer parameter
5.3 Detailed Observer Design Procedure

\( J \) should be obtained as the output of an optimization problem. In other words, for given parameters \( \alpha, h_{iu} \), and \( \mu_i (i = \{2, 3\}) \), the global optimal solution of (5.22) should be calculated in a way that a suitable observer parameter \( J \) that realizes the observer equations (5.7) is achieved. However, the matrix inequality (5.22) is nonlinear, and LMI optimization techniques cannot thus be employed to solve the problem in its current format. This issue is resolved in the following theorem, which is the main result of the chapter.

**Theorem 5.3.4.** Consider the system (5.1), and assume that Conditions I and II hold. Given constant scalar parameters \( h_{iu} > 0, \mu_i < 1 (i = \{2, 3\}) \), and \( \alpha > 0 \), functional observer (5.3) is globally \( \alpha \)-exponentially stable for any delays fulfilling (5.2), if there exists matrix parameters \( P_1 > 0 \in \mathbb{R}^l, P_2 \in \mathbb{R}^{l \times l}, R_i > 0 \in \mathbb{S}^l, S_i > 0 \in \mathbb{S}^l, i = \{2, 3\}, K \in \mathbb{R}^{l \times (3l+n+p)} \), and a tuning gain matrix \( \Gamma \in \mathbb{R}^{l \times l} \), such that the following LMI is feasible.

\[
\begin{bmatrix}
\Pi_{1,1} & \begin{bmatrix} P_1^T N_{12} + KN_{22} \\ \Gamma^T P_2^T N_{12} + \Gamma^T KN_{22} \end{bmatrix} \\
\ast & -(1 - \mu_2) e^{-2\alpha h_{2u}} S_2 \\
\ast & \ast \\
\ast & \ast \\
\ast & \ast \\
\ast & \ast \\
\ast & \ast \\
\ast & \ast \\
\end{bmatrix}
- \begin{bmatrix}
\begin{bmatrix} P_1^T N_{13} + KN_{23} \\ \Gamma^T P_2^T N_{13} + \Gamma^T KN_{23} \end{bmatrix} \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\begin{bmatrix}
\Pi_{1,5} \\
\Pi_{1,6} \\
\Pi_{1,7} \\
\Pi_{1,8} \\
\end{bmatrix}
< 0,
\]

\[(5.33)\]

where

\[
\Pi_{1,1} = \text{sym}\left(\begin{bmatrix}
P_1^T N_{11} + KN_{21} \\
\Gamma^T P_2^T N_{11} + \Gamma^T KN_{21} \\
\end{bmatrix} + 2\alpha EP + \sum_{i=2}^{3} S_i - \eta_i R_i \right)
\]

and \( \Pi_{1,5}, \Pi_{1,6}, \Pi_{1,7}, \Pi_{1,8}, \Pi_{6,5}, \Pi_{6,6}, \Pi_{6,7}, \Pi_{6,8}, \Pi_{5,5}, \Pi_{5,6}, \Pi_{5,7}, \Pi_{5,8}, \Pi_{8,5}, \Pi_{8,6}, \Pi_{8,7}, \Pi_{8,8}, \Pi_{7,5}, \Pi_{7,6}, \Pi_{7,7}, \Pi_{7,8}, \Pi_{8,5}, \Pi_{8,6}, \Pi_{8,7}, \Pi_{8,8} \), are equal to \( \Pi_{1,5}, \Pi_{1,6}, \Pi_{1,7}, \Pi_{1,8}, \Pi_{6,5}, \Pi_{6,6}, \Pi_{6,7}, \Pi_{6,8}, \Pi_{5,5}, \Pi_{5,6}, \Pi_{5,7}, \Pi_{5,8}, \Pi_{8,5}, \Pi_{8,6}, \Pi_{8,7}, \Pi_{8,8} \), respectively. In addition, we have

\[ J = P_2^{-T} K. \]

\[(5.34)\]
5.3 Detailed Observer Design Procedure

**Proof.** Since Conditions I and II are satisfied, Condition (B) of Theorem 5.3.1 and Theorem 5.3.2 are fulfilled. Next, putting $F_i, i = \{1, 2, 3\}$ from (5.18) into the matrix inequality (5.22), defining $K \triangleq P_2^T J$, and assuming $P_3 = P_3 \Gamma$, where $\Gamma$ is a pre-adjusted weighting matrix, the LMI (5.33) is achieved. Hence, the fulfilment of (5.33) results in the statements of Theorem 5.3.1, and consequently Condition (A) of Theorem 5.3.1 also follows. This completes the proof of the theorem.

**Remark 5.3.3.** It is clear that in (5.33), the parameters $h_{iu}$, and $\mu_i, i = \{2, 3\}$, appear in non-linear forms. Hence, the conceivable upper-bounds for each one of these parameters can be sought via linearly incrementing their values, and constantly examining the feasibility of the LMI (5.33) at every step.

5.3.4 FO Design for Non-Differentiable State Delays Scenario

The main result of the section is summarized in the following corollary.

**Corollary 5.3.2.** Consider the system (5.1) and assume that Conditions I and II are realized. Given constant scalar parameters $h_{iu} > 0, i = \{2, 3\}$, $\alpha > 0$, functional observer (5.3) is globally $\alpha$-exponentially stable for any delays fulfilling (5.2) and unknown $\mu_i$s, if there exists matrix parameters $P_1 > 0 \in S^l, P_2 \in \mathbb{R}^{l \times l}, R_i > 0 \in S^l, S_i > 0 \in S^l, i = \{2, 3\}, K \in \mathbb{R}^{l \times (3l+n+p)}$, and a weighting matrix $\Gamma \in \mathbb{R}^{l \times l}$, such that the following LMI is satisfied

\[
\begin{bmatrix}
\Pi^f_{1,1} & \Pi^f_{1,3} & \Pi^f_{1,4} & \Pi^f_{1,5} & \Pi^f_{1,6} \\
* & \Pi^f_{3,3} & 0 & \Pi^f_{3,5} & 0 \\
* & * & \Pi^f_{4,4} & 0 & \Pi^f_{4,6} \\
* & * & * & \Pi^f_{5,5} & 0 \\
* & * & * & * & \Pi^f_{6,6}
\end{bmatrix} \prec 0,
\]

(5.35)
where \( \overline{\Pi}^1_{1,1} = \overline{\Pi}^s_{1,1} \), and the other block parameters are defined in Corollary 5.3.1. Moreover, the relation (5.34) holds.

**Proof.** The proof follows from the same line of the procedure of verifying Theorem 5.3.4, and thus is omitted due to space limitations.

The weighting parameter \( \Gamma \) utilized in LMIs (5.33) and (5.35), is indeed a design parameter that should be pre-selected to avoid the nonlinearity of the LMIs. However, appropriate adjustment of this parameter is essential, in order to achieve a less conservative stability condition. It can be done via a linear searching program. However, since the size of this parameter can be considerably large, an ordinary searching algorithm may incur a significant computational burden. To resolve this issue, an intelligent optimization framework, based on the genetic algorithm is proposed in Table 5.1 to effectively adjust the matrix \( \Gamma \), given the desired values of \( \alpha, h_{iu}, \) and \( \mu_i, i = \{2, 3\} \). This algorithm can be very helpful in reducing the amount numerical calculations, and thus saving the computation time.

**Remark 5.3.4.** If the tuning algorithm advised in Table 5.1 fails in finding an appropriate parameter \( \Gamma \), then adequately reducing the values of the upper-bounds of delays \( h_{iu} \) and the delays derivatives \( \mu_i \) (for the slow-varying delays scenario), might be effective. Hence, the instructions given in Remark 5.3.3, together with applying the tuning schema given in Table 5.1 provide a recursive algorithm to obtain rough upper-limits of the state delays \( h_{iu} \) and their derivatives \( \mu_i \), given the desired performance gain \( \alpha > 0 \), such that the observer (5.3) is exponentially stable.

**Remark 5.3.5.** The problem of solving LMIs (5.33) or (5.35) is a semi-definite programming [122, 126], whence a global optimal solution can be sought using a convex optimization program. However, for the sake of i) using the inequalities (5.28) and (5.29), ii) the specific choice of the LKF in deriving the results of Theorem 5.3.3 and Corollary 5.3.1, and
Detailed Observer Design Procedure

Table 5.1: Adjusting framework for the observer parameter $\Gamma$

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step 1</td>
<td>The fitness function is selected in a way to ensure the feasibility of the LMI (5.33). Let us define a function that $\Gamma$ is its input, and $t_{\text{min}}$ is its output, wherein they are related through the following LMI,</td>
</tr>
<tr>
<td></td>
<td>$\bar{\Pi}^s &lt; t_{\text{min}}$. (5.36)</td>
</tr>
<tr>
<td>Step 2</td>
<td>Choose an arbitrary and appropriate limit for each element of $\Gamma$ ($\Gamma_{i,j}$, $i, j = {1, \ldots, l}$) to be applied to the GA as $\Gamma_{i,j_{\text{min}}} \leq \Gamma_{i,j} \leq \Gamma_{i,j_{\text{max}}}$.</td>
</tr>
<tr>
<td>Step 3</td>
<td>Generate an initial population $\Gamma_j$, $j = {1, 2, \ldots, N_p}$, arbitrarily based on the limits selected in Step 2.</td>
</tr>
<tr>
<td>Step 4</td>
<td>Select the mutation and cross-over parameters of the genetic algorithm (e.g. the defaults of MATLAB genetic algorithm toolbox), and apply the GA to optimally search for an appropriate chromosome. The stopping criteria is first finding an appropriate parameter $\Gamma$, for which the LMI (5.36) is feasible. The second stopping criteria is passing the limit of $N_g$ generations of inadequate populations, which flags the failure of the algorithm.</td>
</tr>
</tbody>
</table>

iii) the nature of the tuning algorithm, which always gives a local optimal parameter $\Gamma$ (due to using GA), the results are still conservative, and for given $\alpha > 0$, larger upper-bounds for the parameters $h_{i,u}$ and $\mu_i$, $i = \{2, 3\}$ are conceivable.

Nevertheless, by virtue of using the descriptor transformation, the advanced weighted matrix inequality (2.13) for calculating a tight upper-bound of the term $\int_{-h_{i,u}}^0 e^{\alpha \theta} \dot{\epsilon}^T (t + \theta) R_i \dot{\epsilon} (t + \theta) d\theta$, and avoiding to put any bounding restrictions on the cross term

$$\sum_{i=2}^3 \dot{\epsilon}^T (t) P^T [0; F_i] \int_{t-h_i(t)}^t \dot{\epsilon} (s) ds,$$

it can be claimed that the constructed stability criteria of this chapter can be strictly less conservative than those of the majority of the existing studies in the field of FO design for time-delay systems (see for example [103–107]). This is based on the observation that most of those works use simple LKFs, and employ the conventional Newton-Leibniz transformation, as well as restrictive Cauchy-Schwartz and Young’s inequalities in simplifying...
the terms in the derivatives of the LKFs that are shown to be more conservative than the mentioned techniques utilized in this chapter [171, 172]. This remark is more justified using a numerical example in the next section.

5.4 Numerical Examples

5.4.1 Example 1, Comprehensive Illustration of the Design Algorithm

In this example, an LTI system comprising two state-delays and two input-delays is investigated. The parameters of the system are

\[ A_1 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0.1 \\ 2 & 3 & -1 & 0 \\ 2 & -1 & 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0_{3 \times 4} \\ 0.1 & 0.21 & 0.2 & 0.1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0.4 \end{bmatrix}, \]

\[ C = \begin{bmatrix} I_2 & 0_{2 \times 2} \end{bmatrix}, \quad B_1 = [0; -1; 2; 0], \quad B_2 = [0; 1; 0; 0], \quad \text{and} \quad B_3 = [0; -1; 0; 1]. \]

A linear combination of the first three states of the system is aimed to be observed as

\[ L = \begin{bmatrix} -3 & 0.1 & 2 & 0 \end{bmatrix}. \]

To this end, both slow-varying and fast-varying delays scenarios are separately investigated. In addition, throughout the example the actuator delays are considered to be unknown for the design perspective, but arbitrary values of \( \tau_2(t) = 2 + \sin(2t) \) and \( \tau_3(t) = 3 + \cos(t) \) are assigned in the simulations. For each scenario of the state-delays, two exponential convergence rates are investigated: \( \alpha = 0.1 \) and \( \alpha = 2.5 \). In addition, the input signals are arbitrarily selected as

\[ u_1(t) = u_2(t) = 2 + 10e^{-4t} \cos(2t), \quad \text{and} \quad u_3(t) = 2H(t - 3) \quad t \geq 0, \]

where \( H(\cdot) \) is the Heaviside step function. The state-delays considered for each scenario are depicted in Fig. 5.1. The top figure illustrates slow-varying
and the lower one shows the fast-varying state-delays, respectively.

![Graphs showing slow-varying and fast-varying state-delays](image)

Figure 5.1: The state-delays associated with the slow-varying (top) and fast-varying (down) scenarios

Evaluating Conditions I and II from (5.16) and (5.21), confirms that under the condition of all zero-state-delays minimum-order FO can be designed for the system. Hence, it is also possible to proceed to the observer design for the general case.

**Slow-varying State-delays**

First, consider the smaller desired convergence rate with $\alpha = 0.1$. As can be seen from Fig. 5.1, $h_{2u} = h_{3u} = 9s$, $\mu_2 = 0.4$ and $\mu_3 = 0.5$. Setting $\Gamma = 1$, it was observed that LMI (5.33) is feasible, and the LMI parameters were attained as: $P_1 = 6.6950$, $P_2 = 4.4361$, $P_3 = 2.2250$. 
\[ R_2 = R_3 = 0.4405, \ S_2 = 1.5195, \ S_3 = 1.5816, \text{ and} \]

\[
J = \begin{bmatrix}
0.6803 & 0_{1 \times 2} & -11.1478 & 0_{1 \times 3} & -9.1953 & 0
\end{bmatrix}.
\]

Thereafter, following the relations (5.18), (5.19), (5.9), (5.10), (5.7d), and (5.11), the observer parameters were obtained as, \( F_1 = -0.5805, \ F_2 = F_3 = 0, \ H_1 = \begin{bmatrix} 3.5196 & 6 \end{bmatrix}, \)

\( H_2 = H_3 = \begin{bmatrix} 0 & 0 \end{bmatrix}, \ G_1 = 4, \text{ and } V = \begin{bmatrix} -2.1610 & 0.1 \end{bmatrix}. \)

Next, the case of \( \alpha = 2.5 \) was studied, considering the same delay parameters as the previous case. However, unlike the preceding scenario, setting \( \Gamma = 1 \) does not result in the feasibility of the LMI (5.33). Hence, the tuning algorithm explained in Table 5.1, was applied to achieve an appropriate gain \( \Gamma = 0.3017 \). Accordingly, the following observer parameters were eventually attained: \( F_1 = -11.4655, \ F_2 = F_3 = 0, \ H_1 = \begin{bmatrix} 243.9830 & 6 \end{bmatrix}, \)

\( H_2 = H_3 = \begin{bmatrix} 0 & 0 \end{bmatrix}, \ G_1 = 4, \text{ and } V = \begin{bmatrix} -23.9309 & 0.1 \end{bmatrix}. \)

Simulation results obtained from the Simulink environment within MATLAB are reported in Fig. 5.2. It appears that the observer with the larger gain \( \alpha = 2.5 \) has an essentially better performance compared with the observer with the smaller gain \( \alpha = 0.1 \). Nevertheless, one should bear in mind that considering a larger gain \( \alpha \) sometimes can result in smaller admissible values of \( h_{iu} \) and \( \mu_i \).

**Non-differentiable State-delays**

Here, the state delays meet the following specifications: \( h_{2u} = h_{3u} = 9s, \mu_2 = 27 >> 1, \) and \( \mu_3 = 45 >> 1. \) Similar to the previous case-study, first the case of \( \alpha = 0.1 \) is addressed. It can be examined that fixing the tuning parameter at \( \Gamma = 1, \) the LMI (5.35) is feasible, and the LMI parameters were obtained as: \( P_1 = 4.0777, \ P_2 = 2.7540, \ R_2 = 0.2155, \)

\( R_3 = 0.3128, \ S_2 = S_3 = 0.8677, \) and \( J = \begin{bmatrix} 17.4447 & 0_{1 \times 2} & 27.4755 & 0_{1 \times 3} & 37.7611 & 0 \end{bmatrix}. \)

Henceforth, the observer parameters were calculated as: \( F_1 = -0.5696, \ F_2 = F_3 = 0, \)
5.4 Numerical Examples

\[ H_1 = \begin{bmatrix} 3.5097 & 6 \end{bmatrix}, \quad H_2 = H_3 = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad G_1 = 4, \quad \text{and} \quad V = \begin{bmatrix} -2.1392 & 0.1 \end{bmatrix}. \]

Finally, the case of \( \alpha = 2.5 \) was studied as the final scenario of the example, assuming similar delay parameters as per the \( \alpha = 0.1 \) case. Our observations show that the LMI (5.35) is infeasible with the assumption of \( \Gamma = 1 \), while it is feasible with \( \Gamma = 0.3017 \). It should be emphasized that the tuning parameter that can be obtained from the algorithm advised in Table 5.1 is not unique, and even under similar initial populations and analogous GA parameters every run of the algorithm can result in a different apt parameter \( \Gamma \). Next, the observer parameters were calculated for this case as: \( F_1 = -12.5441, \quad F_2 = F_3 = 0, \quad H_1 = \begin{bmatrix} 293.6215 & 6 \end{bmatrix}, \quad H_2 = H_3 = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad G_1 = 4, \quad \text{and} \quad V = \begin{bmatrix} -26.0882 & 0.1 \end{bmatrix}. \)

Performing simulations in the MATLAB environment, again exhibited an extremely better performance for the observer with \( \alpha = 2.5 \), as demonstrated in Fig. 5.3. In addition,
a quick comparison of Figs. 5.3 and 5.2 expresses that despite their different calculated observer parameters, there is indistinguishable difference between the observers performances. This is due to their analogous system parameters and similar desired performances (i.e., similar $h_i(t)$ and $\alpha$, $i = \{2, 3\}$). However, it should be emphasized that the observer design framework for the fast-varying delays scenario is thoroughly robust against the rate of variation in the state-delays, which is a crucial factor in practical situations. Furthermore, studying the conditions wherein there exists a minimum-order functional observer for the system considering the slow-varying state-delays assumption, while it does not exist under the fast-varying delays hypothesis is an interesting topic for future research.

Figure 5.3: The convergence errors (left), and the tracking performances (right) of the higher and lower convergence speed cases in the fast-varying state-delays case-study
5.4.2 Example 2, Highlighting the Less Conservatism of the Proposed Approach

This example is focused on highlighting the less conservatism of the proposed observer’s stability criteria compared with the other contributions on this topic, as illustrated in Remark 5.3.5. To this aim, let us consider the system (5.1) with only a single time-varying state delay (without any input delay) and the following distribution matrices

\[
A_1 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 1 \end{bmatrix},
\]

and \(B_1 = \begin{bmatrix} 0 & -1 \end{bmatrix}^T\). To our knowledge, the most recent existing paper in the literature that studies delay-dependent FO design for LTI systems including time-varying state delays is [105], which also only considers the slow-varying delay assumption and asymptotic stability (not exponential stability). Let us further assume that \(h_{2u} = 3s\) and \(\mu_2 = 0.4\). After applying the observer design algorithms of [103, 105] we found that the stability criteria proposed in those papers are infeasible, and those algorithms are thus not applicable to this example.

On the other hand, a simple scrutiny shows that both of the Conditions I and II are satisfied. Moreover, eliminating the third, fifth, and eighth block columns and rows of the matrix \(\bar{\Pi}^s\), and setting \(S_3 = R_3 \equiv 0\), the stability criteria proposed in Theorem 5.3.4 can be directly applied to single-state-delay systems. Next, the observer was designed for the system for two exponential convergence rates: \(\alpha = 0.1\) and \(\alpha = 1.5\). When \(\alpha = 0.1\), setting \(\Gamma = 1\) the modified LMI (5.33) is found feasible, and the following observer parameters were attained: \(F_1 = -3.738\), \(F_2 = -1\), \(G_1 = -1\), \(H_1 = 13.9723\), \(H_2 = 0\), and \(V = -3.7380\). In addition, it was essential to apply the tuning algorithm for
the case $\alpha = 1.5$, which gave $\Gamma = 0.0105$ as an appropriate chromosome. Consequently, the observer parameters for this case were obtained as $F_1 = -192.4258$, $F_2 = -1$, $G_1 = -1$, $H_1 = 37028$, $H_2 = 0$, and $V = -192.4258$. Hence, this example justifies that the proposed observer’s stability criteria of this chapter can be less conservative than the conditions proposed in the related existing papers in the literature.

Simulation results obtained from the Simulink environment are depicted in Fig. 5.4. The convergence of the observer in both scenarios is clear from this figure, and it also highlights the advantage of increasing the desired exponential convergence rate.

The next chapter, inspired from the results of Chapters 3 and 4, proposes an effective algorithm to design minimum order FOs for LTI systems with interval time-varying delays. Although single state delay is considered, the results of this chapter can be helpful to straightforwardly extend the results of the forthcoming chapter to cover systems with mixed multiple state and input delays.

## 5.5 Conclusions

The novel problem of functional observer design for LTI systems with multiple known time-varying state delays and multiple unknown time-varying input delays has been addressed. Two scenarios have been considered for the derivatives of the state-delays: I) values less than one, and II) unknown values. The necessary and sufficient conditions of the asymptotic stability of the system for all zero-state-delay scenario has been obtained, which are also necessary to be satisfied for the general case. Employing the Lyapunov Krasovskii approach, a set of DD sufficient conditions for the $\alpha$-exponential stability of the FO are established in terms of LMIs. The design procedure has also been facilitated with a GA based searching algorithm to adjust one of the observer parameters. Due to using
Figure 5.4: The convergence errors (top), and the tracking performances (bottom) of the proposed FO under different desired exponential convergence rates (Example 2)

more advance techniques, such as the descriptor transformation and a novel weighted integral inequality in the analysis of the LKF, the constructed stability conditions can be less conservative than the majority of the other existing related papers that study FO design for retarded systems. Two illustrative numerical examples have explained the observer design framework, and have highlighted its performance and its superiority compared with the other relevant works in the literature.
Chapter 6

Functional Observer Design for LTI Systems with Interval Time-Varying Delays

6.1 Introduction

As mentioned previously, the problem of full-order observer design for time-delay systems is a well-established field, in a way that even stochastic systems have been studied (see e.g. [159, 160, 164, 173]). However, unlike the wide range of applications of functional observers in time-delay systems, this topic is fairly overlooked and requires more attention. The existing contributions in this area can be classified into three categories. The first group of works attempts to analyze time-delay systems by decoupling the effects of delay on the closed-loop observer dynamics [108, 166]. Although this type of analysis is simple and effective when applicable, it may induce significant structural restrictions on the observer. The second group of approaches, such as those in [68, 103–105], design delay-dependent
observer structures, but choose an independent-of-delay stability analysis framework. This kind of approach can still be conservative, due to ignoring the information on delays, such as their limits and derivatives bounds in the observer design procedure. Finally, the third group of contributions consider delay-dependent observer structures with delay-dependent stability criteria [103, 105–107, 174]. Both cases of constant and time-varying delays have been studied in the papers of this category.

Nevertheless, the problem of FO design for retarded systems with interval time-varying delays in which the delay is both lower-bounded and upper-bounded is still not investigated in particular. Taking into account the lower-limits of the delays values in addition to their upper-bounds is clearly more realistic, and may result in less conservative stability criteria. The problems regarding the stability analysis of interval time delay systems has thus received a considerable attention in recent years (see e.g. [134, 136, 138, 140]).

Moreover, the majority of the existing papers on functional observer design for time-delay systems only consider slowly-varying delays or ignore the information on the delays derivatives (by assuming unknown delay-derivatives). In addition, most of the so far established delay-dependent stability conditions for FOs are obtained using conservative LKFs, as well as restrictive bounding techniques. These drawbacks can result in relatively small observer’s stability region in terms of the admissible upper-bounds of delays.

Motivated by the above shortcomings we propose a methodology to design functional observers for retarded LTI systems with interval time-varying delays. The delay derivative is assumed to be bounded with an upper-bound not limited to be less than one. To the best of the author’s knowledge, this particular problem is addressed here for the first time. Sufficient conditions for the asymptotic stability of the observer are achieved using the Lyapunov Krasovskii approach and are expressed in terms of LMIs. A new augmented LKF including triple integral terms is introduced to the aim. In addition, a new analysis scheme employing Writing-based single and double integral inequalities, convex combination
scheme, reciprocally convex approach, and the descriptor transformation is elaborated to establish an effective stability criterion for the observer. This can result in less conservative delay-dependent stability conditions compared with those of the recent approaches that design FOs for time-delay systems. In addition, a tuning framework based on the genetic algorithm is proposed to adjust a number of weighting parameters of the LMI condition. Two numerical examples and simulation results are given to justify the effectiveness of our approach with regard to the few existing approaches that study this topic.

The rest of this chapter is arranged as follows. The problem, together with some helpful preliminaries are expressed in Section 6.2. Thereafter, the main results of the chapter are given first by deriving the observer equations in Section 6.3.1, next by establishing the new observer’s stability criteria in Section 6.3.2, and finally by presenting the design procedure in Section 6.3.3. Eventually, numerical examples are provided in Section 6.4, and the conclusions are given in Section 6.5.

### 6.2 Problem Formulation and Preliminaries

**Notations:** Throughout the chapter \( \mathbb{R}^n \) denotes the n-dimensional Euclidean space; \( \mathbb{R}^{n \times m} \) is the space of \( n \times m \) real matrices; \( \mathbb{S}^n \) is the space of \( n \times n \) symmetric matrices; \( I_n \) is the \( n \times n \) identity matrix; and \( 0 \) is the zero matrix of appropriate dimension. Moreover, \( \text{sym}(X) = X + X^T \); \( \rho(X) \) is the rank of the matrix \( X \); \( \otimes \) is the Kronecker product; * in a symmetric matrix stands for the symmetric element; \( X^\dag \) is a generalized inverse or a pseudo-inverse of the matrix \( X \); \( Y = X^\bot \) is a right-orthogonal of \( X \), i.e. \( XY = 0 \); and \( |x| \) is the norm of the vector \( x \). In addition, \( C_n(\Omega) \) is the space of continuous functions mapping from \( \Omega \) to \( \mathbb{R}^n \) with the topology of uniform convergence. Finally, \( X \succ 0(\prec 0) \) and \( X \succeq 0(\preceq 0) \) declares that the matrix \( X \) is positive-definite (negative-definite) and positive semi-definite (negative semi-definite), respectively.
Consider the following proper LTI time-delay system

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + A_d x(t - h(t)) + Bu(t) \\
y(t) &= Cx(t) + Du(t) \\
z(t) &= Lx(t) \\
x(\theta) &= \phi(\theta) \quad \forall \theta \in [-h_2, 0],
\end{align*}
\]

where \( x(\cdot) \in \mathbb{R}^n \) is the state vector, \( u(\cdot) \in \mathbb{R}^m \) is the actuator input vector, \( y(\cdot) \in \mathbb{R}^p \) is the output measurement vector, and \( z(\cdot) \in \mathbb{R}^{l} \) is the vector of desired functions to be estimated. Moreover, \( A \in \mathbb{R}^{n \times n}, \ A_d \in \mathbb{R}^{n \times n}, \ B \in \mathbb{R}^{n \times m}, \ C \in \mathbb{R}^{p \times n}, \ D \in \mathbb{R}^{p \times m}, \) and \( L \in \mathbb{R}^{l \times n} \) are constant known matrices. In addition, \( \phi(\cdot) \in \mathcal{C}_n([-h_2, 0]) \) is the initial function of the system.

**Assumptions:** the delay function \( h(t) \) fulfills the following properties

\[
0 < h_1 \leq h(t) \leq h_2 \quad \dot{h}(t) \leq \mu,
\]

where \( h_1 < h_2 \), and \( \mu \) is any real constant. Moreover, for the simplicity of notations let us define \( h_{12} \triangleq h_2 - h_1 \). Throughout this chapter, without the loss of generality it is assumed that \( \rho(C) = p < n \) and \( \rho(L) = l \).

Our aim is to introduce a new practical algorithm to design a delay-dependent functional observer of minimum-order \((l)th\) order that for any initial function \( \phi(\cdot) \in \mathcal{C}_n([-h_2, 0]) \), asymptotically reconstructs the function \( z(t) \).

**Definition 6.2.1.** A functional observer for the system (6.1) is asymptotically stable, if for any \( \phi(\cdot) \in \mathcal{C}_n([-h_2, 0]) \) the estimation error function \( e(\cdot, \phi(\theta)) \triangleq \hat{z}(\cdot) - z(\cdot) \) satisfies

\[
\lim_{t \to \infty} |e(t, \phi(\theta))|_c = 0 \quad \forall \theta \in [-h_2, 0],
\]
where \( \hat{z}(\cdot) \) is the output of the observer dynamics, and

\[
|e(t)|_c \triangleq \sup_{-h_u \leq \theta \leq 0} |e(t + \theta)|.
\]

**Remark 6.2.1.** In the majority of papers studying FO or unknown-input FO design for time-delay systems (see e.g. [103, 105, 175]) it is assumed that the delay derivative is strictly less than one (i.e., \( \mu < 1 \)), or the upper-bound of the delay derivative is considered to be unknown. This restriction is relaxed in the this chapter.

### 6.3 Detailed Observer Design Framework

#### 6.3.1 Deriving Observer Equations

The following \( l \)th order standard observer structure is utilized in the chapter

\[
\dot{\omega}(t) = F_1 \omega(t) + F_2 \omega(t - h(t)) + G_1 u(t) + G_2 u(t - h(t)) + H_1 y(t) + H_2 y(t - h(t))
\]

\[
\dot{\hat{z}}(t) = \omega(t) + V y(t) + V_u u(t)
\]

\[
\omega(\theta) = 0 \quad \forall \theta \in [-h_2, 0],
\]

where \( \omega(\cdot) \in \mathbb{R}^l \) is the state vector of the observer, \( F_1, F_2, G_1, G_2, H_1, H_2, V, \) and \( V_u \) are observer parameters with appropriate dimensions. Let us define the auxiliary error \( \epsilon(\cdot) \triangleq \omega(\cdot) - T x(\cdot) \), where \( T \in \mathbb{R}^{l \times n} \) is an auxiliary observer parameter to be designed. The following theorem summarizes the necessary and sufficient conditions of the asymptotic stability of the observer.
Theorem 6.3.1. The observer (6.4) is an asymptotically stable functional observer for the system (6.1), if and only if the following conditions are satisfied,

i. the following estimation error dynamics is asymptotically stable for any delay satisfying (6.2)

\[
\dot{\epsilon}(t) = F_1\epsilon(t) + F_2\epsilon(t - h(t)) \\
\epsilon(\theta) = -T\phi(\theta) \forall \theta \in [-h_2, 0],
\]

(6.5)

ii. there exist observer parameters \(F_1, F_2, G_1, G_2, H_1, H_2, V, V_u, \) and \(T\) that satisfy the interconnected equations below

\[
T + VC - L = 0, \hspace{1cm} (6.6a)
\]

\[
F_1T - TA + H_1C = 0, \hspace{1cm} (6.6b)
\]

\[
F_2T - TA_d + H_2C = 0, \hspace{1cm} (6.6c)
\]

\[
G_1 = TB - H_1D, \hspace{1cm} (6.6d)
\]

\[
G_2 = -H_2D, \hspace{1cm} (6.6e)
\]

\[
V_u = -VD. \hspace{1cm} (6.6f)
\]

Proof. The proof of the theorem is straightforward, and only the sufficiency part is verified. Differentiating \(\epsilon(t)\), after some manipulations gives
\[ \dot{\epsilon}(t) = F_1\epsilon(t) + F_2\epsilon(t - h(t)) + (F_1T - TA + H_1C)x(t) \]
\[ + (F_2T - TA_d + H_2C)x(t - h(t)) + (G_1 - TB + H_1D)u(t) \]
\[ (G_2 - H_2D)u(t - h(t)) \]  
(6.7)

Hence, upon the satisfaction of conditions (6.6b)-(6.6e), together with Condition i it can be deduced that the auxiliary error \( \epsilon(t) \) asymptotically converges to the origin. In addition, according to the definition of \( e(t) \equiv e(t, \phi(\theta)) \) we have

\[ e(t) = \epsilon(t) + (T + VC - L)x(t) + (VD + V_u)u(t). \]  
(6.8)

Thus, if conditions (6.6b) and (6.6f) are satisfied, then the error \( e(t) \) asymptotically converges to zero. This concludes the sufficiency part of the theorem.

The first three equations of (6.6) are constrained Sylvester equations, which are interconnected matrix equations. The observer design algorithm consists of two major steps:

A. finding a closed-from solution for equations (6.6a)-(6.6c).

B. designing appropriate matrices \( F_1 \) and \( F_2 \) using the solution obtained from Step A, such that Condition i of Theorem 6.3.1 is fulfilled.

Henceforth, first an effective criteria is established to obtain conditions on \( F_1 \) and \( F_2 \), such that Condition i of Theorem 6.3.1 is realized. Next, Steps A and B are more articulated.

### 6.3.2 Stability Analysis

A new criterion for the stability of (6.5) in general is constructed in the following theorem.
**Theorem 6.3.2.** For given delay parameters $h_1$, $h_2$, and $\mu$, the delay differential equation (6.5) is asymptotically stable for any delay satisfying (6.2), if there exist matrices $P_i \succ 0 \in S^l$, $i = \{1,4,6\}$, $P_j \in \mathbb{R}^{l \times l}$, $j = \{2,3,5\}$, $Q_i \succ 0 \in S^l$, $i = \{1,2,3\}$, $R_i \succ 0 \in S^l$, $i = \{1,\cdots,4\}$, $S_i \succ 0 \in S^l$, $i = \{1,2\}$, and free weighting matrices $U_i \in \mathbb{R}^{l \times l}$, $i = \{1,\cdots,4\}$, such that the following LMIs are satisfied

$$\Psi_i \prec 0, \quad i = \{1,2\} \quad (6.9)$$

where $\Psi_1 \triangleq \Psi(h(t))|_{h(t)=h_1}$, $\Psi_2 \triangleq \Psi(h(t))|_{h(t)=h_2}$, and

$$\Psi(h(t)) = \begin{bmatrix}
\Pi(t) & \Pi_9 & \Pi_{10}(t) & \Pi_{11}(t) & \Pi_{12} \\
* & -72\frac{\zeta^2}{\lambda}S_2 & 0 & 0 & 0 \\
* & * & -12R_2 & 0 & 0 \\
* & * & * & -12R_2 & 0 \\
* & * & * & * & \tilde{\Pi}_{12} 
\end{bmatrix}. $$

In addition, $\Pi(t) = [\Pi_{i,j}]_{8 \times 8}$ with non-zero elements as

$$\begin{align*}
\Pi_{1,1} &= \text{sym}(P_3) + Q_4 + h^2_1R_1 + h^4_1R_2 - 6h^2_1S_1 - 12\frac{\zeta}{\lambda}a_{11}S_2 - \frac{4}{h_1}R_3 + \text{sym}(U^TF_1), \\
\Pi_{2,1} &= F^T_2U_1, \\
\Pi_{2,2} &= -(1 - \mu)Q_3 - 4(h_2 - h(t))^2R_4 - 4(h(t) - h_1)^2R_4, \\
\Pi_{3,1} &= P^T_2 - P^T_3 - \frac{2}{h_1}R_3, \\
\Pi_{3,2} &= -2(h(t) - h_1)^2R_4, \\
\Pi_{3,3} &= Q_2 - Q_1 + Q_3 - \frac{4}{h_1}R_3 - 4(h(t) - h_1)^2R_4, \\
\Pi_{4,1} &= -P^T_2, \\
\Pi_{4,2} &= -2(h_2 - h(t))^2R_4, \\
\Pi_{4,4} &= -Q_2 - 4(h(t) - h_1)^2R_4, \\
\Pi_{5,1} &= P_6 + \frac{6}{h_1}R_3 + \frac{24}{h_1}S_1, \\
\Pi_{5,2} &= U^TF_2, \\
\Pi_{5,3} &= -P_6 + \frac{6}{h_1}R_3 + P^T_5, \\
\Pi_{5,4} &= -P^T_5, \\
\Pi_{5,5} &= -4R_1 - \frac{12}{h_1}R_3 - 12S_1, \\
\Pi_{6,1} &= 12S_1, \\
\Pi_{6,5} &= \frac{6}{h_1}R_1 + \frac{24}{h_1}S_1, \\
\Pi_{6,6} &= -\frac{12}{h_1}R_1 - \frac{72}{h_1}S_1, \\
\Pi_{7,1} &= -\frac{12\zeta}{\lambda}a_{17}S_2 + P_5 + U^TF_1, \\
\Pi_{7,2} &= 6(h_2 - h(t))R_4 + U^TF_2, \\
\Pi_{7,3} &= P_4 - P_5, \\
\Pi_{7,4} &= -P_4 + 6(h_2 - h(t))R_4, \\
\Pi_{7,7} &= -12R_4 - 4(h_2 - h(t))^2R_2 - 12\frac{\zeta}{\lambda}S_2, \\
\Pi_{8,1} &= -\frac{12\zeta}{\lambda}a_{17}S_2 + P_5 + U^TF_1, \\
\Pi_{8,2} &= 6(h(t) - h_1)R_4 + U^TF_2, \\
\Pi_{8,3} &= 6(h(t) - h_1)R_4 + P_4 - P_5, \\
\Pi_{8,4} &= -P_4, \\
\Pi_{8,7} &= -12\frac{\zeta}{\lambda}\zeta_4S_2,
\end{align*}$$
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\[ \Pi_{8,8} = -12 \frac{\zeta \dot{\zeta}}{\zeta} S_2 - 12 R_4 - 4(h(t) - h_1)^2 R_2, \]
\[ \Pi_9 = \left[ 12 \frac{\zeta \dot{\zeta}}{\zeta} a_{19} S_2, 0, 0, 0, 0, 0, 24 \frac{\zeta \dot{\zeta}}{\zeta} S_2, 24 \frac{\zeta \dot{\zeta}}{\zeta} S_2 \right]^T, \]
\[ \Pi_{10}(t) = [0, 0, 0, 0, 0, 0, 6(h(t) - h(t)) R_2, 0]^T, \]
\[ \Pi_{11}(t) = [0, 0, 0, 0, 0, 0, 6(h_2 - h(t)) R_2]^T, \]
\[ \Pi_{12} = [P_1 - U_1 + U_4^T F_1, U_4^T F_2, 0, 0, P_3 - U_3, 0, P_2 - U_2, P_2 - U_2]^T, \text{ and} \]
\[ \bar{\Pi}_{12} = h_1 R_3 + h_4^2 R_4 + h_4^2 S_1 + \zeta_2^2 S_2 - \text{sym}(U_4). \]  
Furthermore, \( a_{11} \triangleq h_1^2 \zeta_4 - 2h_1 \zeta_2 \zeta_3 + 1.5 \zeta_2^3, \) \( a_{17} \triangleq \zeta_2 \zeta_3 - h_1 \zeta_4, \) \( a_{19} \triangleq 3 \zeta_2^2 - 2h_1 \zeta_3, \) \( \zeta_i \triangleq \zeta(h_1, h_1), i \in \{2, 3, 4\}, \) \( \text{and} \) \( \bar{\zeta} \triangleq \bar{\zeta}(h_2, h_1). \)

**Proof.** Consider the following Lyapunov Krasovskii functional candidate

\[ V(t, x(t), x_t, \dot{x}_t) = \sum_{i=1}^4 V_i(t), \quad (6.10) \]
where

\[ V_1(t) = \bar{\eta}^T(t) E P \bar{\eta}(t), \]
\[ V_2(t) = \int_{t-h_1}^t \dot{\epsilon}(s) Q_1 \epsilon(s) ds + \int_{t-h_2}^{t-h_1} \dot{\epsilon}(s) Q_2 \epsilon(s) ds + \int_{t-h(t)}^{t-h_1} \dot{\epsilon}(s) Q_3 \epsilon(s) ds, \]
\[ V_3(t) = h_1 \int_{-h_1}^0 \dot{\epsilon}(s) R_1 \epsilon(s) ds + \int_{h_1}^t \dot{\epsilon}(s) R_2 \epsilon(s) ds + \int_{t-h(t)}^{t-h_1} \dot{\epsilon}(s) R_3 \epsilon(s) ds, \]
\[ V_4(t) = h_1^2 \int_{-h_1}^0 \int_{s_1}^0 \dot{\epsilon}(s) S_1 \epsilon(s) ds ds + h_1^3 \int_{-h_2}^{t-h_1} \int_{s_1}^t \dot{\epsilon}(s) S_2 \epsilon(s) ds ds, \]

with

\[ \bar{\eta}(t) = \left[ \epsilon^T(t), \int_{t-h_2}^{t-h_1} \dot{\epsilon}(s) ds, \int_{t-h_2}^{t} \epsilon^T(s) ds, \int_{t-h_2}^{t} \epsilon^T(s) ds, \epsilon^T(t) \right]^T, \quad P \triangleq \left[ \begin{array}{cccc} P_1 & P_2 & P_3 & 0 \\ * & P_4 & P_5 & 0 \\ * & * & P_6 & 0 \\ U_1 & U_2 & U_3 & U_4 \end{array} \right]. \]
and \( E \triangleq \begin{bmatrix} I_{3t} & 0 \\ 0 & 0 \end{bmatrix} \).

The following variables are also defined for the simplicity in notations:
\[
\begin{align*}
z_1 & \triangleq x(t), \quad z_2 \triangleq x(t - h(t)), \quad z_3 \triangleq x(t - h_1), \quad z_4 \triangleq x(t - h_2), \quad z_5 \triangleq \int_{t-h_1}^{t} x(s)ds, \\
z_6 & \triangleq \int_{-h_1}^{0} \int_{s}^{0} x(t + u)duds, \quad z_7 \triangleq \int_{-h_2}^{0} x(t + u)du, \quad z_8 \triangleq \int_{-h(t)}^{0} x(t + u)du, \\
z_9 & \triangleq \int_{-h_2}^{0} \int_{s}^{0} x(t + u)duds, \quad z_{10} \triangleq \int_{-h(t)}^{0} \int_{s}^{0} x(t + u)duds, \\
z_{11} & \triangleq \int_{-h(t)}^{0} \int_{s}^{0} x(t + u)duds, \quad \text{and} \quad z_{12} \triangleq \dot{x}(t). \\
\end{align*}
\]

Differentiating (6.10) along the solution of (6.5), and observing that
\[
\int_{t-h_2}^{t-h(t)} f(s)ds = \int_{t-h_2}^{t-h(t)} f(s)ds + \int_{t-h(t)}^{t-h_2} f(s)ds \text{ holds for any integrable function} \ f : [t - h_2, t - h_1] \to \mathbb{R}, \text{ results in the following equations.}
\]

\[
\begin{align*}
\dot{V}_1(t)|_{(6.5)} & = \text{sym} \left( \begin{bmatrix} z_1 \\ z_7 + z_8 \\ z_5 \\ z_{12} \end{bmatrix} \right)^T \begin{bmatrix} P_1 & P_2 & P_3 & 0 \\ * & P_4 & P_5 & 0 \\ * & * & P_6 & 0 \\ U_1 & U_2 & U_3 & U_4 \end{bmatrix} \begin{bmatrix} z_{12} \\ z_3 - z_4 \\ z_1 - z_3 \\ F_1 z_1 + F_2 z_2 - z_{12} \end{bmatrix}, \\
\dot{V}_2(t)|_{(6.5)} & = z_1^T Q_1 z_1 - z_3^T Q_1 z_3 + z_2^T Q_2 z_2 - z_4^T Q_2 z_4 + z_3^T Q_3 z_3 - (1 - h(t)) z_2^T Q_2 z_2, \\
\dot{V}_3(t)|_{(6.5)} & = h_1^2 z_1^T R_1 z_1 + h_1 z_1^T R_3 z_1 + h_1^2 z_1^T R_2 z_1 + h_1^2 z_1^T R_4 z_1, \\
\end{align*}
\]

\[
\begin{align*}
&= \int_{-h_1}^{0} \epsilon^T(t + \tau) R_3 \epsilon(t + \tau)d\tau - h_1 \int_{-h_1}^{0} x^T(t + \tau) R_1 \epsilon(t + \tau)d\tau \\
&- h_3 \int_{-h_2}^{0} \epsilon^T(t + \tau) R_2 \epsilon(t + \tau)d\tau - h_3 \int_{-h(t)}^{0} \epsilon^T(t + \tau) R_2 \epsilon(t + \tau)d\tau \\
&- h_3 \int_{-h_2}^{0} \epsilon^T(t + \tau) R_4 \epsilon(t + \tau)d\tau - h_3 \int_{-h(t)}^{0} \epsilon^T(t + \tau) R_4 \epsilon(t + \tau)d\tau, \\
\end{align*}
\]
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\[
\dot{V}_4(t)|_{(6.5)} = h_1 z_{12}^T (t) S_1 z_{12} + \zeta_2^2 z_{12}^2 S_2 z_{12} - 2h_1 \int_{-h_1}^{0} \int_{-h_1}^{0} e^T(t + \tau) S_1 e(t + \tau) d\tau ds \\
- 2\zeta_2 \int_{-h_2}^{0} \int_{-h_2}^{0} e^T(t + \tau) S_2 e(t + \tau) d\tau ds,
\]

(6.14)

Let us define \( \delta(t) \triangleq (h_2 - h(t))^3 + (h(t) - h_1)^3 \). Then, using Lemmas 2.2.5 and 3.3.1, Corollary 3.3.1, and the fact that \( \delta(t) \leq h_{12}^3 \), expressions for the upper-bounds of (6.12)-(6.14) can be obtained as follows.

\[
\dot{V}_2(t)|_{(6.5)} \leq z_1^T Q_1 z_1 - z_3^T Q_1 z_3 + z_3^T Q_2 z_3 - z_4^T Q_2 z_4 + z_4^T Q_3 z_3 - (1 - \mu) z_3^T Q_3 z_2, 
\]

\[
\dot{V}_3(t)|_{(6.5)} \leq h_{12}^2 z_{12}^T R_1 z_{12} + h_1 z_{12}^T R_3 z_{12} + h_{12}^2 z_{12}^T R_4 z_{12} + h_{12}^4 T R_2 z_{12} + h_1 z_{12}^T R_3 z_{12}
\]

\[
- 2 \begin{bmatrix}
  z_5 \\
  z_6
\end{bmatrix}^T \begin{bmatrix}
  2 & -\frac{3}{h_1} \\
  -\frac{3}{h_1} & \frac{6}{h_1^2}
\end{bmatrix} \otimes R_1 \begin{bmatrix}
  z_5 \\
  z_6
\end{bmatrix}
\]

\[
- \frac{2}{h_1} \begin{bmatrix}
  z_1 - z_3 \\
  h_1 z_1 - z_5
\end{bmatrix}^T \begin{bmatrix}
  2 & -\frac{3}{h_1} \\
  -\frac{3}{h_1} & \frac{6}{h_1^2}
\end{bmatrix} \otimes R_3 \begin{bmatrix}
  z_1 - z_3 \\
  h_1 z_1 - z_5
\end{bmatrix}
\]

\[
- \frac{2}{(h_2 - h(t))^3} \begin{bmatrix}
  z_7 \\
  z_{10}
\end{bmatrix}^T \begin{bmatrix}
  2(h_2 - h(t))^2 & -3(h_2 - h(t)) \\
  -3(h_2 - h(t)) & 6
\end{bmatrix} \otimes R_2 \begin{bmatrix}
  z_7 \\
  z_{10}
\end{bmatrix}
\]

\[
- \frac{2}{(h(t) - h_1)^3} \begin{bmatrix}
  z_8 \\
  z_{11}
\end{bmatrix}^T \begin{bmatrix}
  2(h(t) - h_1)^2 & -3(h(t) - h_1) \\
  -3(h(t) - h_1) & 6
\end{bmatrix} \otimes R_2 \begin{bmatrix}
  z_8 \\
  z_{11}
\end{bmatrix}
\]

\[
- \frac{2}{(h_2 - h(t))^3} \begin{bmatrix}
  z_2 - z_4 \\
  (h_2 - h(t))z_2 - z_7
\end{bmatrix}^T \begin{bmatrix}
  2(h_2 - h(t))^2 & -3(h_2 - h(t)) \\
  -3(h_2 - h(t)) & 6
\end{bmatrix}
\]
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\[
\begin{align*}
&-2 \delta(t) \left( \frac{1}{(h(t) - h_1)^3} \right) \left[ \begin{array}{c}
  z_3 - z_2 \\
  (h(t) - h_1)z_3 - z_8
\end{array} \right]^T
\times
\left[ \begin{array}{cc}
  2(h(t) - h_1)^2 & -3(h(t) - h_1) \\
  -3(h(t) - h_1) & 6
\end{array} \right] \\
&\times
\left[ \begin{array}{c}
  (h(t) - h_1)z_3 - z_8 \\
  (h_2 - h(t))z_2 - z_7
\end{array} \right], \\
&\dot{V}_4(t)|_{(6.5)} \leq h_1^4 z_{12}^T S_1 z_{12} + \frac{\zeta_2}{2} z_{12}^T S_2 z_{12} \\
&\quad \quad - 12 \left[ h_1 z_1 - z_5 \right]^T \left[ \begin{array}{c}
  \frac{1}{h_1} \\
  \frac{-2}{h_1} \frac{6}{h_1^2}
\end{array} \right] \otimes S_1 \left[ h_1 z_1 - z_5 \right] \\
&\quad \quad - 12 \frac{\zeta_2}{\zeta} \left[ h_{12} z_1 - z_7 - z_8 \right]^T \left[ \begin{array}{cc}
  \frac{\zeta_4}{2} z_1 - z_9 & -2 \zeta_3 \\
  -2 \zeta_3 & 6 \zeta_2
\end{array} \right] \otimes S_2 \\
&\quad \quad \times \left[ h_{12} z_1 - z_7 - z_8 \right] \quad (6.16)
\end{align*}
\]

Let us define

\[
\Theta(a, b) \triangleq \left[ \begin{array}{cc}
  2(a - b)^2 & -3(a - b) \\
  -3(a - b) & 6
\end{array} \right].
\]

Since \( \Theta(h_2, h(t)) \) and \( \Theta(h(t), h_1) \) are positive-definite, one can apply the reciprocally convex approach (Lemma 2.2.7) to (6.16) to obtain

\[
\dot{V}_3(t)|_{(6.5)} \leq h_1^2 z_1^T R_1 z_1 + h_1 z_{12}^T R_3 z_{12} + h_4 z_1^T R_2 z_1 + h_4 z_{12}^T R_4 z_{12} \\
- 2 \left[ \begin{array}{c}
  z_5 \\
  z_6
\end{array} \right]^T \left[ \begin{array}{c}
  \frac{3}{h_1} \\
  \frac{-3}{h_1} \frac{6}{h_1^2}
\end{array} \right] \otimes R_1 \left[ \begin{array}{c}
  z_5 \\
  z_6
\end{array} \right] - 2 \eta_1^T \Gamma_1(t) \eta_1 - 2 \eta_2^T \Gamma_2(t) \eta_2 \quad (6.17)
\]
6.3 Detailed Observer Design Framework

\[-\frac{2}{h_1} \begin{bmatrix} z_1 - z_3 \\ h_1 z_1 - z_5 \end{bmatrix}^T \begin{bmatrix} 2 & \frac{-3}{h_1} \\ \frac{-3}{h_1} & \frac{6}{h_1^2} \end{bmatrix} \otimes R_3 \begin{bmatrix} z_1 - z_3 \\ h_1 z_1 - z_5 \end{bmatrix},\]

where \( \bar{\eta}_1^T \triangleq [z_7^T, z_{10}^T, z_8^T, z_{11}^T] \), \( \bar{\eta}_2^T \triangleq [(z_2 - z_4)^T, ((h_2 - h(t))z_2 - z_7)^T, (z_3 - z_2)^T, ((h(t) - h_1)z_3 - z_8)^T] \),

\[\Gamma_1(t) \triangleq \begin{bmatrix} \Theta(h_2, h(t)) & 0 \\ 0 & \Theta(h(t), h_1) \end{bmatrix} \otimes R_2,\]

\[\Gamma_2(t) \triangleq \begin{bmatrix} \Theta(h_2, h(t)) & 0 \\ 0 & \Theta(h(t), h_1) \end{bmatrix} \otimes R_4,\]

and clearly the inequalities \( \Gamma_i(t) \succeq 0, i = \{1, 2\} \) always hold.

Accordingly, aggregating Relations (6.11), (6.15), (6.17), and (6.18) gives

\[\dot{V}(t) \leq z^T \Psi(h(t)) z, \quad (6.19)\]

where \( z^T \triangleq [z_1^T, z_2^T, \cdots, z_{12}^T] \). Therefore, since \( \Psi(h(t)) \) is a convex function of \( h(t) \), according to the convex combination technique (Lemma 2.2.8), the inequality (6.19) holds if and only if the LMIs (6.9) are feasible. The proof is thus completed according to the Lyapunov Krasovskii stability theorem (Theorem 2.2.3).

\[\square\]

**Remark 6.3.1.** It is noted that the the off-diagonal blocks in the definitions of matrices \( \Gamma_1(t) \) and \( \Gamma_2(t) \) are fixed to be zero matrices. However, this assumption is solely taken to reduce the number of decision variables, and one can consider these blocks as non-zero free slack variables that may reduce from the conservatism of the resulted stability criterion. Nevertheless, our observations show that these terms are not quite effective to this aim, whereas our approach in fixing them at zero block matrices, can significantly lessen the computational burden by decreasing \( 8l^2 \) in the number of decision variables.

In addition, by retaining the off-diagonal blocks of \( \Gamma_i(t), i = \{1, 2\} \), through applying
Lemma 2.2.8, four additional LMIs are also appended to the stability criterion, which replace the positive-definiteness assumption of the parameters \( R_2 \) and \( R_4 \).

**Remark 6.3.2.** The stability analysis proposed in Theorem 6.3.2 possesses the following effective features:

- A new augmented LKF including single, double, and triple integral terms, and the descriptor transformation is used.

- The delay-decomposition technique is employed, by splitting the integral terms

\[
\int_{-h_2}^{-h_1} \dot{c}^T(t + \tau) R_2 c(t + \tau) d\tau \quad \text{and} \quad \int_{-h_2}^{-h_1} \dot{c}^T(t + \tau) R_4 \dot{c}(t + \tau) d\tau.
\]

- Wirtinger-based single and double integral inequalities are utilized that are proved to be less conservative than the conventional Jensen’s inequalities in obtaining tighter upper-bounds for cross integral terms (see Chapter 3).

- The reciprocally convex optimization approach, as well as the convex combination technique are applied in analyzing the terms including \( h(t) \) in the LKF’s derivative.

**Remark 6.3.3.** The stability analysis approach introduced in Theorem 6.3.2 provides a novel practical method to use the delay decomposition technique, along with the Wirtinger-based integral inequality and the reciprocally convex approach to obtain a tighter upper-bound for the interval-delay dependent integral terms of the form \( \int_{t-h_2}^{t-h_1} f(s) ds \). This type of analysis can be similarly extended to the corresponding double-integral terms, as an effective tool in analysing time-delay systems with interval time-varying delays.

### 6.3.3 Observer Design

To design the observer parameters, in a way that the conditions of Theorem 6.3.1 are realized, first a closed-from solution for equations (6.6) is obtained (Step A). The design
procedure commences in an analogous way to our novel method proposed in Section 4.3.1 by defining $W \triangleq [I_n, C^T]^T$, which is always a full column rank matrix. This property of $W$ gives us the opportunity to decompose (6.6a) into the following sub-equations

$$T = LW_1 + Z\bar{W}_1,$$  \hspace{1cm} (6.20)

$$V = LW_2 + Z\bar{W}_2,$$  \hspace{1cm} (6.21)

where $[W_1, W_2] \triangleq W^\dagger$, and $[\bar{W}_1, \bar{W}_2] \triangleq (I_{n+p} - WW^\dagger)$. In addition, post-multiplying both sides of (6.6b) and (6.6c) by the non-singular matrix $\bar{U} \triangleq [C^\dagger, C^\perp]$, gives

$$H_1 = -F_1TC^\dagger + TAC^\dagger$$  \hspace{1cm} (6.22)

$$H_2 = -F_2TC^\dagger + TA_dC^\dagger$$  \hspace{1cm} (6.23)

$$F_1TC^\perp = TAC^\perp$$  \hspace{1cm} (6.24)

$$F_2TC^\perp = TA_dC^\perp$$  \hspace{1cm} (6.25)

Thereafter, substituting $T$ from (6.20) into (6.24) and (6.25), and considering the relation $\bar{W}_1C^\perp = 0$ (see Section 4.3.1), one obtains

$$F_1LW_1C^\perp = LW_1AC^\perp + Z\bar{W}_1A_1C^\perp$$  \hspace{1cm} (6.26)

$$F_2LW_1C^\perp = LW_1A_dC^\perp + Z\bar{W}_1A_dC^\perp$$  \hspace{1cm} (6.27)

Hence, equations (6.26) and (6.27) can be reformulated as

$$\left[ \begin{array}{cc} F_1 & F_2 & -Z \end{array} \right] \Omega = \Phi$$  \hspace{1cm} (6.28)
where \( \Omega \triangleq \begin{bmatrix} LW_1C^\perp & 0 \\ 0 & LW_1C^\perp \\ \overline{W}_1AC^\perp & \overline{W}_1A_dC^\perp \end{bmatrix} \), and \( \Phi \triangleq \begin{bmatrix} LW_1AC^\perp & LW_1A_dC^\perp \end{bmatrix} \). It is well-known that (6.28) has a solution if and only if the range space of \( \Phi \) is in the space spanned by the eigenvectors of \( \Omega \). Mathematically speaking, this condition is equivalent to the following rank condition,

\[
\rho \left( \begin{bmatrix} \Phi \\ \Omega \end{bmatrix} \right) = \rho (\Omega) \quad (6.29)
\]

**Corollary 6.3.1.** The necessary and sufficient condition for the fulfilment of Condition ii of Theorem 6.3.1 is the realization of the following rank condition

**Condition I**

\[
\rho \left( \begin{bmatrix} LW_1A & LW_1A_d \\ LW_1 & 0 \\ 0 & LW_1 \\ \overline{W}_1A & \overline{W}_1A_d \\ C & 0 \\ 0 & C \end{bmatrix} \right) = \rho \left( \begin{bmatrix} LW_1 & 0 \\ 0 & LW_1 \\ \overline{W}_1A & \overline{W}_1A_d \\ C & 0 \\ 0 & C \end{bmatrix} \right) \quad (6.30)
\]

**Proof.** To prove the corollary, it is sufficient to show that Condition I is equivalent to the rank condition (6.29). To this aim, first consider the left-hand-side of (6.30). Apprehending that the matrix \( [C^\dagger, C^\perp] \) is a square non-singular matrix, we have
Similarly, the right-hand-side of (6.30) can be written as

\[ \rho \begin{bmatrix} LW_1 & LW_1 A_d \\ LW_1 & 0 \\ 0 & LW_1 \\ \bar{W}_1 A & \bar{W}_1 A_d \\ C & 0 \\ 0 & C \end{bmatrix} = \rho \begin{bmatrix} LW_1 & LW_1 A_d \\ LW_1 & 0 \\ 0 & LW_1 \\ \bar{W}_1 A & \bar{W}_1 A_d \\ C & 0 \\ 0 & C \end{bmatrix} = 2p + \rho \begin{bmatrix} \Phi \\ \Omega \end{bmatrix} \] (6.31)

Therefore, it is deduced from (6.31) and (6.32) that conditions (6.29) and (6.30) are equivalent. This completes the proof of the corollary. \[\square\]

Upon the fulfilment of Condition \( I \) it is readily obtained from (6.28) that
6.3 Detailed Observer Design Framework

\[
\begin{bmatrix}
F_1 & F_2 & -Z
\end{bmatrix} = M_1 + JM_2
\]  
(6.33)

where \( M_1 \triangleq \Phi \Omega^\dagger \in \mathbb{R}^{l \times (2l+n+p)} \), \( M_2 \triangleq (I_{2l+n+p} - \Omega \Omega^\dagger) \), and \( J \in \mathbb{R}^{l \times (2l+n+p)} \) is an arbitrary matrix. By appropriately decomposing \( M_1 \) and \( M_2 \) as

\[
M_1 = \begin{bmatrix}
M_{11} & M_{12} & M_{13}
\end{bmatrix}, \quad M_2 = \begin{bmatrix}
M_{21} & M_{22} & M_{23}
\end{bmatrix},
\]

the following is achieved from (6.33)

\[
F_i = M_{1i} + JM_{2i}, \quad i = \{1, 2\}
\]
(6.34)

\[-Z = M_{13} + JM_{23}
\]
(6.35)

Now, the main result of the chapter is summarized in the following theorem.

**Theorem 6.3.3.** The observer (6.4) is an asymptotically stable functional observer for System (6.1) if Condition I is satisfied and there exist parameters

\[
P_i \succ 0 \in \mathbb{S}^l, \quad i = \{1, 4, 6\}, \quad P_j \in \mathbb{R}^{l \times l}, \quad j = \{2, 3, 5\}, \quad Q_i \succ 0 \in \mathbb{S}^l, \quad i = \{1, 2, 3\}, \quad R_i \succ 0 \in \mathbb{S}^l, \quad i = \{1, \cdots, 4\}, \quad S_i \succ 0 \in \mathbb{S}^l, \quad i = \{1, 2\}, \quad \text{scalars } \alpha_i, \quad i = \{1, 2, 3\}, \quad \text{and free-weighting matrices}
\]

\[
U_4 \in \mathbb{R}^{l \times l}, \quad \Lambda \in \mathbb{R}^{l \times (2l+n+p)}, \quad \text{and } T_1 \in \mathbb{R}^{l \times l}, \quad \text{in a way that the following matrix inequalities are feasible}
\]

\[
\bar{\Psi}_i \prec 0, \quad i = \{1, 2\}
\]
(6.36)

where \( \bar{\Psi}_1 \triangleq \bar{\Psi}(t)|_{h(t)=h_1}, \quad \bar{\Psi}_2 \triangleq \bar{\Psi}(t)|_{h(t)=h_2} \), and

\[
\bar{\Psi}(t) = \begin{bmatrix}
\bar{\Pi}(t) & \Pi_9 & \Pi_{10}(t) & \Pi_{11}(t) & \bar{\Pi}_{12}
\end{bmatrix}
\]

\[
\begin{bmatrix}
* & -72 \frac{\varsigma_2}{\varsigma} S_2 & 0 & 0 & 0
* & * & -12 R_2 & 0 & 0
* & * & * & -12 R_2 & 0
* & * & * & * & \bar{\Pi}_{12}
\end{bmatrix},
\]
wherein $\Pi_9$, $\Pi_{10}$, $\Pi_{11}$, and $\bar{\Pi}_{12}$ are as defined in Theorem 6.3.2,

\[
\bar{\Pi}_{12} = \begin{bmatrix}
P_1 - \alpha_4 U_4 + U_4^T M_{11} + \Lambda M_{21}, U_4^T M_{12} + \Lambda M_{22}, \\
0, 0, P_3 - \alpha_3 U_4, 0, P_2 - \alpha_2 U_4, P_2 - \alpha_2 U_4 \end{bmatrix}^T,
\]

and $\bar{\Pi}(t)$ is identical to $\Pi(t)$ defined in the same theorem except in the following elements:

\[
\bar{\Pi}_{1,1} = \text{sym}(P_3) + Q_1 + h_2^2 R_1 + h_1 R_2 - 6 h_1^2 S_1 - 12 \zeta_2 \alpha_1 a_1 S_2 - \frac{3}{h_1} R_3 + \alpha_1 \text{sym}(U_4 M_{11} + \Lambda M_{21}), \\
\bar{\Pi}_{2,1} = \alpha_1 M_{22}^T \Lambda^T + \alpha_1 M_{12}^T U_4, \\
\bar{\Pi}_{5,1} = P_6 + \frac{6}{n^2} R_3 + \alpha_3 U_4^T M_{11} + \alpha_3 \Lambda M_{21}, \\
\bar{\Pi}_{5,2} = \alpha_3 U_4^T M_{12} + \alpha_3 \Lambda M_{22}, \\
\bar{\Pi}_{7,1} = -\frac{12 \zeta_2}{\zeta} a_1 S_2 + P_5 + \alpha_2 U_4^T M_{11} + \alpha_2 \Lambda M_{21}, \\
\bar{\Pi}_{7,2} = 6(h_2 - h(t)) R_4 + \alpha_2 U_4^T M_{12} + \alpha_2 \Lambda M_{22}, \\
\bar{\Pi}_{8,1} = -\frac{12 \zeta_2}{\zeta} a_1 S_2 + P_5 + \alpha_2 U_4^T M_{11} + \alpha_2 \Lambda M_{21}, \text{ and } \bar{\Pi}_{8,2} = 6(h(t) - h_1) R_4 + \alpha_2 U_4^T M_{12} + \alpha_2 \Lambda M_{22}.
\]

In addition,

\[
J = U_4^{-T} \Lambda.
\]  

\[\text{(6.37)}\]

**Proof.** The theorem can be directly concluded from Theorems 6.3.1 and 6.3.2, and Corollary 6.3.1. According to Corollary 6.3.1, upon the accomplishment of Condition I, Condition ii of Theorem 6.3.1 is attained. In addition, after substituting $F_1$ and $F_2$ from (6.34) into $\Psi(t)$, setting $U_i = \alpha_i U_4$ for $i = \{1, 2, 3\}$, and defining $\Lambda \triangleq U_4^T J$, $i = \{1, 2, 3, 4\}$, it is clear that the inequalities (6.36) are equivalent to (6.9). Subsequently, Condition i of Theorem 6.3.1 follows, which completes the proof of the theorem. \(\square\)

**Remark 6.3.4.** The justification behind setting $U_i = \alpha_i U_4$ for $i = \{1, 2, 3\}$ in Theorem 6.3.3, is to avoid facing with the problem of solving the following matrix equation for $J$,

\[
\bar{U} J = \bar{\Lambda},
\]

where $\bar{U} = \begin{bmatrix} U_1 & U_2 & U_3 & U_4 \end{bmatrix}^T \in \mathbb{R}^{4 \times l}$, and $\bar{\Lambda} = \begin{bmatrix} \Lambda_1^T & \Lambda_2^T & \Lambda_3^T & \Lambda_4^T \end{bmatrix}^T \in \mathbb{R}^{4 \times (2l+n+p)}$. Unfortunately, due to the size of $\bar{U}$, (6.38) generally has no solution. However, $\alpha_i$s can be
replaced by $A_i \in \mathbb{R}^{l \times l}$, $i = \{1, 2, 3\}$, as arbitrary weighting matrices, instead of the scalar parameters. This can help in further reducing from the conservatism created from the assumption $U_i = \alpha_i U_4$, $i = \{1, 2, 3\}$.

### Tuning Based on the Genetic Algorithm

Since the matrix inequalities (6.36) contain nonlinear terms $\alpha_i U_4$ for $i = \{1, 2, 3\}$, they cannot be solved using the available effective LMI solvers. Hence, the scalars $\alpha_i$ are pre-defined to resolve this issue. However, finding appropriate values for these parameters remains as an important problem. To this aim, similar to the framework given in Table 5.1 (see Section 5.3), a genetic algorithm (GA) based approach is proposed in the sequel to automatically seek for convenient tuning parameters $\alpha_i$, $i = \{1, 2, 3\}$, given the values of $h_1$, $h_2$, and $\mu$.

**Step 1** Consider the LMI feasibility problem given in Theorem 6.3.3 as the fitness function with the following modification of the LMIs (6.36)

$$
\bar{\Psi}_i \prec t_{\min}, \quad i = \{1, 2\},
$$

where $t_{\min} < 0$ is an arbitrary scalar. In this fitness function, $\alpha_i$, $i = \{1, 2, 3\}$ are considered as the variables, and $t_{\min}$ is the output of the function that needs to be minimized.

**Step 2** Choose an arbitrary and workable searching limit for each variable, as $\alpha_i \in [\alpha_i \text{ min}, \alpha_i \text{ max}]$.

**Step 3** Generate $N_p$ set of $\alpha_i$s with arbitrary values as the chromosomes of the initial population, and define $\bar{\alpha}_j \triangleq [\alpha_1^j, \alpha_2^j, \alpha_3^j]$, $j = \{1, \cdots, N_p\}$.

**Step 4** Employ the genetic algorithm to optimally search for a chromosome that satisfy the LMIs (6.39). If even after $N_g$ generations the algorithm was unable to find an apt
6.4 Numerical Examples

chromosome, which results in the fulfilment of the constraint $t_{\min} < \eta < 0$, then the algorithm terminates without any solution.

**Remark 6.3.5.** If Step 4 of the algorithm could not find an applicable chromosome, even after careful increasing of the searching limits, the number of chromosomes in each population, and the number of generations, then for given $h_i$, $i = 1, 2$, and $\mu$, the algorithm does not converge, and thus the stability region should be appropriately shrunk.

After applying Theorem 6.3.3, and the tuning algorithm, which calculate a convenient matrix $J$ from (6.37), the remaining observer parameters $F_i$, $i = \{1, 2\}$, $Z$, $V$, $T$, $H_1$, $H_2$, $G_1$, $G_2$, and $V_u$ can be directly obtained from (6.34), (6.35), (6.21), (6.20), (6.22), (6.23), (6.6d), (6.6e), and (6.6f), respectively.

**Remark 6.3.6.** It is conceivable to relax Condition I to a less conservative condition. This can be done either by changing the observer architecture via adding more delayed output terms as in [107], or by increasing the order of the observer via appending auxiliary functions to the desired functional (i.e. adding extra rows to the matrix $L$) as in [63, 65]. By using the the latter method, the observer will not be of minimum-order. Studying this problem is one of our future concerns.

6.4 Numerical Examples

In the following examples, for the aim of simulation studies, an arbitrary control input $u(t) = 5 + 10e^{-0.4t}\cos(2t)$, $t \geq 0$ is applied in order to appropriately excite the states of the system. In addition, the delay function in simulations has been arbitrarily considered to be $h_i(t) = \frac{h_{12}}{2}\sin\left(\frac{2\mu}{h_{12}}t\right) + \frac{h_1 + h_2}{2}$, which satisfies the delay properties described in (6.2). To the best of our knowledge, the most recent paper that solves the problem of delay-dependent FO design for retarded systems with time-varying delays is [105]. We have
made comparisons with this work wherever applicable.

### 6.4.1 Example 1

Consider the system (6.1) with the following parameters

\[
A = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 1 \end{bmatrix},
\]

\[
B = \begin{bmatrix} 1 & -1 \end{bmatrix}^T, \quad D = 1.
\]

It can be observed that Condition I is realized. Considering \(h_1 = 0.1s, h_2 = 2s\), and \(\mu = 0.5\), the stability criteria proposed in [105] does not work. Moreover, setting \(\alpha_i = 1, i = \{1, 2, 3\}\), results in infeasible LMIs (6.36) for the given delay specifications. Nevertheless, applying the tuning algorithm, \(\alpha_1 = 0.9058, \alpha_2 = 0.1576, \alpha_3 = 0.9595\) were acquired, such that the LMI problem of Theorem 6.3.3 is feasible. Hence, the LMI parameters were obtained as

\[
P_1 = 352.5309, \quad P_2 = 14.4241, \quad P_3 = 15.6456, \quad P_4 = 30.6869, \quad P_5 = 41.6672, \quad P_6 = 141.3947, \quad Q_1 = 336.6891, \quad Q_2 = 83.9544, \quad Q_3 = 179.4113, \quad R_1 = 0.0708, \quad R_2 = 8.1681, \quad R_3 = 0.0345, \quad R_4 = 4.3666, \quad S_1 = 0.0138, \quad S_2 = 2.1656, \quad U_4 = 91.2981, \quad \Lambda = 10^4 \begin{bmatrix} 1.9561 & 0 & -3.6397 & 0 & -1.5833 \end{bmatrix}.
\]

Thereafter, the observer parameters were calculated as \(F_1 = -3.6606, \quad F_2 = -1, \quad H_1 = 13.4, \quad H_2 = 0, \quad G_1 = -10.7394, \quad G_2 = 0, \quad \text{and} \quad V = -V_u = -3.6606\).

This example justifies that the stability criteria in [105] can be more conservative than the criteria proposed in this chapter, and the effectiveness of the genetic algorithm in tuning the nonlinear free variables of the inequalities (6.36) is also highlighted.

Simulation results obtained from the MATLAB/Simulink environment are depicted in Fig. 6.1. The initial conditions of the states of the system were considered as \(\phi(\theta) = [1, -1]^T, \forall \theta \in [-2, 0]\). As can be seen from Fig. 6.1 the estimated state asymptotically tracks its real values with an agreeable performance. It is observed in the figure
that around the time \( t = 2s \), a slight deviation occurs that quickly diminishes. This error appears, when the delayed state dynamic starts to affect the system, which is reasonable.

Figure 6.1: The estimation of the second state of the system in Example 1

6.4.2 Example 2

In this example, a 4th order system with the following parameters is studied
6.4 Numerical Examples

\[ A = \begin{bmatrix} -10 & 1 & 0 & 0 \\ -41 & -2 & 4 & 0 \\ 0 & 0 & -2 & 1 \\ 2 & 0 & -20 & -6 \end{bmatrix}, \quad A_d = \begin{bmatrix} 2 & 0 & 0 & -5 \\ -1 & 3 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ -2 & 1 & -5 & 0 \end{bmatrix}, \]

\[ C = \begin{bmatrix} 1.5 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & I_2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & -1 & 0.2 \end{bmatrix}^T, \quad \text{and} \quad D = \begin{bmatrix} 0 & 1 \end{bmatrix}^T. \]

Therefore, the last two states of the system are desired to be asymptotically reconstructed.

It can be testified that Condition I is satisfied. Let us assume that \( h_1 = 1s, h_2 = 9s, \) and \( \mu = 2. \) Since \( \mu > 1, \) the methodologies proposed in papers like [103, 105, 175] are not applicable. Moreover, assuming \( \alpha_i = 1, i = \{1, 2, 3\}, \) the LMIs (6.36) were found to be infeasible. Accordingly, applying the tuning algorithm a suitable chromosome were found as \( \bar{\alpha} = [0.8356, 0.0085, 0.8071]. \) Therefore, Theorem 6.3.3 was successfully applied to obtain an appropriate parameter \( J, \) and the observer parameters were subsequently obtained as

\[ F_1 = \begin{bmatrix} -18.8499 & 1 \\ 36.1344 & -6 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 & 0 \\ -5 & 0.0714 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 115.1506 & 42.5067 \\ -382.2136 & -104.1678 \end{bmatrix}, \]

\[ H_2 = \begin{bmatrix} 3.4750 & 6.8189 \\ -10.7074 & -10.5181 \end{bmatrix}, \quad G_1 = \begin{bmatrix} -51.9317 \\ 132.4207 \end{bmatrix}, \quad G_2 = \begin{bmatrix} -6.8189 \\ 10.5181 \end{bmatrix}, \]

\[ V = \begin{bmatrix} 0 & -2.1062 \\ 0.0095 & 7.0168 \end{bmatrix}, \quad \text{and} \quad V_u = \begin{bmatrix} 2.1062 \\ -7.0168 \end{bmatrix}. \]

In addition, the behaviour of the system and the closed-loop observer was simulated in the Simulink environment, assuming \( \phi(\theta) = [-2, 0.01, 0, 10]^T, \forall \theta \in [-9, 0]. \) The results depicted in Fig. 6.2, show a good performance of the observer, such that both of the estimated states converge to their real values in less than two seconds of the simulation.
time. Similar to the previous example, the slight deviation around $t = 6s$ comes from the delayed part of the system’s dynamic that shows its effect after a time lag.

Next chapter considers *unknown* time-varying state delays in the observer design for hereditary systems. A new sliding mode functional observer structure is proposed to this

Figure 6.2: The estimation of the third and the fourth states of the system in Example 2
aim, and the design method is established based on the achievements of this and the previous chapters.

6.5 Conclusions

A methodology for delay-dependent partial state estimation of LTI retarded systems with interval time-varying delays has been proposed. The delay derivative has been assumed to be bounded, but the upper-limit is not confined to values less than one. The asymptotically stable minimum-order observer has been designed via solving a set of interconnected matrix equations together with a semi-definite programming in the form of LMIs. Due to using a novel augmented LKF with triple integral terms, and employing effective techniques, such as the descriptor transformation and convex combination approaches, the proposed observer’s stability criteria can achieve larger stability regions with regard to the few other available approaches that design FOs for time-delay systems. In addition, novel Wirtinger-based integral inequalities have been applied instead of Jensen’s inequalities, or other alternative conservative schemes such as Young’s inequality and Cauchy-Schwartz inequality. An effective searching algorithm based on GA has been proposed to tune a number of LMIs’ parameters. Two numerical examples and simulation results have emphasized the efficacy and the supremacy of the proposed observer design framework.
Chapter 7

Functional Observer Design for LTI Systems With Unknown Time-Varying State Delays

7.1 Introduction and Problem Statement

As mentioned in the previous chapters, observer design for time-delay systems has recently become very compelling, and a diverse range of problems related to full-order Luenberger observer design for time-delay systems have been solved in the existing literature (see e.g. [159, 176–180]). However, the majority of these filters take the knowledge of the real delay values into their structure, which is usually not applicable, particularly when the delay is time-varying or stochastic. There are a few existing works that study the observer design problem for unknown time-delay systems [163, 173, 181, 182]. In addition, sliding mode full-order observers have been developed to comply with the system’s uncertainties
and disturbances, and it is shown that these observers have advantages over unknown-input observers [85, 93, 183, 184]. The concept of sliding mode functional observers was first introduced in [185], and it was successfully applied to the speed control of sensorless synchronous machines. Kee et al. [85] studied two-sliding mode unknown-input FOs as the latest work on this topic.

The problem of delay-dependent full-order observer design for unknown time-varying delay systems was first studied in [181]. In this work, a sliding mode observer is designed in combination with the Lyapunov Krasovskii approach to deal with the delayed terms, and a constant auxiliary delay is employed in the observer structure. Xie and Ji [182] improved this approach by further considering arbitrarily large and bounded delay derivative. Ghanes et al. [173] investigate the problem for a class of nonlinear systems, and employ the Lyapunov Krasovskii approach to ensure the practical stability of the observer error dynamics. Sayyeddelshad and Gustafsson [163] study robust $H_{\infty}$ observer design for a class of discrete nonlinear systems with unknown interval time-varying delays and parameter uncertainties, using a delay-free observer structure, and assuming the stability of the nominal system.

Although partial state estimation of linear and nonlinear time-delay systems has been recently well-developed (see e.g. [103, 105, 106, 174, 175] and the previous two chapters), to the best of the author’s knowledge, delay-dependent functional observer design for linear time-delay systems, under the assumption of the uncertainty of delay is still an open problem. Only a few number of papers have designed delay-free FOs for LTI systems with unknown time-delays, by treating the delayed terms as unknown-inputs [105, 108, 174]. Hence, those works design unknown-input functional observers to address the delay uncertainty problem with the expense of assuming restrictive structural conditions. The main dilemma that is raised from the delay uncertainty is the fact that it cannot be used in the observer structure.
The main contribution of this chapter is to address the above limitation by proposing a novel practical algorithm. To this aim, a new sliding mode functional observer structure is proposed that exploits an auxiliary time-varying delay within its architecture. The real delay is assumed to be interval time-varying with an arbitrary upper-bound of its rate. Nevertheless, the bounds of the delay can be unknown, and even unbounded. The necessary and sufficient conditions of the stability of the proposed observer are derived, and different characteristics of the those conditions are discussed. Based on the latter findings, the observer parameters are designed via solving an LMI problem, which guarantees the asymptotic stability of the delayed differential equation of the error dynamics. The LMI is established using the Lyapunov Krasovskii approach, together with solving a number of constrained equations. To escape the nonlinearity of the matrix inequality derived from the Lyapunov approach, and to reduce the conservatism of the stability criteria, the genetic algorithm-based searching framework, proposed in Section 6.3.3, is employed to adjust a number of the observer parameters. In addition, the necessary and sufficient conditions of the stability of the delay-free observer structure, which is attained by nullifying the auxiliary delay in the observer dynamics, are obtained. Moreover, a critical comparison between the delay-free and delay-dependent observers is demonstrated. Two illustrative examples and simulation results delineate the proposed observer design framework, and highlight its superb features.

The rest of this chapter is organized as follows. The problem and some preliminaries are given in Section 7.2. The novel observer structure is introduced in Section 7.3.1, its stability is analysed in Section 7.3.2, and the design algorithm is developed and explained in Section 7.3.3. In addition, numerical examples are given in Section 7.4, and the chapter is concluded in Section 7.5.
7.2 Problem Formulation

**Notations:** $\mathbb{R}^n$ is the n-dimensional vector space of real numbers; $\mathbb{R}^{n \times m}$ is the space of $n \times m$ real matrices; $\mathbb{S}^{n \times n}$ is the space of symmetric real matrices; $I_n$ is the $n \times n$ identity matrix; 0 is the zero matrix of appropriate dimension; and $C_n(\Omega)$ is the space of continuous functions mapping from $\Omega$ to $\mathbb{R}^n$ with the topology of uniform convergence. Moreover, $|x|$ denotes the Euclidean norm of the vector $x$, $\text{sign}(\cdot)$ is the signum function, and $\star$ in a symmetric block-diagonal matrix stands for the symmetric element. Furthermore, $X^T$ is the transpose, $X^\dagger$ is the generalized inverse, $X^\perp$ is the right orthogonal, $\rho(X)$ is the rank, $\|X\|_2$ is the 2-norm, and $\mathcal{R}(X)$ is the range space of the matrix $X$, and $\text{sym}(X) = X + X^T$. In addition, for a matrix $Y \in \mathbb{S}^{n \times n}$, $Y \prec 0$ and $Y \prec \gamma$ indicate that the matrices $Y$ and $Y - \gamma I_n$ are negative semi-definite, respectively.

Consider the following LTI system with a single time-varying state delay

\begin{align*}
\dot{x}(t) &= Ax(t) + A_dx(t - h(t)) + Bu(t) \\
y(t) &= Cx(t) \\
z(t) &= Lx(t) \\
x(\theta) &= \phi(\theta) \quad \forall \theta \in [-h_2, 0],
\end{align*}

(7.1)

where $x(\cdot) \in \mathbb{R}^n$, $y(\cdot) \in \mathbb{R}^p$, $u(\cdot) \in \mathbb{R}^m$, and $z(\cdot) \in \mathbb{R}^l$ are the vectors of states, outputs, inputs, and the functions of the states that are aimed to be estimated, respectively. Moreover, $\phi(\cdot) \in C_n([-h_2, 0])$ is the vector of the initial functions of the states of the system. In addition, $A$, $A_d$, $B$, $C$, and $L$ are constant distribution matrices of the system, with appropriate dimensions. Furthermore, the unknown delay function $h(t)$, satisfies the following conditions

\begin{align*}
0 \leq h_1 \leq h(t) \leq h_2, \quad \dot{h}(t) \leq \mu,
\end{align*}

(7.2)
where \( h_1 < h_2 \), and \( \mu \) are constant real scalars. Throughout the chapter without the loss of generality it is assumed that the distribution matrices \( C \) and \( L \) are of full ranks, and \( n > p \).

Our objective is to propose a delay-dependent functional observer structure and design its parameters to asymptotically reconstruct the functions \( z(t) \), given any initial function \( \phi(\cdot) \in C_n([-h_2, 0]) \). The designed FO must be robust against the uncertainties in the delay values. To this aim, the observer should satisfy the asymptotic stability criterion given in Definition 6.2.1.

### 7.3 New Sliding-Mode Observer Structure and Its Design Algorithm

#### 7.3.1 Observer Structure

As the first step, let us augment the output to the function \( z(t) \) to form a new variable as

\[
\bar{z}(t) = \bar{L}x(t),
\]

where \( \bar{L}^T \triangleq [L^T, C^T] \). Hence, in order to achieve the objective of this chapter it is sufficient to design a functional observer for \( \bar{z}(t) \), such that the estimation error vector \( \bar{e}(t) \triangleq \hat{\bar{z}}(t) - \bar{z}(t) \) asymptotically converges to zero, in the sense of Definition 6.2.1. We propose the following observer structure:
\[ \dot{\omega}(t) = F\omega(t) + N\omega(t - \hat{h}(t)) + G_u u(t) + H_1 y(t) + H_2 y(t - \hat{h}(t)) \]
\[ - \begin{bmatrix} X_2 \\ N_{22} \end{bmatrix} \epsilon_2(t - \hat{h}(t)) - \begin{bmatrix} X_1 \\ F_{22} - G \end{bmatrix} \epsilon_2(t) + \begin{bmatrix} R \\ -I_p \end{bmatrix} \nu(t) \]
\[ \hat{\pi}(t) = \omega(t) + V y(t) \]
\[ \omega(\theta) = 0 \quad \forall \theta \in [-\bar{h}_2, 0], \]

where \( \omega(\cdot) \in \mathbb{R}^{l+p} \) is the states of the observer dynamics, and \( \hat{\pi}(\cdot) \) is the estimated vector of \( \pi(\cdot) \). Moreover, for a design parameter \( T \in \mathbb{R}^{(l+p) \times n} \), \( \epsilon_2(\cdot) \in \mathbb{R}^p \) is the second partition of the vector \( \epsilon(\cdot) \triangleq \omega(\cdot) - Tx(\cdot) \in \mathbb{R}^{l+p} \), i.e., \( \epsilon^T(\cdot) = [\epsilon_1^T(\cdot), \epsilon_2^T(\cdot)] \); and \( \nu(t) \) is the sliding mode robustifying term, defined as
\[ \nu(t) = \begin{cases} \alpha(t) \frac{P\epsilon_2(t)}{|P\epsilon_2(t)|} & \epsilon_2(t) \neq 0 \\ 0 & \epsilon_2(t) = 0 \end{cases} \]

where \( \hat{P} \in \mathbb{S}^{p \times p} \), is a positive-definite matrix, and \( \alpha(t) \) is a real-time upper-limit of a perturbation vector, which is articulated later in Section 7.3.2. In addition, \( F, N, G_u, H_1, H_2, X_1, X_2, G, V \), and \( R \) are constant matrices as observer parameters that should be appropriately designed. Furthermore, \( \hat{h}(t) \) is an arbitrary delay function, which satisfies the following conditions:
\[ 0 \leq \bar{h}_1 \leq \hat{h}(t) \leq \bar{h}_2, \quad \dot{\hat{h}}(t) \leq \bar{\mu} < 1, \]

where \( \bar{h}_1, \bar{h}_2, \) and \( \bar{\mu} \) are constant scalars. Moreover, the matrices \( F, N, T, \) and \( H_i, i = \{1, 2\} \) are adequately partitioned as
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\[ F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \quad N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}, \]
\[ T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}, \quad H_i = \begin{bmatrix} H_{i1} \\ H_{i2} \end{bmatrix}. \] (7.7)

**Theorem 7.3.1.** Consider the system (7.1). The observer (7.4) is a globally asymptotically stable functional observer for the system if and only if

(i) there exist matrices \( F, N, G_u, H_1, H_2, V, \) and \( T \) that satisfy the following constrained equations

\[ T + VC - L = 0 \] (7.8a)
\[ H_1C + FT - TA = 0 \] (7.8b)
\[ NT + H_2C - TA_d = 0 \] (7.8c)
\[ G_u = TB \] (7.8d)

(ii) for any initial function \( \phi(\cdot) \in \mathcal{C}_n([-h_2, 0]) \) the following error dynamics is globally asymptotically stable

\[ \dot{\epsilon}(t) = F\epsilon(t) + N\epsilon(t - \hat{h}(t)) - H_2C \int_{t-h(t)}^{t} \dot{x}(s)ds - N \int_{t-h(t)}^{t} \dot{\omega}(s)ds - \begin{bmatrix} X_2 \\ N_{22} \end{bmatrix} \epsilon_2(t - \hat{h}(t)) - \begin{bmatrix} X_1 \\ F_{22} - G \end{bmatrix} \epsilon_2(t) + \begin{bmatrix} R \\ -I_p \end{bmatrix} \nu(t) \] (7.9)

\[ \epsilon(\theta) = -T\phi(\theta) \quad \forall \theta \in [-h_2, 0]. \]

**Proof.** The sufficiency part is proved in the sequel and the proof of the necessity part is omitted due to its simplicity. Differentiating \( \epsilon(t) \) along the solutions of (7.1) and (7.4) and
employing the Leibniz-Newton formula results in the following dynamic equations

\[
\dot{\epsilon}(t) = F\epsilon(t) + N\epsilon(t - h(t)) + (H_1 C + FT - TA)x(t) + (G_u - TB)u(t) + (NT + H_2 C - TA_d)x(t - h(t)) + N\left(\omega(t - \hat{h}(t)) - \omega(t - h(t))\right) \\
+ H_2 C\left(x(t - \hat{h}(t)) - x(t - h(t))\right) + \begin{bmatrix} R \\ -I_p \end{bmatrix}\nu(t) \\
- \begin{bmatrix} X_2 \\ N_{22} \end{bmatrix}\epsilon_2(t - \hat{h}(t)) - \begin{bmatrix} X_1 \\ F_{22} - G \end{bmatrix}\epsilon_2(t)
\]

\[
\epsilon(\theta) = -T\phi(\theta) \quad \forall \theta \in [-h_2, 0].
\]

Hence, the fulfilment of Condition (ii) and equations (7.8b)-(7.8d), concludes that the auxiliary error \( \epsilon(t) \) is globally asymptotically stable in the sense of Definition 6.2.1. Moreover, after the direct substitution into the definition of \( \dot{\epsilon}(t) \), it is deduced that if in addition (7.8a) is satisfied, then the FO (7.4) is globally asymptotically stable.

Henceforth, we design the observer parameters such that Condition (ii) is realized, with the fulfilment of the constrained Sylvester equations (7.8). Based on the definition of \( \epsilon(t) \), the following equation is always true

\[
N\left(\omega(t - \hat{h}(t)) - \omega(t - h(t))\right) = N\left(\epsilon(t - \hat{h}(t)) - \epsilon(t - h(t))\right) \\
+ NT\left(x(t - \hat{h}(t)) - x(t - h(t))\right)
\]

Therefore, taking (7.7) and (7.11) into account and considering the Leibniz-Newton
where \( \epsilon \) (7.12), it is obtained that

\[
\dot{\epsilon}_1(t) = F_{11}\epsilon_1(t) + F_{12}\epsilon_2(t) + N_{11}\epsilon_1(t-h(t)) + N_{12}\epsilon_2(t-h(t)) + Rv(t)
\]
\[+ N_{11} \left( \epsilon_1(t-\hat{h}(t)) - \epsilon_1(t-h(t)) \right) + N_{12} \left( \epsilon_2(t-\hat{h}(t)) - \epsilon_2(t-h(t)) \right)
\]
\[- H_{21}C \int_{t-h(t)}^{t-h(h(t))} \dot{x}(s)ds - N_1T \int_{t-h(t)}^{t-h(h(t))} \dot{x}(s)ds - X_2\epsilon_2(t-h(t)) - X_1\epsilon_2(t)
\]
\[= F_{11}\epsilon_1(t) + (F_{12} - X_1)\epsilon_2(t) + N_{11}\epsilon_1(t-\hat{h}(t)) - (H_{21}C + N_1T) \int_{t-h(h(t))}^{t-h(h(t))} \dot{x}(s)ds
\]
\[+ (N_{12} - X_2)\epsilon_2(t-h(t)) + Rv(t),
\]

\[
\dot{\epsilon}_2(t) = F_{21}\epsilon_1(t) + F_{22}\epsilon_2(t) + N_{21}\epsilon_1(t-h(t)) + N_{22}\epsilon_2(t-h(t)) - F_{22}\epsilon_2(t) + G\epsilon_2(t)
\]
\[+ N_{21} \left( \epsilon_1(t-\hat{h}(t)) - \epsilon_1(t-h(t)) \right) + N_{22} \left( \epsilon_2(t-\hat{h}(t)) - \epsilon_2(t-h(t)) \right)
\]
\[- H_{22}C \int_{t-h(h(t))}^{t-h(h(t))} \dot{x}(s)ds - N_2T \int_{t-h(h(t))}^{t-h(h(t))} \dot{x}(s)ds - N_{22}\epsilon_2(t-h(t)) - \nu(t)
\]
\[= G\epsilon_2(t) + F_{21}\epsilon_1(t) + N_{21}\epsilon_1(t-\hat{h}(t)) - (H_{22}C + N_2T) \int_{t-h(h(t))}^{t-h(h(t))} \dot{x}(s)ds - \nu(t),
\]

where \( \epsilon^T(t) = [\epsilon_1^T(t), \epsilon_2^T(t)] \), \( \epsilon_1(t) \in \mathbb{R}^l \) and \( \epsilon_2(t) \in \mathbb{R}^p \). Augmenting the last equations in (7.12) and (7.13), it is obtained that

\[
\dot{\epsilon}(t) = \begin{bmatrix}
F_{11} & F_{12} - X_1 \\
F_{21} & G
\end{bmatrix} \epsilon(t) + \begin{bmatrix}
R \\
-I_p
\end{bmatrix} \nu(t)
\]
\[+ \begin{bmatrix}
N_{11} & N_{12} - X_2 \\
N_{21} & 0
\end{bmatrix} \epsilon(t-\hat{h}(t)) - \begin{bmatrix}
H_{21}C + N_1T \\
H_{22}C + N_2T
\end{bmatrix} \int_{t-h(h(t))}^{t-h(h(t))} \dot{x}(s)ds,
\]
Now, let us map the system (7.14) through the transformation matrix \( \tilde{T} \triangleq \begin{bmatrix} I_l & R \\ 0 & I_p \end{bmatrix} \),

and assume that the following relations hold:

\[
\begin{bmatrix}
F_{11} & F_{12} - X_1 \\
F_{21} & G
\end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
\tilde{T}^{-1} = 
\begin{bmatrix}
\bar{F}_{11} & 0 \\
F_{21} & G - F_{21}R
\end{bmatrix}
\end{bmatrix},
\]

(7.15)

\[
\begin{bmatrix}
N_{11} & N_{12} - X_2 \\
N_{21} & 0
\end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
\tilde{T}^{-1} = 
\begin{bmatrix}
\bar{N}_{11} & 0 \\
N_{21} & -N_{21}R
\end{bmatrix}
\end{bmatrix}.
\]

(7.16)

Accordingly, the system (7.14) can be represented as

\[
\begin{aligned}
\dot{\bar{\epsilon}}_1(t) &= \bar{F}_{11} \bar{\epsilon}_1(t) + \bar{N}_{11} \bar{\epsilon}_1(t - \hat{h}(t)) \\
&\quad - (H_{21}C + N_1T + RH_{22}C + RN_2T) \int_{t-\hat{h}(t)}^{t-h(t)} \dot{x}(s)ds,
\end{aligned}
\]

(7.17)

\[
\begin{aligned}
\dot{\bar{\epsilon}}_2(t) &= (G - F_{21}R) \bar{\epsilon}_2(t) + \eta(t, \bar{\epsilon}(t), x(t)) - \nu(t),
\end{aligned}
\]

(7.18)

where \( N_1 \triangleq [N_{11}, N_{12}] \), \( N_2 \triangleq [N_{21}, N_{22}] \)

\[
\bar{\epsilon}(t) = \tilde{T} \epsilon(t),
\]

and \( \bar{\epsilon}_1(t) \) and \( \bar{\epsilon}_2(t) \) are associated with \( \epsilon_1(t) \) and \( \epsilon_2(t) \), respectively. Moreover, \( \bar{F}_{11} \triangleq F_{11} + RF_{21}, \bar{N}_{11} \triangleq N_{11} + RN_{21} \), and

\[
\begin{aligned}
\eta(t, \bar{\epsilon}(t), x(t)) &\triangleq F_{21} \bar{\epsilon}_1(t) + N_{21} \bar{\epsilon}_1(t - \hat{h}(t)) - N_{21} R \bar{\epsilon}_2(t - \hat{h}(t)) \\
&\quad - (H_{22}C + N_2T) \int_{t-\hat{h}(t)}^{t-h(t)} \dot{x}(s)ds.
\end{aligned}
\]

(7.19)
Furthermore, for the validity of the relations (7.15) and (7.16) it is necessary that

\[ X_1 = -F_{11}R + F_{12} + R G - RF_{21}R, \]
\[ X_2 = -N_{11}R - RN_{21}R + N_{12}. \]

(7.20)  
(7.21)

Hence, the following theorem is achieved.

**Theorem 7.3.2.** Consider the system (7.1). The observer (7.4) is a globally asymptotically stable functional observer for the system if and only if Condition (i) of Theorem 7.3.1 is satisfied, and

(a) the following constraint equation holds

\[ H_{21}C + N_{11}T_1 + N_{12}T_2 + RH_{22}C + RN_{21}T_1 + RN_{22}T_2 = 0, \]

(7.22)

(b) for any initial function \( \phi(\cdot) \in C_n([-h_2, 0]) \), the following error dynamics is globally asymptotically stable

\[ \dot{\bar{\epsilon}}_1(t) = \bar{F}_{11}\bar{\epsilon}_1(t) + \bar{N}_{11}\bar{\epsilon}_1(t - \hat{h}(t)) \]
\[ \dot{\bar{\epsilon}}_2(t) = (G - F_{21}R)\bar{\epsilon}_2(t) + \eta(t, \bar{\epsilon}(t), x(t)) - \nu(t) \]
\[ \bar{\epsilon}(\theta) = -\hat{T}T\phi(\theta). \]

(7.23a)  
(7.23b)

**Remark 7.3.1.** It is noteworthy to mention that the dynamics of \( \bar{\epsilon}_1(t) \) is decoupled from that of \( \bar{\epsilon}_2(t) \), and further it is independent of the real delay values \( h(t) \). This property is very helpful in the stability analysis of the system (7.23) and can result in less restrictive stability conditions of the observer.
7.3 New Sliding-Mode Observer Structure and Its Design Algorithm

7.3.2 Stability Analysis

It is clear from (7.19) that upon the boundedness of the norms of $\bar{\epsilon}_i(t), i = \{1, 2\}$ and the difference vector $x(t - h(t)) - x(t - \hat{h}(t))$, then $\eta(t, \bar{\epsilon}(t), x(t))$ is bounded, i.e., there exists a bounded function $\alpha(t)$ that for every time $t > 0$, we have

$$|\eta(t, \bar{\epsilon}(t), x(t))| \leq \alpha(t) \tag{7.24}$$

Now, we investigate delay-dependent stability conditions for the error dynamics (7.23). Let us define

$$G = \bar{G} + F_{21}R, \tag{7.25}$$

with $\bar{G}$ as a stable matrix with arbitrary negative eigenvalues.

**Theorem 7.3.3.** Given (7.5) and Relations (7.24) and (7.25), the dynamics associated with the auxiliary error vector $\bar{\epsilon}_2(t)$, described in (7.23b), is finite time stable.

**Proof.** Consider the following Lyapunov candidate function,

$$V_s(t) = \bar{\epsilon}_2^T(t) \bar{\epsilon}_2(t), \tag{7.26}$$

where $\bar{P} > 0$ is any symmetric matrix. Differentiating (7.26), taking into account (7.24) and substituting from (7.5) and (7.25), one obtains

$$\dot{V}_s(t) = 2\bar{\epsilon}_2^T(t) \bar{P} \left( \bar{G}\bar{\epsilon}_2(t) + \eta(t, \bar{\epsilon}(t), x(t)) - \alpha(t) sign \left( \bar{P}\bar{\epsilon}_2(t) \right) \right) \leq 2 \parallel \bar{P} \parallel_2 \bar{\epsilon}_2^T(t) \bar{G}\bar{\epsilon}_2(t) \leq -2\bar{\alpha}V_s(t) \tag{7.27}$$
where \( \bar{\alpha} \) is a positive scalar that satisfies

\[
\| \tilde{P} \|_2 \bar{G} \leq -\bar{\alpha} \tilde{P}.
\]

Accordingly, \( V_s(t) \) is exponentially stable and this results in the ideal sliding of \( \bar{\epsilon}_2(t) \) to the origin [186]. The theorem is now verified. \hfill \Box

Now, sufficient conditions for the stability of (7.23a) are established in the following theorem.

**Theorem 7.3.4.** Consider the error dynamics (7.23). Given the auxiliary delay parameters \( 0 \leq \bar{h}_1 \leq \bar{h}_2 \) and \( \bar{\mu} < 1 \), the system (7.23a) is globally asymptotically stable for any auxiliary delay satisfying (7.6), if there exist \( l \times l \) symmetric positive definite matrices \( P_{ii}, Q_i, i = \{1, 2, 3\} \), \( R_j, j = \{1, \ldots, 4\} \), and arbitrary \( l \times l \) real matrices \( P_{12}, P_{13}, P_{23}, \) and \( E_i, i = \{1, \ldots, 4\} \) that satisfy the following matrix inequality:

\[
\begin{bmatrix}
\Gamma_{1,1} & * & * & * & * & * & * \\
P_{12}^T - P_{13}^T + R_3 & \Gamma_{2,2} & * & * & * & * & * \\
-P_{12}^T & R_4 & -Q_2 - R_4 & * & * & * & * \\
P_{33} + E_3^T \bar{F}_{11} & P_{23}^T - P_{33} & -P_{23}^T & -R_1 & * & * & * \\
P_{23} + E_2^T \bar{F}_{11} & P_{22} - P_{23} & -P_{22} & 0 & -R_2 & * & * \\
N_{11}^T E_1 & 0 & 0 & N_{11}^T E_3 & N_{11}^T E_2 & -(1 - \bar{\mu})Q_3 & * \\
\Gamma_{7,1} & 0 & 0 & P_{13} - E_3 & P_{12} - E_2 & E_4^T \bar{N}_{11} & \Gamma_{7,7}
\end{bmatrix} < 0
\]

(7.28)

where \( \Gamma_{1,1} \triangleq \text{sym}(P_{13}) + Q_1 + \bar{h}_1^2 R_1 + \bar{h}_2^2 R_2 + \text{sym}(E_1^T \bar{F}_{11}), \Gamma_{2,2} \triangleq Q_2 + Q_3 - Q_1 - R_3 - R_4, \Gamma_{7,1} \triangleq E_4^T \bar{F}_{11} + P_{11} - E_1, \) and \( \Gamma_{7,7} \triangleq -\text{sym}(E_4) + \bar{h}_1^2 R_3 + \bar{h}_2^2 R_4. \)

**Proof.** Consider the following Lyapunov Krasovskii functional candidate equipped with
the descriptor transformation
\[ V(t) = \sum_{i=1}^{3} V_i(t) \]  

where

\[ V_1(t) = \Upsilon^T(t) E P \Upsilon(t) \], \quad \tag{7.30} \]
\[ V_2(t) = \int_{t-h_1}^{t} \tilde{e}_1^T(s) Q_1 \tilde{e}_1(s) ds + \int_{t-h_2}^{t} \tilde{e}_1^T(s) Q_2 \tilde{e}_1(s) ds + \int_{t-h(t)}^{t} \tilde{e}_1^T(s) Q_3 \tilde{e}_1(s) ds, \]  
\[ \tag{7.31} \]
\[ V_3(t) = h_1 \int_{t-s_1}^{t} \tilde{e}_1^T(s) R_1 \tilde{e}_1(s) ds + \int_{t-h_2}^{t} \tilde{e}_1^T(s) R_2 \tilde{e}_1(s) ds + \int_{t-h_3}^{t} \tilde{e}_1^T(s) R_3 \tilde{e}_1(s) ds + \int_{t-h_4}^{t} \tilde{e}_1^T(s) R_4 \tilde{e}_1(s) ds, \]  
\[ \tag{7.32} \]

\[ E = \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & 0 \\ * & P_{22} & P_{23} & 0 \\ * & * & P_{33} & 0 \\ E_1 & E_2 & E_3 & E_4 \end{bmatrix}, \quad \text{and} \]

\[ \Upsilon^T(t) = \left[ \tilde{e}_1(t), \int_{t-h_1}^{t} \tilde{e}_1^T(s) ds, \int_{t-h_2}^{t} \tilde{e}_1^T(s) ds, \int_{t-h_3}^{t} \tilde{e}_1^T(s) ds, \int_{t-h_4}^{t} \tilde{e}_1^T(s) ds, \tilde{e}_1(t) \right]. \]

Differentiating (7.29) along the solution of (7.23a), results

\[ \dot{V}_1(t) = 2 \begin{bmatrix} \tilde{e}_1(t) \\ \int_{t-h_1}^{t} \tilde{e}_1(s) ds \\ \int_{t-h_2}^{t} \tilde{e}_1(s) ds \\ \int_{t-h_3}^{t} \tilde{e}_1(s) ds \\ \int_{t-h_4}^{t} \tilde{e}_1(s) ds \\ \tilde{e}_1(t) \end{bmatrix}^T \begin{bmatrix} P_{11} & P_{12} & P_{13} & E_1^T \\ * & P_{22} & P_{23} & E_2^T \\ * & * & P_{33} & E_3^T \\ 0 & 0 & 0 & E_4^T \end{bmatrix} \]
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Hence, if the matrix inequality (7.28) is feasible (i.e., 

\[ \hat{V}(t) \leq \hat{\zeta}^T(t) \Gamma \hat{\zeta}(t), \]  

(7.37)

where \( \Gamma \) is the matrix on the left-hand-side of (7.28), and

\[ \hat{\zeta}(t) = \begin{bmatrix} \hat{\xi}_1(t) \\ \hat{\xi}_2(t) \end{bmatrix} \]

(7.38)

Hence, if the matrix inequality (7.28) is feasible (i.e., \( \Gamma < 0 \)), then according to the
Lyapunov Krasovskii theorem, the system (7.23a) is asymptotically stable in the large. This concludes the proof of the theorem.

7.3.3 Observer Design Algorithm

Summing up the results of the previous sections, to design the observer parameters, Equations (7.8) together with (7.22) should be simultaneously solved, such that the matrix inequality (7.28) is satisfied. Direct elaborations on (7.8b) and (7.8c), gives the following equations

\begin{align}
H_{11}C + F_{11}T_1 + F_{12}T_2 - T_1 A &= 0 \quad (7.38a) \\
H_{12}C + F_{21}T_1 + F_{22}T_2 - T_2 A &= 0 \quad (7.38b) \\
N_{11}T_1 + N_{12}T_2 + H_{21}C - T_1 A_d &= 0 \quad (7.38c) \\
N_{21}T_1 + N_{22}T_2 + H_{22}C - T_2 A_d &= 0 \quad (7.38d)
\end{align}

In addition, let us rewrite Equation (7.8a) as

\[
\begin{bmatrix}
T & V
\end{bmatrix} \bar{M} = \bar{L}. \quad (7.39)
\]

where \( \bar{M} \triangleq \begin{bmatrix} I_n & C \end{bmatrix} \). It can always be obtained from (7.39) that

\[
T = \bar{L}W_1 + Z\bar{W}_1, \quad (7.40)
\]

\[
V = \bar{L}W_2 + Z\bar{W}_2, \quad (7.41)
\]

where \( W_i \) and \( \bar{W}_i, i = \{1, 2\} \) are defined with appropriate dimensions as
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\[
\begin{bmatrix}
W_1 & W_2
\end{bmatrix} \triangleq \tilde{M}^\dagger \quad \text{and} \quad \begin{bmatrix}
\tilde{W}_1 & \tilde{W}_2
\end{bmatrix} \triangleq I_{n+p} - \tilde{M}\tilde{M}^\dagger. \quad \text{Moreover, } Z^T = [Z_1^T, Z_2^T] \text{ is an arbitrary design matrix, where } Z_1 \in \mathbb{R}^{l \times (n+p)}, \text{ and } Z_2 \in \mathbb{R}^{p \times (n+p)}. \quad \text{Now, post-multiplying equations (7.38) by } [C^\dagger, C^\perp] \text{ results in the following equations}
\]

\[
H_{11} = T_1 AC^\dagger - F_{12} T_2 C^\dagger - F_{11} T_1 C^\dagger \quad (7.42)
\]

\[
H_{12} = T_2 AC^\dagger - F_{22} T_2 C^\dagger - F_{21} T_1 C^\dagger \quad (7.43)
\]

\[
H_{21} = T_1 A_d C^\dagger - N_{12} T_2 C^\dagger - N_{11} T_1 C^\dagger \quad (7.44)
\]

\[
H_{22} = T_2 A_d C^\dagger - N_{22} T_2 C^\dagger - N_{21} T_1 C^\dagger \quad (7.45)
\]

Moreover, as it is shown in Section 4.3.1, \( \tilde{W}_1 C^\perp = 0 \) is always true. Hence, after substituting from (7.40), the remaining equations obtained from splitting the sub-equations of (7.38) can be written as

\[
F_{11} LW_1 C^\perp + F_{12} CW_1 C^\perp - LW_1 AC^\perp - Z_1 \tilde{W}_1 AC^\perp = 0 \quad (7.46)
\]

\[
F_{21} LW_1 C^\perp + F_{22} CW_1 C^\perp - CW_1 AC^\perp - Z_2 \tilde{W}_1 AC^\perp = 0 \quad (7.47)
\]

\[
N_{11} LW_1 C^\perp + N_{12} CW_1 C^\perp - LW_1 A_d C^\perp - Z_1 \tilde{W}_1 A_d C^\perp = 0 \quad (7.48)
\]

\[
N_{21} LW_1 C^\perp + N_{22} CW_1 C^\perp - CW_1 A_d C^\perp - Z_2 \tilde{W}_1 A_d C^\perp = 0 \quad (7.49)
\]

Now, augmenting the unknown matrices of (7.46)-(7.49) yields

\[
\begin{bmatrix}
F_{11} & F_{12} & N_{11} & N_{12} & Z_1
\end{bmatrix} \Phi = \Psi_1, \quad (7.50)
\]

\[
\begin{bmatrix}
F_{21} & F_{22} & N_{21} & N_{22} & Z_2
\end{bmatrix} \Phi = \Psi_2, \quad (7.51)
\]

where
\[ \Phi \triangleq \begin{bmatrix}
LW_1 C^\perp & 0 \\
CW_1 C^\perp & 0 \\
0 & LW_1 C^\perp \\
0 & CW_1 C^\perp \\
-W_1 AC^\perp & -W_1 A_d C^\perp
\end{bmatrix}, \]

\[ \Psi_1 \triangleq \begin{bmatrix}
LW_1 AC^\perp & LW_1 A_d C^\perp
\end{bmatrix}, \text{ and } \Psi_2 \triangleq \begin{bmatrix}
CW_1 AC^\perp & CW_1 A_d C^\perp
\end{bmatrix}. \]

In addition, it can be deduced from (7.48), (7.49), and (7.40) that the constraint equation (7.22) is equivalent to

\[ \begin{bmatrix}
Z_1 RZ_2 & \bar{W}_1 A_d
\end{bmatrix} = -(L + RC)W_1 A_d \quad (7.52) \]

Henceforth, it is shown that equation (7.51) always has a solution.

**Proposition 7.3.1.** For any distribution matrices \( A, A_d, C, \) and \( L, \) Equation (7.51) has always at least one solution.

**Proof.** It is well-known that (7.51) has at least one solution if and only if the following rank condition is satisfied

\[ \rho \left( \begin{bmatrix}
\Phi \\
\Psi_2
\end{bmatrix} \right) = \rho(\Phi) \quad (7.53) \]

However, it can be obtained from the definitions of \( W_1 \) and \( \bar{W}_1 \) that

\[ W_1 = (I_n + C^T C)^{-1}, \quad (7.54) \]
and
\[
\bar{W}_1 = \begin{bmatrix}
I_n - W_1 \\
-CW_1
\end{bmatrix}.
\] (7.55)

Therefore, \( CW_1 \in \mathcal{R}(\bar{W}_1) \), and thus \( \Psi_2 \in \mathcal{R}\left( \begin{bmatrix} \bar{W}_1 A C^\perp & \bar{W}_1 A_d C^\perp \end{bmatrix} \right) \). Hence, \( \Psi_2 \) is in the space spanned by the row space of \( \Phi \), and this concludes the proposition. \( \square \)

The solution of (7.51) can thus be expressed as
\[
\begin{bmatrix}
F_{21} & F_{22} & N_{21} & N_{22} & Z_2
\end{bmatrix} = M_2 + J_2 \Theta
\] (7.56)
where \( M_2 \triangleq \Psi_2 \Phi^\dagger \), \( \Theta \triangleq (I_x - \Phi \Phi^\dagger) \), \( x = 2(l + p) + n + p \), and \( J_2 \in \mathbb{R}^{p \times x} \) is an arbitrary matrix to be defined.

In a similar manner (7.50) has a solution if and only if the following rank condition is satisfied:
\[
\rho \left( \begin{bmatrix}
\Phi \\
\Psi_1
\end{bmatrix} \right) = \rho(\Phi)
\] (7.57)

Moreover, after some algebraic manipulations, as in Corollary 6.3.1 in the previous chapter, condition (7.57) is found to be equivalent to the following rank criterion
### Condition I

\[
\rho \left( \begin{bmatrix}
LW_1 A & LW_1 A_d \\
LW_1 & 0 \\
CW_1 & 0 \\
0 & LW_1 \\
0 & CW_1 \\
-\bar{W}_1 A & -\bar{W}_1 A_d \\
C & 0 \\
0 & C
\end{bmatrix} \right) = \rho \left( \begin{bmatrix}
LW_1 & 0 \\
CW_1 & 0 \\
0 & LW_1 \\
0 & CW_1 \\
-\bar{W}_1 A & -\bar{W}_1 A_d \\
C & 0 \\
0 & C
\end{bmatrix} \right)
\]

(7.58)

Thereafter, assuming that Condition I is satisfied, we have

\[
\begin{bmatrix}
F_{11} & F_{12} & N_{11} & N_{12} & Z_1
\end{bmatrix} = M_1 + J_1 \Theta
\]

(7.59)

where \(M_1 \triangleq \Psi_1 \Phi^\dagger\) and \(J_1 \in \mathbb{R}^{l \times x}\) is an arbitrary matrix as a design degree of freedom.

Analogously, the following necessary condition is concluded from (7.52)

### Condition II

\[
\rho \left( \begin{bmatrix}
\bar{W}_1 A_d \\
(L + RC)W_1 A_d
\end{bmatrix} \right) = \rho(\bar{W}_1 A_d)
\]

(7.60)

**Proposition 7.3.2.** For any distribution matrices \(A_d\), \(C\), and \(L\), provided that the condition \(\rho([L^T, C^T]) = l + p\) is fulfilled, the rank condition (7.60) is equivalent to the following matching condition

\[
\rho \left( \begin{bmatrix}
L A_d \\
C A_d
\end{bmatrix} \right) = \rho(C A_d)
\]

(7.61)

**Proof.** Referring back to (7.54) and (7.55), and since the matrix \(W_1\) is of full row rank, the left and right hand-sides of (7.60) can be written as
\[ \rho(W_1A_d) = \rho \left( \begin{bmatrix} C^T C \\ -C \end{bmatrix} W_1A_d \right) \]
\[ = \rho(CA_d) \] (7.62)

\[ \rho \left( \begin{bmatrix} W_1A_d \\ (L + RC)W_1A_d \end{bmatrix} \right) = \rho \left( \begin{bmatrix} W_1A_d \\ LW_1A_d \end{bmatrix} \right) \]
\[ = \rho \left( \begin{bmatrix} CA_d \\ LA_d \end{bmatrix} \right) \] (7.63)

Hence, the statement of the proposition holds. \( \square \)

**Remark 7.3.2.** If \( L \) is not in the orthogonal space of the output distribution matrix \( C \), then the matrix \( R \) can be designed in a way to minimize the rank of \( (L + RC)W_1A_d \). It indicates that the matrix \( R \) plays the role of a design degree of freedom that in general can relax the restrictive matching condition (7.61) to a less conservative condition.

After appropriately partitioning Matrices \( M_i \) and \( \Theta, i = \{1, 2\} \), based on the dimensions of the unknown observer parameters in (7.56) and (7.59), as \( M_i = [M_{i,1}, M_{i,2}, \ldots, M_{i,5}] \) and \( \Theta = [\Theta_1, \Theta_2, \ldots, \Theta_5] \), we have

\[ F_{ij} = M_{i,j} + J_i \Theta_j \quad i, j = \{1, 2\}, \] (7.64)

\[ N_{ij} = M_{i,j+2} + J_i \Theta_{j+2} \quad i, j = \{1, 2\}, \] (7.65)

\[ Z_1 = M_{1,5} + J_1 \Theta_5, \] (7.66)

\[ Z_2 = M_{2,5} + J_2 \Theta_5. \] (7.67)

Now, substituting from (7.66) and (7.67) into the matching condition (7.52) results
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\[(J_1 + RJ_2)\bar{\Theta}_5 = U, \quad (7.68)\]

where \(\bar{\Theta}_5 \triangleq \bar{\Theta}_5 \bar{W}_1 A_d \in \mathbb{R}^{x \times (n+p)}\) and \(U \triangleq -(L + RC)W_1 A_d - (M_{15} + RM_{25})\bar{W}_1 A_d\) can be simply calculated. Now assuming that Condition II holds, it is obtained from (7.68) that

\[J_1 + RJ_2 = U_1 + \bar{J}\bar{U}, \quad (7.69)\]

where \(U_1 \triangleq U\bar{\Theta}_5^\dagger\) and \(\bar{U} \triangleq (I_x - \bar{\Theta}_5 \bar{\Theta}_5^\dagger)\), and \(\bar{J} \in \mathbb{R}^{l \times x}\) is an arbitrary matrix to be designed.

Thereafter, taking the relations (7.64), (7.65), and (7.69) into account, the parameters \(\bar{F}_{11}\) and \(\bar{N}_{11}\) can be calculated from

\[\bar{F}_{11} = \bar{U}_f + \bar{J}\bar{U}\Theta_1, \quad (7.70)\]
\[\bar{N}_{11} = \bar{U}_n + \bar{J}\bar{U}\Theta_3, \quad (7.71)\]

where \(\bar{U}_f \triangleq M_{11} + RM_{21} + U_1 \Theta_1\) and \(\bar{U}_n \triangleq M_{13} + RM_{23} + U_1 \Theta_3\).

Upon the satisfaction of the conditions of Theorem 7.3.4, it is clear from Theorems 7.3.3 and 7.3.4 that the system (7.23) is globally asymptotically stable. Nevertheless, the matrix inequality (7.28) is nonlinear, due to the existence of nonlinear terms \(E_i^T \bar{F}_{11}\) and \(E_i^T \bar{N}_{11}\), \(i = \{1, \cdots 4\}\). This point stays against using the promising convex optimization methods in solving the inequality (7.28) to attain appropriate unknown observer parameters \(\bar{F}_{11}\) and \(\bar{N}_{11}\). This problem is resolved in the following theorem, which is the main result of the chapter.

**Theorem 7.3.5.** Upon the fulfillment of Conditions I and II, there exist an asymptotically stable functional observer (7.4) for the system (7.1), if for given parameters \(0 \leq \bar{h}_1 \leq \bar{h}_2\) and \(\bar{\mu}\), there exist positive definite matrices \(P_{ii} \in \mathbb{S}^{l \times l}, Q_i \in \mathbb{S}^{l \times l}, i = \{1, 2, 3\}, R_j \in \mathbb{S}^{l \times l}\),
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\( j = \{1, \cdots, 4\} \), scalars \( \gamma_i \), \( i = \{1, 2, 3\} \), and arbitrary real matrices \( P_{12} \in \mathbb{R}^{l \times l} \), \( P_{13} \in \mathbb{R}^{l \times l} \), \( P_{23} \in \mathbb{R}^{l \times l} \), \( E_4 \in \mathbb{R}^{l \times l} \), and \( \Lambda \in \mathbb{R}^{l \times x} \), in a way that the following LMI is feasible:

\[
\begin{bmatrix}
\Pi_{1,1} & * & * & * & * & * & * & * \\
\Pi_{2,1} & \Pi_{2,2} & * & * & * & * & * & * \\
-P_{12}^T & R_4 & -Q_2 - R_4 & * & * & * & * & * \\
P_{23} - P_{33} & P_{23}^T & -P_{23}^T & -R_1 & * & * & * & * \\
P_{22} - P_{23} & 0 & -R_2 & * & * & * & * & * \\
0 & 0 & \Pi_{6,4} & \Pi_{6,5} & -(1 - \bar{\mu})Q_3 & * \\
0 & 0 & P_{13} - \gamma_3 E_4 & P_{12} - \gamma_2 E_4 & \Pi_{7,6} & \Pi_{7,7}
\end{bmatrix} < 0,
\]

(7.72)

where \( \Pi_{1,1} = \text{sym}(P_{13}) + Q_1 + \bar{h}_2^2 R_1 + \bar{h}_2^2 R_1 + \gamma_1 \text{sym}(E_4^T U_f) + \gamma_1 \text{sym}(\Lambda U \Theta_1) \),
\( \Pi_{2,1} = -P_{12}^T - P_{13}^T + R_3 \), \( \Pi_{2,2} = Q_2 + Q_3 - Q_1 - R_3 - R_4 \), \( \Pi_{4,1} = P_{33} + \gamma_3 E_4^T U_f + \gamma_3 \text{sym}(\Lambda U \Theta_1) \), 
\( \Pi_{5,1} = P_{23} + \gamma_2 E_4^T U_f + \gamma_2 \text{sym}(E_4^T U_f) + \gamma_2 \text{sym}(\Lambda U \Theta_1) \), \( \Pi_{6,1} = \gamma_1 U_1^n E_4 + \gamma_1 \Theta_3^T U^T \Lambda^T \), \( \Pi_{6,4} = \gamma_3 U_1^T E_4 + \gamma_3 \Theta_3^T U^T \Lambda^T \), \( \Pi_{6,5} = \gamma_2 U_1^n E_4 + \gamma_2 \Theta_3^T U^T \Lambda^T \), \( \Pi_{7,1} = E_4^T U_f + \Lambda U \Theta_1 + P_{11} - \gamma_1 E_4 \), 
\( \Pi_{7,6} = E_4^T U_n + \Lambda U \Theta_3 \), and \( \Pi_{7,7} = -\text{sym}(E_4) + \bar{h}_1^2 R_3 + \bar{h}_2^2 R_4 \).

In addition, we have

\[
\bar{J} = \Lambda E_4^{-T}
\]

(7.73)

**Proof.** The theorem is a direct consequence of Theorems 7.3.1-7.3.4. Only the operations performed to obtain (7.72) are summarized here. First, \( F_{11} \) and \( N_{11} \) are substituted from (7.70) and (7.71) into (7.28), respectively. Next, it is assumed that \( E_i = \gamma_i E_4, i = \{1, 2, 3\} \), and the variable \( \Lambda \triangleq E_4^T J \) is defined. This transforms the nonlinear matrix inequality into Inequality (7.72). Furthermore, the investigation of \( \Pi_{7,7} \) shows that for the feasibility of LMI (7.72), the term \( E_4 \) must be nonsingular. Hence, Relation (7.73) always follows from the definition of \( \Lambda \). \( \square \)
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Inequality (7.72) is still nonlinear due to the terms $\gamma_i E_4$. However, pre-assigning the scalar variables $\gamma_i$, $i = \{1, 2, 3\}$ resolves this issue. To find apt parameters $\gamma_i$, the genetic algorithm (GA) based tuning scheme proposed in Section 6.3.3 is applied.

**Remark 7.3.3.** If the tuning algorithm was unsuccessful in finding a suitable chromosome, we can always modify the auxiliary delay parameters $h_1$, $h_2$, and $\bar{\mu} < 1$, for example by reducing $\bar{h}_2$ and/or $\bar{\mu}$, in a way to resolve this issue. It is of value to remind that the auxiliary delay function can also be fixed at the origin, i.e., $\hat{h}(t) \equiv 0$.

**Theorem 7.3.6.** Consider the system (7.1) and the observer (7.4) with $\hat{h}(t) \equiv 0$. Upon the achievement of Conditions I and II, functional observer (7.4) is globally asymptotically stable if and only if the following condition is satisfied,

**Condition III**

$$\rho \left( \begin{bmatrix} sI_l - \bar{U}_f - \bar{U}_n \\ \bar{U}(\Theta_1 + \Theta_3) \end{bmatrix} \right) = l, \quad \forall s \in \mathbb{C}_+ \quad (7.74)$$

**Proof.** If we set $\hat{h}(t) \equiv 0$, then from (7.70) and (7.71) the system (7.23a) is asymptotically stable, if and only if the pair $(\bar{U}_f + \bar{U}_n, \bar{U}(\Theta_1 + \Theta_3))$ is detectable. The latter is equivalent to the rank condition (7.74). In addition, the observable eigenvalues of $\bar{F}_{11} + \bar{N}_{11}$ can be arbitrarily assigned through designing an appropriate matrix $\bar{J}$. The rest of the proof is straightforward, and is thus omitted.

**Remark 7.3.4.** The major justification on the superiority of the delay-dependent observer structure over the delay-free structure with $\hat{h}(t) \equiv 0$ can be given as follows:

- the disturbance term $\eta(t, \bar{\epsilon}(t), x(t))$, defined in (7.19), can be made smaller relying on our information on the real-time delay $h(t)$. This results in a smaller sliding gain $\alpha(t)$. Accordingly, the performance of the sliding mode FO can be improved,
particularly under conditions that due to relatively large sampling periods or noisy output measurements the ideal sliding does not occur.

- it is well-known that adding a delay into an unstable ordinary system sometimes can stabilize it (see e.g. [9, 52]). This point can also be utilized in designing less conservative FOs for ordinary linear systems without delay. In other words, in the FO design problem for an ordinary LTI system (i.e. system (7.1) with $A_d \equiv 0$), adding auxiliary delayed terms of the form $N\omega(t-\hat{h}(t)) + H_2 y(t-\hat{h}(t)) - [X^T_2, N^T_{212}] e_2(t-\hat{h}(t))$ to the sliding mode functional observer structure can lessen the observability requirements under special situations. Studying these conditions is a topic for further research.

Table 7.1 is given to sum up the whole observer design procedure.

**Table 7.1: The summary of the observer design algorithm**

<table>
<thead>
<tr>
<th>Step</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>i.</td>
<td>Evaluate Conditions I and II. If both of them are satisfied, proceed to the next step. Otherwise, the algorithm terminates without a solution.</td>
</tr>
<tr>
<td>ii.</td>
<td>Based on the knowledge on the real delay parameters, select appropriate values for $\bar{h}_1$, $\bar{h}_2$, and $\bar{\mu}$.</td>
</tr>
<tr>
<td>iii</td>
<td>Based on Remark 7.3.2, assign an appropriate value for the matrix $R$. In addition, fix $J_2$ at an arbitrary value, and calculate $F_{21}$ and $F_{22}$ from (7.64); $N_{21}$ and $N_{22}$ from (7.65); and $Z_2$ from (7.67).</td>
</tr>
<tr>
<td>iv.</td>
<td>Employing the tuning algorithm, solve the LMI (7.72) for the parameter $\bar{J}$. Under the failing situation, modify $\bar{h}_2$ as the maximum allowable upper-bound of delay for given $\bar{h}_1$ and $\bar{\mu}$. The other auxiliary delay parameters $\bar{h}_1$ and $\bar{\mu}$ can also be modified in a similar way. If the procedure given in this step was successful, jump to Step vi. Otherwise, proceed to the next step.</td>
</tr>
<tr>
<td>v.</td>
<td>Examine Condition III and upon its fulfilment, calculate $\bar{J}$ as indicated in Theorem 7.3.6. Otherwise, the algorithm fails and stops accordingly.</td>
</tr>
<tr>
<td>vi.</td>
<td>Calculate $J_1$ from (7.68); $F_{11}$ and $F_{12}$ from (7.64); $N_{11}$ and $N_{12}$ from (7.65); and $Z_1$ from (7.66). Moreover, the parameters $T$, $V$, $H_{11}$, $H_{12}$, $H_{21}$, $H_{22}$, $G_u$, $G$, $X_1$, and $X_2$ can be obtained from (7.40), (7.41), (7.42), (7.43), (7.44), (7.45), (7.8d), (7.25), (7.20), and (7.21), respectively.</td>
</tr>
</tbody>
</table>
Remark 7.3.5. In a particular situation, when \( L = I_n \), full-order sliding-mode observer for unknown time-delay systems is enclosed. This problem has been previously studied in papers like [181, 182]. However, the observer structure of those contributions is different from the structure proposed in this chapter. In addition, the advantages of the proposed observer compared with those works are threefold:

- in [181, 182] the matching condition is expressed as
  \[
  \rho \left( \begin{bmatrix} A_d & \cdot \\ CA_d & \cdot \end{bmatrix} \right) = \rho(CA_d). \tag{7.75}
  \]

  However, due to the existence of a degree of freedom created by the matrix \( R \), (7.75) is apparently more conservative than the associated matching condition of this chapter, i.e., Condition II (see Remark 7.3.2).

- it is further assumed as a structural constraint in [181, 182] that the invariant zeros of \((A, A_d, C)\) must lie in the negative complex plane. However, this assumption is relaxed in this chapter.

- the observer design frameworks in [181, 182] depends on the real-time delay values \( h(t) \), whereas in this chapter it depends on the auxiliary delay \( \hat{h}(t) \), which is clearly profitable.

Remark 7.3.6. The proposed observer design algorithm can be extended to LTI systems with mixed unknown time-varying state and input delays, and/or unknown disturbances. The investigation of this problem is a subject of our future studies.
7.4 Numerical Examples

7.4.1 Example 1, Non-Observable System

Consider the system (7.1) with the following distribution matrices:

\[
A = \begin{bmatrix}
0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0.1 \\
2 & 3 & -1 & 0 \\
2 & -1 & 0 & -1
\end{bmatrix},
\]

\[
A_d = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0.1 & 0.21 & 0.2 & 0.1
\end{bmatrix},
\]

\[
B = \begin{bmatrix} 0 & -1 & 2 & 0 \end{bmatrix}^T,
\]

\[
C = \begin{bmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix},
\]

and \(L = \begin{bmatrix} 0 & 0 & 2 & 0 \end{bmatrix}\). The delay parameters are \(h_1 = 2s\), \(h_2 = 9s\), and \(\mu = 2\). Investigating the matching condition (7.75), indicates that this condition is violated, and thus there is no full-order observer for the system. However, Condition II (as the matching condition of this chapter) and Condition I are satisfied. Thus, Steps (i)-(iii) of the observer design framework (given in Table 7.1) can be passed. This shows the superiority of functional observers over full-order Luenberger observers in lessening the observability requirements to less demanding conditions.

Assuming the uncertainty of the real delay parameters, the auxiliary delay parameters were arbitrarily chosen as \(\bar{h}_1 = 1s\), \(\bar{h}_2 = 5s\), and \(\bar{\mu} = 0.5\). Now, setting \(\bar{\gamma} = [1, 1, 1]\) it is found that the LMI (7.72) it not feasible. Hence, applying the tuning algorithm, which is based on the genetic algorithm, gave an appropriate chromosome \(\bar{\gamma} = \begin{bmatrix} 30 & 0 & -10 \end{bmatrix}\). Therefore, applying Step (iv) of the design algorithm the LMI parameters were attained, where some of them are reported as follows: \(Q_1 = 241.8529\), \(Q_2 = 92.2120\), \(Q_3 = 92.1267\), \(R_1 = 194.0026\), \(R_2 = 27.0737\), \(R_3 = 5.6776\), and \(R_4 = 0.5453\). Thus, Step (vi) of the algorithm resulted in the following observer parameters:
7.4 Numerical Examples

F = \[
\begin{bmatrix}
-1.3468 & -0.5598 & -0.0523 \\
-0.1459 & 0.5598 & 0.0523 \\
-0.0811 & 0.2088 & 0.3069
\end{bmatrix},
\]
N = \[
\begin{bmatrix}
0 & -0.8781 & -0.4812 \\
0 & 0.8781 & 0.4812 \\
0 & 0.9494 & 0.4761
\end{bmatrix},
\]
\[
V = \begin{bmatrix} 0.7082 \\ 0.1621 \\ -2.0671 \end{bmatrix},
H_1 = \begin{bmatrix} 5.1059 \\ -0.0707 \\ -0.0545 \end{bmatrix},
H_2 = \begin{bmatrix} -0.3343 \\ -0.3343 \\ -0.3542 \end{bmatrix},
\]
\[
G = \begin{bmatrix} -3.1459 \\ -0.0811 \end{bmatrix},
G_u = \begin{bmatrix} 4 & 0 & 0 \end{bmatrix}^T,
X_1 = \begin{bmatrix} -0.8781 \\ -0.4812 \end{bmatrix},
\]
\[
X_2 = \begin{bmatrix} -0.2366 \\ -0.2339 \end{bmatrix}.
\]

In addition, a simple examination shows that Condition III is also fulfilled. Hence, Step (v) of the observer design algorithm can also be applied according to Theorem 7.3.6, as an alternative approach. As a result, the sequel observer parameters were attained, in order to place the eigenvalue of the delay-free error dynamics \( \dot{\epsilon}_1(t) = (\bar{F}_{11} + \bar{N}_{11})\epsilon_1(t) \) at the arbitrary desired location \( \lambda = -4 \):

\[
F = \begin{bmatrix}
-3.9951 & -0.7634 & -0.2339 \\
-0.0049 & 0.7634 & 0.2339 \\
0.0482 & 0.5218 & 0.4828
\end{bmatrix},
N = \begin{bmatrix}
0 & -0.5239 & -0.0073 \\
0 & 0.5239 & 0.0073 \\
0 & 0.3987 & 0.6752
\end{bmatrix},
\]
\[
V = \begin{bmatrix} 0.9901 \\ 0.9901 \\ 0.0965 \end{bmatrix},
H_1 = \begin{bmatrix} -5.9901 \\ 27.9159 \\ 0.0446 \end{bmatrix},
H_2 = \begin{bmatrix} -0.0045 \\ -0.0045 \\ 0.0612 \end{bmatrix},
\]
\[
G = \begin{bmatrix} -3.0049 \\ 0.0482 \\ -0.0049 \end{bmatrix},
G_u = \begin{bmatrix} 4 & 0 & 0 \end{bmatrix}^T,
X_1 = \begin{bmatrix} -0.5239 \\ -0.0073 \end{bmatrix},
\]
\[
X_2 = \begin{bmatrix} 0.2366 \\ -0.2339 \end{bmatrix}.
\]

Based on the designed parameters, simulations were performed in the Simulink environment of MATLAB, arbitrarily assuming \( \phi(\theta) = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^T, \forall \theta \in [-9, 0] \), as the initial condition of the system. In addition, the sliding mode parameters were fixed at...
\( \alpha(t) = 2 \) and \( \tilde{P} = I_2 \). The results are demonstrated in Fig. 7.1, where \( e_i(t) \) and \( \tilde{z}_i(t) \), \( i = \{1, 2, 3\} \), are the \( i \)th elements of \( e(t) \) and \( \tilde{z}(t) \), respectively. It is clear from the figure that the output estimation error converges to zero extremely fast (less than \( t = 0.15s \)), which is due to the sliding mode convergence of \( \tilde{e}_2(t) \). Moreover, the desired function asymptotically converges to the origin, in less than \( t = 4s \), which also exhibits a good performance despite the uncertainty of the real-time delay.

Simulating the delay-free observer, without changing the common parameters shows even a better performance in estimating the desired function \( z(t) = \tilde{z}_1(t) \), \( t \geq 0 \). This point, which is highlighted in Fig. 7.2, is due to our degree of freedom in assigning the eigenvalue location of the auxiliary error \( \tilde{e}_1(t) \) in the delay-free observer structure. This is literally an advantage of the delay-free observer over the delay-dependent observer that should be emphasised.

### 7.4.2 Example 2, Delay-free Observer Not Working

Here, we have considered the following parameters for the system:

\[
A = \begin{bmatrix}
-10 & 1 & 0 & 0 \\
-41 & -2 & 4 & 0 \\
0 & 0 & -2 & 1 \\
2 & 0 & -20 & -6
\end{bmatrix}, \quad A_d = \begin{bmatrix}
2 & 0 & 0 & -5 \\
0 & 0 & 0 & 9 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
1.5 & 0 & 0 & 0 \\
0 & -2 & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 2 & -1 & 0.2
\end{bmatrix}^T, \quad \text{and } L = \begin{bmatrix}
0_2 & I_2
\end{bmatrix}.
\]

Moreover, the delay parameters are \( h_1 = 0.1s \), \( h_2 = 3s \), and \( \mu = 1.2 \).

A simple evaluation confirms that both Conditions I and II are satisfied. Hence, we can go through Steps (i)-(iii) of the observer design algorithm. Let us assume that we only
know the upper-bound of delay \( h_2 \). Hence, we set \( \bar{h}_2 = h_2 = 3s, \) and \( \bar{h}_1 = 0s \) and \( \bar{\mu} = 0.4 \) were arbitrarily assigned. Nevertheless, fixing \( \bar{\gamma} \) at \([1, 1, 1]\), it is found that the LMI (7.72) is infeasible. Accordingly, the tuning algorithm was employed to obtain \( \bar{\gamma} = \begin{bmatrix} 25 & 4 & 5 \end{bmatrix} \) as a suitable set, which after applying Step (vi) of the algorithm summarized Table 7.1, resulted in the following observer parameters:
Figure 7.2: A comparison between the delay-dependent observer and the delay-free observer in estimating the desired function in Example 1

\[ F = \begin{bmatrix} -2.6497 & 1 & -0.0708 & -0.4450 \\ -20.7052 & -6 & -0.6865 & -0.1848 \\ 0.6497 & 0 & 0.0708 & 0.4450 \\ 0.7052 & 0 & 0.6865 & 0.1848 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & -0.0325 & -0.0420 & -0.6113 \\ 0 & -0.6191 & -0.8463 & -0.5057 \\ 0 & 0.0325 & 0.0420 & 0.6113 \\ 0 & 0.6191 & 0.8463 & 0.5057 \end{bmatrix}, \]

\[ V = \begin{bmatrix} 0.1906 & -0.0812 \\ 0.1290 & -0.0881 \\ 0.8094 & 0.0812 \\ -0.1290 & 1.0881 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 6.0406 & 0.0626 \\ 2.8765 & 2.0589 \\ -6.2927 & 0.0117 \\ -6.1288 & 0.0943 \end{bmatrix}, \quad H_2 = \begin{bmatrix} -0.2985 & -0.0544 \\ -0.1113 & -0.0587 \\ 0.2985 & 0.0544 \\ 0.1113 & 0.0587 \end{bmatrix}, \]

\[ G = \begin{bmatrix} -2.3503 & 0 \\ 0.7052 & -3 \end{bmatrix}, \quad G_u = \begin{bmatrix} -1.6107 & -0.3461 & 0.6107 & 0.5461 \end{bmatrix}. \]
7.4 Numerical Examples

\[ X_1 = \begin{bmatrix} -0.0420 & -0.6113 \\ -0.8463 & -0.5057 \end{bmatrix}, \text{ and } X_2 = \begin{bmatrix} -1.0708 & -1.4450 \\ 19.3135 & 2.8152 \end{bmatrix}. \]

On the other hand, as can be testified, Condition III is \textit{not} satisfied. In consequence, a delay-free FO with structure (7.4) cannot be designed for this system. This example explicitly highlights the second comment of Remark 7.3.4 on the advantages of the delay-dependent over the delay-free observer.

Figure 7.3: The estimation of the desired functions in Example 2 
\((\alpha(t) = 0.1)\)
Simulations were performed in the Simulink environment, assuming

\[
\phi(\theta) = [-2, 0.01, 0, 10]^T, \quad \theta \in [-3, 0], \quad \bar{P} = I_2,
\]

and two scenarios for the sliding gain: 1- \( \alpha(t) = 0.1 \) and 2- \( \alpha(t) = 5 \). The results are depicted in Figs. 7.3 and 7.4, which follow similar notations as the figures in Example 1. It is clear from Fig. 7.3 that \( \alpha(t) = 0.1 \) is not big enough for the finite-time convergence of

Figure 7.4: The estimation of the desired functions in Example 2

(\( \alpha(t) = 5 \))
the outputs estimation errors. This has itself created errors in the estimation of the desired functions $z(t)$. On the other hand, as Fig. 7.4 exhibits, increasing the sliding gain results in the finite-time convergence of the outputs errors and the asymptotic convergence of the desired functions.

### 7.5 Conclusions

A novel practical delay-dependent sliding mode FO design algorithm for LTI systems with unknown time-varying state delays has been proposed for the first time. The real time-delay has been assumed to be interval time-varying with either bounded or unbounded delay derivative. The new observer structure depends on an auxiliary delay parameter, which its difference from the real delay values, can result in better ideal sliding performance of the output filter. Sufficient conditions of the stability of the FO have been derived using the Lyapunov Krasovskii methodology, and are expressed in terms of an LMI and two structural rank conditions. A number of the LMI weighting parameters have been updated using a genetic algorithm-based tuning scheme to avoid the nonlinearity of the LMI, as well as increasing the conservativeness of the stability criteria. Furthermore, the necessary and sufficient conditions of the stability of the observer, under zero auxiliary delay values (delay-free observer), have been obtained. Numerical examples and simulation results illustrated the design procedure, and confirmed the effectiveness of the proposed theoretical findings.
Chapter 8

Conclusions and Future Work

8.1 Summary

This dissertation has explored the important problem of functional observer design for time-delay systems. Firstly, a novel less restrictive stability criterion has been proposed for interval time-varying delay systems, employing Wirtinger-based integral inequalities and other advanced techniques. Secondly, using the concept of functional observability, an algorithm has been proposed to systematically increase the order of a functional observer in the minimum possible number to design an asymptotic FO for the system. In addition, the transformation-based and direct approaches for designing functional observers have been revisited and improved.

Thirdly, an algorithm has been developed for designing exponentially stable FOs for LTI systems with multiple mixed time-varying state and input delays. The state delays have been assumed to be known, bounded, with small or unknown derivatives, while the input delays have been considered as unknown (or arbitrary) values. Fourthly, the problem of functional observer design for interval time-varying delay systems under the assumption
of bounded delay-derivatives not limited to be less than one have been addressed. A new LKF and advanced techniques have been used to obtain less restrictive observer parameters to this aim. Finally, the crucial problem of FO design for LTI systems with unknown time-varying state delays has been investigated. A novel sliding-mode functional observer has been proposed to this aim that employs an auxiliary delay in its structure. The observer parameters have been calculated from an auxiliary delay-dependent stability criterion, obtained from the Lyapunov Krasovskii approach.

8.2 Conclusions

Henceforth, the key conclusions presented by each chapter are as follows.

1. In Chapter 3:

   I. A novel Wirtinger-based double integral inequality has been derived, and the analytical amount of improvement to the corresponding Jensen’s inequality brought by using the new inequality, has been calculated.

   II. New criteria have been established using the Lyapunov Krasovskii approach for the stability analysis of LTI systems with interval time-varying delay. It is shown that the proposed criteria can improve the stability region with respect to the recent state-of-the art existing criteria.

   III. In constructing the novel LMI stability condition, a new augmented LKF including triple-integral terms has been employed. In addition, Wirtinger-based integral inequalities, positive-definiteness of the LKF, and a wise combination of splitting the integrals and effective zero equations, are considered in analyzing the LKF.
2. In Chapter 4:

   I. A recursive algorithm has been proposed to increase the order of an FO in the minimum possible number.

   II. A new methodology to solve the constrained observer equations has been proposed, which can give more numerical robustness with respect to the other existing transformation-based design approaches.

   III. The direct approach has been revisited, and has been improved to make it workable to design minimal multi-functional observers for LTI systems.

   IV. Illustrative numerical examples have supported the theoretical claims of this chapter.

3. In Chapter 5:

   I. The novel problem of delay-dependent functional observer design for LTI systems with multiple mixed time-varying state and input delays has been addressed.

   II. The state delays are bounded and slow-varying, or have unknown rates. The input delays are considered to be unknown.

   III. The proposed observer design methodology can be directly applied to design unknown-input functional observers for LTI systems with multiple mixed time-varying state delays.

   IV. The exponential stability of the observer is ensured, rather than its asymptotic stability.

   V. Less conservative LKFs are employed, and more effective techniques in the delay-dependent stability analysis of the observer are utilized with regard to the other existing papers on this particular topic.
VI. A GA-based tuning scheme for adjusting a weighting matrix is proposed, in the established LMI matrix.

4. In Chapter 6:

I. The new problem of delay-dependent FO design for LTI systems with interval time-varying state delays has been considered, wherein the delay rate is assumed to be bounded with an arbitrary upper-limit.

II. Novel sufficient conditions for the asymptotic stability of the observer are elaborated in terms of linear matrix inequalities, which can be less conservative than the existing criteria on this topic.

III. A novel LKF has been considered to this aim and contemporary techniques such as Wirtinger-based integral inequalities and reciprocally convex approach are employed.

IV. A number of LMI parameters have been adjusted via a GA-based framework, similar to the one proposed in the previous chapter.

5. In Chapter 7:

I. The crucial open problem of delay-dependent FO design for linear systems with unknown time-varying delays has been addressed.

II. A new sliding mode functional observer structure has been proposed to this aim, and the necessary and sufficient conditions of its stability have been obtained. The observer structure employs an auxiliary time-varying delay, which can be adjusted based on the knowledge on the true delay’s limits, and the stability zone that can be provided by the observer’s stability criterion.
III. Auxiliary delay-dependent sufficient conditions for the asymptotic stability of the observer are obtained using the Lyapunov Krasovskii approach, and are expressed in terms of an LMI and two structural rank conditions.

IV. The delay-free observer structure that is attained after fixing the auxiliary delay at the origin, has been additionally discussed. It is shown via a numerical example that this particular case is more conservative than the general delay-dependent observer design algorithm.

Overall, it can be concluded that the topic of time delay systems is a broad and interesting research area with several challenges that need to be addressed even for simple linear time-invariant systems. It can also be deduced that the theory of functional observer design, as a useful generalized version of ordinary observers, creates complicated mathematical problems, but relaxes some critical limitations that are subject to Luenberger observers. In this thesis extensive mathematical knowledge in the areas of linear algebra, functional differential equations, convex optimisation, variable structure systems theory, intelligent systems, and an important extension of the Lyapunov theory, have been effectively employed to solve some new state-of-the-art application-oriented fundamental problems in the theory of state estimation of ordinary and time-delay systems. The solutions provided to the complicated problems in each chapter are supported by deep mathematical manipulations, and can be applied in real-world online applications with low computational costs and complexities. In particular, Chapter 7 provides a reliable and systematic solution algorithm for an important problem of observer design theory, which is founded on and supported by the results of the previous chapters.

Nevertheless, there are still several problems that need to be addressed. For example, more advanced knowledge and research is demanded to consider the same investigated problems for nonlinear, uncertain, and stochastic systems. The next section of the chapter
sheds some light on the potential future directions of this research.

8.3 Future Work

In the sequel a few number of possible future directions of this research are proposed.

1. The problem of stability analysis of time-delay systems using the Lyapunov Krasovskii approach is prone to further studies and improvements. This can be done by improving the LKF and the estimation methods that are employed in establishing the stability criteria. In addition, more particular realistic assumptions on the delays, such as considering additive delays, or assuming a lower-bound for the delay-derivative (as well as its upper-limit) have recently been the subjects of new studies in this area.

2. The problem of functional observer-based controller design is an interesting and practical topic, which needs further attention by the research community. This idea is crucial and highly applicable, when we need to design an observer-based controller for the system, but the system is not observable. Hence, the maximum functional observable subspace of the system can be attained in a way, and the controller gain can be calculated with the constraint of being in that row space. Hence, the control designer has essentially more degrees of freedom to stabilize the system, or improve its performance, compared to when solely the output measurements are used by the control signal.

3. Several realistic problems such as robotics can be modelled as linear time-varying (LTV) systems. However, the problem of observer and more importantly functional observer design for LTV systems have been fairly overlooked, and there are very few contributions on this topic. More advanced mathematical techniques and knowledge
is required for designing functional observers for this class of systems. The author might focus on this important topic in the future.

4. The problem of sliding mode functional observers, absolutely requires further investigations. This includes studying higher-order sliding modes for further relaxing the structural matching conditions that need to be satisfied. Relaxing these conditions are extremely important in a wide range of practical applications such as robust FO design for electro-mechanical systems.

5. More importantly, focusing on the applications of the proposed theoretical findings is an important research direction of the author. These applications are broad and include observer-based bilateral teleoperation of surgery robots or ground/aerial vehicles under delayed communication networks, or the stabilization of Maglev systems that possess internal delays in their behaviour.
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