EFFORT CHOICE WITH
INTERTEMPORAL LINKAGES

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Practice.
1. CHAPTER

Everyday market and non-market activities require spending effort. In economics it is common to assume that exerting effort yields a positive productive outcome but also yields disutility for the individual, both typically within the same period; likewise, there can also be a diminishing marginal productivity of effort, again within the same period.

There have been, however, influential models in the early 2000s - primarily designed to explain the ‘Equity Premium Puzzle’ - that consider DMs with non-time-separable utility functions in consumption. Barberis, Huang & Santos (2001) considered DMs who “derive direct utility not only from consumption but also [utility] from fluctuations in the value of their ... wealth. They are loss averse over these fluctuations” (p. 1). Fuhrer (2000) considered a model “in which consumers’ utility depends in part on current consumption relative to past consumption” (p. 367). This approach in general is related to the “prospect theory” of Kahneman and Tversky (1979), which demonstrated that a typical individual in real-life evaluates her outcomes as a gain or loss rather than just the value of outcome. That is, in addition to just acquiring the period’s outcome, it is important for an individual to understand how good or bad she performs in comparison to what she expects. This expectation is defined as a reference point.¹

¹ In descriptive theories of decision-making under uncertainty, the distinction between gains and losses has received considerable attention. DMs do not appear to integrate outcomes with their wealth or existing consumption level, as normally assumed in expected utility theory. Rather individuals appear to react to events as changes, relative to some natural reference point. This observation was first made by Markowitz (1952).
It is needless to mention that loss aversion and gain fondness also induces preferences for particular patterns of decision-making over time. Significant attention in literature has been paid to the concept of loss aversion which, among other reasons, explains why people deviate from expected utility. People not only interpret outcomes as a gain or loss relative to a reference point but are also more sensitive to losses than to gains. Many studies have found evidence of loss aversion (e.g. Kahneman et al. 1990; Tversky and Kahneman 1991; Barberis, Huang & Santos 2001). In this thesis, I continue to consider the case of loss aversion together with gain fondness, i.e., when the DM is more sensitive to gain than to loss. For example, when a young tennis player is just at the beginning of her career, she knows that the path will be strewn with obstacles and losses, and therefore she is more likely to suffer less from the loss but experience more enjoyment from even a small win. The gain-loss asymmetry has figured prominently in theories of motivation, goal pursuit and decision-making. Focusing on preventing a loss versus achieving a gain or on achieving a gain versus preventing a loss activates very different kinds of psychological states, behaviours, decisions and choices.

In addition, many decisions and choices have a time dimension, and reference points are also important in intertemporal choice (Loewenstein and Prelec, 1992). While the standard theory based on expected utility assumes that choices are time invariant in a routine setup, Kahneman and Thaler (1991) demonstrated that individuals actually base their decisions on experience(s) that they had in the past. As is the case in Barberis & Huang (2001) and Fuhrer (2000), such an analysis typically involves comparisons to past performance. The general framework that is analysed

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2 Loewenstein (1988) pointed out that in situations when past consumption levels set reference points for future consumption, individuals may prefer an increasing consumption profile. The preference for a rising consumption profile helps explain an anomaly in labor markets, for example, such as wages rising with age even when productivity is not (Medoff and Abraham, 1980).
in this thesis also involves comparisons to past performance, and therefore is a reference-dependent one in nature.

Another direction not fully adhering to time-separability of utility functions is considered by Dragone (2009). He based his analysis on evidence that, in addition to disutility of work, many tasks exerting effort is also costly because it is fatiguing and that the current fatigue negatively affects the performance of the tasks to be performed in the future.\(^3\) The evidence that Dragone (2009) cites supports this view both for physical and for cognitive activities. In physiology, for instance, well-established literature shows that muscle contraction involves the depletion of energy signalling the onset of fatigue, and that energy deficiency is a major cause of low performance (see, among others, Sahlin, Tonkonogi and Söderlund, 1998). Similar conclusions have been reached with respect to high-level cognitive activities as well (Dorrian, Lamond and Dawson, 2000; Rogers, Dorrian and Dinges, 2003). Nevertheless, in many situations people must exert effort for prolonged periods of time and empirical evidence shows that this seriously influences performance. Similar degrading patterns in performance when prolonged effort is exerted are common in many activities people do. In competitive sports, for instance, competitive athletes who exert a lot of effort in a particular game are not expected to maintain the same effort intensity in the next game that follows in a few days; there is already literature which studies the effect of recovery strategies on such demanding physical activities.

\(^3\) Ryvkin (2011) considers fatigue in dynamic tournaments. In his model too current fatigue affects future performance.
performance (see Montgomery et al, 2008). Interestingly, although fatigue affecting future performance was already noted at the beginning of the 20th century by economists such as Chapman (1909), Florence (1924), Marshall (1920) and Vernon (1921), this approach has been abandoned for a long time afterwards.  

In economic literature, as alluded to before, effort is generally taken into account by assuming that it is a productive input, but it yields disutility for the DM. Within this standard approach, Marchetti and Nucci (2001) combined the usual assumption that effort yields disutility with another usual assumption that the marginal disutility of hourly effort rises with the number of hours, showing that, once the number of hours per worker reaches a critical threshold value, fatigue sets in and the marginal hour is worked with decreasing effort. Garcia-Sanchez and Vazquez-Mendez (2005), with a different focus, observed that fatigue can affect the instantaneous production function of a firm (through influencing the effective working time), with the consequence that different work schedules could be optimal depending on the individual preferences for leisure and on whether the fatigue constraint really binds or not. On the other hand, in a more behavioral setup, Akerlof et al (1990) introduced the fair wage-effort hypothesis in which workers proportionately withdraw effort as their actual wage falls short of their fair wage.

In this thesis, on the other hand, in a nutshell I will incorporate the directions mentioned above into a very basic dynamic effort choice model

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4 Parallel to the physiological argument on physical resource depletion, the psychological literature explains this evidence on the basis of a limited stock of cognitive resources (Kahneman, 1973). In either case, the exhaustion of the stock of physical resources or cognitive resources impairs performance. Thus, these resources in a sense resemble a car battery. Once they are depleted, they need to be recharged for future performance.

5 Nevertheless, an enormous number of research articles have been published on the topic of fatigue and its effect on physical and physiological functions in other literatures (see, for example, Abd-Elfattah et al, 2015).
in a general way, where effort is costless. In doing so, I will consider a forward-looking DM, who discounts the future, and who is choosing an effort level in each period - where as usual a higher effort level generates a higher level of productive outcome - in a fully dynamic history-dependent environment by simultaneously considering past, present and future, mainly in a finite-horizon setup. The remainder of this introductory chapter will provide a summary of the basics of my framework, its main concepts and its main results, which will be pursued in detail in Chapters 2, 3 and 4. Chapter 5 will conclude.

In Chapter 2, I will consider the case that the DM’s effort in one period does not have any negative or positive bearing on the following period’s effort, which I will term the Zero Intertemporal Effort Interdependence (ZIEI). In that chapter the DM does not compare her consecutive outcomes in adjacent periods, and thus will not encounter any loss aversion or any gain fondness between adjacent periods due to fluctuations in the values of her outcomes in consecutive periods, which I will term the Intertemporal Reference Independence (IRI) (I will elaborate on loss aversion and gain fondness as well as reference dependence shortly). Chapter 2 will only consider “IRI”. Then I will show that in the ZIEI and IRI case, the DM will choose the maximum possible effort level every period. I will then consider “fatigue” due to higher effort in one period affecting the effort level of the next period negatively, which I will term the Negative Intertemporal Effort Interdependence (NIEI). Note that the opposite situation would be a higher effort in one period affecting the effort level in the next period positively - possibly due to the learning-by-doing effect - which I will term the Positive Intertemporal Effort Interdependence (PIEI). \(^6\) In the NIEI and IRI case, I will

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\(^6\) The concept of learning-by-doing has played a central role in economics since Arrow (1963) introduced the concept. According to learning-by-doing, higher productivity is achieved through practice, self-perfection as well as minor innovations. An example is a factory that increases output by learning how to use equipment/machines better
show that the DM will start with the highest possible effort level and follow up with the lowest possible effort level the next period, following this alternating up-and-down pattern throughout. In the PIEI and IRI case, however, it turns out that the DM will still choose the maximum possible effort level for every period as in the ZIEI and IRI case.  

In Chapters 3 and 4, I consider a DM who - as opposed to Intertemporal Reference Independence (IRI) - has Intertemporal Reference Dependence (IRD) instead, i.e., the DM will compare her consecutive outcomes in adjacent periods, and thus, apart from her standard direct utility from the outcome of her choice of effort in each period, the DM will also get a reference payoff gain from an increase in the outcome or suffer a reference payoff loss from a decrease in the outcome (unless her outcomes are exactly the same in consecutive periods). I will call it Intertemporal Gain Fondness. Specifically, the gain-fondness coefficient $\lambda^+ > 0$ of the DM will determine the extent of the reference utility increase she will experience from her intertemporal outcome increment. Likewise, the DM will experience a reference utility without adding workers or significant amounts of physical capital/equipment (For example, Lundberg, 1961, demonstrated that, the Horndal iron works plant in Sweden, even without any new investment over a period of 15 years, raised output per worker hour by 2% per annum). Doing refers to the capability of workers to improve their productivity by regularly repeating the same type of action. Lucas (1988) considered learning-by-doing as a main catalyst of productivity growth and increasing returns to human capital.  

Another motivation for the PIEI (though perhaps less effective than the learning-by-doing effect) could be found in the following vignette that is familiar to academics in economics: It would not be an exaggeration to state that many aspiring economists would want to work and/or co-author with star economists, even if these aspiring economists may have to do most of the work, so that the star economists would agree to put their magic touch - or at least their name - on their joint work (and presumably such a situation would not be unique to economics). In other words, once one gets very prominent (e.g., one wins the Nobel Prize or comes close to it), there will be other academics who would be willing to share co-authorship with them at almost any cost and consequently these prominent academics can easily produce more papers.
decrease from any loss, i.e., from any decline in her outcome compared to the previous period. I will call it Intertemporal Loss Aversion. In that case, her loss-aversion coefficient \( \lambda^- > 0 \) will determine the extent of the reference utility decrease she will experience from that intertemporal outcome drop. IRD will be present if \( \lambda^- > 0 \) and/or \( \lambda^+ > 0 \).

In Chapter 3, I first consider the ZIEI and IRD case with \( \lambda^- > \lambda^+ \) (i.e., the “\( \lambda^- \)-dominant” case) which is the typical or most probable case considered in the reference-dependence literature. In this setup, the DM behaves like a standard DM, i.e., just like the one in the ZIEI and IRI, when \( \lambda^+ \) is relatively low in that she chooses the highest effort level every period. When \( \lambda^+ \) is relatively high and she does not discount the future much, however, the DM starts with a low level of effort - and thus a low level of outcome - in the first period, but behaves just like a standard DM thereafter, by choosing the highest possible effort level in every period from then on. (I also show that this behaviour of the DM is robust to full commitment - and thus it coincides with a social planner’s optimal plan). I then consider the ZIEI and IRD case with \( \lambda^+ > \lambda^- \) , (i.e., the “\( \lambda^+ \)-dominant” case). In this case, my results turn out to be the same as those of the \( \lambda^+ \leq \lambda^- \) case whenever \( \lambda^+ \) does not exceed \( \lambda^- \) by much. When \( \lambda^+ \) exceeds \( \lambda^- \) significantly, however, a much richer dynamic may arise, and fluctuating or volatile effort levels can become standard under some parameters: thus, the DM may even prefer first taking a loss so that she can then enjoy a gain in the next period when her discounting is strong. Further she may prefer following this up-and-down effort path throughout, as long as she finishes everything on a good note, i.e., with the highest possible effort and thus outcome. (However, in this case the DM’s behaviour is no longer robust to any level of commitment, and it does not coincide with a social planner’s optimal path any longer.)

In Chapter 4, I continue considering a DM who has Intertemporal Reference Dependence (IRD). I consider the NIEI (Negative Intertemporal
Efforts Interdependence) and the PIEI (Positive Intertemporal Effort Interdependence) cases along with IRD. Via NIEI, this chapter introduces fatigue in a dynamic behavioral model of intertemporal decision-making. The critical assumption is that the DM again has a certain amount of maximum effort potential that is depleted or recovered depending on effort exertion. The analysis in this NIEI and IRD case shows that it is not optimal to provide a constant effort path or an up-and-down effort path, as this would neglect the endogenous cost of effort on the dynamics of fatigue, where this cost could also be magnified through IRD. Specifically, in the NIEI and IRD case it turns out that regardless of the extent of her Intertemporal Loss Aversion and/or the extent of her Intertemporal Gain Fondness (i.e., regardless of $\lambda^- \gtrless \lambda^+$), any DM’s choice of effort path is the same and strongly monotonic over the entire time horizon. In the PIEI and IRD case, I first consider a $\lambda^-$-dominant DM (i.e., with $\lambda^- > \lambda^+$). I find that the DM behaves like an IRI agent (i.e., with $\lambda^- = \lambda^+ = 0$), who always chooses the highest effort level in every period; that is, her effort path always involves the highest possible effort. Next, I consider the case $\lambda^+ > \lambda^-$. In this case too, the DM chooses a path where her effort level is always at the maximum possible level after the first period, while her first period effort (1) too is at the maximum possible level if her discount factor is not sufficiently high, and (2) is less than the maximum possible level when her discount factor is sufficiently high.

To sum it up, when we have IRI, there is already significant variation between the DM’s effort paths depending on ZIEI, PIEI (the DM choosing the maximum effort level all the way) vs NIEI (the DM alternating between the highest and lowest possible levels of effort). When we have IRD, even within ZIEI there is a significant but even richer variation between the DM’s effort paths, depending on whether $\lambda^- > \lambda^+$ or not. With $\lambda^- > \lambda^+$ the DM always chooses the maximum effort level, while with $\lambda^+ > \lambda^-$ basically anything goes, i.e., the DM may always choose the maximum effort level
or alternate between the highest and lowest possible effort levels (and also choose some other variations of those effort paths). When we have IRD with NIEI, however, surprisingly the DM’s effort path becomes strictly monotone increasing, and this is so regardless of \( \lambda^- \preceq \lambda^+ \). On the other hand, such a strictly monotone-increasing effort path is not exhibited by the DM when we have IRD with PIEI. In that latter case, there is no full convergence of the effort paths of the \( \lambda^- \)-dominant and \( \lambda^+ \)-dominant DMs as opposed to the NIEI and IRD case; nevertheless, there is very strong resemblance between the effort paths of the \( \lambda^- \)-dominant and \( \lambda^+ \)-dominant DMs: the \( \lambda^- \)-dominant DM exhibits an effort path of the maximum possible effort level in each period (i.e., just like that of the DM in the ZIEI and IRI case) while the \( \lambda^+ \)-dominant DM exhibits either the very same effort path or an effort path where she starts with a less than maximum effort level but then quickly moves to the maximum possible effort level in all remaining periods (i.e., almost like that of the DM in some effort paths in the ZIEI and IRI case, but this time not starting with the lowest possible effort level).
2. CHAPTER

2.1 INTRODUCTION

In this chapter, as I have mentioned before in the introductory chapter, first I will consider the most basic case, namely the Zero Intertemporal Effort Interdependence (ZIEI) and the Intertemporal Reference Independence (IRI) case. As also mentioned before, I show that the DM will pick the maximum possible effort level every period. This behavior is the same whether the DM is the social planner or a decentralised agent. I also show that incorporating disutility of effort does not affect the outcome qualitatively and that neither does any diminishing marginal productivity of effort.

Then I extend this simplest possible framework slightly by keeping the IRI feature but considering the Negative Intertemporal Effort Interdependence (NIEI) and the Positive Intertemporal Effort Interdependence (PIEI). I show that in the NIEI and IRI case, the DM starts with the highest possible effort level and then follows up with the lowest possible effort level, following this alternating up-and-down pattern throughout. In the PIEI and IRI case, the DM still chooses the maximum possible effort level every period. I also note that incorporating any disutility of effort or any diminishing marginal productivity of effort makes the NIEI and PIEI case intractable. With NIEI and PIEI, I thus only consider the case where there is no disutility of effort or no diminishing marginal productivity of effort.
2.2 Preliminaries

In this section, I introduce the basic definitions and notions that I will use throughout the whole discussion.

First, I use the following notations throughout the discussion: \( e_t \) is an effort (i.e., an effort level) at any period \( t \). \( u(e_t) \) is the standard direct utility, \( \lambda^+ \) and \( \lambda^- \) are positive numbers. Next, I introduce the definitions that I will use further.

The following definition introduces the case when the DM is a standard agent who does not have any intertemporal linkages of efforts.

**Definition 1:** Zero Intertemporal Effort Interdependence (ZIEI) is defined as DM’s effort in a period which does not have any negative or positive bearing on the following period’s effort, i.e., the standard direct utility is defined as \( u(e_t) = e_t \forall t \geq 1 \) and \( e_t \in [0,1] \).

The following definition introduces intertemporal linkage of efforts between periods through the existence of fatigue that the DM can experience when exerting a high level of efforts.

**Definition 2:** Negative Intertemporal Effort Interdependence (NIEI) is defined as DM’s effort in a period which does have a negative effect on the following period’s effort, i.e., the standard direct utility is defined as \( u(e_t) = (1 - e_{t-1})e_t \forall t \geq 1 \) and \( e_t \in [0,1] \).

**Definition 3:** Positive Intertemporal Effort Interdependence (PIEI) is defined as DM’s effort in a period which does have a positive effect on the following period’s effort, i.e., the standard direct utility is defined as \( u(e_t) = e_{t-1}e_t \forall t \geq 1 \) and \( e_t \in [0,1] \).

The above definition introduces the case when exerting a high level of effort brings a positive effect or additional benefit in the next period, such
as, for example, pure peer effect (i.e., co-authorship). If an academic is very productive in terms of research output, it creates a positive reputation and attracts others for future co-authorships.

**Definition 4:** The DM is $\lambda^-$-dominant if $\lambda^- > \lambda^+$, where $\lambda^+$ and $\lambda^-$ are positive real numbers.

**Definition 5:** Intertemporal Loss Aversion (ILA) is defined as the case where $\lambda^- > \lambda^+$.

In the remainder of this thesis, I will refer to such an agent as a $\lambda^-$-dominant DM.

The above definition introduces loss aversion which refers to the tendency for the DM to strongly prefer avoiding losses to acquiring gains. The DM compares her consecutive outcomes in adjacent periods and suffers more from a decrease in the payoffs than she enjoys from an increase in the outcomes.

The following definitions introduce gain fondness that refers to the tendency for the DM to strongly prefers acquiring gain than avoiding losses. In this case, the DM also compares her consecutive outcomes in adjacent periods, but she enjoys an increase in the outcomes more than suffers from a decrease in the payoffs.

**Definition 6:** The DM is $\lambda^+$-dominant if $\lambda^- \leq \lambda^+$, where $\lambda^+$ and $\lambda^-$ are positive real numbers.

**Definition 7:** Intertemporal gain fondness (IGF) is defined as the case when $\lambda^- < \lambda^+$.

In the remainder of this thesis, I will refer to such an agent as a $\lambda^+$-dominant DM.

The following definition introduces the case where the DM does not compare her consecutive outcomes in adjacent periods, and thus will not encounter any loss aversion or any gain fondness between adjacent periods.
DEFINITION 8: If $\lambda^- = \lambda^+ = 0$, there is **Intertemporal Reference Independence (IRI)**.

The following definition introduces the case where the DM will compare her consecutive outcomes in adjacent periods, and thus, apart from her standard direct utility from the outcome of her choice of effort in each period, the DM will also get a reference payoff gain and/or suffer a reference payoff loss unless her outcomes are exactly the same in consecutive periods.

DEFINITION 9: If $\lambda^- > 0$ and/or $\lambda^+ > 0$, then there is **Intertemporal Reference Dependence (IRD)**.

### 2.3 Model

I consider a DM who takes part in the following **finite dynamic task**. Time is *discrete* and there are $T \geq 2$ periods, with $T < \infty$. In each period $t = 1, \ldots, T$, the DM has to take a *costless action* (i.e., an effort level) $e_t \in [0, 1]$ that yields a *direct utility* of $u(e_t) > 0$ in period $t$ with certainty.\(^1\) An effort *path* is denoted by $e = (e_1, \ldots, e_T) \in [0, 1]^T$. Given an effort path $e = (e_1, \ldots, e_T)$, I say that the DM has an *intertemporal gain* in period $t \geq 2$ if $u(e_t) > u(e_{t-1})$ and has an *intertemporal loss* if $u(e_t) < u(e_{t-1})$. Given an intertemporal gain, the DM experiences a *reference utility gain* of $\lambda^+(u(e_t) - u(e_{t-1}))$ while with an intertemporal loss, the DM experiences a *reference utility loss* of $\lambda^-(u(e_{t-1}) - u(e_t))$, where $\lambda^+$ and $\lambda^-$ are positive real numbers. Thus, $\lambda^-$ is the *loss-aversion coefficient* of the DM while $\lambda^+$ is her *gain-fondness coefficient*.

Clearly when $t = 1$, the notions of *intertemporal loss* or *intertemporal gain* are not applicable. Thus, the DM’s period $t \geq 2$ utility from the effort

\(^1\) The total DM’s outcome is defined as a sum of monetary or/and non-pecuniary payoffs (i.e., $u(e_t) = u_{mon}(e_t) + u_{nonp}(e_t)$). However, I am not going to discuss these payoffs separately.
path \( e = (e_1, \ldots, e_T) \) is \( v_t(e) = u(e_t) + \lambda^+(u(e_t) - u(e_{t-1})) \) if \( u(e_t) > u(e_{t-1}) \) and \( v_t(e) = u(e_t) - \lambda^-(u(e_{t-1}) - u(e_t)) \) if \( u(e_t) < u(e_{t-1}) \). Given \( v(\cdot) \), the DM chooses \( e \in [0,1]^T \) to maximise her discounted sum of utility given by

\[
V^\delta(e) = u(e_1) + \sum_{t=2}^{T} \delta^{t-1} v_t(e),
\]

(2.1)

where \( \delta \in (0,1) \) is her time discount factor.

2.3.1 THE ZIEI CASE: \( u(e_t) = e_t \)

First, I consider the case of **Zero Intertemporal Effort Interdependence** (ZIEI) (i.e., \( u(e_t) = e_t \)) with **Intertemporal Reference Independence** (IRI) (i.e., \( \lambda^- = \lambda^+ = 0 \)); that is, there exists no intertemporal linkage between periods.

Consider two cases: (1) the DM is a social planner and (2) she is a decentralised agent.\(^2\) In the first case, with the DM as the social planner, then her discounted payoff over time is \( V^\delta(e) = \sum_{t=1}^{T} \delta^{t-1} u(e_t) \). As a result, it is easy to verify that the DM’s payoff is at maximum only if the effort each period is \( e_t = 1, \forall t \). Suppose that the DM decides to deviate from this effort path and her choice is to take \( e_t = x_t \), where \( x_t \in [0,1] \). As a result, the DM’s payoff is \( V^\delta(e_t = x_t) = x_1 + \delta x_2 + \ldots + \delta^{T-1} x_T \). Comparing these two payoffs, I have \( V^\delta(e_t = 1) > V^\delta(e_t = x_t) \). To summarise, if the DM is the social planner, her optimal effort at each period is \( e^*_t = 1 \forall t \).

Next, I consider the case when the DM is a decentralised agent. Suppose that she is at period \( T \), then the DM’s payoff is \( V(e_T) = e_T \). As a result, maximum payoff at period \( T \) is reached only if the effort is \( e_T = 1 \). Consider

\(^2\) The following term “social planner” is used to describe the DM who makes a decision by choosing - and committing to - effort levels over all periods of time to achieve the best result for all periods involved, maximising her total payoff. She follows this optimal effort path over all periods without any deviations. The term “decentralized agent” is used to describe the DM who makes a decision about optimal effort level that maximizes payoff of the period \( t \) and can reconsider her choice every period.
the DM at period $T - 1$. Hence, her payoff is $V_\delta(e_{T-1}) = e_{T-1} + \delta e_T = e_{T-1} + \delta 1$. As a result, her action (i.e., an effort) at period $T - 1$ is $e_{T-1} = 1$, as well. Continuing the same logic, the equilibrium effort path is $e = (1, 1, \ldots, 1, 1)$. To prove it, let’s consider the following lemmas.

**Lemma 1:** In the case of ZIEI and IRI, for any $t > 1$ and $\delta \in (0, 1)$, the optimal effort is $e_t^* = 1$.

**Proof** The proof invokes an induction method. Let $e^* = (e_1^*, e_2^*, \ldots, e_T^*)$ be an equilibrium. Consider the statement $S(n)$ such that $e_{T-n} = 1, e_{T-n-1} = 1, \ldots, e_T = 1$. From that we infer that $S(0)$ and $S(1)$ are true. Suppose that the statement $S(n)$ is true as well; that is, $e_{T-n} = 1, e_{T-n-1} = 1, \ldots, e_T = 1$. Let’s now consider the statement $S(n+1)$. The DM’s payoff at period $T - (n+1)$ is $V_\delta(e_{T-(n+1)}) = e_{T-(n+1)} + \delta e_{T-n} + \delta^2 e_{T-n-1} + \ldots + \delta^{n-1} e_T = e_{T-(n+1)} + \delta \cdot 1 + \delta^2 \cdot 1 + \ldots + \delta^n 1$. As a result, the effort at period $T - (n+1)$ is $e_{T-(n+1)} = 1$, therefore, the statement $S(n+1)$ is true as well. To summarise, the effort in each period $t > 1$ is $e_t = 1$. This concludes the proof. □

Now let’s assume that the DM is at period $t = 1$.

**Lemma 2:** In the case of ZIEI and IRI, for $t = 1$ and any $\delta \in (0, 1)$, the optimal effort is $e_1^* = 1$.

**Proof** Consider the DM’s payoff at period 1, $V_\delta(e) = \sum_{t=1}^{T} \delta^{t-1} u(e_t)$. According to Lemma 1, we have $V_\delta(e) = e_1 + \delta 1 + \delta^2 1 + \ldots + \delta^{T-1} 1$. As a result, $e_1 = 1$. This concludes the proof. □

The following proposition provides the general result of this section:

**Proposition 1:** In the case of ZIEI and IRI, for any $\delta \in (0, 1)$, $T, k \in \{1, T\}$, the equilibrium effort path is $e^* = (1, 1, \ldots, 1, 1)$. \(^3\)

\(^3\) I say that the DM has $k$ - commitment power, $k = 1, \ldots, T$, if at any period $t$ she can implement any $k$-length effort path $e_t, \ldots, e_{\min\{T,T+k\}}$. When $k = T$ I say the DM has full commitment power while if $k = 1$ she has no commitment power.
This means that when the DM has Zero Intertemporal Effort Interdependence (ZIEI) and Intertemporal Reference Independence (IRI), then irrespective of her time discounting, the DM will exert full effort whether she has full or no commitment.

2.3.2 THE NIEI CASE: \( u(e_t) = (1 - e_{t-1})e_t \)

Now, I consider the case of Negative Intertemporal Effort Interdependence (NIEI) (i.e., \( u(e_t) = (1 - e_{t-1})e_t \)) with Intertemporal Reference Independence (IRI) (i.e., \( \lambda^- = \lambda^+ = 0 \)); that is, there exists an intertemporal linkage of efforts but there is no intertemporal reference dependence. In other words, this is the case where exerting high level of effort in one period leads to accumulation of fatigue in the subsequent period.

Let’s consider the following example of \( T = 2 \). Suppose that the DM is the social planner, and for the sake of simplicity, I assume that at the beginning, her horizon is only one period, that is \( T = 1 \). As a result, the DM’s payoff is \( V(e_1) = e_1 \). Therefore, the DM’s effort is \( e_1 = 1 \). Now suppose that the DM’s horizon is two periods, i.e., \( T = 2 \). The DM’s payoff is \( V_\delta(e_t) = e_1 + \delta(e_2(1 - e_1)) = e_1(1 - \delta e_2) + \delta e_2 \). The DM’s choice of efforts \( e_t \) is interdependent, that is her choice of \( e_1 \) affects the effort of the next period, that is \( e_2 \) is a function of \( e_1 \), with \( (1 - e_1)e_2 \). If \( \delta < \frac{1}{e_2} \), the DM’s effort in the first period is \( e_1 = 1 \). However, if \( \delta = 1 \), the effort path could be any of \( e = (1, 0) \) or \( e = (0, 1) \).

Next, suppose that the DM is a decentralised agent, and she is at the last period. Her payoff at this period is \( V(e_2) = e_2(1 - e_1) \). For any \( e_1 \neq 1 \) and \( \forall \delta \in (0, 1) \), the DM’s effort is \( e_2 = 1 \). If DM is at the period \( t = 1 \), the payoff is \( V_\delta(e_t) = e_1 + \delta(e_2(1 - e_1)) = e_1(1 - \delta 1) + \delta 1 \). If \( \delta = 1 \), that is when \( V = e_1 + e_2(1 - e_1) \), her first period payoff could be \( e_1 = 0 \) and then the effort path is \( e = (0, 1) \), or it could be \( e_1 = 1 \) and then her effort path is \( e = (1, 0) \).

Consider the DM who is the social planner. Then I have the following lemma:
Lemma 3: In the case of NIEI and IRI, for any $\delta \in (0, 1)$, if $T$ is even and $T > 2$, the effort path $e$ is such that $e_t = 1$ if $t$ is odd and $e_t = 0$ if $t$ is even.

Proof: If $T$ is even, it can be considered as a combination of two-period strings. As we have gone over the $T = 2$ case, for now suppose that $T = 4$, therefore it consists of a combination of two two-period strings. The DM’s payoff is $V = e_1 + \delta e_2 (1 - e_1) + \delta^2 e_3 (1 - e_2) + \delta^3 e_4 (1 - e_3) = e_1 (1 - \delta e_2) + \delta e_2 (1 - \delta e_3) + \delta^2 e_3 (1 - \delta e_4) + \delta^3 e_4$. Her effort $e_t$ affects her choice of the level of effort of the next period $e_{t+1}$ in the form of $(1 - e_t)e_{t+1}$. As a result, if $\delta < 1$ and if the effort is $e_1 = 1$, then in the next period the fatigue state is $1 - e_1 = 1 - 1 = 0$ and regardless of her effort in the second period, the utility in that period is 0. Her effort choice in $t = 2$ affects her effort in period $t = 3$, and because the fatigue state at this period is $1 - e_2 = 1 - 0 = 1$, therefore, the DM’s effort choice at period 3 is $e_3 = 1$. Consequently, in the next period the DM experiences fatigue; that is $1 - e_3 = 1 - 1 = 0$, and the level of effort that is chosen by the DM is $e_4 = 0$

Now suppose that the DM wants to deviate and choose another effort $x_t$ at period $t$, i.e., $e_t = x_t$, where $x_t \in (0, 1)$. As a result, her payoff is $V_\delta(x_1, x_2, x_3, x_4) = x_1 + \delta x_2 (1 - x_1) + \delta^2 x_3 (1 - x_2) + \delta^3 x_4 (1 - x_3)$. Comparing this payoff to $V(1, 0, 1, 0) = 1 + \delta 0 + \delta^2 1 + \delta^3 0$, we have $V(1, 0, 1, 0) > V_\delta(x_1, x_2, x_3, x_4)$. Consequently, the combination of two-period strings $\{1, 0\}$ dominates any others. This concludes the proof. □

Remark 1: If $\delta = 1$, the effort path could be any of $\{0, 1\}$ or $\{1, 0\}$. The DM’s payoff is $V = e_1 + e_2 (1 - e_1) + e_3 (1 - e_2) + e_4 (1 - e_3) = e_1 (1 - e_2) + e_2 (1 - e_3) + e_3 (1 - e_4) + e_4$. In the case of $\{0, 1\}$ the payoff is $V(\{0, 1\}) = 0 + 1 + 0 + 1$. In the case of $\{1, 0\}$, the payoff is $V(\{1, 0\}) = 1 + 0 + 1 + 0$. These payoffs are the same.

Lemma 4: In the case of NIEI and IRI, for any $\delta \in (0, 1)$ if $T$ is odd, the effort path represents the combination of two periods plus one, i.e., such that
\(e_t = 1\) if \(t\) is odd and \(e_t = 0\) if \(t\) is even.

**Proof** If \(T\) is the odd number of periods, we can consider that as a combination of two-period strings plus one-period string. According to Lemma 3, the effort path for any two periods is \(\{1, 0\}\). Let’s consider the case \(T = 3\). The DM’s payoff is

\[
V = e_1 + \delta e_2 (1 - e_1) + \delta^2 e_3 (1 - e_2) = e_1 (1 - \delta e_2) + \delta e_2 (1 - \delta e_3) + \delta^2 e_3.
\]

If \(\delta < 1\), the effort \(e_1 = 1\), then in the next period the fatigue state is \(1 - e_1 = 1 - 1 = 0\) and because the fatigue state at this period is \(1 - e_2 = 1 - 0 = 1\), regardless of her effort in the second period, the utility in that period is 0. Her effort choice in \(t = 2\) affects her effort in period \(t = 3\), and because the fatigue state at this period is \(1 - e_2 = 1 - 0 = 1\), therefore, the DM’s effort choice at period 3 is \(e_3 = 1\).

Suppose that the DM wants to deviate and choose the effort \(x_t\) at period \(t\), i.e., \(e_t = x_t\), where \(x_t \in (0, 1)\). As a result, her payoff is \(V_\delta(x_1, x_2, x_3) = x_1 + \delta x_2 (1 - x_1) + \delta^2 x_3 (1 - x_2)\). Comparing this payoff to \(V(1, 0, 1) = 1 + \delta 0 + \delta^2 1\), we have \(V(1, 0, 1) > V_\delta(x_1, x_2, x_3)\). As a result, this path \(\{1, 0\} + \{1\}\) dominates. This concludes the proof. \(\square\)

**Lemma 5:** In the case of NIEI and IRI, for any \(\delta \in (0, 1)\), the effort path \(e^* = (1, 0, 1, 0 \ldots)\) dominates any others such as \(e^{**} = (x_1, x_2, x_3, x_4, \ldots)\), \(\forall x_t \in [0, 1)\).

**Proof** We are going to demonstrate the following case by selecting the effort path that starts with the highest effort and in the subsequent period the lowest level of effort, 0, and then vacillating between the two efforts thereafter. My contention is that this effort path dominates any other effort path. To that end, suppose that \(e^* = (e_1^*, e_2^*, \ldots, e_T^*)\) is an equilibrium effort path, such that \(e^* = (1, 0, 1, 0 \ldots)\). Then the DM’s payoff is \(V(1, 0, 1, 0 \ldots) = 1 + \delta 0 + \delta^2 1 + \delta^3 0 + \ldots\). Now suppose that there exists other equilibrium paths, such as, for example, \(0, 1, 0, 1, \ldots\) or \(x, y, z, w, \ldots\), where \(x, y, z, w \in (0, 1)\). The DM’s payoff of \(0, 1, 0, 1, \ldots\) is \(V(0, 1, 0, 1, \ldots) = 0 + \delta 1 + \delta^2 0 + \delta^3 1 + \ldots\). The DM’s payoff
of \(x, y, z, w, \ldots\) is

\[ V(x, y, z, w, \ldots) = x + \delta y(1 - x) + \delta^2 z(1 - y) + \delta^3 w(1 - y) + \ldots. \]

We will now compare payoffs \(V(1, 0, 1, 0, \ldots) \preceq V(0, 1, 0, 1, \ldots)\). Consequently, \(1 + \delta 0 + \delta^2 1 + \delta^3 0 + \ldots - [0 + \delta 1 + \delta^2 0 + \delta^3 1 + \ldots] > 0\). Therefore, \(e^* = (1, 0, 1, 0, \ldots)\) dominates \(0, 1, 0, 1, \ldots\). Now let’s check \(V(1, 0, 1, 0, \ldots) \preceq V(x, y, z, w, \ldots)\). As a result, we have \(V(1, 0, 1, 0, \ldots) - V(x, y, z, w, \ldots) = 1 + \delta 0 + \delta^2 1 + \delta^3 0 + \ldots - [x + \delta y(1 - x) + \delta^2 z(1 - y) + \delta^3 w(1 - y) + \ldots] > 0\). Hence, the effort path \(e^* = (1, 0, 1, 0, \ldots)\) dominates any other effort path for the DM. This concludes the proof. □

To summarise these findings, I provide the following proposition:

**PROPOSITION 2:** In the case of NIEI and IRI, for any \(\delta \in (0, 1)\) and for \(T > 2\), the optimal effort path presents either the combination of \(\{1, 0\}\) if \(T\) is even, or the combination of two periods plus one, i.e., \(\{1, 0\} + \{1\}\), if \(T\) is odd. This effort path \(e^* = (1, 0, 1, 0, \ldots)\) dominates any others.

This means that when the DM has Negative Intertemporal Effort Interdependence (NIEI) and Intertemporal Reference Independence (IRI), then irrespective of her time discounting, the DM will start with the highest possible effort level and follow up with the lowest possible effort level in the next period, and following this alternating up-and-down pattern throughout.

### 2.3.3 The PIEI Case: \(u(e_t) = e_{t-1}e_t\)

In this section I consider the case of **Positive Intertemporal Effort Interdependence** (PIEI) (i.e., \(u(e_t) = e_{t-1}e_t\)) with **Intertemporal Reference Independence** (IRI) (i.e., \(\lambda^- = \lambda^+ = 0\)); that is, there exists an intertemporal linkage of efforts, but there is no intertemporal reference dependence. In other words, this is the case where a high level of effort in one period brings about a positive impact on the DM’s effort in the subsequent period.
I consider two cases, (1) the DM is a social planner and (2) she is a decentralised agent. First, suppose that the DM is the social planner, then her expected payoff over time is 

\[ V_\delta(e) = \sum_{t=1}^{T} \delta^{t-1} u(e_t) = e_1 + \sum_{t=2}^{T} \delta^{t-1} e_{t-1} e_t. \]

As a result, it is easy to verify that the DM’s payoff is at maximum only if the effort at each period is \( e_t = 1 \) \( \forall t \). Suppose that the DM decides to deviate from this effort path and her choice is \( e_t = x_t \) \( \forall t \), where \( x_t \in [0, 1) \). Consequently, the DM’s payoff is 

\[ V_\delta(e_t = x_t) = x_1 + \delta x_1 x_2 + \ldots + \delta^{T-1} x_{T-1} x_T. \]

Comparing these two payoffs, I obtain 

\[ V_\delta(e_t = 1) > V_\delta(e_t = x_t). \]

To summarise, if the DM is the social planner, her optimal effort at each period is \( e_t^* = 1 \) \( \forall t \).

Now, suppose that the DM is a decentralised agent and she is at period \( T \). Then the DM’s payoff is \( V(e_T) = e_T e_{T-1} \). As a result, the DM’s effort at period \( T \) is \( e_T = 1 \). Now assume the DM is at period \( T - 1 \). Hence, her payoff is 

\[ V_\delta(e_{T-1}) = e_{T-1} e_{T-2} + \delta e_T e_{T-1} = e_{T-1} e_{T-2} + \delta e_{T-1}. \]

Consequently, her effort at period \( T - 1 \) is \( e_{T-1} = 1 \). Now let’s assume that the DM is at period \( T - 2 \). Hence, her payoff is 

\[ V(e_{T-2}) = e_{T-2} e_{T-3} + \delta e_{T-1} e_{T-2} + \delta^2 e_T e_{T-1} = e_{T-2} e_{T-3} + \delta e_{T-2} + \delta^2 1 = e_{T-2} (e_{T-3} + \delta) + \delta^2. \]

As a result, the DM’s effort at period \( T - 2 \) is \( e_{T-2} = 1 \). Continuing the same logic, one can see that her equilibrium effort path will be \( e = (1, 1, \ldots, 1, 1) \). To prove it, let’s consider the following lemmas.

**Lemma 6:** In the case of PIEI and IRI, for any \( \delta \in (0, 1) \), \( T \) and \( t > 1 \), \( e_t^* = 1 \).

**Proof** To prove this lemma, we invoke an induction approach. Let \( e^* = (e_1^*, e_2^*, \ldots, e_T^*) \) be an equilibrium. Consider the statement \( S(n) \) such that \( e_{T-n} = 1, e_{T-n-1} = 1, \ldots, e_T = 1 \). We infer that \( S(0) \) and \( S(1) \) are true. Suppose that the statement \( S(n) \) is true as well. Consider the statement \( S(n+1) \). Then the DM’s payoff at period \( T - (n+1) \) is 

\[ V(e_{T-(n+1)}) = e_{T-(n+1)} e_{T-(n+2)} + \delta e_{T-n} e_{T-(n+1)} + \delta^2 e_{T-n-1} e_{T-(n)} + \ldots + \delta^{n-1} e_{T-1} e_{T-n} = e_{T-(n+1)} e_{T-(n+2)} + \delta e_{T-n} + \delta^2 1 + \ldots + \delta^n 1 = e_{T-(n+1)} (e_{T-(n+2)} + \delta) + \delta^2 1 + \delta^3 1 \ldots + \delta^{n-1} 1. \]

Consequently, the DM’s effort is \( e_{T-(n+1)} = 1 \), therefore the statement \( S(n+1) \) is true. The effort path \( \forall t > 1 \) is \( e_t = 1 \).
This concludes the proof. □

**Lemma 7:** In the case of PIEI and IRI, for any \( \delta \in (0, 1) \), \( e_1^* = 1 \).

**Proof** Consider the DM’s payoff at period 1, \( V_\delta(e_1) = e_1 + \delta e_2 e_1 + \delta^2 e_3 e_2 + \ldots + \delta^{T-1} e_T e_{T-1} = e_1 + \delta e_1 + \delta^2 e_2 + \ldots + \delta^{T-1} e_{T-1} = e_1 (1 + \delta) + \delta^2 e_2 + \ldots + \delta^{T-1} e_{T-1} \). The DM’s effort at period 1 is \( e_1 = 1 \). This concludes the proof. □

To summarise these findings, I provide the following proposition:

**Proposition 3:** In the case of PIEI and IRI, for any \( \delta \in (0, 1) \), \( T, k \in \{1, T\} \), \( e^* = (1, 1, \ldots, 1, 1) \).

This means that when the DM has Positive Intertemporal Effort Interdependence (PIEI) and Intertemporal Reference Independence (IRI), then irrespective of her time discounting, the DM will exert full effort whether she has full or no commitment.

### 2.4 Discussion: Robustness Checks

Until now I have assumed that the direct utility is a function of costless effort that was characterised by constant marginal productivity. However, the results could be generalised for the case of decreasing marginal productivity of effort and disutility. Marginal productivity theory is a cornerstone in the analysis of labor or other factors markets and the input side of short-run production. It provides insight into the demand for factors of production based on the notion, for example, that a DM exerts an effort based on a comparison between the productivity of the effort and the cost of the effort. However, as more of an effort is exerted, marginal productivity basically declines. It means that diminishing marginal productivity of efforts affects the optimal effort level that should be chosen by the DM to maximise her payoff.

Another concept that should be taken in consideration together with productivity of effort is disutility of effort. In economics it is common to
assume that exerting effort on a given task is a costly activity that positively influences the performance on the task to be done. As a result, exerting of effort induces disutility. The DM will choose an effort so as to maximize expected utility minus effort disutility. In general, disutility can be considered as the cost of exerting effort, and can be formalized by assuming that the agent does not like working, i.e., exerting effort yields disutility, and that the agent will exert effort on the task only if appropriately motivated and/or compensated. Disutility of efforts includes the evidence that, for example, mental tasks require attention effort but people have finite mental processing speeds. Consequently, exerting effort is costly because working on a specific activity prevents the agent from paying attention to another one. As a result, it is important to introduce disutility of effort in my findings.

Now I consider diminishing marginal productivity of effort within each period. As a result, the direct utility is defined as $u(e_t) = e_t^\alpha$, where $\alpha \in [0, 1)$. In the case of ZIEI with IRI, where there is no intertemporal linkage between periods, it turns out that introducing $\alpha \in [0, 1)$ does not change the results at all. The DM still chooses the highest effort level, i.e., 1, in each period to maximize her utility, regardless of whether she is the social planner or a decentralized agent.

**Claim 1**: In the case of ZIEI and IRI, for any $\delta \in (0, 1)$, $T$, $k=1$, $k=T$, the the optimal effort path is $e^* = (1, 1, \ldots, 1, 1)$.

All proofs of claims are in Appendix 1. Each proof is organised as a set of proved facts.

The results remain the same even in the presence of intertemporal linkage between periods. In the case of PIEI and IRI, with $\alpha \in [0, 1)$, the DM still chooses the effort level of 1 in each period to maximize her utility regardless of whether she is a social planner or a decentralized agent.
CLAIM 2: In the case of PIEI and IRI, for any $\delta \in (0, 1)$, $T$, $k=1$, and $k=T$, $e^* = (1, 1, \ldots, 1, 1)$.

In the case of NIEI and IRI, with $\alpha \in [0, 1)$, the DM also follows the same pattern in the presence of intertemporal linkage. She prefers to choose the highest effort level at the beginning and then the lowest effort level in the subsequent period, and following this up and down pattern thereafter.

CLAIM 3: In the case of NIEI and IRI, for any $\delta \in (0, 1)$ for $T > 2$, the optimal effort path presents either the combination of $\{1, 0\}$ if $T$ is even, or the combination of two periods plus one, i.e., $\{1, 0\} + \{1\}$, if $T$ is odd. This effort path $e^* = (1, 0, 1, 0 \ldots)$ dominates any others.

Such a robustness is not observed when I consider the disutility of effort within each period. Consider the disutility (or cost) of effort $C(x) = \beta e_t^\gamma$, where $\beta \geq 0$ and $\gamma \geq 1$. In the case of ZIEI with IRI, with $\beta > 0$ and $\gamma > 1$, the DM will choose the effort level $0 < e_{max} < 1$ to maximize her utility. Suppose the DM tries to maximize the direct utility $U(e_t) = u(e_t) - C(e_t)$, then the DM will choose the effort level $0 < e_{max} < 1$ to maximize her utility. Thus, the results will remain qualitatively the same.

CLAIM 4: In the case of ZIEI and IRI, for any $\delta \in (0, 1)$, $T$, $k=1$, $k=T$, the optimal effort path is $e^* = (e_{max}, e_{max}, \ldots, e_{max}, e_{max})$.

The same results can be observed in the case of PIEI and IRI.

CLAIM 5: In the case of PIEI and IRI, for any $\delta \in (0, 1)$, $T$, $k=1$, and $k=T$, $e^* = (e_{max}, e_{max}, \ldots, e_{max}, e_{max})$.

The analysis of the case NIEI and IRI with disutility turns out to be intractable.
2.5 Conclusion

In this chapter, the analysis fully pertains to the Intertemporal Reference Independence (IRI) case, i.e., without involving any Intertemporal Loss Aversion or Intertemporal Gain Fondness. I found that the DM behaves the same way in the ZIEI and PIEI cases by choosing the maximum possible effort level. In the NIEI case, however, the DM alternates between the highest possible effort level and zero effort level. It also turns out that these cases are robust to the inclusion of diminishing marginal productivity of effort. Although the ZIEI and PIEI cases are robust to the incorporation of disutility of effort within a period, the NIEI case is not.
APPENDIX 1: ROBUSTNESS CHECK

Case: Diminishing marginal productivity for IRI

1 The ZIEI Case: \( u(e_t) = e_t^\alpha \)

Consider the DM who is the social planner, then her expected payoff over time is \( V_\delta(e) = \sum_{t=1}^{T} \delta^{T-1} u(e_t) = \sum_{t=1}^{T} \delta^{T-1} e_t^\alpha \). As a result, it is easy to verify that the DM’s payoff is at maximum only if the effort each period is \( e_t = 1, \forall t \). Suppose that the DM decides to deviate from this effort path and her choice is to take \( e_t = x_t \), where \( x_t \in [0,1) \). As a result, the DM’s payoff is \( V_\delta(e_t = x_t) = x_1^\alpha + \delta x_2^\alpha + \ldots + \delta^{T-1} x_T^\alpha \). Comparing these two payoffs, I have \( V_\delta(e_t = 1) > V_\delta(e_t = x_t) \). To summarise, if the DM is the social planner, her optimal effort at each period is \( e_t^* = 1, \forall t \) and \( \alpha \in [0,1) \).

Now, assume that the DM is a decentralised agent. Suppose that she is at period \( T \), then the DM’s payoff is \( V(e_T) = e_T^\alpha \). Clearly, \( e_T = 1 \). Now consider the DM at period \( T-1 \), where her payoff will be \( V_\delta(e_{T-1}) = e_{T-1}^\alpha + \delta e_T^\alpha = e_{T-1}^\alpha + \delta 1^\alpha \). Hence, her effort at period \( T-1 \) too is \( e_{T-1} = 1 \). Continuing the same logic, the optimal effort path is \( e = (1,1,\ldots,1,1) \).

Proof of Claim 1

Claim 1 will be proved through the following facts.

FACT 1: In the case of ZIEI and IRI, for any \( t > 1 \) and \( \delta \in (0,1) \), the equilibrium effort is \( e_t^* = 1 \).

Proof The proof invokes an inductive approach. Let \( e^* = (e_1^*, e_2^*, \ldots, e_T^*) \) be an equilibrium. Consider the statement \( S(n) \) such that \( e_{T-n} = 1, e_{T-n-1} = 1, \ldots, e_T = 1 \). From this we know that \( S(0) \) and \( S(1) \) are true. Suppose that statement \( S(n) \) is true as well, that is
\[ e_{T-n} = 1, e_{T-n-1} = 1, \ldots, e_T = 1. \]

Let’s consider the statement \( S(n+1) \).

The DM’s payoff at period \( T - (n + 1) \) is \( V_\delta(e_{T-(n+1)}) = e_\alpha^{T-(n+1)} + \delta e_\alpha^{T-n} + \delta^2 e_\alpha^{T-n-1} + \ldots + \delta^{n-1} e_\alpha^T = e_\alpha^{T-(n+1)} + \delta 1^\alpha + \delta^2 1^\alpha + \ldots + \delta^n 1^\alpha \). As a result, the effort at period \( T - (n + 1) \) is \( e_{T-(n+1)} = 1 \); therefore, the statement \( S(n+1) \) is true as well. To summarise, the effort at each period \( t > 1 \) is \( e_t = 1 \). This concludes the proof. □

**Fact 2:** In the case of ZIEI and IRI, for \( t = 1 \) and any \( \delta \in (0, 1) \), the optimal effort is \( e_1^* = 1 \).

**Proof** Consider the DM’s payoff at period 1, \( V_\delta(e) = \sum_{t=1}^T \delta^{t-1} u(e_t) \).

According to Fact 1, we have the \( V_\delta(e) = e_1^* + \delta 1^\alpha + \delta^2 1^\alpha + \ldots + \delta^{T-1} 1^\alpha \). As a result, \( e_1 = 1 \). This concludes the proof. □

This completes the proof of claim. ■

2 **The PIEI Case:** \( u(e_t) = e_{t-1} e_\alpha^t \)

Suppose that the DM is the social planner. Then her expected payoff over time is \( V_\delta(e) = \sum_{t=1}^T \delta^{t-1} u(e_t) = e_1^* + \sum_{t=2}^T \delta^{t-1} e_{t-1} e_\alpha^t \). As a result, it is easy to verify that the DM’s payoff is at maximum only if the effort at each period is \( e_t = 1, \forall t \). Suppose that the DM decides to deviate from this effort path and her choice is to take \( e_t = x_t \), where \( x_t \in [0, 1) \). Thus, the DM’s payoff is \( V_\delta(e_t = x_t) = x_1^\alpha + \delta x_1 x_2^\alpha + \ldots + \delta^{T-1} x_{T-1} x_T^\alpha \). Comparing these two payoffs, I have \( V_\delta(e_t = 1) > V_\delta(e_t = x_t) \). To summarise, if the DM is the social planner, then her optimal effort at each period is \( e_t^* = 1 \) \( \forall t \).

Now, suppose that the DM is a decentralised agent and she is at period \( T \). Then the DM’s payoff is \( V(e_T) = e_\alpha^T e_{T-1} \). Consequently, the DM’s effort at period \( T \) is \( e_T = 1 \). Now consider the DM at period \( T - 1 \). Hence, her payoff is

\[
V_\delta(e_{T-1}) = e_\alpha^{T-1} e_{T-2} + \delta e_\alpha^{T-2} e_{T-2} = e_\alpha^{T-1} e_{T-2} +
\]
\(\delta 1^\alpha e_{T-1}\). As a result, her effort at period \(T - 1\) is \(e_{T-1} = 1\). Continuing the same logic, the optimal effort path is \(e = (1, 1, \ldots, 1, 1)\).

*Proof of Claim 2*

Claim 2 will be proved through the following facts.

**FACT 3:** In the case of PIEI and IRI, for any \(\delta \in (0, 1)\), \(T\) and \(t > 1\), \(e^*_t = 1\).

*Proof* The proof invokes an inductive approach. Let \(e^* = (e^*_1, e^*_2, \ldots, e^*_T)\) be an optimal solution. Consider the statement \(S(n)\) such that \(e_{T-n} = 1, e_{T-n-1} = 1, \ldots, e_T = 1\). We infer that \(S(0)\) and \(S(1)\) are true. Suppose that the statement \(S(n)\) is true as well. Let’s consider the statement \(S(n+1)\).

Then the DM’s payoff at period \(T - (n + 1)\) is 
\[
V(T - (n + 1)) = e^T_{T-(n+1)}e^T_{T-(n+2)} + \delta e^T_{T-n}e^T_{T-(n+1)} + \delta^2 e^T_{T-n-1}e^T_{T-(n)} + \ldots + \delta^{n-1} e^T_{T-1}e_{T-1} = e^T_{T-(n+1)}e^T_{T-(n+2)} + \delta 1^\alpha e_{T-(n+1)} + \delta^2 1^\alpha + \ldots + \delta^{n-1} 1^\alpha. 
\]

Consequently, the DM chooses the effort level \(e_{T-(n+1)} = 1\), therefore the statement \(S(n+1)\) is true. Thus, the effort path \(\forall t > 1\) is \(e_t = 1\).

This concludes the proof. \(\square\)

**FACT 4:** In the case of PIEI and IRI, for any \(\delta \in (0, 1)\), \(e^*_1 = 1\).

*Proof* Consider the DM’s payoff at period 1, \(V_\delta(e_1) = e^\alpha_1 + \delta e^\alpha_2 e_1 + \delta^2 e^\alpha_3 e_2 + \ldots + \delta^{T-1} e^\alpha_T e_{T-1} = e^\alpha_1 + \delta 1^\alpha e_1 + \delta^2 1^\alpha + \ldots + \delta^{T-1} 1^\alpha.\)

Hence, \(e_1 = 1\). The DM chooses the effort level \(e_1\) to be 1. This concludes the proof. \(\square\)

This completes the proof of claim. •

3 **The NIEI Case:** \(u(e_t) = (1 - e_{t-1})e_t^\alpha\)
The direct utility is \( u(e_t) = (1 - e_{t-1})e_t^\alpha \), where \( \alpha \in [0, 1) \). The DM’s expected payoff is \( V_\delta(e) = \sum_{t=1}^T \delta^{t-1}u(e_t) = e_1^\alpha + \sum_{t=2}^T \delta^{t-1}(1 - e_{t-1})e_t^\alpha \).

**Proof of Claim 3**

Claim 3 will be proved through the following facts.

**FACT 5:** In the case of NIEI and IRI, for any \( \delta \in (0, 1) \), if \( T \) is even and \( T > 2 \), the effort path represents the combination of \( \{1, 0\} \).

**Proof** If \( T \) is even, it can be considered a combination of two-period strings. For now suppose that \( T = 4 \). Therefore it consists of a combination of two two-period strings. The DM’s payoff is \( V = e_1^\alpha + \delta e_2^\alpha (1 - e_1) + \delta^2 e_3^\alpha (1 - e_2) + \delta^3 e_4^\alpha (1 - e_3) \). Hence, if \( \delta < 1 \), the effort will be \( e_1 = 1 \). Then in the next period, the fatigue state is \( 1 - e_1 = 1 - 1 = 0 \), and regardless of her effort in the second period, her utility in that period is 0. The fatigue state at period 3 is \( 1 - e_2 = 1 - 0 = 1 \). Therefore, the DM’s choice at period 3 is \( e_3 = 1 \). Consequently, in the next period, the DM encounters the fatigue state \( 1 - e_3 = 1 - 1 = 0 \), and the choice of level of effort by the DM is \( e_4 = 0 \).

Now suppose that the DM wants to deviate and choose another effort \( x_t \) at period \( t \), i.e., \( e_t = x_t \), where \( x_t \in (0, 1) \). Hence, her payoff is \( V_\delta(x_1, x_2, x_3, x_4) = x_1^\alpha + \delta x_2^\alpha (1 - x_1) + \delta^2 x_3^\alpha (1 - x_2) + \delta^3 x_4^\alpha (1 - x_3) \). Compare this payoff to \( V(1, 0, 1, 0) = 1^\alpha + \delta 0 + \delta^2 1^\alpha + \delta^3 0 \). Then we have \( V(1, 0, 1, 0) > V_\delta(x_1, x_2, x_3, x_4) \). Consequently, a combination of two-period strings \( \{1, 0\} \) dominates any others. This concludes the proof. \( \square \)

**FACT 6:** In the case of NIEI and IRI, for any \( \delta \in (0, 1) \) if \( T \) is odd, the effort path represents the combination of two periods plus one, i.e., \( \{1, 0\} + \{1\} \).
Proof Suppose $T$ is odd. Then we can consider a combination of two-period strings plus one-period. According to Fact 5, the two-period strings are \{1,0\}. Now suppose $T = 3$. Then it consists of a combination of two-period strings plus one-period. The DM’s payoff is $V = e_1^\alpha + \delta e_2^\alpha (1 - e_1) + \delta^2 e_3^\alpha (1 - e_2)$. Hence, if $\delta < 1$, the effort $e_1 = 1$. Then in the next period, the fatigue state is $1 - e_1 = 1 - 1 = 0$, and regardless of her effort in the second period, her utility in that period will be 0. The fatigue state at period 3 is $1 - e_2 = 1 - 0 = 1$. Therefore, the DM’s choice at period 3 will be $e_3 = 1$.

Now suppose that the DM wants to deviate and take another effort $x_t$ at period $t$, that is $e_t = x_t$, where $x_t \in (0,1)$. Hence, her payoff is $V_\delta(x_1,x_2,x_3) = x_1^\alpha + \delta x_2^\alpha (1 - x_1) + \delta^2 x_3^\alpha (1 - x_2)$. Compare this payoff to $V(1,0,1) = 1^\alpha + \delta 0 + \delta^2 1^\alpha$. Then we have $V(1,0,1) > V_\delta(x_1,x_2,x_3)$. Consequently, a combination of two-period strings \{1,0\} plus one-period \{1\} will dominate any other path. This concludes the proof. □

FACT 7: In the case of NIEI and IRI, for any $\delta \in (0,1)$, the effort path $e^* = (1,0,1,0\ldots)$ dominates any others such as $e^{**} = (x_1,x_2,x_3,x_4,\ldots)$, $\forall x_t \in [0,1)$.

Proof The DM’s payoff from the effort path $e^* = (1,0,1,0\ldots)$ is $V(1,0,1,0,\ldots) = x^\alpha + \delta x_1^\alpha + \delta^2 x_2^\alpha (1 - x) + \delta^3 x_3^\alpha (1 - y) + \delta^4 x_4^\alpha (1 - y) + \ldots$ The DM’s payoff from the effort path $e^{**} = (x,y,z,w,\ldots)$ is $V(x,y,z,w,\ldots) = x^\alpha + \delta y^\alpha (1 - x) + \delta^2 z^\alpha (1 - y) + \delta^3 w^\alpha (1 - y) + \ldots$

Comparing these two payoffs, we have $V(1,0,1,0,\ldots) - V(x,y,z,w,\ldots) = x^\alpha + \delta x_1^\alpha + \delta^2 x_2^\alpha (1 - x) + \delta^3 x_3^\alpha (1 - y) + \delta^4 x_4^\alpha (1 - y) + \ldots > 0$. Hence, $e^* = (1,0,1,0\ldots)$ dominates any other effort path. This concludes the proof. □

This completes the proof of claim. ■
Case: Disutility

1 The ZIEI and IRI Case

Consider the DM who is the social planner. Then her expected payoff is 
\( V_\delta(e) = \sum_{t=1}^{T} \delta^{t-1} U(e_t) \). Then, it is easy to verify that the DM’s payoff is at maximum only if her effort at each period is \( e_t = e^*, \forall t \).

Now, assume that the DM is a decentralised DM. Suppose that she is at period \( T \), then the DM’s payoff is \( V(e_T) = U(e_T) \). Clearly, \( e_T = e_{\text{max}} \). Next, consider the DM at period \( T - 1 \), where her payoff is \( V_\delta(e_{T-1}) = U(e_{T-1}) + \delta U(e_{\text{max}}) \). Hence, her effort at period \( T - 1 \) too is \( e_{T-1} = e_{\text{max}} \). Continuing the same logic, the optimal effort path is \( e = (e_{\text{max}}, e_{\text{max}}, \ldots, e_{\text{max}}, e_{\text{max}}) \).

Proof of Claim 4

Claim 4 will be proved through the following facts.

FACT 8: In the case of ZIEI and IRI, for any \( t > 1 \) and \( \delta \in (0, 1) \), the equilibrium effort is \( e_t^* = e_{\text{max}} \).

Proof The proof invokes an inductive approach. Let \( e^* = (e_1^*, e_2^*, \ldots, e_T^*) \) be an equilibrium path. Consider the statement \( S(n) \) such that \( e_{T-n} = e^*, e_{T-n-1} = e^*, \ldots, e_T = e_{\text{max}} \). We infer that \( S(0) \) and \( S(1) \) are true. Suppose the statement \( S(n) \) is true as well, that is \( e_{T-n} = e^*, e_{T-n-1} = e^*, \ldots, e_T = e_{\text{max}} \). Let’s consider the statement \( S(n+1) \). The DM’s payoff at period \( T - (n + 1) \) is 
\[ V_\delta(T - (n + 1)) = U(e_{T-(n+1)}) + \delta U(e_{T-n}) + \delta^2 U(e_{T-n-1}) + \ldots + \delta^{n-1} U(e_T) = U(e_{T-(n+1)}) + \delta U(e^*) + \delta^2 U(e^*) + \ldots + \delta^n U(e^*) \]
As a result, the effort at the period \( T - (n + 1) \) is \( e_{T-(n+1)} = e_{\text{max}} \), therefore, the statement \( S(n+1) \) is true as well. To summarise, the effort at each period \( t > 1 \) is \( e_t = e_{\text{max}} \). This concludes the proof. \( \square \)
FACT 9: In the case of ZIEI and IRI, for \( t = 1 \) and any \( \delta \in (0, 1) \), the equilibrium effort is \( e_1^* = e_{\text{max}} \).

Proof Consider the DM’s payoff at period 1, \( V_\delta(e) = \sum_{t=1}^{T} \delta^{t-1} U(e_t) = U(e_1) + \delta U(e^*_1) + \delta^2 U(e^*_1) + \ldots + \delta^{T-1} U(e^*_1) \). According to Fact 8, we have

\[
V_\delta(e) = \sum_{t=1}^{T} \delta^{t-1} U(e_t) = U(e_1) + \delta U(e_{\text{max}}) + \delta^2 U(e_{\text{max}}) + \ldots + \delta^{T-1} U(e_{\text{max}})
\]

Thus, \( e_1 = e_{\text{max}} \). This concludes the proof. □

This completes the proof of claim.

2 The PIEI and IRI Case

Suppose that the DM is a social planner. Then her expected payoff is \( V_\delta(e) = \sum_{t=1}^{T} \delta^{t-1} U(e_t) \). Hence, it is easy to verify that the DM’s payoff is at maximum only if the effort at each period is \( e_t = e_{\text{max}} \), \( \forall t \). Suppose that the DM decides to deviate from this effort path and her choice is \( e_t = x_t \), where \( x_t \in [0, e_{\text{max}}) \). Thus, the DM’s payoff is

\[
V_\delta(e_t = x_t) = U(x_1) + \delta U(x_2) + \ldots + \delta^{T-1} U(x_T)
\]

Comparing these two payoffs, I have \( V_\delta(e_t = e_{\text{max}}) > V_\delta(e_t = x_t) \). To summarise, if the DM is a social planner, then her optimal effort at each period is \( e_t^* = e_{\text{max}} \), \( \forall t \).

Now, suppose that the DM is a decentralised agent and she is at period \( T \). Then the DM’s payoff is \( V(e_T) = U(e_T) \). Consequently, the DM’s effort at period \( T \) is \( e_T = e_{\text{max}} \). Now, consider the DM at period \( T - 1 \). Hence, her payoff is \( V_\delta(e_{T-1}) = U(e_{T-1}) + \delta U(e_T) = U(e_{T-1}) + \delta U(e_{\text{max}}) \). Hence, her effort at \( T - 1 \) will be \( e_{T-1} = e_{\text{max}} \). Continuing the same logic, the equilibrium effort path will be \( e = (e_{\text{max}}, e_{\text{max}}, \ldots, e_{\text{max}}, e_{\text{max}}) \).

Proof of Claim 5

Claim 5 will be proved through the following facts.
FACT 10: In the case of PIEI and IRI, for any $\delta \in (0,1)$, $T$ and $t > 1$, $e_t^* = e_{\text{max}}$.

Proof We will invoke an inductive approach. Let $e^* = (e_1^*, e_2^*, \ldots, e_T^*)$ be an equilibrium. Consider the statement $S(n)$ such that $e_{T-n} = e_{\text{max}}$, $e_{T-n-1} = e_{\text{max}}$, $\ldots$, $e_T = e_{\text{max}}$. We infer that $S(0)$ and $S(1)$ are true. Suppose that the statement $S(n)$ is true as well. Let’s consider the statement $S(n+1)$. The DM’s payoff at period $T - (n + 1)$ is $V(T - (n + 1)) = U(e_{T-(n+1)}) + \delta U(e_{T-n}) + \delta^2 U(e_{T-n-1}) + \ldots + \delta^{n-1} U(e_T) = U(e_{T-(n+1)}) + \delta U(e_{\text{max}}) + \delta^2 U(e_{\text{max}}) + \ldots + \delta^n U(e_{\text{max}})$. Consequently, the effort is $e_{T-(n+1)} = e_{\text{max}}$. Therefore, the statement $S(n+1)$ is true. Thus, the effort $\forall t > 1$ is $e_t = e_{\text{max}}$. This concludes the proof. □

FACT 11: In the case of PIEI and IRI, for any $\delta \in (0,1)$, $e_1^* = e_{\text{max}}$.

Proof Consider the DM’s payoff at period 1, $V_\delta(e_1) = U(e_1) + \delta U(e_2) + \delta^2 U(e_3) + \ldots + \delta^{T-1} U(e_T) = U(e_1) + \delta U(e_{\text{max}}) + \delta^2 U(e_{\text{max}}) + \ldots + \delta^{T-1} U(e_{\text{max}})$. Hence, $e_1 = e_{\text{max}}$. This ends the proof. □

This completes the proof of claim. ■

3 The NIEI and IRI Case

The analysis of the case for NIEI and IRI with disutility is intractable.
3. CHAPTER

3.1 INTRODUCTION

In this chapter I consider a DM who has Intertemporal Reference dependence (IRD), i.e., the DM will compare her consecutive outcomes in adjacent periods, and thus, apart from her standard direct utility from the outcome of her choice of effort in each period, the DM can also get a reference payoff gain from an increase in the outcome and/or suffer a reference payoff loss from a decrease in the outcome - unless her outcomes are exactly the same in consecutive periods. Specifically, the gain-fondness coefficient $\lambda^+ \geq 0$ of the DM determines the extent of the reference utility increase she will experience from her intertemporal outcome increment. Likewise the DM experiences a reference utility decrease from any loss, i.e., any decline in her outcome compared to last period. In that case, her loss-aversion coefficient $\lambda^- \geq 0$ determines the extent of the reference utility decrease she will experience from that intertemporal outcome drop.

I extend my analysis of the case of ZIEI (Zero Intertemporal Effort Interdependence) with $\lambda^- > 0$ and/or $\lambda^+ > 0$. I first consider the finite-horizon case with $\lambda^- > \lambda^+$, which is the typical case considered in the reference-dependence literature. In this setup, the DM behaves like a standard DM (as in the ZIEI and IRI case) when $\lambda^+$ is relatively low in that she chooses the highest possible effort level for every period. However, when $\lambda^+$ is relatively high and she does not discount the future much, the DM starts with a low level of effort (and thus a low level of outcome) at $t = 1$, but behaves just like a standard DM thereafter, by picking the highest effort level
for every period from then on. I show that this behaviour of the DM is robust
to full commitment, and thus it coincides with a social planner’s optimal plan.

I then consider the case \( \lambda^+ > \lambda^- \). My results turn out to be the same as the
\( \lambda^+ \leq \lambda^- \) case whenever \( \lambda^+ \) does not exceed \( \lambda^- \) by much. When \( \lambda^+ \) exceeds
\( \lambda^- \) significantly, however, a much richer dynamic may arise, and fluctuating
or volatile effort levels and thus outcome levels can become standard under
some parameters: the DM may even prefer first taking a loss so that she can
then enjoy a gain later when \( \lambda^+ > \lambda^- \) and her discounting is strong. Further
she may prefer following this up-and-down pattern throughout, as long as she
finishes with the highest possible effort and outcome. However, in this case
the DM’s behaviour is no longer robust to full commitment, and it does not
coincide with a social planner’s optimal path any longer. Finally, I also show
that incorporating disutility of effort does not affect the outcome qualitatively.
Neither does any diminishing marginal productivity of effort.

3.2 Model

In this chapter too I will follow the same definitions and notations from
Chapter 2. Briefly, here too I consider the DM who takes part in the finite
dynamic task, where time is discrete and \( T < \infty \). In each period \( t = 1, \ldots, T \),
the DM has to take a costless effort \( e_t \in [0, 1] \) that yields a direct utility
of \( u(e_t) > 0 \) in period \( t \) with certainty. The DM’s period \( t \geq 2 \) utility
from the effort path \( e = (e_1, \ldots, e_T) \) is
\[
\nu_t(e) = u(e_t) + \lambda^+(u(e_t) - u(e_{t-1}))
\]
if \( u(e_t) > u(e_{t-1}) \) and
\[
\nu_t(e) = u(e_t) - \lambda^-(u(e_{t-1}) - u(e_t))
\]
if \( u(e_t) < u(e_{t-1}) \),
where \( \lambda^+(u(e_t) - u(e_{t-1})) \) is a reference utility gain, and \( \lambda^-(u(e_{t-1}) - u(e_t)) \)
is a reference utility loss. The DM chooses \( e \in [0, 1]^T \) to maximise her
discounted sum of utility given by

\[
V^*_\delta(e) = u(e_1) + \sum_{t=2}^T \delta^{t-1} \nu_t(e),
\]

where \( \delta \in (0, 1) \) is her time discount factor.
3.3 INTERTEMPORAL REFERENCE DEPENDENCE: $\lambda^- > 0$

AND/OR $\lambda^+ > 0$

3.3.1 THE ZIEI CASE WITH A $\lambda^-$-DOMINANT DM

Here, I will study the **Zero Intertemporal Effort Interdependence** (ZIEI) (i.e., $u(e_t) = e_t$) with $\lambda^+ < \lambda^-$. The following Proposition 4 summarizes the main findings of this section. (For proofs of lemmas, claims and propositions, see Appendix 1, unless stated otherwise.)

**PROPOSITION 4:** For any $\delta \in (0, 1)$, $\lambda^+ < \lambda^-$ and $T$, $e^*$ is an optimal path if and only if $e^*$ is fully commitment proof where $e^*_t = 1$ for all $t > 1$ while $e_1 = 1$ if $\lambda^+ < \frac{1}{\delta}$ and $e_1 = 0$ otherwise.

Thus, Proposition 4 states that if $\lambda^+ \delta < 1$, i.e., both the gain-fondness coefficient and her discount factor are relatively small, the DM will behave just like the DM in the ZIEI and IRI case choosing an effort path of $(1, 1, 1, ..., 1, 1)$ or one with a slightly different beginning $(0, 1, 1, ..., 1, 1)$. If both $\lambda^+$ and $\delta$ are less than one, then $(1, 1, 1, ..., 1, 1)$ will take place.

If, however, $\lambda^+ \delta > 1$, (i.e., $\lambda^+ > \frac{1}{\delta}$ or $\delta > \frac{1}{\lambda^+}$), then our DM will exhibit a big increase in her effort initially and then follow up with the maximum effort afterwards. This requires that both $\lambda^+$ and $\delta$ are substantial. A low $\delta$ would prevent the DM from doing so since obtaining a 0 payoff at period 1 would be unacceptable for such a DM; likewise, a low $\lambda^+$ would not induce the DM to experience such an gain or improvement at the expense of a 0 payoff at period 1.

Proposition 4 will be proved through the following lemmas and propositions.

**LEMMA 8:** For both $k=1$ and $k=T$, $e^*_T = 1$. 

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**Lemma 9:** Suppose \( k = T \). There exists an optimal solution such that for each \( t \geq 2 \), we have \( e^*_t = 1 \).

While Lemma 9 allows us to clearly show that if the equilibrium set is non-empty, then there is an equilibrium such that the DM provides the effort level 1 in each of the last \( T - 1 \) periods. I can use Lemma 9 to prove that the equilibrium set is non-empty. In what follows, I will provide a more direct approach and show that the above-mentioned equilibrium is unique. In that result too, I will keep assuming that \( k = T \). But before that, I will give an indication of the direction I will take by showing that for any \( t > 1 \) it cannot be that \( e_{t+1} = 0 \).

**Lemma 10:** For \( k = 1 \) and \( k = T \), no optimal effort path can have a period \( t > 1 \) such that \( e_{t+1} = 0 \).

This Lemma 10 states that when the DM has Zero Intertemporal Effort Interdependence (ZIEI) and Intertemporal Reference Dependence (IRD) with \( \lambda^- > \lambda^+ \), then the DM never chooses the lowest possible effort in any subsequent period.

**Lemma 11:** Suppose \( k = T \). Then in any equilibrium, the following must be true: there exists \( 0 < x \leq 1 \) such that \( e_2 = \ldots = e_{T-1} = x \leq 1 = e_T \) and \( e_1 = x \) if \( \delta < \frac{1}{\lambda^+} \) and \( e_1 = 0 \) otherwise.

**Lemma 12:** Suppose \( k = T \). Then the effort path \( e^x = (0, x, x, \ldots, 1) \) is strictly dominated by \( e^1 \) and the effort path \( e^x = (x, x, x, \ldots, 1) \) is strictly dominated by \( e^1 \).

This means that when the DM has Zero Intertemporal Effort Interdependence (ZIEI) and Intertemporal Reference Dependence (IRD) with \( \lambda^- > \lambda^+ \), then if the coefficient of gain-fondness is relatively low, the effort path (where the DM chooses the highest possible effort at each period) strictly
dominates any other effort paths. If the coefficient of gain fondness is relatively high and the DM does not discount the future much, the effort path (where the DM starts with the lowest level of effort at the first period, but behaves just like a standard DM thereafter by picking the highest effort level at every period) strictly dominates any other effort paths as well.

Lemma 13: The optimal effort path with $k = 1$ is identical to that with $k = T$.

This completes the proof of the Proposition 4. ■

As mentioned above again, if the DM was a standard utility person as in the ZIEI and IRI case, she would choose the effort ‘1’ in each of the $T$ periods. The above analysis then yields the following interesting proposition (and thus the proof will be skipped).

Proposition 5: For $k=1$ and $k=T$, the efforts of a standard utility maximizing DM are identical to a loss-averse/gain-fond utility maximizing DM irrespective of the value of $\lambda^-$ that exceeds $\lambda^+$ but if and only if $\lambda^+ < \frac{1}{\delta}$.

The above analysis yields a “Boolean” world in the effort space as the DM essentially remains fully active except in the initial period where she may either be fully active or fully inactive.

3.3.2 The ZIEI Case with a $\lambda^+$-Dominant DM

Now, I will study the ZIEI case where $\lambda^+ \geq \lambda^-$, i.e., the DM has a higher gain-fondness coefficient than her loss-aversion coefficient. In the above section with $\lambda^+ < \lambda^-$, the DM would never choose an effort that leads to an intertemporal loss when $\lambda^- > \lambda^+$. However, as I will show, this need not hold with $\lambda^+ \geq \lambda^-$ (although it can still continue to hold when $\lambda^+ \geq \lambda^-$ with $\lambda^+$ and $\lambda^-$ being very close).
In a case of a $\lambda^+$-dominant DM, she may prefer first taking a loss so that she can then enjoy a gain when $\lambda^+ \geq \lambda^-$ and her discounting is strong. Further, she may prefer following this up-and-down pattern throughout, as long as she finishes everything on a good note, i.e., with an effort level $e_T = 1$.

The following facts will be useful. I will need the following definition:

**Definition 10:** (Prior) Effort history $h_{t-1}$ for period $t$ is defined as an effort that was exerted by the DM in period $t-1$.\(^1\)

**Fact 12:** For any $\delta \in (0, 1)$, $T, k=1, k=T$, and $\lambda^+ \geq \lambda^-$, $e_T = 1$.

*Proof* Suppose the DM is at period $T$. Then $e_T = 1$ regardless of $\lambda^+ \geq \lambda^-$. To see this let us fix any immediate effort history $h_{T-1} = e_{T-1} \in [0, 1]$ for period $T$. Then given $u(\cdot)$ is monotonic increasing, it cannot be that $e_T < h_{T-1}$. Given that, it also follows that $e_T$ takes its maximum value on $[0, 1]$, i.e., $e_T = 1$. As this argument does not use commitment power, the result follows.\(\Box\)

**Fact 13:** For $T = 2$, the effort path is $(1, 1)$ if $\delta < \frac{1}{\lambda^+}$; or $(0, 1)$ otherwise for any $\delta$ and $k$.

*Proof* Suppose that $T = 2$. The last period effort is $e_T = 1$. The DM payoff from $e_1$ is $u(e_1) + \delta(1 + \lambda^+(1 - u(e_1)))$. Then $C(e_1)) = 1 - \delta \lambda^+$.\(^2\) This implies that if $C(e_1) = 1 - \delta \lambda^+ > 0$ then $e_1 = 1$, otherwise $e_1 = 0$. This proves Fact 13.\(\Box\)

In the next section I will first proceed with examples.

### 3.3.3 Example with $T = 3$ and $T = 4$

Here, I will provide only outcomes of the cases $T = 3$ and $T = 4$. For details of these examples, see Appendices 2 and 3. These examples provide a wide

\(^1\) This definition will be useful in proofs
\(^2\) I use this notation $C(e_i)$ throughout this chapter whenever I need to analyse the coefficient of $u(e_i)$ in $V_\delta(e)$. 

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spectrum of all possible effort paths that the DM can follow. Moreover, these examples demonstrate that regardless of \( k = 1 \) or \( k = T \), the effort path depends on the relation between \( \lambda^+ \) and \( \lambda^- \).

For \( T = 3 \), I have:

When \( \lambda^+ < \lambda^-(1 + \lambda^-) \), the optimal effort path is \( e^* = (1, 1, 1) \) if \( \delta < \frac{1}{\lambda^+} \) and \( \delta < \frac{1 + \lambda^-}{\lambda^+} \).

If \( \delta > \frac{1}{\lambda^+} \) and \( \delta < \frac{1 + \lambda^-}{\lambda^+} \), the optimal effort path is \( e^* = (0, 1, 1) \).

When \( \lambda^+ > \lambda^-(1 + \lambda^-) \), however, the DM’s optimal effort path could be

\( e^* = (1, 1, 1) \) if \( \delta < \frac{1}{\lambda^+} \);

\( e^* = (0, 1, 1) \) if \( \delta > \frac{1}{\lambda^+} \) and \( \delta < \frac{1 + \lambda^-}{\lambda^+} \);

and \( e^* = (1, 0, 1) \) if \( \delta < \frac{1}{\lambda^+} \) and \( \delta > \frac{1 + \lambda^-}{\lambda^+} \).

Next, I introduce outcomes for the example of \( T = 4 \). This example is also important because it provides effort paths different from those in the case of \( T = 3 \).

When \( \lambda^+ < \lambda^-(1 + \lambda^-) \), the equilibrium effort path is \( e^* = (1, 1, 1, 1) \) if \( \delta < \frac{1}{\lambda^+} \) and \( \delta < \frac{1 + \lambda^-}{\lambda^+} \).

If \( \delta > \frac{1}{\lambda^+} \) and \( \delta < \frac{1 + \lambda^-}{\lambda^+} \) the optimal effort path is \( e^* = (0, 1, 1, 1) \).

If \( \delta > \frac{1}{\lambda^+} \) and \( \delta > \frac{1 + \lambda^-}{\lambda^+} \) the optimal effort path is \( e^* = (0, 1, 0, 1) \).

When \( \lambda^+ > \lambda^-(1 + \lambda^-) \), however, the DM’s optimal effort path could be

\( e^* = (1, 1, 1, 1) \) if \( \delta < \frac{1}{\lambda^+} \) and \( \delta < \frac{1 + \lambda^-}{\lambda^+} \);

\( e^* = (0, 1, 1, 1) \) if \( \delta > \frac{1}{\lambda^+} \) and \( \delta < \frac{1 + \lambda^-}{\lambda^+} \);

\( e^* = (1, 0, 1, 1) \) if \( \delta < \frac{1}{\lambda^+} \) and \( \delta > \frac{1 + \lambda^-}{\lambda^+} \);

and \( e^* = (0, 1, 0, 1) \) if \( \delta > \frac{1}{\lambda^+} \) and \( \delta > \frac{1 + \lambda^-}{\lambda^+} \).

Based on examples for \( T = 3 \) and \( T = 4 \), the following Claim 6 is immediate:

**Claim 6:** For \( \lambda^+ \geq \lambda^- \), \( T = 3 \) and \( T = 4 \) the optimal effort path is determined by the relation between \( \delta \) and \( \lambda^+ \) and \( \lambda^-(1 + \lambda^-) \) and \( T \).

The above Claim 6 is a short-cut of the following:
(i) If $\lambda^+ < \lambda^-(1 + \lambda^-)$:

$$e^* = \begin{cases}
(1, 1, 1, \{1\}), & \text{if } \delta < \frac{1}{\lambda^+} \\
(0, 1, 1, \{1\}), & \text{if } \delta > \frac{1}{\lambda^+} \text{ and } \delta < \frac{1+\lambda^-}{\lambda^+} \\
(0, 1, 0, \{1\}), & \text{if } \delta > \frac{1+\lambda^-}{\lambda^+}
\end{cases}$$

(ii) if $\lambda^+ > \lambda^-(1 + \lambda^-)$:

$$e^* = \begin{cases}
(1, 1, 1, \{1\}), & \text{if } \delta < \frac{1}{\lambda^+} \\
(0, 1, 1, \{1\}), & \text{if } \delta > \frac{1}{\lambda^+} \text{ and } \delta < \frac{1+\lambda^-}{\lambda^+} \\
(1, 0, 1, \{1\}), & \text{if } \delta < \frac{1}{\lambda^+} \text{ and } \delta > \frac{1+\lambda^-}{\lambda^+} \\
(0, 1, 0, \{1\}), & \text{if } \delta > \frac{1}{\lambda^+} \text{ and } \delta > \frac{1+\lambda^-}{\lambda^+}
\end{cases}$$

Now, I provide the general result in the case of $T$ periods. The DM’s optimal effort path is given by the following proposition.
PROPOSITION 6: For $\lambda^+ > \lambda^-$ the equilibrium effort path is:
\[ e = (1, 1, \ldots, 1, 1) \text{ if } \delta < \frac{1}{\lambda^+}, \]
\[ e = (0, 1, 1, \ldots, 1, 1) \text{ if } \frac{1 + \lambda^-}{\lambda^+} > \delta > \frac{1}{\lambda^+}; \]
Or there is $e = (0, 1, 0, 1, \ldots, 0, 1)$ if $\frac{1 + \lambda^-}{\lambda^+} < \delta$.

When the DM has Zero Intertemporal Effort Interdependence (ZIEI) and Intertemporal Reference Dependence (IRD) with $\lambda^+ > \lambda^-$ (i.e., "$\lambda^+$-dominant" case), then the DM may even prefer taking a loss so that she can then enjoy a gain later. Further, the DM may prefer following this up-and-down pattern throughout, as long as she finishes with the highest possible effort and outcome.

See Appendix 1 for the proof.

3.4 DISCUSSION: ROBUSTNESS CHECKS

In this chapter too, until now, I have considered the direct utility which is a function of costless effort that was characterised by constant marginal productivity. Now I will consider diminishing marginal productivity of effort within each period. As a result, the direct utility is determined as $u(e_t) = e_t^\alpha$, where $\alpha \in [0, 1)$. In the case of ZIEI and IRD, where there exists intertemporal linkage between periods in terms involving intertemporal loss aversion or intertemporal gain fondness, it turns out that introducing a $\alpha \in [0, 1)$ does not change the results at all in both cases: $\lambda^-$-dominant DM or $\lambda^+$-dominant DM.

In the case of ZIEI and $\lambda^- -$ dominant DM the following claim describes the effort path:

CLAIM 7: In the case of ZIEI and a $\lambda^- -$ dominant DM, for any $\delta \in (0, 1)$, $\lambda^+ < \lambda^-$ and $T$, $e^*$ is an optimal path if and only if $e^*$ is fully commitment proof where $e_t^* = 1$ for all $t > 1$ while $e_1 = 1$ if $\lambda^+ < \frac{1}{\delta}$ and $e_1 = 0$ otherwise.
The proof of claim mimics the related proofs in Chapter 2; therefore I omit it here.

In the case of ZIEI and a $\lambda^+$-dominant DM, the following claim describes the effort path under the condition of diminishing marginal productivity of effort:

**CLAIM 8:** For $\lambda^+ \geq \lambda^-$ the optimal effort path is:

- If $\delta < \frac{1}{\lambda^+}$, $e = (1,1,\ldots,1,1)$;
- If $\frac{1+\lambda^-}{\lambda^+} > \delta > \frac{1}{\lambda^+}$, $e = (0,1,1,\ldots,1,1)$;
- Or there is $e = (0,1,0,1,\ldots,0,1)$ if $\frac{1+\lambda^-}{\lambda^+} < \delta$.

Again, the proof of claim mimics the related proofs in Chapter 2; therefore I omit it here.

Such a robustness is not observed when I consider disutility of effort within each period. Consider the disutility (or cost) of effort $C(x) = \beta e^\gamma_t$, where $\beta \geq 0$ and $\gamma \geq 1$. In the case of ZIEI with IRD, with $\beta > 0$ and $\gamma > 1$, the DM will choose the effort level $0 < e_{\text{max}} < 1$ to maximize her utility. Suppose the DM tries to maximize the direct utility $U(e_t) = u(e_t) - C(e_t)$, then the DM will choose the effort level $0 < e_{\text{max}} < 1$ to maximize her utility. Thus, the results will remain qualitatively the same. To prove it, I replace $u(e_t)$ by $U_t$ in the case of ZIEI and $\lambda^-$-dominance and in the case of ZIEI and $\lambda^+$-dominance to obtain the following claims.

**CLAIM 9:** In the case of ZIEI and a $\lambda^-$-dominant DM, for any $\delta \in (0,1)$, $\lambda^+ < \lambda^-$ and $T$, $e^*$ is an optimal path if and only if $e^*$ is fully commitment proof where $e^*_t = e_{\text{max}}$ for all $t > 1$ while $e_1 = e_{\text{max}}$ if $\lambda^+ < \frac{1}{\delta}$ and $e_1 = 0$ otherwise.

Again, as the proof of claim mimics the related proofs in Chapter 2, I omit it here.

**CLAIM 10:** For $\lambda^+ \geq \lambda^-$ the equilibrium effort path is:
If $\delta < \frac{1}{\lambda^+}$, $e = (e_{\text{max}}, e_{\text{max}}, \ldots, e_{\text{max}}, e_{\text{max}})$;

If $\frac{1+\lambda^-}{\lambda^+} > \delta > \frac{1}{\lambda^+}$, $e = (0, e_{\text{max}}, e_{\text{max}}, \ldots, e_{\text{max}}, e_{\text{max}})$;

Or there is $e = (0, e_{\text{max}}, 0, e_{\text{max}}, \ldots, 0, e_{\text{max}})$ if $\frac{1+\lambda^-}{\lambda^+} < \delta$.

Likewise, as the proof of claim mimics the related proofs in Chapter 2, I omit it here.

### 3.5 Conclusion

In this chapter I analyse the Intertemporal Reference Dependence (IRD) case, i.e., the case that involves Intertemporal Loss Aversion or Intertemporal Gain Fondness. Unlike Chapter 2, in this chapter the DM compares her consecutive outcomes in adjacent periods, and thus, her payoff at each period gets either a gain due to an increase in her output or a payoff loss due to an decrease in her outcome, unless her outcomes are exactly the same in consecutive periods.

I found that in the framework of “$\lambda^-$-dominance” the DM will behave just like the DM in the ZIEI and IRI case choosing an effort path of $(1, 1, 1, \ldots, 1, 1)$ or another effort path which has a slightly different beginning $(0, 1, 1, \ldots, 1, 1)$. This behaviour is robust to full or no commitment, and thus it also coincides with a social planner’s optimal plan. It is also robust to diminishing marginal productivity of effort and disutility of effort.

In the case of “$\lambda^+$-dominance”, when $\lambda^+$ does not exceed $\lambda^-$ by much, the DM behaves just like the way she did in the $\lambda^-$-dominant case. In the case where $\lambda^+$ exceeds $\lambda^-$ significantly, however, the DM exhibits different effort paths even with an up-and-down pattern throughout, as long as she finishes with an effort level $e_T = 1$. It is robust to diminishing marginal productivity of effort and disutility of effort.

Thus, overall, when the DM moves from the ZIEI and IRI case to the ZIEI and IRD case, whether or not $\lambda^- > \lambda^+$ starts mattering a lot. The DM with $\lambda^- > \lambda^+$ still behaves very much like the DM with $\lambda^- = \lambda^+ = 0$, while the
DM with $\lambda^- \leq \lambda^+$ may start behaving very differently especially as $\lambda^-$ and $\lambda^+$ diverge.
APPENDIX 1: PROOFS

Proof of Lemma 8

Let us fix any immediate history $h_{T-1} \in [0, 1]$ for period $T$. Then, given $u(\cdot)$ is monotonic increasing, it cannot be that $e_T < h_{T-1}$. Given that, it also follows that $e_T$ takes its maximum value on $[0, 1]$. As this argument does not use commitment power. Lemma 8 has been proved. □

Proof of Lemma 9

The proof will involve an induction argument and invoke the One-Step-Deviation Principle (OSDP). Let $e^* = (e^*_1, e^*_2, \ldots, e^*_T)$ be an equilibrium and consider the statement $S(n), T-1 \geq n \geq 0$ that $e^*_{T-n} = e^*_{T-(n-1)}, \ldots, e^*_{T-1}, e^*_T = 1$.

From Lemma 4 we know that $e^*_T = 1$ and hence $S(0)$ is true. So consider $S(1)$ and note that to prove it is true, OSDP ensures that blocking any deviation at period $T-1$ will suffice. So given $e^*$, the utility of the DM from $e^*$ (given $S(1)$ holds) is

$$V_\delta(e^*|S(1)) = u(e^*_1) + \sum_{t=2}^{T-2} \delta^{t-1} v_t(e^*|S(1)) + \delta^{T-2} [1 + \lambda^+(1 - u(e^*_{T-2}))] + \delta^{T-1}.$$ 

Suppose the DM considers a OSD to alter her original plan at period $T-1$ and let that choice be $x \in [0, 1)$. We need to consider two cases separately. First assume $x \leq e^*_{T-2}$. In this case, her utility is

$$V_\delta(e^*|x) = u(e^*_1) + \sum_{t=2}^{T-2} \delta^{t-1} v_t(e^*|S(1)) + \delta^{T-2} [u(x) - \lambda^-(u(e^*_{T-2}) - u(x))] + \delta^{T-1} [1 + \lambda^+(1 - u(x))]$$

Let $C(x)$ be the coefficient of $u(x)$ in $V_\delta(e^*|x)$. Then

$$C(x) = \delta^{T-2} u(x)[1 + \lambda^- - \delta \lambda^+] > 0.$$
Hence, \( x = e_{T-2}^* \).

Next assume \( x \geq e_{T-2}^* \). In this case, her utility is

\[
V_\delta(e^*|x) = u(e_1^*) + \sum_{t=2}^{T-2} \delta^{t-1} v_t(e^*|S(1)) + \delta^{T-2} [u(x) + \lambda^+(u(x) - u(e_{T-2}^*))] + \\
+ \delta^{T-1} [1 + \lambda^+(1 - u(x))].
\]

Here,

\[
C(x) = \delta^{T-2} u(x)[1 + \lambda^+ - \delta \lambda^+] > 0.
\]

Hence, \( x = 1 \). We next show that \( V_\delta(e^*|x = 1) > V_\delta(e^*|e_{T-2}^*) \) for each \( e^* \).

Observe that

\[
V_\delta(e^*|x = 1) - V_\delta(e^*|e_{T-2}^*) = (1 - u(e_{T-2}^*)) [1 + \lambda^+ (1 - \delta)] > 0
\]

whenever \( e_{T-2}^* < 1 \). If \( e_{T-2}^* = 1 \) then the result is trivially true. Hence we have shown that \( S(1) \) is true. So now assume \( S(n) \) is true and check \( S(n+1) \). But OSDP and the proof above for \( S(1) \) render this argument in a straightforward way. So see this, suppose \( S(n) \) is true. This means irrespective of \( e_{n-1}^* \), it must be that \( e_n^* = e_{n+1}^* = \ldots e_T^* = 1 \). From the proof for \( S(1) \), we know that irrespective of the value of \( e_{n-2}^* \), it must be that \( e_{n-1}^* = 1 \). Hence \( S(n+1) \) is true. This completes the proof. \( \square \)

**Proof of Lemma 10**

Assume \( k = T \). Pick any arbitrary \( t \in \{2, \ldots, T-1\} \) and consider the path

\[
e(e_t, e_{t+1}) = (e_1, \ldots, e_{t-1}, e_t, e_{t+1}, e_{t+2}, \ldots, e_{T-1}, 1).
\]

Let

\[
V_\delta^j(e(e_t, e_{t+1})) := \sum_{i=j}^{T} \delta^{i-j} v(e(e_t, e_{t+1})).
\]

Then,

\[
V_\delta(e(1, 1)) := 1 + \lambda^+[1 - u(e_{t-1})] + \delta + \delta^2 [u(e_{t+2}) - \lambda^- (1 - u(e_{t+2})] + \\
+ \delta V_\delta^{t+3}(e(e_t, e_{t+1})),
\]

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\[
V_\delta(e(1,0)) := 1 + \lambda^+ [1 - u(e_{t-1})] - \delta\lambda^- + \delta^2 [u(e_{t+2}) + \lambda^+ u(e_{t+2})] + \\
\quad + \delta V_\delta^{t+3}(e(e_t,e_{t+1})),
\]
and
\[
V_\delta(e(0,0)) := -\lambda^- [u(e_{t-1})] + \delta^2 [u(e_{t+2}) + \lambda^+ u(e_{t+2})] + \delta V_\delta^{t+3}(e(e_t,e_{t+1})).
\]

It is easy to verify that \(V_\delta(e(1,1)|e_{t-1} = 1) \leq V_\delta^t(e(1,1)|e_{t-1})\) for all \(e_{t-1} \in [0,1]\). Now, \(V_\delta^t(e(1,1)|e_{t-1} = 1) > V_\delta^t(e(0,0))\) if and only if \(1 + \lambda^- - \delta \lambda^- > \delta(\lambda^+ - \lambda^-)u(e_{t+2})\). With \(\lambda^- > \lambda^+\), the RHS is negative while the LHS is positive. Hence \(V_\delta^t(e(1,1)|e_{t-1} = 1) > V_\delta^t(e(0,0))\) for all effort paths. Also, \(V_\delta^t(e(1,1)|e_{t-1} = 1) > V_\delta^t(e(0,0))\) if and only if \(1 + \delta + \lambda^- (1 - \delta^2) > \delta^2 (\lambda^+ - \lambda^-)u(e_{t+2})\). With \(\lambda^- < \lambda^+\). Again the RHS is negative while the LHS is positive. Hence \(V_\delta^t(e(1,1)|e_{t-1} = 1) > V_\delta^t(e(0,0))\) for all effort paths.

Put together we have have proved the lemma. □

**Proof of Lemma 11.**

Consider an arbitrary equilibrium effort path \(e = e_1, e_2, e_3, \ldots, e_T\), assuming an equilibrium exists. We will first show that \(e_1 \in \{0,e_2\}\). Given \(e\), we consider two cases. In the first, we assume that \(e_1 \leq e_2\). Then

\[
V_\delta(e) = u(e_1) + \delta[u(e_2) + \lambda^+(u(e_2) - u(e_1))] + \delta^2 V_\delta^{t+3}
\]

where \(V_\delta^{t+3} = \sum_{t=3}^{T} \delta^{t-3} v_t(\cdot)\). Then by OSDP it suffices to optimize w.r.t. \(e_1\) to see the best choice in period 1. Here \(C(e_1) = 1 - \delta \lambda^+\). Hence, \(e_1 = 0\) if \(\delta > \frac{1}{\lambda^+}\) and \(e_1 = e_2\) if \(\delta < \frac{1}{\lambda^+}\).

Similarly, if \(e_1 \geq e_2\) then

\[
V_\delta(e) = u(e_1) + \delta[u(e_2) - \lambda^-(u(e_1) - u(e_2))] + \delta^2 V_\delta^{t+3}
\]

where \(V_\delta^{t+3}\) is as before. Then by OSDP it again suffices to optimize w.r.t. \(e_1\) to see the best choice in period 1. Here \(C(e_1) = 1 - \delta \lambda^-\). Hence, \(e_1 = e_2\) if \(\delta > \frac{1}{\lambda^-}\) and \(e_1 = 1\) if \(\delta < \frac{1}{\lambda^-}\). Given \(\lambda^+ < \lambda^-\), there are a maximum of three cases to consider. First, suppose \(\delta > \frac{1}{\lambda^+}\). By choosing
the effort \( e_2 \) the DM’s payoff is \( u(e_2) + \delta u(e_2) + \delta^2 V_\delta^{\geq 3} \) while from effort 0 it is \( \delta[u(e_2) + \lambda^+ u(e_2)] + \delta^2 V_\delta^{\geq 3} \). Since \( \delta > \frac{1}{\lambda^+} \), it follows that the effort 0 strictly dominates \( e_2 \). Next suppose \( \delta < \frac{1}{\lambda^+} \). By taking the effort 1 the DM’s payoff is \( 1 + \delta[u(e_2) - \lambda^- (1 - u(e_2))] + \delta^2 V_\delta^{\geq 3} \) while from effort \( e_2 \) it is \( u(e_2) + \delta u(e_2) + \delta^2 V_\delta^{\geq 3} \). Since \( \delta < \frac{1}{\lambda^+} \), it follows that the effort 1 strictly dominates \( e_2 \). Finally if \( \frac{1}{\lambda^+} < \delta < \frac{1}{\lambda^-} \) the effort is always \( e_2 \).

Now we pin down \( e_2 \) in relation to \( e_3 \).

- **Case \( \delta > \frac{1}{\lambda^+} \):**
  - \( 0 \leq e_2 \leq e_3 \): The DM’s payoff from \( e_2 \) is \( \delta[u(e_2) + \lambda^+ u(e_2)] + \delta^2[u(e_3) + \lambda^+(u(e_3) - u(e_2))] \). Then \( C(e_2) = \delta(1 + \lambda^+ - \delta \lambda^+) > 0 \). This means \( e_2 = e_3 \).
  - \( 0 \leq e_2 \geq e_3 \): The DM’s payoff from \( e_2 \) is \( \delta[u(e_2) + \lambda^+ u(e_2)] + \delta^2[u(e_3) - \lambda^-(u(e_2) - u(e_3))] \). Then \( C(e_2) = \delta(1 + \lambda^+ - \delta \lambda^-) \). If \( C(e_2) > 0 \), we have \( e_2 = 1 \). If \( C(e_2) < 0 \), then \( e_2 = e_3 \).
  - There exists \( x, y \in [0, 1] \) such that \( (e_1, e_2, e_3) \in \{(0, x, x), (0, 1, y)\} \).

- **Case \( \frac{1}{\lambda^+} < \delta < \frac{1}{\lambda^-} \):**
  - \( e_2 \leq e_3 \): The DM’s payoff from \( e_2 \) is \( u(e_2) + \delta u(e_2) + \delta^2[u(e_3) + \lambda^+(u(e_3) - u(e_2))] \). Then \( C(e_2) = 1 + \delta - \delta^2 \lambda^+ > 0 \) since \( \delta < \frac{1}{\lambda^+} \). This means \( e_2 = e_3 \).
  - \( e_2 \geq e_3 \): The DM’s payoff from \( e_2 \) is \( u(e_2) + \delta u(e_2) + \delta^2[u(e_3) - \lambda^-(u(e_2) - u(e_3))] \). Then \( C(e_2) = 1 + \delta - \delta^2 \lambda^- \). So if \( C(e_2) > 0 \), then we have \( e_2 = e_3 \). If \( C(e_2) < 0 \), then \( e_2 = 0 \).
  - There exists \( x \in [0, 1] \) such that \( (e_1, e_2, e_3) = (x, x, x) \).

- **Case \( \frac{1}{\lambda^-} > \delta \):**
- $e_2 \leq e_3$: The DM’s payoff from $e_2$ is 
\[1 + \delta [u(e_2) - \lambda^-(1 - u(e_2))] + \delta^2 [u(e_3) + \lambda^+(u(e_3) - u(e_2))] + \delta^3 [u(e_4) + \lambda^+(u(e_4) - u(y))] + \ldots.\]
Then, $C(y) = \delta [1 + \lambda^- - \lambda^+] > 0$. Hence $y = 1$. On the other hand if $e_4 \leq y$ then the payoff is 
\[1 + \delta (u(y) - \lambda^-(1 - u(y))) + \delta^2 u(y) + \delta^3 [u(e_4) - \lambda^-(u(y) - u(e_4))] + \ldots.\]
Then, $C(y) = \delta^2 [1 + \lambda^- - \delta \lambda^-] > 0$. Hence $y = 1$.

CLAIM 11: $(e_1,e_2,e_3) = (0,1,y)$ if and only if $y = 1$.

To see this for any effort path with $(e_1,e_2,e_3) = (0,1,y)$, the payoff of the DM with $e_4 \geq y$ is 
\[\delta [1 + \lambda^+] + \delta^2 [u(y) - \lambda^-(1 - u(y))] + \delta^3 [u(e_4) + \lambda^+(u(e_4) - u(y))] + \ldots.\]

Then, $C(y) = \delta^2 [1 + \lambda^- - \delta \lambda^+] > 0$. Hence $y = 1$. On the other hand if $e_4 \leq y$ then the payoff is 
\[1 + \delta (u(y) - \lambda^-(1 - u(y))) + \delta^2 u(y) + \delta^3 [u(e_4) - \lambda^-(u(y) - u(e_4))] + \ldots.\]
Then, $C(y) = \delta^2 [1 + \lambda^- - \delta \lambda^-] > 0$. Hence $y = 1$.

CLAIM 12: $(e_1,e_2,e_3) = (1,y,y)$ if and only if $y = 1$.

To see this, given any effort path with $(e_1,e_2,e_3) = (1,y,y)$, the payoff of the DM with $e_4 \geq y$ is 
\[1 + \delta (u(y) - \lambda^-(1 - u(y))) + \delta^2 u(y) + \delta^3 [u(e_4) - \lambda^+(u(e_4) - u(y))] + \ldots.\]
Then $C(y) = \delta [1 + \lambda^- - \delta - \delta \lambda^+] > 0$. Hence $y = 1$. If $e_4 \leq y$ is 
\[1 + \delta (u(y) - \lambda^-(1 - u(y))) + \delta^2 u(y) + \delta^3 [u(e_4) - \lambda^-(u(y) - u(e_4))] + \ldots.\]
Then $C(y) = \delta [1 + \lambda^- + \delta - \delta \lambda^-] > 0$. Hence $y = 1$. 

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So we know that any effort path must respect the following:

\((e_1, e_2, e_3) \in \{(0, x, x), (x, x, x)\}\).

Now we show that then \(e_4 = x\). To see this consider the path

\(x, x, x, e_4, e_5, \ldots, 1\). We have shown that irrespective of \(e_4\), the path should start \(x, x, x\). Now consider the choice of \(e_4\). This is equivalent to choosing \(e_4\) at \(e_1, x, x, e_4, e_5, \ldots, 1\). Hence the two problems are equivalent. Now suppose the path is \(0, x, x, e_4, e_5, \ldots, 1\) and consider the choice of \(e_4\). This is equivalent to choosing \(e_4\) at \(e_1, x, x, e_4, e_5, \ldots, 1\). Hence the two problems are equivalent again.

This completes the proof of the lemma. □

**Proof of Lemma 12.**

Given Lemma 4, we know that in any equilibrium we have \(e_T = 1\). So for any arbitrary effort path \(e = e_1, \ldots, e_{T-2}, e_{T-1}, 1\). The payoff of the DM at time \(T - 1\) from her effort \(e_{T-1}\) is as follows. Suppose \(e_{T-2} \leq e_{T-1}\). Then it is \(u(e_{T-1}) + \lambda^+[u(e_{T-1}) - u(e_{T-2})] + \delta[1 + \lambda^+(1 - u(e_{T-1}))]\). Hence \(C(e_{T-1}) = 1 + \lambda^+ - \delta \lambda^+ > 0\). Hence \(e_{T-1} = 1\). Suppose \(e_{T-2} \geq e_{T-1}\). Then it is \(u(e_{T-1}) - \lambda^-[u(e_{T-2}) - u(e_{T-1})] + \delta[1 + \lambda^+(1 - u(e_{T-1}))]\). Hence \(C(e_{T-1}) = 1 + \lambda^- - \delta \lambda^+ > 0\). Hence \(e_{T-1} = e_{T-2}\). But it is easy to verify that payoff from 1 dominates payoff from \(e_{T-2}\). This implies that as long as there is history of efforts, the DM’s future is the constant effort path of \(1, 1, \ldots, 1\). Given this, her period 1 problem is identical to the case with \(k = 1\) and the result follows. □

**Proof of Proposition 6**

Consider the last two periods pattern \(e_{T-1}, e_T\). According to, Fact 12 \(e_T = 1\). Assume that \(e_{T-1} \geq e_{T-2}\). The DM’s outcome payoff from \(e_{T-1}\) is \(u(e_{T-1}) + \lambda^+[u(e_{T-1}) - u(e_{T-2})] + \delta[1 + \lambda^+(1 - u(e_{T-1}))]\). There is \(C(e_{T-1}) = 1 + \lambda^+ - \delta \lambda^+ > 0\) always. It means that \(e_{T-1} = 1\).

Now suppose that \(e_{T-1} \leq e_{T-2}\). The DM’s payoff from \(e_{T-1}\) is \(u(e_{T-1}) - \lambda^-[u(e_{T-2}) - u(e_{T-1})] + \delta(1 + \lambda^+(1 - u(e_{T-1}))\). There is \(C(e_{T-1}) = (1 + \lambda^- - \delta \lambda^+ > 0\) always. It means that \(e_{T-1} = 1\).
\( \lambda^- - \delta \lambda^+ \). So if \( C(e_{T-1}) > 0 \), then the DM’s effort at period \( T - 1 \) is equal to \( e_{T-2} \), otherwise, \( e_{T-1} = 0 \).

We analyse an effort that will be chosen by the DM. First, we compare \( e_{T-1} = 1 \) and \( e_{T-1} = e_{T-2} \).

\[
C(e_{T-1}|e_{T-1} = 1) \leq C(e_{T-1}|e_{T-1} = e_{T-2})
\]

It is always true \( C(e_{T-1}|e_{T-1} = 1) > C(e_{T-1}|e_{T-1} = e_{T-2}) \). As a result, if \( \delta < \frac{1+\lambda^-}{\lambda^+} \), the DM’s effort is \( e_{T-1} = 1 \). However, the result changes when \( \delta > \frac{1+\lambda^-}{\lambda^+} \). In this case, we compare \( C(e_{T-1}|e_{T-1} = 1) \) and \( C(e_{T-1}|e_{T-1} = 0) \).

\[
C(e_{T-1}|e_{T-1} = 1) \leq C(e_{T-1}|e_{T-1} = 0)
\]

\[
u(e_{T-1}) + \lambda^+(u(e_{T-1}) - u(e_{T-2})) + \delta[1 + \lambda^+(1 - u(e_{T-1}))] \leq \\
u(e_{T-1}) - \lambda^-(u(e_{T-2}) - u(e_{T-1})) + \delta(1 + \lambda^+(1 - u(e_{T-1})) \]

\[
1 + \lambda^+(1 - u(e_{T-2})) \leq -\lambda^- u(e_{T-2}) + \delta \lambda^+
\]

For any \( \delta > \frac{1+\lambda^+ + u(e_{T-2})(\lambda^- - \lambda^+)}{\lambda^+} \) the DM’s effort at period \( T - 1 \) is 0, otherwise it is \( e_{T-1} = 1 \). The above implies that the action (i.e., the effort) path \((e_{T-1}, e_T) = (0, 1)\) if \( \delta > \frac{1+\lambda^+ + u(e_{T-2})(\lambda^- - \lambda^+)}{\lambda^+} \) and \( \delta > \frac{1+\lambda^-}{\lambda^+} \).

We can now state that if \( \delta > \frac{1+\lambda^-}{\lambda^+} \), the effort path is \((e_{T-1}, e_T) = (0, 1)\), otherwise \((e_{T-1}, e_T) = (1, 1)\).

Now let’s consider the effort path that the DM has at \( T - 3 \) and \( T - 2 \). Now suppose that the DM is at period \( T - 2 \):

\[
\ldots, e_{T-3}, e_{T-2}, e_{T-1}, e_T
\]

Suppose the effort path is \((e_{T-1} = 1, e_T = 1)\), under condition \( \delta < \frac{1+\lambda^-}{\lambda^+} \).

Let’s consider the case where \( e_{T-2} \geq e_{T-3} \), then the DM’s payoff from \( e_{T-2} \) is \( u(e_{T-2}) + \lambda^+(u(e_{T-2}) - u(e_{T-3})) + \delta[1 + \lambda^+(1 - u(e_{T-2}))] + \delta^2(1) \).

Then \( C(e_{T-2}) = 1 + \lambda^+ - \delta \lambda^+ > 0 \). This means that \( e_{T-2} = 1 \).
Now consider the case where \( e_{T-2} \leq e_{T-3} \). The DM’s payoff from \( e_{T-2} \) is \( u(e_{T-2}) - \lambda^-(u(e_{T-3}) - u(e_{T-2})) + \delta(1 + \lambda^+(1 - u(e_{T-2})) + \delta^2(1) \). Then \( C(e_{T-2}) = (1 + \lambda^- - \delta \lambda^+) \). So if \( C(e_{T-2}) > 0 \), we have \( e_{T-2} = e_{T-3} \), or if \( C(e_{T-2}) < 0 \), then \( e_{T-2} = 0 \). The last case is impossible, because it contradicts to the initial condition \( \delta < \frac{1 + \lambda^-}{\lambda^+} \).

We analyse which effort will be chosen by the DM. For this purpose, we compare \( e_{T-2} = 1 \) and \( e_{T-2} = e_{T-3} \) the DM’s payoffs. We have that \( C(e_{T-2}|e_{T-2} = 1) > C(e_{T-2}|e_{T-2} = e_{T-3}) \). It is always true because \( 1 + \lambda^+(1 - u(e_{T-3})) > e_{T-3} + \delta \lambda^+(1 - u(e_{T-3})) \) for \( \delta < \frac{1 + \lambda^-}{\lambda^+} \). Therefore, in this case, \( e_{T-2} = 1 \). As a result, \( (e_{T-2}, e_{T-1}, e_T) = (1, 1, 1) \), if \( \delta < \frac{1 + \lambda^-}{\lambda^+} \).

Suppose the effort path is still \( (e_{T-1} = 1, e_T = 1) \), but under conditions \( \delta > \frac{1 + \lambda^-}{\lambda^+} \) and \( \delta < \frac{1 + \lambda^+ + u(e_{T-2})(\lambda^- - \lambda^+)}{\lambda^+} \).

Let’s consider the case where \( e_{T-2} \geq e_{T-3} \). The DM’s payoff from \( e_{T-2} \) is \( u(e_{T-2}) + \lambda^+(u(e_{T-2}) - u(e_{T-3})) + \delta [1 + \lambda^+(1 - u(e_{T-2}))] + \delta^2(1) \). Then \( C(e_{T-2}) = 1 + \lambda^- - \delta \lambda^+ > 0 \), which is always true, therefore \( e_{T-2} = 1 \). This case contradicts to the condition of \( \delta > \frac{1 + \lambda^-}{\lambda^+} \).

Now consider the case where \( e_{T-2} \leq e_{T-3} \). The DM’s payoff from \( e_{T-2} \) is \( u(e_{T-2}) - \lambda^-(u(e_{T-3}) - u(e_{T-2})) + \delta(1 + \lambda^+(1 - u(e_{T-2})) + \delta^2(1) \). Then \( C(e_{T-2}) = (1 + \lambda^- - \delta \lambda^+) \). So if \( C(e_{T-2}) > 0 \), this case is excluded from discussion because it contradicts to the initial condition. If \( C(e_{T-2}) < 0 \), then \( e_{T-2} = 0 \). As a result, \( (e_{T-2}, e_{T-1}, e_T) = (0, 1, 1) \), if \( \delta > \frac{1 + \lambda^-}{\lambda^+} \) and \( \delta < \frac{1 + \lambda^+}{\lambda^+} \).

Now consider the case where the DM’s effort path is \( (e_{T-1} = 0, e_T = 1) \), under condition \( \delta > \frac{1 + \lambda^-}{\lambda^+} \) and \( \delta > \frac{1 + \lambda^+ + u(e_{T-2})(\lambda^- - \lambda^+)}{\lambda^+} \).

First, let’s consider \( e_{T-2} \geq e_{T-3} \). The DM’s payoff from \( e_{T-2} \) is \( u(e_{T-2}) + \lambda^+(u(e_{T-2}) - u(e_{T-3})) + \delta[0 - \lambda^- (u(e_{T-2}) - 0)] + \delta^2(1) \). Then \( C(e_{T-2}) = 1 + \lambda^+ - \delta \lambda^- > 0 \). This means that \( e_{T-2} = 1 \).

Now, let’s consider that \( e_{T-2} \leq e_{T-3} \), and the DM’s payoff from \( e_{T-2} \) is \( u(e_{T-2}) - \lambda^-(u(e_{T-3}) - u(e_{T-2})) + \delta[0 - \lambda^- (u(e_{T-2}) - 0)] + \delta^2(1) \). Then \( C(e_{T-2}) = (1 + \lambda^- - \delta \lambda^-) > 0 \). So we have \( e_{T-2} = e_{T-3} \).
We analyse which effort will be chosen by the DM. Let’s compare $e_{T - 2} = 1$ and $e_{T - 2} = e_{T - 3}$. We show that $C(e_{T - 2}|e_{T - 2} = 1) > C(e_{T - 2}|e_{T - 2} = e_{T - 3})$. It is true because $1 + \lambda^+(1 - u(e_{T - 3})) - \delta \lambda^- > u(e_{T - 3})(1 - \delta \lambda^-) \Rightarrow 1 + \lambda^+ - \delta \lambda^- > u(e_{T - 3})(1 + \lambda^+ - \delta \lambda^-)$. Hence, there is $e_{T - 2} = 1$. As a result, $(e_{T - 2}, e_{T - 1}) = (1, 0, 1)$, if $\delta > \frac{1 + \lambda^-}{\lambda^+}$.

Comparing the DM’s payoffs from $(e_{T - 2}, e_{T - 1}, e_T) = (1, 0, 1)$ and $(e_{T - 2}, e_{T - 1}, e_T) = (0, 1, 1)$ under condition $\delta > \frac{1 + \lambda^-}{\lambda^+}$, we have $V(1, 0, 1) - V(0, 1, 1) = 1 + \lambda^+(1 - u(e_{T - 4})) + \delta (0 - \lambda^-) + \delta^2(1 + \lambda^+) - [0 - \lambda^-(u(e_{T - 4})) + \delta(1 + \lambda^+) + \delta^2(1)] > 0$, if $\lambda^+$ is significantly higher than $\lambda^-$ and the discount factor is not strong.

Now the DM is at $T - 3$, and the last three period effort path is $(e_{T - 2}, e_{T - 1}) = (1, 1, 1)$, if $\delta, \frac{1 + \lambda^-}{\lambda^+}$.

Let’s consider the case where $e_{T - 4} \leq e_{T - 3}$ then the DM’s payoff from $e_{T - 3}$ is $u(e_{T - 3}) + \lambda^+(u(e_{T - 3}) - u(e_{T - 4})) + \delta(1 + \lambda^+(1 - u(e_{T - 3}))) + \delta^2 + \delta^3$. There is $C(e_{T - 3}) = 1 + \lambda^+ - \delta \lambda^+ > 0$ always, and $e_{T - 3} = 1$.

Now consider the case where $e_{T - 4} \geq e_{T - 3}$. The DM’s payoff from $e_{T - 3}$ is $u(e_{T - 3}) - \lambda^-(u(e_{T - 4}) - u(e_{T - 3})) + \delta (1 + \lambda^+(1 - u(e_{T - 3}))) + \delta^2 + \delta^3$. There is $C(e_{T - 3}) = 1 + \lambda^- - \delta \lambda^+$. Then $C(e_{T - 3}) > 0$ and $e_{T - 3} = e_{T - 4}$ if $\delta < \frac{1 + \lambda^-}{\lambda^+}$, otherwise $C(e_{T - 3}) < 0$ and $e_{T - 3} = 0$ if $\delta > \frac{1 + \lambda^-}{\lambda^+}$. The last case $e_{T - 3} = 0$ if $\delta > \frac{1 + \lambda^-}{\lambda^+}$ is impossible because it contradicts to the initial conditions.

Let’s compare the DM’s payoffs for $e_{T - 3} = 1$ and $e_{T - 3} = e_{T - 4}$. Then $1 + \lambda^+(1 - u(e_{T - 4})) + \delta + \delta^2 + \delta^3 > u(e_{T - 4}) + \delta (1 + \lambda^+(1 - u(e_{T - 4})))$. There is $1 + \lambda^+(1 - u(e_{T - 4})) > u(e_{T - 4}) + \delta \lambda^+(1 - u(e_{T - 4}))$ and this is always true, therefore, $e_{T - 3} = 1$. Therefore, the effort path is $(e_{T - 3}, e_{T - 2}, e_{T - 1}, e_T) = (1, 1, 1, 1)$, if $\delta < \frac{1 + \lambda^-}{\lambda^+}$.

Suppose the following effort path for the last three periods is $(e_{T - 2}, e_{T - 1}, e_T) = (1, 0, 1)$, if $\delta > \frac{1 + \lambda^-}{\lambda^+}$.

Let’s consider the case where $e_{T - 4} \leq e_{T - 3}$ then the DM’s payoff of $e_{T - 3}$ is
As a result, \( e(T-3) + \lambda (e(T-3) - u(e_{T-4})) + \delta (1 + \lambda (1 - u(e_{T-3}))) + \delta^2 (0 - \lambda^-) + \delta^3 (1 + \lambda^+) \). Then \( C(e_{T-3}) = 1 + \lambda^+ - \delta \lambda^+ > 0 \) is always. There is \( e_{T-3} = 1 \).

Now consider the case where \( e_{T-4} \geq e_{T-3} \) then the payoff from \( e_{T-3} \) is \( u(e_{T-3}) - \lambda^- (u(e_{T-4}) - u(e_{T-3})) + \delta (1 + \lambda^+ (1 - u(e_{T-3}))) + \delta^2 (0 - \lambda^-) + \delta^3 (1 + \lambda^+) \). There is \( C(e_{T-3}) = 1 + \lambda^- - \delta \lambda^+ \). Then if \( C(e_{T-3}) > 0 \), \( e_{T-3} = e_{T-4} \), otherwise if \( C(e_{T-3}) < 0 \), and \( e_{T-3} = 0 \). The case \( e_{T-3} = e_{T-4} \) if \( \delta < \frac{1 + \lambda^+}{\lambda^+} \) is impossible because contradicts initial condition.

Let’s compare the DM’s payoffs for \( e_{T-3} = 1 \) and \( e_{T-3} = 0 \). \( 1 + \lambda^+ (1 - u(e_{T-4})) + \delta - \delta^2 (\lambda^-) + \delta^3 (1 + \lambda^+)) \leq - \lambda^- u(e_{T-4}) + \delta (1 + \lambda^+) - \delta^2 \lambda^- + \delta^3 (1 + \lambda^+) \). There is \( 1 + \lambda^+ (1 - u(e_{T-4})) + \delta \leq - \lambda^- u(e_{T-4}) + \delta (1 + \lambda^+) \).

As a result, \( e_{T-3} = 1 \), if \( \delta < \frac{1 + \lambda^+ + u(e_{T-4})(\lambda^- - \lambda^+)}{\lambda^+} \), or \( e_{T-3} = 0 \), if \( \delta > \frac{1 + \lambda^+ + u(e_{T-4})(\lambda^- - \lambda^+)}{\lambda^+} \).

As a result, \( (e_{T-3}, e_{T-2}, e_{T-1}, e_{T}) = (0, 1, 0, 1) \), if \( \delta > \frac{1 + \lambda^-}{\lambda^+} \). However if \( \delta < \frac{1 + \lambda^-}{\lambda^+} \), the effort path is \( (e_{T-3}, e_{T-2}) = (1, 1) \).

The next step involves the proof of the following statements:

1. If \( \delta > \frac{1 + \lambda^-}{\lambda^+} \), the effort path is \( e^* = (0, 1, \ldots, 0, 1) \);
2. If \( \delta < \frac{1}{\lambda^+} \) and \( \delta < \frac{1 + \lambda^-}{\lambda^+} \), the effort path is \( e^* = (1, 1, 1, \ldots, 1, 1) \);
3. If \( \delta > \frac{1}{\lambda^+} \) and \( \delta < \frac{1 + \lambda^-}{\lambda^+} \), the effort path is \( e^* = (0, 1, 1, \ldots, 1, 1) \);

(1) First, let \( e^* = (e^*_1, e^*_2, \ldots, e^*_T) \) be an equilibrium and consider the statement \( S(n), n \geq 0 \) that \( e^* = (e^*_{T-n}, e^*_{T-(n-1)}, \ldots, e^*_{T-1}, e^*_T) = (0, 1, \ldots, 0, 1) \) if \( \delta > \frac{1 + \lambda^-}{\lambda^+} \). Now, we aim to prove that if the following statement \( S(n) \) is true, the same path should be observed in the case of \( S(n+2) \).

So now assume \( S(n) \) is true and check \( S(n+2) \). We infer from above that \( S(1) \) and \( S(3) \) are true. OSDP and the proof above for \( S(1) \) render this
argument in a straightforward way. So to see this, suppose \( S(n) \) is true. This means irrespective of \( e_{n-1}^*, e_{n-2}^* \), it must be that \((e_{n-1}^*, e_{n+1}^*, \ldots , e_{T-1}^*, e_T^*) = (0, 1, \ldots , 0, 1)\). From the proof for \( S(1) \), I know that irrespective of the value of \( e_{n-3}^* \), it must be that \((e_{n-2}^*, e_{n-1}^*) = 0, 1)\). Hence, \( S(n + 2) \) is true.

For now, \( \forall t > 2 \) the effort path is \( e = (0, 1, \ldots , 0, 1) \). Suppose that the DM is at \( t = 2 \). Consider the case, where \( e_1 \leq e_2 \). The DM’s payoff from \( e_2 \) is \( u(e_2) + \lambda^+(u(e_2) - u(e_1)) + \delta(0 - \lambda^-(u(e_2))) + \delta^2(0 + \lambda^+) + \delta^3(0 - \lambda^-) + V_\delta(\{0, 1\}) \). There is \( C(e_2) = 1 + \lambda^+ - \delta \lambda^- > 0 \). It means that \( e_2 = 1 \).

Next, we consider \( e_1 \geq e_2 \). The DM’s payoff from \( e_2 \) is \( u(e_2) - \lambda^- (u(e_1) - u(e_2)) + \delta(0 - \lambda^-(u(e_2))) + \delta^2(0 + \lambda^+) + \delta^3(0 - \lambda^-) + V_\delta(\{0, 1\}) \). There is \( C(e_2) = 1 + \lambda^- - \delta \lambda^- > 0 \). It means that \( e_2 = 1 \).

Comparing the DM’s payoffs from \( e_2 = 1 \) and \( e_2 = e_1 \), we have \( 1 + \lambda^+(1 - u(e_1)) + \delta(0 - \lambda^-(1)) + \delta^2(1 + \lambda^+) + \delta^3(0 - \lambda^-) + V_\delta(\{0, 1\}) \) \( - [u(e_1) + \delta(0 - \lambda^- (u(e_1)) + \delta^2(1 + \lambda^+) + \delta^3(0 - \lambda^-) + V_\delta(\{0, 1\})] = 1 + \lambda^- (1 - u(e_1)) + \delta(0 - \lambda^- (u(e_1))) - [u(e_1) + \delta(0 - \lambda^- u(e_1))] > 0 \). Therefore, \( e_2 = 1 \).

Suppose the DM is at \( t = 1 \). Let consider the effort \( e_1 \). The DM’s payoff from \( e_1 \) is \( u(e_1) + \delta(1 + \lambda^+(1 - u(e_1))) + \delta^2(0 - \lambda^- (1)) + \delta^3(1 + \lambda^+) + \delta^- (0 - \lambda^-) + V_\delta(\{0, 1\}) \). There are \( C(e_1) = 1 - \delta \lambda^+ \). It means that \( e_1 = 0 \) if \( \delta > \frac{1}{\lambda^+} \). As a result, we proved that if \( \delta > \frac{1}{\lambda^+} \), the effort path is \( e = (0, 1, 0, 1, \ldots , 0, 1) \).

(2) Next, let \( e^* = (e_1^*, e_2^*, \ldots , e_T^*) \) be an equilibrium and consider the statement \( S^*(n), T \geq n \geq 0 \) such that \( e^* = (e_{T-n}^*, C_{T-(n-1)}, \ldots , e_{T-1}^*, e_T^*) = (1, 1, \ldots , 1, 1) \) if \( \delta < \frac{1 + \lambda^-}{\lambda^+} \). We are going to prove that if the following statement \( S^*(n) \) is true, the same path we observe in the case of \( S^*(n + 1) \).

So now assume \( S^*(n) \) is true and check \( S^*(n + 1) \). From above it is true for \( S^*(0) \) and \( S^*(1) \). OSDP and the proof above for \( S^*(1) \) render this argument in a straightforward way. Suppose the statement \( S^*(n) \) is true. This means irrespective of \( e_{n-1}^*, e_{n-2}^* \), it must be that \((e_{n-1}^*, e_{n+1}^*, \ldots , e_{T-1}^*, e_T^*) = (1, 1, \ldots , 1, 1) \). From the proof of \( S^*(1) \), we know that irrespective of the
value of $e_{n-2}^*$, it must be that $e_{n-1}^* = 1$ as long as $\delta < \frac{1 + \lambda^-}{\lambda^+}$. Hence, $S^*(n + 1)$ is true.

Now suppose that the DM is at period $t = 1$, then the DM’s payoff is $V_\delta(e_1) = u(e_1) + \delta(1 + \lambda^+(1 - u(e_1))) + \delta^2(1) + V_{\delta}^{t \geq 3}(1) = u(e_1)(1 - \delta \lambda^+) + V_{\delta}^{t \geq 3}(1)$. As a result, if $\delta < \frac{1}{\lambda^+}$, $e_1 = 1$, otherwise $e_1 = 0$. To summarise, if $\delta < \frac{1}{\lambda^+}$, the effort path is $e = (1, 1, \ldots, 1, 1)$, or if $\frac{1}{\lambda^+} < \delta < \frac{1 + \lambda^-}{\lambda^+}$ the effort path is $e = (0, 1, \ldots, 1, 1)$. Putting all finding together, we have for $\lambda^+ > \lambda^-$ the optimal effort path is $e = (1, 1, \ldots, 1, 1)$ if $\delta < \frac{1}{\lambda^+}$; or $e = (0, 1, 1, \ldots, 1, 1)$ if $\frac{1 + \lambda^-}{\lambda^+} > \delta > \frac{1}{\lambda^+}$; or $e = (0, 1, 0, 1, \ldots, 0, 1)$ if $\frac{1 + \lambda^-}{\lambda^+} < \delta$.

This completes the proof of proposition. □
APPENDIX 2: EXAMPLE: $T = 3$

Case $k = 1$

I suppose that the DM is a decentralized agent. The DM is at period $T$, then there is $e_T = 1$ regardless of loss or gain (see Fact 12).

Now suppose that the DM is at period $T - 1$:

$$\ldots, e_{T-2}, e_{T-1}, e_T$$

Assume that $e_{T-1} \geq e_{T-2}$. The DM’s payoff from $e_{T-1}$ is $u(e_{T-1}) + \lambda^+(u(e_{T-1}) - u(e_{T-2})) + \delta[1 + \lambda^+(1 - u(e_{T-1}))]$. There is $\mathcal{C}(e_{T-1}) = 1 + \lambda^+ - \delta \lambda^+ > 0$ always. It means that $e_{T-1} = 1$.

Suppose that $e_{T-1} \leq e_{T-2}$. The DM’s payoff from $e_{T-1}$ is $u(e_{T-1}) - \lambda^-(u(e_{T-2}) - u(e_{T-1})) + \delta(1 + \lambda^+(1 - u(e_{T-1})))$. There is $\mathcal{C}(e_{T-1}) = (1 + \lambda^- - \delta \lambda^+)$. So if $\mathcal{C}(e_{T-1}) > 0$, then the DM’s effort at period $T - 1$ is $e_{T-2}$, otherwise, $e_{T-1} = 0$. I analyse which effort will be chosen by the DM. First, I compare $e_{T-1} = 1$ and $e_{T-1} = e_{T-2}$.

$$\mathcal{C}(e_{T-1}|e_{T-1} = 1) \leq \mathcal{C}(e_{T-1}|e_{T-1} = e_{T-2})$$

It is always true. As a result, if $\delta < \frac{1 + \lambda^-}{\lambda^+}$, the DM effort is $e_{T-1} = 1$. However, the result changes when $\delta > \frac{1 + \lambda^-}{\lambda^+}$. In this case, I compare $\mathcal{C}(e_{T-1}|e_{T-1} = 1)$ and $\mathcal{C}(e_{T-1}|e_{T-1} = 0)$.

$$\mathcal{C}(e_{T-1}|e_{T-1} = 1) \leq \mathcal{C}(e_{T-1}|e_{T-1} = 0)$$

$$u(e_{T-1}) + \lambda^+(u(e_{T-1}) - u(e_{T-2})) + \delta[1 + \lambda^+(1 - u(e_{T-1}))] \leq$$

$$u(e_{T-1}) - \lambda^-(u(e_{T-2}) - u(e_{T-1})) + \delta(1 + \lambda^+(1 - u(e_{T-1})))$$

$$1 + \lambda^+(1 - u(e_{T-2})) \leq -\lambda^- u(e_{T-2}) + \delta \lambda^+$$

For any $\delta > \frac{1 + \lambda^+ + u(e_{T-2})(\lambda^- - \lambda^+)}{\lambda^+}$ the DM’s effort $e_{T-1}$ is 0, otherwise it is $e_{T-1} = 1$. The above implies that the effort path $e_{T-1}, e_T$ could be any of:
\[(e_{T-1}, e_T) = \begin{cases} 
(1,1), & \text{if } \delta < \frac{1+\lambda^-}{\lambda^+} \quad \text{or} \quad \delta < \frac{1+\lambda^+ + u(e_{T-2})(\lambda^- - \lambda^+)}{\lambda^+} \\
(0,1), & \text{if } \delta > \frac{1+\lambda^- + u(e_{T-2})(\lambda^- - \lambda^+)}{\lambda^+} \quad \text{and} \quad \delta > \frac{1+\lambda^-}{\lambda^+} 
\end{cases} \]

Now suppose that the DM is at period \(T = 1:\)

\[
e_1, e_2, e_3
\]

Based on above discussions, suppose that at periods \(t = 2\) and \(t = 3\) the effort path is \((1,1)\).

The DM’s payoff at period \(T = 1\) is \(u(e_1) + \delta (1 + \lambda^+ (1 - u(e_1))) + \delta^2 (1)\).

There is \(C(e_1) = 1 - \delta \lambda^-\). So if \(C(e_1) > 0\), the effort path is \(e_1 = 1\), otherwise \(e_1 = 0\).

If the effort path at periods 2 and 3 is \((0,1)\), the DM’s payoff at \(T = 1\) is \(u(e_1) + \delta (0 - \lambda^- (u(e_1))) + \delta^2 (1 + \lambda^+ (1))\). Then \(C(e_1) = 1 - \delta \lambda^-\). So if \(C(e_1) > 0\) then \(e_1 = 1\), otherwise \(e_1 = 0\). Combining all possible effort paths I have:

\[
(e_1, e_2, e_3) = \begin{cases} 
(1,1,1), & \text{if } \delta < \frac{1}{\lambda^+} \quad \text{and} \quad \delta < \frac{1+\lambda^-}{\lambda^+} \\
(0,1,1), & \text{if } \delta > \frac{1}{\lambda^+} \quad \text{and} \quad \delta < \frac{1+\lambda^-}{\lambda^+} \quad \text{or} \quad \\
(1,0,1), & \text{if } \delta < \frac{1}{\lambda^+} \quad \text{and} \quad \delta > \frac{1+\lambda^- + u(e_1)(\lambda^- - \lambda^+)}{\lambda^+} \\
(0,0,1), & \text{if } \delta > \frac{1}{\lambda^+} \quad \text{and} \quad \delta > \frac{1+\lambda^- + u(e_1)(\lambda^- - \lambda^+)}{\lambda^+}.
\end{cases} \]
The effort path \((0,0,1)\) is impossible, because it self-contradicts. Therefore,

\[
(e_1, e_2, e_3) = \begin{cases} 
(1,1,1), & \text{if } \delta < \frac{1}{\lambda^+} \text{ and } \delta < \frac{1+\lambda^-}{\lambda^+} \\
(0,1,1), & \text{if } \delta > \frac{1}{\lambda^+} \text{ and } \delta < \frac{1+\lambda^-}{\lambda^+} \\
(1,0,1), & \text{if } \delta < \frac{1}{\lambda^+} \text{ and } \delta > \frac{1+\lambda^-}{\lambda^+} 
\end{cases}
\]

Further discussion about the effort path depends on the relation between \(\lambda^+\) and \(\lambda^-.\) To summarize:

If \(\lambda^+ < \lambda^-(1+\lambda^-)\), the equilibrium effort path is \(e^* = (1,1,1)\) if \(\delta < \frac{1}{\lambda^+}\) and \(\delta < \frac{1+\lambda^-}{\lambda^+}\). When \(\delta > \frac{1}{\lambda^+}\) and \(\delta < \frac{1+\lambda^-}{\lambda^+}\), the equilibrium effort path is \(e^* = (0,1,1)\).

As soon as \(\lambda^+ > \lambda^-(1+\lambda^-)\), the DM’s equilibrium effort path could be: \(e^* = (1,1,1)\) if \(\delta < \frac{1}{\lambda^+}\); \(e^* = (0,1,1)\) if \(\delta > \frac{1}{\lambda^+}\) and \(\delta < \frac{1+\lambda^-}{\lambda^+}\); and \(e^* = (1,0,1)\) if \(\delta < \frac{1}{\lambda^+}\) and \(\delta > \frac{1+\lambda^-}{\lambda^+}\).

Case \(k = T\)

Now let consider the case where the DM is a social planner.

\(e_1, e_2, e_3\)

Suppose that \(e_1 \leq e_2\). The DM’s payoff from \(e_1\) is

\[
V_\delta = u(e_1) + \delta(u(e_2) + \lambda^+(u(e_2) - u(e_1))) + \delta^2(1+\lambda^+(1-u(e_2))) = \\
u(e_1)(1-\delta\lambda^+) + u(e_2)\delta(1+\lambda^+ - \delta\lambda^+) + \delta^2(1+\lambda^+)
\]

Then \(C(e_1) > 0\) if \(\delta < \frac{1}{\lambda^+}\), otherwise \(C(e_1) < 0\). The second term \(C(e_2)\) is always positive. As a result, it means that if \(\delta < \frac{1}{\lambda^+}\), the effort path is \((1,1,1)\). Otherwise, the effort path is \((0,1,1)\) if \(\delta > \frac{1}{\lambda^+}\).

Suppose that \(e_1 \geq e_2\). The DM’s payoff from \(e_1\) is

\[
V_\delta = u(e_1) + \delta(u(e_2) - \lambda^-(u(e_1) - u(e_2))) + \delta^2(1+\lambda^+(1-u(e_2))) = \\
u(e_1)(1-\delta\lambda^-) + u(e_2)\delta(1+\lambda^- - \delta\lambda^+) + \delta^2(1+\lambda^+)
\]
Then $C(e_1) > 0$ if $\delta < \frac{1}{\lambda^+}$, otherwise $C(e_1) < 0$. At the same time, $C(e_2) > 0$ if $\delta < \frac{1 + \lambda^-}{\lambda^+}$, and $C(e_2) < 0$ if $\delta > \frac{1 + \lambda^-}{\lambda^+}$. Combine them together:

$$e_1 = \begin{cases} 1, & \text{if } \delta < \frac{1}{\lambda^+} \text{ and } \delta < \frac{1 + \lambda^-}{\lambda^+} \\ 0, & \text{if } \delta > \frac{1}{\lambda^+} \text{ and } \delta > \frac{1 + \lambda^-}{\lambda^+} \end{cases} \quad \text{and} \quad e_2 = \begin{cases} e_1, & \text{if } \delta < \frac{1 + \lambda^-}{\lambda^+} \\ 0, & \text{if } \delta > \frac{1 + \lambda^-}{\lambda^+} \end{cases}$$

The above gives us the possible pattern of the effort paths:

$$ (e_1, e_2, e_3) = \begin{cases} (1, 1, 1), & \text{if } \delta < \frac{1}{\lambda^+} \text{ and } \delta < \frac{1 + \lambda^-}{\lambda^+} \\ (1, 0, 1), & \text{if } \delta < \frac{1}{\lambda^+} \text{ and } \delta > \frac{1 + \lambda^-}{\lambda^+} \\ (0, 0, 1), & \text{if } \delta > \frac{1}{\lambda^+} \text{ and } \delta < \frac{1 + \lambda^-}{\lambda^+} \\ (0, 0, 1), & \text{if } \delta > \frac{1}{\lambda^+} \text{ and } \delta > \frac{1 + \lambda^-}{\lambda^+} \end{cases}$$

Now, I compare possible patterns to find true $k = 3$ commitment effort path:

$$ (e_1, e_2, e_3)^{e_1 \leq e_2} = \begin{cases} (1, 1, 1), & \text{if } \delta < \frac{1}{\lambda^+} \\ (0, 1, 1), & \text{if } \delta > \frac{1}{\lambda^+} \end{cases} \quad \text{vs} \quad (e_1, e_2, e_3)^{e_1 \geq e_2} = \begin{cases} (1, 1, 1), & \text{if } \delta < \frac{1}{\lambda^+} \text{ and } \delta < \frac{1 + \lambda^-}{\lambda^+} \\ (1, 0, 1), & \text{if } \delta < \frac{1}{\lambda^+} \text{ and } \delta > \frac{1 + \lambda^-}{\lambda^+} \\ (0, 0, 1), & \text{if } \delta > \frac{1}{\lambda^+} \text{ and } \delta < \frac{1 + \lambda^-}{\lambda^+} \\ (0, 0, 1), & \text{if } \delta > \frac{1}{\lambda^+} \text{ and } \delta > \frac{1 + \lambda^-}{\lambda^+} \end{cases}$$

Further discussion depends on the relationship between $\lambda^+$ and $\lambda^-$. Suppose that $\lambda^+ > \lambda^- (1 + \lambda^-)$. If $\delta < \frac{1}{\lambda^+}$, then $(e_1, e_2, e_3) = (1, 1, 1)$. If $\delta < \frac{1}{\lambda^+}$ and $\delta > \frac{1 + \lambda^-}{\lambda^+}$, two possible effort paths are possible $(1, 0, 1)$ and $(0, 1, 1)$. Then $V_\delta(1, 0, 1) \leq V_\delta(0, 1, 1)$.
As can be seen, along this interval $\delta$ has non-linear representation (in this case: quadratic). It means that both effort paths are possible. For simplicity, take $\delta = 1$, as a result, the DM effort path is $(0, 1, 1)$. If $\delta = 0$, then the DM’s effort path is $(1, 0, 1)$.

If $\delta > \frac{1}{\lambda^+}$ and $\delta > \frac{1+\lambda^-}{\lambda^+}$, two possible effort paths are possible $(0, 0, 1)$ and $(0, 1, 1)$.

As a result, the following $\delta^2 \lambda^+ < \delta (1 + \lambda^+)$ is true. It means that the DM takes $(0, 1, 1)$.

Suppose that $\lambda^+ < \lambda^-(1 + \lambda^-)$.

If $\delta < \frac{1}{\lambda^+}$, then $(e_1, e_2, e_3) = (1, 1, 1)$. The effort path $(e_1, e_2, e_3) = (1, 0, 1)$ is impossible because of self-contradiction.

As a result, $\delta^2 \lambda^+ < \delta (1 + \lambda^+)$. It means that the DM takes $(0, 1, 1)$, if $\delta > \frac{1}{\lambda^+}$.

Further discussion about the effort path depends on the relation between $\lambda^+$ and $\lambda^-$. To summarize:

If $\lambda^+ < \lambda^-(1 + \lambda^-)$, the equilibrium effort path is $e^* = (1, 1, 1)$ if $\delta < \frac{1}{\lambda^+}$ and $\delta < \frac{1+\lambda^-}{\lambda^+}$. When $\delta > \frac{1}{\lambda^+}$ and $\delta < \frac{1+\lambda^-}{\lambda^+}$ the equilibrium effort path is $e^* = (0, 1, 1)$.

As soon as $\lambda^+ > \lambda^-(1 + \lambda^-)$, the DM’s optimal effort paths could be: $e^* = (1, 1, 1)$ if $\delta < \frac{1}{\lambda^+}$; $e^* = (0, 1, 1)$ if $\delta > \frac{1}{\lambda^+}$ and $\delta < \frac{1+\lambda^-}{\lambda^+}$; and $e^* = (1, 0, 1)$ if $\delta < \frac{1}{\lambda^+}$ and $\delta > \frac{1+\lambda^-}{\lambda^+}$.
APPENDIX 3: EXAMPLE: $T = 4$

Case: $k = 1$

From Appendix 2 the following effort path for $T - 1, T$ periods is:

$$(e_{T-1}, e_T) = \begin{cases} 
(1, 1), & \text{if } \delta < \frac{1+\lambda^-}{\lambda} \\
(0, 1), & \text{if } \delta > \frac{1+\lambda^- + u(e_{T-2}) (\lambda^- - \lambda^+)}{\lambda^+} \\
(0, 1), & \text{if } \delta > \frac{1+\lambda^+ + u(e_{T-2}) (\lambda^- - \lambda^+)}{\lambda^+} \quad \text{and } \delta > \frac{1+\lambda^-}{\lambda^+} 
\end{cases}$$

Now the DM is at period $T - 2$ and the previous effort path is $(e_{T-1} = 1, e_T = 1)$. Suppose $e_{T-2} \geq e_{T-3}$, then the DM’s payoff from $e_{T-2}$ is $u(e_{T-2}) + \lambda^+(u(e_{T-2}) - u(e_{T-3})) + \delta[1 + \lambda^+(1-u(e_{T-2}))] + \delta^2(1)$. Then $C(e_{T-2}) = 1 + \lambda^+ - \delta \lambda^+ > 0$. This means that $e_{T-2} = 1$.

Suppose that $e_{T-2} \leq e_{T-3}$, then the DM’s payoff from $e_{T-2}$ is $u(e_{T-2}) - \lambda^- (u(e_{T-3}) - u(e_{T-2})) + \delta(1 + \lambda^+(1-u(e_{T-2})) + \delta^2(1)$. Then $C(e_{T-2}) = (1 + \lambda^- - \delta \lambda^+)$. So if $C(e_{T-2}) > 0$, then I have $e_{T-2} = e_{T-3}$. If $C(e_{T-2}) < 0$, then $e_{T-2} = 0$.

Now I analyse which effort will be chosen by the DM. First, I compare $e_{T-2} = 1$ and $e_{T-2} = e_{T-3}$. I show that $C(e_{T-2}|e_{T-2} = 1) > C(e_{T-2}|e_{T-2} = e_{T-3})$. It is always true because $1 + \lambda^+(1-u(e_{T-3})) > u(e_{T-3}) + \delta \lambda^+(1-u(e_{T-3})$ for $\delta < \frac{1+\lambda^-}{\lambda^+}$. Therefore, in the case of $\delta < \frac{1+\lambda^-}{\lambda^+}$, $e_{T-2} = 1$.

The result changes when $\delta > \frac{1+\lambda^-}{\lambda^+}$. In this case I compare $C(e_{T-2}|e_{T-2} = 1)$ and $C(e_{T-2}|e_{T-2} = 0)$, that is $C(e_{T-2}|e_{T-2} = 1) \leq C(e_{T-2}|e_{T-2} = 0)$: $1 + \lambda^+(1-u(e_{T-3})) \leq -\lambda^-(u(e_{T-3})) + \delta \lambda^+$. The DM’s effort is $e_{T-2} = 1$ if $\delta < \frac{1+\lambda^+ + u(e_{T-2}) (\lambda^- - \lambda^+)}{\lambda^+}$, otherwise $e_{T-2} = 0$.

Now suppose that the DM’s previous effort path is $(e_{T-1} = 0, e_T = 1)$. Suppose $e_{T-2} \geq e_{T-3}$, then the DM’s payoff from $e_{T-2}$ is $u(e_{T-2}) + \lambda^+(u(e_{T-2}) - u(e_{T-3})) + \delta[0 - \lambda^-(u(e_{T-2}) - 0)] + \delta^2(1)$. Then $C(e_{T-2}) = 1 + \lambda^+ - \delta \lambda^- > 0$. This means that $e_{T-2} = 1$.

Suppose that $e_{T-2} \leq e_{T-3}$, then the DM’s payoff from $e_{T-2}$ is $u(e_{T-2}) -$
\(
\lambda^-(u(e_{T-3}) - u(e_{T-2})) + \delta(0 - \lambda^-(u(e_{T-2}) - 0) + \delta^2(1). \text{ Then } C(e_{T-2}) = (1 + \lambda^- - \delta \lambda^-) > 0. \text{ So I have } e_{T-2} = e_{T-3}.
\)

Now I analyse which effort will be chosen by the DM. First, I compare \(e_{T-2} = 1\) and \(e_{T-2} = e_{T-3}\). I show that \(C(e_{T-2} | e_{T-2} = 1) > C(e_{T-2} | e_{T-2} = e_{T-3})\). Then I have \(1 + \lambda^+(1 - u(e_{T-3})) - \delta \lambda^- > u(e_{T-3})(1 - \delta \lambda^-) \Rightarrow 1 + \lambda^+ - \delta \lambda^- > u(e_{T-3})(1 + \lambda^+ - \delta \lambda^-). \text{ As a result, } e_{T-3} = 1.\)

\[
(e_{T-2}, e_{T-1}, e_T) = \begin{cases} 
(1, 1, 1), & \text{if } \delta < \frac{1+\lambda^-}{\lambda^+} \\
(0, 1, 1), & \text{if } \delta > \frac{1+\lambda^+ + u(e_{T-2})(l^- - l^+)}{\lambda^+}, \\
(1, 0, 1), & \text{if } \delta > \frac{1+\lambda^- + u(e_{T-2})(l^- - l^+)}{\lambda^+} \text{ and } \delta > \frac{1+\lambda^+}{\lambda^+}
\end{cases}
\]

Suppose that the DM is at period \(T = 1\). The effort path at periods 2, 3 and 4 is \((1, 1, 1)\). The DM’s payoff at period \(T = 1\) is \(u(e_1) + \delta(1 + \lambda^+(1 - u(e_1))) + \delta^2(1) + \delta^3(1)\). Then \(C(e_1) = 1 - \delta \lambda^+\). So if \(C(e_1) > 0\), then \(e_1 = 1\), otherwise \(e_1 = 0\).

Suppose the effort path at periods 2, 3 and 4 is \((0, 1, 1)\), then the DM’s payoff at period \(T = 1\) is \(u(e_1) + \delta(0 - \lambda^-(u(e_1))) + \delta^2(1 + \lambda^+(1)) + \delta^3\). Hence \(C(e_1) = 1 - \delta \lambda^-\). So if \(C(e_1) > 0\) then \(e_1 = 1\), otherwise \(e_1 = 0\).

Suppose the effort path at periods 2, 3 and 4 is \((1, 0, 1)\), then the DM’s payoff at period \(T = 1\) is \(u(e_1) + \delta(1 + \lambda^+(1 - u(e_1))) + \delta^2(0 - \lambda^-(1)) + \delta^3\). If \(C(e_1) = 1 - \delta \lambda^+ > 0\), \(e_1 = 1\), otherwise \(e_1 = 0\).

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\[
\begin{align*}
(e_1, e_2, e_3, e_4) &= \begin{cases} 
(1, 1, 1, 1), & \text{if } \delta < \frac{1}{\lambda^+} \quad \delta < \frac{1+\lambda^-}{\lambda^-} \\
(0, 1, 1, 1), & \text{if } \delta > \frac{1}{\lambda^+} \quad \delta < \frac{1+\lambda^-}{\lambda^-} \\
(1, 0, 1, 1), & \text{if } \delta < \frac{1}{\lambda^+} \quad \delta > \frac{1+\lambda^-}{\lambda^-} \\
(1, 1, 0, 1), & \text{if } \delta > \frac{1}{\lambda^+} \quad \delta < \frac{1+\lambda^-}{\lambda^-} \\
(0, 1, 0, 1), & \text{if } \delta > \frac{1}{\lambda^+} \quad \delta > \frac{1+\lambda^-}{\lambda^-} \\
\end{cases}
\end{align*}
\]

The effort paths \((0, 0, 1, 1)\) and \((1, 1, 0, 1)\) are impossible.

Further discussion about effort path depends on the relation between \(\lambda^+\) and \(\lambda^-\). To summarize:
If $\lambda^{+} < \lambda^{-}(1 + \lambda^{-})$, the equilibrium effort pattern is $e^{*} = (1, 1, 1, 1)$ if $\delta < \frac{1}{\lambda^{+}}$ and $\delta < \frac{1+\lambda^{-}}{\lambda^{+}}$. When $\delta > \frac{1}{\lambda^{+}}$ and $\delta < \frac{1+\lambda^{-}}{\lambda^{+}}$ the equilibrium effort path is $e^{*} = (0, 1, 1, 1)$. At the same time, if $\delta > \frac{1}{\lambda^{+}}$ and $\delta > \frac{1+\lambda^{-}}{\lambda^{+}}$, the equilibrium effort path is $e^{*} = (0, 1, 0, 1)$.

As soon as $\lambda^{+} > \lambda^{-}(1 + \lambda^{-})$, the DM’s equilibrium effort path could be: $e^{*} = (1, 1, 1, 1)$ if $\delta < \frac{1}{\lambda^{+}}$ and $\delta < \frac{1+\lambda^{-}}{\lambda^{+}}$; $e^{*} = (0, 1, 1, 1)$ if $\delta > \frac{1}{\lambda^{+}}$ and $\delta < \frac{1+\lambda^{-}}{\lambda^{+}}$; $e^{*} = (1, 0, 1, 1)$ if $\delta < \frac{1}{\lambda^{+}}$ and $\delta > \frac{1+\lambda^{-}}{\lambda^{+}}$; and $e^{*} = (0, 1, 0, 1)$ $\delta > \frac{1}{\lambda^{+}}$ and $\delta > \frac{1+\lambda^{-}}{\lambda^{+}}$.

**Case: $k = T$**

Since according to Fact 12 $e_{4} = 1$, I need to look only at $e_{1}, e_{2}, e_{3}$. Suppose that $e_{1} \leq e_{2} \leq e_{3}$. The DM’s payoff from $e_{1}$ is

$$V_{\delta} = u(e_{1}) + \delta(u(e_{2}) + \lambda^{+}(u(e_{2}) - u(e_{1}))) +$$

$$+ \delta^{2}(u(e_{3}) + \lambda^{+}(u(e_{3}) - u(e_{2}))) + \delta^{3}(1 + \lambda^{+}(1 - u(e_{3}))) =$$

$$= u(e_{1})(1 - \delta\lambda^{+}) + u(e_{2})\delta(1 + \lambda^{+} - \delta\lambda^{+}) +$$

$$+ u(e_{3})\delta^{2}(1 + \lambda^{+} - \delta\lambda^{+}) + \delta^{3}(1 + \lambda^{+})$$

Then $C(e_{1}) > 0$ if $\delta < \frac{1}{\lambda^{+}}$, otherwise $C(e_{1}) < 0$. At the same time, $C(e_{2}) > 0$ and $C(e_{3}) > 0$ always. If $\delta < \frac{1}{\lambda^{+}}$, the effort path is $(1, 1, 1, 1)$, otherwise the effort path is $0, 1, 1, 1$.

Suppose that $e_{1} \geq e_{2} \geq e_{3}$. The DM’s payoff from $e_{1}$ is

$$V_{\delta} = u(e_{1}) + \delta(u(e_{2}) - \lambda^{-}(u(e_{1}) - u(e_{2}))) + \delta^{2}(u(e_{3}) - \lambda^{-}(u(e_{2}) - u(e_{3}))) +$$

$$+ \delta^{3}(1 + \lambda^{+}(1 - u(e_{3}))) = u(e_{1})(1 - \delta\lambda^{-}) +$$

$$+ u(e_{2})\delta(1 + \lambda^{-} - \delta\lambda^{-}) +$$

$$+ u(e_{3})\delta^{2}(1 + \lambda^{-} - \delta\lambda^{+}) + \delta^{3}(1 + \lambda^{+})$$

Then $C(e_{1}) > 0$ if $\delta < \frac{1}{\lambda^{+}}$, otherwise $C(e_{1}) < 0$. At the same time, $C(e_{2}) > 0$ is true always. There is $C(e_{3}) > 0$ if $\delta < \frac{1+\lambda^{-}}{\lambda^{+}}$ and $C(e_{3}) < 0$ if $\delta > \frac{1+\lambda^{-}}{\lambda^{+}}$.  

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If $\delta < \frac{1}{\lambda}$ and $\delta < \frac{1+\lambda^-}{\lambda}$, the effort path is $(1,1,1,1)$. If $\delta < \frac{1}{\lambda}$ and $\delta > \frac{1+\lambda^-}{\lambda}$, the effort path is $(1,0,1,1)$.

If $\delta > \frac{1}{\lambda}$ and $\delta < \frac{1+\lambda^-}{\lambda}$, the effort path is $(0,1,1,1)$. If $\delta > \frac{1}{\lambda}$ and $\delta > \frac{1+\lambda^-}{\lambda}$, the effort path is $(0,0,1,1)$.

Suppose that $e_1 \leq e_2 \geq e_3$. The DM’s payoff from $e_1$ is

$$V_\delta = u(e_1) + \delta (u(e_2) + \lambda^+ (u(e_2) - u(e_1))) + \delta^2 (u(a_3) - \lambda^- (u(e_2) - u(e_3))) + \delta^3 (1 + \lambda^+ (1 - u(e_3))) =
$$

$$= u(e_1) (1 - \delta \lambda^+) + u(e_2) \delta (1 + \lambda^+ - \delta \lambda^-) + u(e_3) \delta^2 (1 + \lambda^- - \delta \lambda^+) + \delta^3 (1 + \lambda^+)$$

Then $C(e_1) > 0$ if $\delta < \frac{1}{\lambda}$, otherwise $C(e_1) < 0$. At the same time, $C(e_2) > 0$ is always. Then $C(e_3) > 0$ if $\delta < \frac{1+\lambda^-}{\lambda}$, otherwise $C(e_3) < 0$.

If $\delta < \frac{1}{\lambda}$ and $\delta < \frac{1+\lambda^-}{\lambda}$, the effort path is $(1,1,1,1)$. If $\delta < \frac{1}{\lambda}$ and $\delta > \frac{1+\lambda^-}{\lambda}$, the effort path is $(1,1,0,1)$.

If $\delta > \frac{1}{\lambda}$ and $\delta < \frac{1+\lambda^-}{\lambda}$, the effort path is $(0,1,1,1)$. If $\delta > \frac{1}{\lambda}$ and $\delta > \frac{1+\lambda^-}{\lambda}$, the effort path is $(0,1,0,1)$.

Suppose that $e_1 \geq e_2 \leq e_3$. The DM’s payoff from $e_1$ is

$$V_\delta = u(e_1) + \delta (u(e_2) - \lambda^- (u(e_1) - u(e_2))) + \delta^2 (u(e_3) + \lambda^+ (u(e_3) - u(e_2))) + \delta^3 (1 + \lambda^+ (1 - u(e_3))) =
$$

$$= u(e_1) (1 - \delta \lambda^-) + u(e_2) \delta (1 + \lambda^- - \delta \lambda^+) + u(e_3) \delta^2 (1 + \lambda^+ - \delta \lambda^-) + \delta^3 (1 + \lambda^+)$$

Then $C(e_1) > 0$ if $\delta < \frac{1}{\lambda}$, otherwise $C(e_1) < 0$. At the same time, $C(e_2) > 0$ if $\delta > \frac{1+\lambda^-}{\lambda}$, otherwise $C(e_2) < 0$. Then $C(e_3) > 0$ is always.

If $\delta < \frac{1}{\lambda}$ and $\delta < \frac{1+\lambda^-}{\lambda}$, the effort path is $(1,1,1,1)$. If $\delta < \frac{1}{\lambda}$ and $\delta > \frac{1+\lambda^-}{\lambda}$, the effort path is $(1,0,1,1)$.

If $\delta > \frac{1}{\lambda}$ and $\delta < \frac{1+\lambda^-}{\lambda}$, the effort path is $(0,1,1,1)$. If $\delta > \frac{1}{\lambda}$ and $\delta > \frac{1+\lambda^-}{\lambda}$, the effort path is $(0,0,1,1)$.

As a result, the equilibrium effort path could be any of:
\[ (e_1, e_2, e_3, e_4) = \begin{cases} 
(1, 1, 1, 1), & \text{if } \delta < \frac{1}{\lambda^+} \\
(0, 1, 1, 1), & \text{if } \delta > \frac{1}{\lambda^+} \\
(1, 1, 1, 1), & \text{if } \delta < \frac{1}{\lambda^-} \quad \text{and} \quad \delta < \frac{1 + \lambda^-}{\lambda^+} \\
(1, 0, 1, 1), & \text{if } \delta < \frac{1}{\lambda^-} \quad \text{and} \quad \delta > \frac{1 + \lambda^-}{\lambda^+} \\
(0, 1, 1, 1), & \text{if } \delta > \frac{1}{\lambda^-} \quad \text{and} \quad \delta < \frac{1 + \lambda^-}{\lambda^+} \\
(0, 1, 0, 1), & \text{if } \delta > \frac{1}{\lambda^-} \quad \text{and} \quad \delta > \frac{1 + \lambda^-}{\lambda^+} \\
(1, 1, 1, 1), & \text{if } \delta < \frac{1}{\lambda^-} \quad \text{and} \quad \delta > \frac{1 + \lambda^-}{\lambda^+} \\
(1, 0, 1, 1), & \text{if } \delta > \frac{1}{\lambda^-} \quad \text{and} \quad \delta > \frac{1 + \lambda^-}{\lambda^+} \\
(0, 1, 0, 1), & \text{if } \delta > \frac{1}{\lambda^-} \quad \text{and} \quad \delta > \frac{1 + \lambda^-}{\lambda^+} \
\end{cases} \]

The further discussion about effort path depends on the relation between \( \lambda^+ \) and \( \lambda^- \). To summarize:

If \( \lambda^+ < \lambda^- (1 + \lambda^-) \), the optimal effort path is \( e^* = (1, 1, 1, 1) \) if \( \delta < \frac{1}{\lambda^+} \) and \( \delta < \frac{1 + \lambda^-}{\lambda^+} \). When \( \delta > \frac{1}{\lambda^+} \) and \( \delta < \frac{1 + \lambda^-}{\lambda^+} \) the optimal effort path is \( e^* = (0, 1, 1, 1) \). At the same time, if \( \delta > \frac{1}{\lambda^-} \) and \( \delta > \frac{1 + \lambda^-}{\lambda^+} \), the optimal effort path is \( e^* = (0, 1, 0, 1) \).

As soon as \( \lambda^+ > \lambda^- (1 + \lambda^-) \), the DM’s optimal effort path could be: \( e^* = (1, 1, 1, 1) \) if \( \delta < \frac{1}{\lambda^-} \) and \( \delta < \frac{1 + \lambda^-}{\lambda^+} \); \( e^* = (0, 1, 1, 1) \) if \( \delta > \frac{1}{\lambda^-} \) and \( \delta < \frac{1 + \lambda^-}{\lambda^+} \); \( e^* = (1, 0, 1, 1) \) if \( \delta < \frac{1}{\lambda^-} \) and \( \delta > \frac{1 + \lambda^-}{\lambda^+} \); and \( e^* = (0, 1, 0, 1) \) if \( \delta > \frac{1}{\lambda^-} \) and \( \delta > \frac{1 + \lambda^-}{\lambda^+} \).
4. CHAPTER

4.1 INTRODUCTION

In this chapter, I continue considering a DM who has Intertemporal Reference Dependence (IRD). In addition I consider the NIEI (i.e., Negative Intertemporal Efforts Interdependence) and the PIEI (i.e., Positive Intertemporal Effort Interdependence) cases along with IRD. Recall that in the NIEI case, the DM’s effort in one period affects her effort level in the next period. I am going to show that, surprisingly, regardless of “$\lambda^- -$ dominance” and/or “$\lambda^+ -$ dominance”, any DM’s choice of effort path is the same and strongly monotonic over the whole time horizon.

In the PIEI case, I first consider a $\lambda^- -$dominant DM. I find that the DM behaves like an IRI agent, who always chooses the highest possible effort level every period, i.e., her effort path will be $(1, 1, ..., 1, 1)$. Next, I consider the case $\lambda^+ > \lambda^-$. In this case too, the DM chooses a path where her effort level is always at the maximum possible level after the first period, while her first period effort $(1)$ too is at the maximum possible level if her discount factor is not sufficiently high, and $(2)$ is less than the maximum possible level when her discount factor is sufficiently high.

4.2 MODEL

In this chapter too, I follow the definitions and notations from Chapter 2 and I continue to consider a DM who takes part in the finite dynamic task, where time is discrete and $T < \infty$, and for each period $t = 1, ..., T$, the
DM has to take a *costless effort* \( e_t = [0, 1] \) that yields a *payoff* of \( u(e_t) \), where \( u(e_t) \) is a direct utility. The DM’s period utility from the effort path \( e = (e_1, \ldots, e_T) \) is \( v(e_t) = u(e_t) + \lambda^+(u(e_t) - u(e_{t-1})) \) if \( u(e_t) > u(e_{t-1}) \) and \( v(e_t) = u(e_t) - \lambda^-(u(e_{t-1}) - u(e_t)) \) if \( u(e_t) < u(e_{t-1}) \), where \( \lambda^+(u(e_t) - u(e_{t-1})) \) is a reference utility gain, and \( \lambda^-(u(e_{t-1}) - u(e_t)) \) is a reference utility loss. Combining all these elements, the *discounted sum of utility* that incorporates reference utility gain and reference utility loss is defined as follows:

\[
V_\delta(e_t, e_{t-1}) = u(e_1) + \sum_{t=2}^{T} v(e_t, e_{t-1})
\]

where \( \delta \in (0, 1) \) is her time discount factor.

I will continue to consider the IRD case, i.e., \( \lambda^- > 0 \) and/or \( \lambda^+ > 0 \).

4.3 THE NIEI CASE: \( u(e_t) = (1 - e_{t-1})e_t \)

The following lemma is immediate.

**Lemma 14:** In the NIEI and IRD case, for any \( T \) and \( \delta \in (0, 1) \), the last period effort is \( e_T = 1 \).

**Proof** Suppose the DM is at period \( T \). Then we have that \( e_T = 1 \) regardless of \( \lambda^+ \) and/or \( \lambda^- \). To see this fix any immediate history \( h_{T-1} = e_{T-1} \in [0, 1] \) for period \( T \).\(^1\) Then, given \( u(\cdot) \) is monotone increasing, it cannot be that \( e_T < h_{T-1} \). Therefore, it also follows that \( e_T \) takes its maximum value on \([0, 1]\). This completes the proof of lemma.\( \square \).

The following proposition characterises the DM’s effort path.

**Proposition 7:** In the NIEI and IRD case, for any \( t > 1 \) and \( \delta \in (0, 1) \) the DM’s effort path is \( e_{t-1} \leq e_t \leq e_{t+1} \).

\(^1\) See the definition *Prior effort history* in Chapter 3
Proof The proof of Proposition consists of two steps and invokes an inductive argument.

1. Step 1 for any \( t > 1 \), if \( e_{t-1} \leq e_{t+1} \), then \( e_t \in [e_{t-1}, e_{t+1}] \);

2. Step 2 for any \( \tau \in [t, t+1] \), \( e_\tau \in [e_t, e_{t+1}] \), where \( t \geq 2 \) and \( \tau < T \).

Step 1: Suppose that the DM is at period \( t \), and assume that the following is true: \( \ldots \leq e_{t-2} \leq e_{t-1} \leq e_{t+1} \leq \ldots \). We want to show that the effort of current period \( t \) is \( e_t \in [e_{t-1}, e_{t+1}] \).

First, let’s consider the DM’s payoff at period \( t \):

\[
V(e_t) = e_t (1 - e_{t-1}) + \begin{cases} 
\lambda^+ (e_t (1 - e_{t-1}) - e_{t-1} (1 - e_{t-2})) \\
-\lambda^- (e_{t-1} (1 - e_{t-2}) - e_t (1 - e_{t-1})) 
\end{cases}
\]

Suppose the DM chooses the effort \( e_t \) such that her current period outcome is \( V(e_t) = e_t (1 - e_{t-1}) + \lambda^+ (e_t (1 - e_{t-1}) - e_{t-1} (1 - e_{t-2})) \). As long as \( e_t (1 - e_{t-1}) - e_{t-1} (1 - e_{t-2}) > 0 \), there is \( e_t > \frac{e_{t-1} (1 - e_{t-2})}{(1 - e_{t-1})} \). Since \( e_{t-1} \leq e_{t+1} \), there is \( \frac{(1 - e_{t-2})}{(1 - e_{t-1})} \geq 1 \). Hence, we have \( e_t \geq e_{t-1} \).

Now consider the DM’s payoff in the next period, \( t + 1 \) : \( V(e_{t+1}) = \delta (e_{t+1} (1 - e_t) + \lambda^+ (e_{t+1} (1 - e_t) - e_t (1 - e_{t-1}))) \). Since \( e_{t+1} (1 - e_t) - e_t (1 - e_{t-1}) > 0 \), we have \( e_{t+1} > \frac{e_t (1 - e_{t-1})}{(1 - e_t)} \). We know that \( e_t > e_{t-1} \); therefore, \( \frac{(1 - e_{t-1})}{(1 - e_t)} \geq 1 \). Thus, we have \( e_{t+1} \geq e_t \).

Next, suppose that the DM decides to deviate and chooses her effort such that the current period payoff is \( V(e_t) = e_t (1 - e_{t-1}) - \lambda^- (e_{t-1} (1 - e_{t-2}) - e_t (1 - e_{t-1})) \). This choice affects the DM’s next period payoff too as follows:

\[
V(e_{t+1}) = \delta (e_{t+1} (1 - e_t) + \begin{cases} 
\lambda^+ (e_{t+1} (1 - e_t) - e_t (1 - e_{t-1}))) \\
-\lambda^- (e_t (1 - e_{t-1}) - e_{t+1} (1 - e_t))) 
\end{cases}
\]
CLAIM 13: In two consequent periods, the following combination \( \{ \lambda^-, \lambda^- \} \) is impossible.

Proof Suppose that the DM’s choice of \( e_t \) is such that the next period DM’s payoff is \( V(e_{t+1}) = \delta(e_{t+1}(1 - e_t) - \lambda^-(e_t(1 - e_{t-1}) - e_{t+1}(1 - e_t))) \). To choose her effort \( e_t \) the DM compares her marginal benefit of that effort to her marginal cost of it. To find the marginal benefit of effort \( e_t \) in period \( t \), we need to take the derivative of \( V(e_t) \) with respect to \( e_t \). Then we obtain \( \frac{\partial V(e_t)}{\partial e_t} = (1 - e_{t-1}) + \lambda^- (1 - e_{t-1}) \). Next, to find marginal cost of effort \( e_t \), we need to take the derivative of \( V(e_{t+1}) \) with respect to \( e_t \). Then we obtain \( \frac{\partial V(e_{t+1})}{\partial e_t} = \delta(1 - e_{t-1}) + \lambda^- (1 - e_{t-1}) \). Comparing marginal benefit to marginal cost of chosen effort \( e_t \) reduces to \( \frac{e_{t+1}}{e_{t-1}-1} = \frac{1+\lambda^-+\delta\lambda^-}{\delta(1+\lambda^-)} \). With \( \frac{1+\lambda^-+\delta\lambda^-}{\delta(1+\lambda^-)} > 0 \), the RHS is positive while the LHS is negative. As this is impossible, the DM cannot choose such an effort level in period \( t \) and thus encounter \( \lambda^- \) in both periods \( t \) and \( t+1 \). □

CLAIM 14: In two consequent periods, the following combination \( \{ \lambda^-, \lambda^+ \} \) is impossible.

Proof Suppose that the DM’s choice of \( e_t \) is such that the next period DM’s payoff is \( V(e_{t+1}) = \delta(e_{t+1}(1 - e_t) + \lambda^+(e_t(1 - e_{t-1}) - e_{t+1}(1 - e_t))) \). As in the above claim’s proof, to choose her effort \( e_t \) the DM compares her marginal benefit of that effort to her marginal cost of it. To find the marginal benefit of effort \( e_t \) in period \( t \), we need to take the derivative of \( V(e_t) \) with respect to \( e_t \). Then we obtain \( \frac{\partial V(e_t)}{\partial e_t} = (1 - e_{t-1}) + \lambda^- (1 - e_{t-1}) \). Likewise, next, to find marginal cost of effort \( e_t \), we need to take the derivative of \( V(e_{t+1}) \) with respect to \( e_t \). Then we obtain \( \frac{\partial V(e_{t+1})}{\partial e_t} = \delta(e_{t+1}) - \lambda^-(e_t + \lambda^+(1 - e_{t-1})) \). Comparing marginal benefit to marginal cost of chosen effort \( e_t \) reduces to \( \frac{e_{t+1}}{e_{t-1}-1} = \frac{1+\lambda^-+\delta\lambda^-}{\delta(1+\lambda^-)} \). With \( \frac{1+\lambda^-+\delta\lambda^-}{\delta(1+\lambda^-)} > 0 \), the RHS is positive while the LHS is negative. As this is impossible, the DM cannot choose such effort in period \( t \) and thus encounter \( \lambda^- \) in period \( t \) and \( \lambda^+ \) in the subsequent period. □
As a result, for any period $t$, if $e_{t-1} \leq e_t \leq e_{t+1}$, this completes the proof of Step 1.

Now we will move to the next step of proof.

**Step 2**: In this step we will show that for any $\tau \in [t, t+1]$, $e_\tau \in [e_t, e_{t+1}]$, where $t \geq 2$ and $\tau < T$.

Suppose that $e^* = (e_1^*, e_2^*, \ldots, e_T^*)$ is an equilibrium. Consider the statement $S(n)$ such that $e_{t+1} \leq e_{t+2} \leq \ldots \leq e_{t+n}$. From Lemma 14 we know that $e_T^* = 1$ and this statement is true for $n = 3$, that is $S(3)$ holds (see Step 1). So, now suppose that if $S(n)$ is true, then $S(n+1)$ is also true $\forall \tau \in [t, t+1]$, where $t \geq 2$.

Since $S(n)$ is true, it means that there is $e_t \leq e_{t+1} \forall t \in [2, 3, \ldots, n]$, where $n < T$. Let’s take any $\tau$ such that $\tau \in [t, t+1]$. According to the proof of Step 1, we have that $e_\tau \in [e_t, e_{t+1}]$. Hence, $S(n+1)$ is true. This completes the proof of Step 2.

Putting all findings together, we establish that the DM’s effort path is $e_{t-1} \leq e_t \leq e_{t+1}$ for any $t > 1$. This concludes the proof. □

Proposition 7 leads to the following interesting remark.

**Remark 2**: In the NIEI with IRD case, regardless of $\lambda^-$ - dominance and/or $\lambda^+$ - dominance, any DM type’s choice of effort path is the same and weakly monotonic over the whole time horizon.

The following lemma establishes that the DM’s choice of the last two periods’ efforts is strictly monotonic: $e_{T-1} < e_T = 1$. In other words, the DM chooses her effort at period $T - 1$ in order to make a very strong finish.

**Lemma 15**: In the NIEI and IRD case, for any $\delta \in (0, 1)$, $e_{T-1} < e_T = 1$.

**Proof** The proof invokes OSDP (i.e., One-Step-Deviation Principle). Suppose that $e_{T-1} = e_T = 1$, that is the DM’s effort at period $T - 1$ is 1. The DM’s payoff is $V(e_{T-1} = 1) = 1(1 - e_{T-2}) + \lambda^+(1(e_{T-2}) - e_{T-2}(1 - e_{T-3})) +$
\(\delta(0 - \lambda^- (1 - e_{T-2}) - 0))\). According to Proposition 7 this combination \(\{\lambda^+, \lambda^-\}\) is impossible. Hence, there is \(e_{T-1} < e_T = 1\). This concludes the proof. \(\square\)

The following Theorem 1 characterizes the DM’s effort path in the case of NIEI and IRD.

**Theorem 1:** In the NIEI and IRD case, \(\forall t\) regardless of \(\lambda^+\) and \(\lambda^-\) as long as either \(\lambda^+ > 0\) and/or \(\lambda^- > 0\), \(e_t < e_{t+1}\).

**Proof** We already know from previous results that the DM’s effort path is at least weakly monotonic, i.e., \(e_1 \leq e_2 \leq \ldots \leq e_{t-2} \leq e_{t-1} \leq e_t \leq e_{t+1} \ldots \leq e_{T-1} \leq e_T\) (by Proposition 7). From Lemma 15 it is known that the last two periods’ efforts are strongly monotonic, i.e., \(e_{T-1} < e_T = 1\).

Suppose that the DM is at period \(T - 2\). And the effort path is such that: 
\[\ldots e_{T-4} \leq e_{T-3} \leq e_{T-2} = e_{T-1} < e_T = 1\]. We will show that this effort path is impossible. Consider three different cases which partition all possibilities:
1) \(e_t = \frac{1}{2}\); 2) \(0 < e_t < \frac{1}{2}\); and 3) \(\frac{1}{2} < e_t < 1\).

First, we consider the case \(e_{T-2} = \frac{1}{2}\). We know that \(e_{T-3} \leq e_{T-2} = \frac{1}{2}\). It means that \(1 - e_{T-3} \geq 1 - e_{T-2} = \frac{1}{2}\). Thus, the DM’s payoff is \(V_\delta(e_{T-2}) = \frac{1}{2}(1 - e_{T-3}) + \lambda^+((\frac{1}{2})(1 - e_{T-3})) + \lambda^-((\frac{1}{2})(1 - e_{T-3}))\). However, this combination of \(\{\lambda^-, \lambda^+\}\) contradicts Proposition 7. Therefore, \(e_t \neq e_{t+1}\) if \(e_t = \frac{1}{2}\).

Next, let’s consider the case \(\frac{1}{2} < e_{T-2} < 1\). Suppose that \(e_{T-2} = \frac{1}{2} + \varepsilon\). It means that \(e_{T-3} \leq e_{T-2} = \frac{1}{2} + \varepsilon\), where \(\varepsilon \in (0, \frac{1}{2})\); therefore \(1 - e_{T-3} \geq 1 - e_{T-2} = \frac{1}{2} - \varepsilon\). Hence, the DM’s payoff is \(V(e_{T-2}) = (\frac{1}{2} + \varepsilon)((1 - e_{T-3}) + \lambda^+((\frac{1}{2} + \varepsilon)(1 - e_{T-3})) + \lambda^-((\frac{1}{2} - \varepsilon)(1 - e_{T-3}))\). However, this combination of \(\{\lambda^-, \lambda^+\}\) contradicts Proposition 7. Therefore, \(e_t \neq e_{t+1}\) if \(\frac{1}{2} < e_t < 1\).
Finally, we consider the case $0 < e_{T-2} < \frac{1}{2}$. That is, the effort at period $T-2$ is $\frac{1}{2} - \varepsilon$, where $\varepsilon \in (0, \frac{1}{2})$. It means that $e_{T-3} \leq e_{T-2} = \frac{1}{2} - \varepsilon$; therefore, $1 - e_{T-3} \geq 1 - e_{T-2} = \frac{1}{2} + \varepsilon$. Thus, the DM’s payoff is $V(e_{T-2}) = (\frac{1}{2} - \varepsilon)(1 - e_{T-3}) + \lambda^+(\frac{1}{2} - \varepsilon)(1 - e_{T-3} - e_{T-3}(1 - e_{T-4})) + \delta((\frac{1}{2} - \varepsilon)(\frac{1}{2} + \varepsilon) - \lambda^-((\frac{1}{2} - \varepsilon)(1 - e_{T-3}) - (\frac{1}{2} - \varepsilon)(\frac{1}{2} + \varepsilon)) + \delta^2(1(\frac{1}{2} + \varepsilon) + \lambda^+((1(\frac{1}{2} + \varepsilon) - (\frac{1}{2} - \varepsilon)(\frac{1}{2} + \varepsilon))$. However, this combination of $\{\lambda^-, \lambda^+\}$ contradicts Proposition 7. Therefore, $e_t \neq e_{t+1}$ if $0 < e_t < \frac{1}{2}$.

Putting all of these findings together, we conclude that the DM’s equilibrium effort path is strictly monotone increasing regardless of $\lambda^- \gtrless \lambda^+$ as long as either $\lambda^-$ or $\lambda^+$ is greater than zero. This completes the proof. □

Thus, Theorem 1 states that when the DM has Negative Intertemporal Effort Interdependence (NIEI) and Intertemporal Reference Dependence (IRD), then irrespective of “$\lambda^-$ - dominance” and/or “$\lambda^+$ - dominance” the DM’s choice of effort path is the same and strictly monotone increasing over
the whole time horizon.

4.4 THE PIEI CASE: \( u(e_t) = e_{t-1}e_t \)

4.4.1 \( \lambda^- \) - DOMINANT DM

Here, I will study PIEI when \( \lambda^+ < \lambda^- \). The following lemma is immediate.

**Lemma 16**: In the PIEI and \( \lambda^- \) - dominant DM case, for any \( T \) and \( \delta \in (0,1) \), the last period effort is \( e_T = 1 \).

**Proof** Suppose the DM is at period \( T \). Then we have \( e_T = 1 \). To see this, fix any immediate history \( h_{T-1} = e_{T-1} \in [0,1] \) for period \( T \). Then, given \( u(\cdot) \) is monotone increasing, it cannot be that \( e_T < h_{T-1} \). Therefore, it also follows that \( e_T \) takes its maximum value on \( [0,1] \). The DM’s payoff is \( V(e_T = 1) = 1e_{T-1} + \lambda^+ (1e_{T-1}^* - e_{T-2}^* e_{T-2}^*) \). Now suppose that the DM wants to deviate and choose the effort \( e_T = 1 - \varepsilon \), where \( \varepsilon \in (0,1) \). Then the DM’s outcome is either \( V(e_T = 1 - \varepsilon)^+ = (1 - \varepsilon)e_{T-1} + \lambda^+ ( (1 - \varepsilon)e_{T-1}^* - e_{T-1}^* e_{T-2}^* ) \) or \( V(e_T = 1 - \varepsilon)^- = (1 - \varepsilon)e_{T-1} - \lambda^+ ( e_{T-1}^* e_{T-2}^* - (1 - \varepsilon)e_{T-1}^* ) \).

First, we compare \( V(e_T = 1 - \varepsilon)^+ \) and \( V(e_T = 1 - \varepsilon)^- \). It is easy to verify that \( V(e_T = 1 - \varepsilon)^+ > V(e_T = 1 - \varepsilon)^- \). Comparing \( V(e_T = 1) \) and \( V(e_T = 1 - \varepsilon)^+ \), we have \( V(e_T = 1) - V(e_T = 1 - \varepsilon)^+ = 1e_{T-1} + \lambda^+ (1e_{T-1}^* - e_{T-1}^* e_{T-2}^*) - \left[ (1 - \varepsilon)e_{T-1} + \lambda^+ ( (1 - \varepsilon)e_{T-1}^* - e_{T-1}^* e_{T-2}^* ) \right] = \varepsilon e_{T-1} + \lambda^+ \varepsilon e_{T-1}^* > 0 \). Consequently, the DM’s effort at \( T \) is \( e_T = 1 \). This completes the proof. \( \square \)

**Lemma 17**: In the PIEI and \( \lambda^- \) - dominant DM case, no optimal effort path can have a period \( t > 1 \), such that \( e_{t+1} = 1 - \varepsilon \), where \( \varepsilon \in (0,1] \).

**Proof** The proof invokes OSDP (i.e., One-Step-Deviation-Principle). Pick any arbitrary \( t \in \{2, \ldots, T - 1\} \) and consider the effort path \( e(e_t, e_{t+1}) = (\ldots, \{e_{t-1}\}, e_t, e_{t+1}, \{e_{t+2}\}, \ldots) \). The DM’s payoff is
\[ V^j_\delta(e(t, e_{t+1})) = \sum_{i=j}^{T} \delta^{i-j} V(e(t, e_{t+1})). \] More specifically, her payoff is \( V^j_\delta(e(\{e_{t-1}\}, 1, 1, \{1\})) = 1e_{t-1} + \lambda^+(1e_{t-1} - e_{t-1}e_{t-2}) + \delta(1 + \lambda^+(1 - 1e_{t-1})) + \delta^2(1 + \lambda^+(1 - 1e_{t-1})) + \delta^3V^t_\delta(1 - e_{t-1}e_{t-2}). \]

Suppose that the DM’s effort at \( t + 1 \) is such that \( e_{t+1} = 1 - \varepsilon \), where \( \varepsilon \in (0, 1) \). Then, the DM payoff is either \( V^j_\delta(e(\{e_{t-1}\}, 1, 1 - \varepsilon, \{1\}))^+ = 1e_{t-1} + \lambda^+[1e_{t-1} - e_{t-1}e_{t-2} + \delta((1 - \varepsilon) + \lambda^+(1 - \varepsilon) - 1e_{t-1})) + \delta^2(1 + \lambda^+(1 - (1 - \varepsilon))) + \delta^3V^t_\delta(e(e_{t-1}e_{t+1})) \) or \( V^j_\delta(e(\{e_{t-1}\}, 1, 1 - \varepsilon, \{1\}))^- = 1e_{t-1} + \lambda^+[1e_{t-1} - e_{t-1}e_{t-2} + \delta((1 - \varepsilon) - \lambda^-(1e_{t-1} - 1 - \varepsilon)) + \delta^2(1 + \lambda^+(1 - (1 - \varepsilon))) + \delta^3V^t_\delta(e(e_{t-1}e_{t+1})) \] is \( \varepsilon \delta(1 + \lambda^+ - \varepsilon \delta^2 \lambda^+) > 0 \). With \( \lambda^- > \lambda^+ \), the LHS and RHS are positive. Hence, \( V^j_\delta(e(\{e_{t-1}\}, 1, 1 - \varepsilon, \{1\}))^+ > V^j_\delta(e(\{e_{t-1}\}, 1, 1 - \varepsilon, \{1\}))^- \) for all \( e_{t-1} \in [0, 1] \).

Next, let’s compare \( V^j_\delta(e(\{e_{t-1}\}, 1, 1, \{1\})) \) and \( V^j_\delta(e(\{e_{t-1}\}, 1, 1 - \varepsilon, \{1\}))^- \). Then we have \( V^j_\delta(e(\{e_{t-1}\}, 1, 1, \{1\})) - V^j_\delta(e(\{e_{t-1}\}, 1, 1 - \varepsilon, \{1\}))^- = 1e_{t-1} + \lambda^+[1e_{t-1} - e_{t-1}e_{t-2} + \delta(1 + \lambda^+(1 - 1e_{t-1})) + \delta^2(1 + \lambda^+(1 - 1e_{t-1})) + \delta^3V^t_\delta(e(e_{t-1}e_{t+1})) \]

Putting all findings together, we conclude that no equilibrium effort path can have a period \( t > 1 \), such that \( e_{t+1} = 1 - \varepsilon \). This completes the proof. \( \Box \)

**Lemma 18:** In the PIEI and \( \lambda^- \) - dominant DM case, for any \( T \) and \( \delta \in (0, 1) \), \( e^*_1 = 1 \).

**Proof** The proof uses One-Step-Deviation-Principle (OSDP). Suppose that
\( e^* = (e_1, 1, 1, \ldots, 1, 1) \) is an equilibrium. If \( e_1 = 1 \), the DM’s payoff is \( V(e_1 = 1) = 1 + \delta(1) + \delta^2(1) + V^{t \geq 3} \). Now suppose that the DM wants to deviate and chooses the effort \( e_1 = 1 - \varepsilon \), where \( \varepsilon \in (0, 1) \). The DM’s payoff is \( V(e_1 = 1 - \varepsilon) = 1 - \varepsilon + \delta(1 - \varepsilon) + V^{t \geq 3} \). Comparing these two payoffs we have that 
\[
V(e_1 = 1) - V(e_1 = 1 - \varepsilon) = 1 + \delta(1) - (1 - \varepsilon) > 0.
\]
Thus, there is \( e_1 = 1 \). This completes the proof. \( \Box \)

To generalise findings, the following proposition is immediate:

**Proposition 8:** In the PIEI and a \( \lambda^- \) - dominant DM case, for any \( T \) the equilibrium effort path is \( e^* = (1, 1, \ldots, 1, 1) \).

This means that when the DM has Positive Intertemporal Effort Interdependence (PIEI) and Intertemporal Reference Dependence (IRD) with "\( \lambda^- \) - dominance", then irrespective of her time discounting, the DM will exert full effort over the whole time horizon.

### 4.4.2 \( \lambda^+ \) - Dominant DM

In this section I will study PIEI when \( \lambda^+ > \lambda^- \). The following lemmas will be useful.

**Lemma 19:** In the PIEI and \( \lambda^+ \) - dominant DM case, for any \( T \) and \( \delta \in (0, 1) \), \( e_T = 1 \).

**Proof** Suppose that the DM is at period \( T \). Then we have \( e_T = 1 \). To see this, fix any immediate history \( h_{T-1} = e_{T-1} \in [0, 1] \) for period \( T \). Then, given \( u(\cdot) \) is monotone increasing, it cannot be that \( e_T < h_{T-1} \). Therefore, it also follows that \( e_T \) takes its maximum value on \( [0, 1] \). The DM’s payoff is \( V(e_T = 1) = 1e_{T-1} + \lambda^+(1e_{T-1} - e_{T-1}^* e_{T-2}^*) \). Now suppose that the DM decides to choose an effort \( e_T = 1 - \varepsilon \), where \( \varepsilon \in (0, 1) \). Hence, the DM’s outcome is either 
\[
V(e_T = 1 - \varepsilon)^+ = (1 - \varepsilon)e_{T-1} + \lambda^+((1 - \varepsilon)e_{T-1} - e_{T-1}^* e_{T-2}^*)
\]
or
\[
V(e_T = 1 - \varepsilon)^- = (1 - \varepsilon)e_{T-1} - \lambda^-(e_{T-1}^* e_{T-2}^* - (1 - \varepsilon) e_{T-1}^*).
\]
Comparing $V(e_T = 1 - \varepsilon)^+$ and $V(e_T = 1 - \varepsilon)^-$, we have $V(e_T = 1 - \varepsilon)^+ > V(e_T = 1 - \varepsilon)^-$. Next, we compare $V(e_T = 1)$ and $V(e_T = 1 - \varepsilon)^+$.

Then we obtain $1e_{T-1} + \lambda^+(1e_{T-1}^* - e_{T-2}^*) - (1 - \varepsilon)e_{T-1} + \lambda^+((1 - \varepsilon)e_{T-1}^* - e_{T-2}^*) \Rightarrow \varepsilon e_{T-1} + \varepsilon^+ e_{T-1} > 0$. Hence, we have $e_T = 1$. This completes the proof.

**Lemma 20:** In the PIEI and $\lambda^+$-dominant DM case, for $T = 2$, the effort path is $e = (1, 1)$.

**Proof** Suppose that $T = 2$. According to Lemma 19, the DM’s effort at $T$ is $e_T = 1$. Also, the DM’s payoff from $e_1$ is $V(e_1) = e_1 + \delta(1e_1 + \lambda^+(1e_1 - e_1)) = e_1(1 + \delta)$. Hence, $e_1 = 1$.

Suppose that the DM wants to deviate and choose an effort $e_1 = (1 - \varepsilon)$, where $\varepsilon \in (0, 1]$. Then the DM’s payoff is $V(e_1 = 1 - \varepsilon) = (1 - \varepsilon) + \delta(1 - \varepsilon)$. It is easy to verify that $V(e_1 = 1) > V(e_1 = 1 - \varepsilon)$. To see this, let’s consider the following difference $V(e_1 = 1) - V(e_1 = 1 - \varepsilon) = 1 + \delta - [(1 - \varepsilon) + \delta(1 - \varepsilon)] = (1 + \delta) \varepsilon > 0$. Therefore, the effort path is $e = (1, 1)$. This concludes the proof.

In the next section I will first proceed with examples.

### 4.4.3 Example with $T = 3$ and $T = 4$

Here, I will provide only outcomes of the cases $T = 3$ and $T = 4$. For details of these examples, see Appendix 1. These examples provide a wide spectrum of all possible effort paths that the DM can follow.

For $T = 3$, I have:

$$(e_1, e_2, e_3) = \begin{cases} 
(1, 1, 1), & \text{if } \delta < \frac{1 + \sqrt{1 + 4\lambda^+}}{2\lambda^+} \\
(1 - \varepsilon, 1, 1), & \text{if } \delta > \frac{1 + \sqrt{1 + 4\lambda^+}}{2\lambda^+} 
\end{cases}$$

where $\varepsilon \in (0, 1]$. 

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Next, I consider the example with $T = 4$. This example is important because it highlights effort paths that are different from those of the case $T = 3$.

Regardless of value of $\lambda^+$ and $\lambda^-$, if $\varepsilon < \delta$, the effort path is:

$$
(e_1, e_2, e_3, e_4) = \begin{cases} 
(1,1,1,1), & \text{if } \begin{cases} 1 + \lambda^- + \delta - \delta^2 \lambda^+ > 0 \\
1 + \delta - \delta^2 \lambda^+ > 0 \end{cases} \\
(1-\varepsilon,1,1,1), & \text{if } \begin{cases} (1-\varepsilon)(1 + \lambda^-) + \delta(1 + \lambda^+ \varepsilon) - \delta^2 \lambda^+ > 0 \\
1 + \delta - \delta^2 \lambda^+ < 0 \end{cases} \\
(1-\varepsilon,1-\varepsilon,1,1), & \text{if } \begin{cases} (1-\varepsilon)(1 + \lambda^-) + \delta(1 + \lambda^+ \varepsilon) - \delta^2 \lambda^+ < 0 \\
1 + \delta(1 - \varepsilon(1 + \lambda^-)) - \delta^2 \lambda^+(1 - \varepsilon) < 0 \end{cases} 
\end{cases}
$$

In the case where, $\varepsilon > \delta$, the effort path is:

$$
(e_1, e_2, e_3, e_4) = \begin{cases} 
(1,1,1,1), & \text{if } \begin{cases} 1 + \lambda^- + \delta - \delta^2 \lambda^+ > 0 \\
1 + \delta - \delta^2 \lambda^+ > 0 \end{cases} \\
(1-\varepsilon,1,1,1), & \text{if } \begin{cases} (1-\varepsilon)(1 + \lambda^-) + \delta(1 + \lambda^+ \varepsilon) - \delta^2 \lambda^+ < 0 \\
1 + \delta(1 - \varepsilon(1 + \lambda^-)) - \delta^2 \lambda^+(1 - \varepsilon) < 0 \end{cases} \\
(1-\varepsilon,1-\varepsilon,1,1), & \text{if } \begin{cases} (1-\varepsilon)(1 + \lambda^-) + \delta(1 + \lambda^+ \varepsilon) - \delta^2 \lambda^+ < 0 \\
1 + \delta(1 - \varepsilon(1 + \lambda^-)) - \delta^2 \lambda^+(1 - \varepsilon) < 0 \end{cases} 
\end{cases}
$$

To generalise these findings, I introduce the following lemma:

**Lemma 21**: In the PIEI and "$\lambda^+$ - dominance" case, if $\lambda^+ < \frac{1+\delta}{\delta^2}$ the optimal effort path is $e^* = (1,1,\ldots,1,1)$.

**Proof** The proof involves an inductive argument and also invokes the One-Step-Deviation Principle (OSDP). Let $e^* = (e^*_1, e^*_2, \ldots, e^*_T)$ be an optimal solution and consider the statement $S(n)$, $T - 1 \geq n \geq 0$ that

$$
e^*_{T-n} = e^*_{T-(n-1)}, \ldots, e^*_{T-1}, e^*_T = 1.$$

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From Lemma 19 we know that $e_T^* = 1$ and hence $S(0)$ is true. Consider $S(1)$.
To prove it is true, OSDP ensures that blocking any deviation at period $T - 1$ will suffice. So given $e^*$, the utility of the DM from $e^*$ given $S(1)$ holds is

$$V_S(e_{T-1} = 1|S(1)) = 1e_{T-2} + \lambda^+(1e_{T-2} - e_{T-2}e_{T-3}) - \delta(1 + \lambda^+(1 - 1e_{T-2})).$$

Suppose that the DM wants to deviate and chooses $e_{T-1} = 1 - \varepsilon$, where $\varepsilon \in (0, 1]$. We consider two cases. First, $V_S(e_{T-1} = 1 - \varepsilon|S(1))^+ = (1 - \varepsilon)e_{T-2} + \lambda^+((1 - \varepsilon)e_{T-2} - e_{T-2}e_{T-3}) - \delta(1 - \varepsilon) + \lambda^+((1 - \varepsilon) - (1 - \varepsilon)e_{T-2})$. Second, $V_S(e_{T-1} = 1 - \varepsilon|S(1))^+ = (1 - \varepsilon)e_{T-2} - \lambda^+(e_{T-2}e_{T-3} - (1 - \varepsilon)e_{T-2}) + \delta(1 - \varepsilon) + \lambda^+((1 - \varepsilon) - (1 - \varepsilon)e_{T-2})$. Comparing $V_S(e_{T-1} = 1|S(1))$ and $V_S(e_{T-1} = 1 - \varepsilon|S(1))^+$, we have $V_S(e_{T-1} = 1|S(1)) - V_S(e_{T-1} = 1 - \varepsilon|S(1))^+ = \varepsilon e_{T-2}(1 + \lambda^+) + \varepsilon^2(1 + \lambda^+(1 - 1e_{T-2})) > 0$. It is easy to verify that $V_S(e_{T-1} = 1|S(1)) > V_S(e_{T-1} = 1 - \varepsilon|S(1))^+$. Hence, $S(1)$ is true. Now, assume $S(n)$ is true and check $S(n + 1)$. OSDP and the proof above for $S(1)$ render this argument in a straightforward way. This means irrespective of $e_{n-1}^*$, it must be that $e_n^* = e_{n+1}^* = \ldots e_T^* = 1$.

Now, let’s analyse $S(n + 1)$. For simplicity let’s denote $T - (n + 1) = m$. Then, $V_S(e_m = 1|S(n+1)) = 1e_{m-1} + \lambda^+(e_{m-1} - e_{m-1}e_{m-2}) + \delta(1 + \lambda^+(1 - 1e_{m-1})) + \delta^2(1) + \delta^3(1) + V$. Suppose that $e_m = 1 - \varepsilon$. Then the DM’s payoff is either $V_S(e_m = 1 - \varepsilon|S(n+1))^+ = (1 - \varepsilon)e_{m-1} - \lambda^-(e_{m-1}e_{m-2} - (1 - \varepsilon)e_{m-1}) + \delta((1 - \varepsilon) + \lambda^+((1 - \varepsilon))(1 - 1e_{m-1}) - \lambda^+(1 - 1e_{m-1})) + \delta^2(1 + \lambda^+(1 - 1e_{m-1})) + \delta^3(1) + V$ or $V_S(e_m = 1 - \varepsilon|S(n+1))^+ = (1 - \varepsilon)e_{m-1} + \lambda^+((1 - \varepsilon)e_{m-1} - e_{m-1}e_{m-2}) + \delta((1 - \varepsilon) + \lambda^+((1 - \varepsilon))(1 - 1e_{m-1})) + \delta^2(1 + \lambda^+(1 - 1e_{m-1})) + \delta^3(1) + V$. It is easy to verify that $V_S(e_m = 1|S(n+1)) > V_S(e_m = 1 - \varepsilon|S(n+1))^+$. Comparing $V_S(e_m = 1|S(n+1))$ and $V_S(e_m = 1 - \varepsilon|S(n+1))^+$, we have $V_S(e_m = 1|S(n+1)) > V_S(e_m = 1 - \varepsilon|S(n+1))^+$ if $\lambda^+ < \frac{e_{m-1} + \delta}{\delta^2 - \delta(1 - e_{m-1})}$. Hence, $S(n + 1)$ is true.

Now, consider the DM’s effort at period $t = 1$. Then her payoff specifically is $V(e_1 = 1) = 1 + \delta(1) + \delta^2(1) + V_T^{\geq 4}$. Suppose the DM wants to deviate
and choose \( e_1 = 1 - \varepsilon \). The DM’s payoff then is \( V(e_1 = 1 - \varepsilon) = (1 - \varepsilon) + \delta(1 - \varepsilon) + \delta^2(1 + \lambda^+(1 - (1 - \varepsilon))) + V_\delta^{t\geq4} \). Comparing \( V(e_1 = 1) \) and \( V(e_1 = 1 - \varepsilon) \), we have \( e_1 = 1 \) if \( \lambda^+ < \frac{1+\delta}{\delta^2} \). This completes the proof. □

This means that when the DM has Positive Intertemporal Effort Interdependence and Intertemporal Reference Dependence (IRD) with "\( \lambda^+ - \) dominance", then if her discount factor is not sufficiently high, the DM will exert full effort over the whole time horizon.

The following Lemma 22 characterises the DM’s effort path in the PIEI and "\( \lambda^+ - \) dominance" case.

**LEMMA 22:** In the PIEI and "\( \lambda^+ - \) dominance" case, the effort \( e_{t+1} \) never goes below any effort level \( e_t \).

**Proof** The proof invokes One-Step-Deviation-Principle (OSDP). Pick any arbitrary \( t \in \{2, \ldots, T-1\} \) and consider the effort path

\[
e(e_t, e_{t+1}) = (e_1, \ldots, e_{t-1}, e_t, e_{t+1}, e_{t+2}, \ldots, e_{T-1}, 1).
\]

Let \( V_\delta^i(\{e_{t-1}\}, e(e_t, e_{t+1}), \{e_{t+2}\}) = \sum_{i=0}^{T} \delta^{i-j} v(e(e_t, e_{t+1})) \) be the DM’s payoff at period \( t \).

Then specifically the DM’s payoff is \( V_\delta^i(\{1\}, e(1, 1), \{1\}) = 1 + \delta(1) + \delta^2(1) + V_\delta^{t\geq(t+3)} \). Suppose that the DM decides to deviate from this path at period \( t+1 \) with a choice of effort at this period \( e_{t+1} = 1 - \varepsilon \), where \( \varepsilon \in (0, 1] \). Consequently, the DM’s payoff is \( V_\delta^i(\{1\}, e(1, 1-\varepsilon), \{1\}) = 1 + \delta((1-\varepsilon)1 - \lambda^-(1 - (1 - \varepsilon)1)) + \delta^2(1(1 - \varepsilon)) + V_\delta^{t\geq(t+3)} \). Comparing these two payoffs we have \( V_\delta^i(\{1\}, e(1, 1), \{1\}) > V_\delta^i(\{1\}, e(1, 1-\varepsilon), \{1\}) \).

Then \( 1 + \delta(1) + \delta^2(1) + V_\delta^{t\geq(t+3)} = \left[ 1 + \delta((1-\varepsilon)1 - \lambda^-(1 - (1 - \varepsilon)1)) + \delta^2(1(1 - \varepsilon)) + V_\delta^{t\geq(t+3)} \right] = \delta(\varepsilon + \lambda^+ - \varepsilon + \delta^2\varepsilon > 0 \). Therefore, the effort path is \( e(e_t, e_{t+1}) = (\{1\}, e(1, 1), \{1\}) \).

Now consider the effort path \( (\{e_{t-1}\}, e(e_t, e_{t+1}), \{e_{t+2}\}) = (\{1 - \varepsilon\}, e(1, 1), \{1\}) \). The DM’s payoff then is \( V_\delta^i(\{1 - \varepsilon\}, e(1, 1), \{1\}) = 1(1 - \varepsilon) + \lambda^+(1(1 - \varepsilon) - (1 - \varepsilon)^2) + \delta(1 + \lambda^+(1 - (1 - \varepsilon))) + \delta^2(1) + V_\delta^{t\geq(t+3)} \).
Suppose the DM wants to deviate and considers two options to choose at period $t+1$: the effort $e_{t+1} = 1 - \varepsilon$ or an even lower effort $e_{t+1} = 1 - \eta$, where $\eta > \varepsilon$ and $\eta \in (0,1]$. If her effort is $e_{t+1} = 1 - \varepsilon$, then the DM’s payoff is $V^t_\delta(\{1 - \varepsilon\}, e(1 - \varepsilon), \{1\}) = 1(1 - \varepsilon) + \lambda^+(1(1 - \varepsilon) - (1 - \varepsilon)^2) + \delta(1(1 - \varepsilon)) + \delta^2(1(1 - \varepsilon)) + V^\geq_\delta(t+3)$. If her effort is $e_{t+1} = 1 - \eta$, then her payoff is $V^t_\delta(\{1 - \varepsilon\}, e(1, 1 - \eta), \{1\}) = 1(1 - \varepsilon) + \lambda^+(1(1 - \varepsilon) - (1 - \varepsilon)^2) + \delta(1(1 - \eta) - \lambda^-(1(1 - \varepsilon) - 1(1 - \eta))) + \delta^2(1(1 - \eta)) + V^\geq_\delta(t+3)$.

Comparing $V^t_\delta(\{1 - \varepsilon\}, e(1, 1), \{1\})$ and $V^t_\delta(\{1 - \varepsilon\}, e(1, 1 - \varepsilon), \{1\})$, we have $1(1 - \varepsilon) + \lambda^+(1(1 - \varepsilon) - (1 - \varepsilon)^2) + \delta(1 + \lambda^+(1(1 - \varepsilon))) + \delta^2(1) + V^\geq_\delta(t+3) - \left[1(1 - \varepsilon) + \lambda^+(1(1 - \varepsilon) - (1 - \varepsilon)^2) + \delta(1(1 - \eta) - \lambda^-(1(1 - \varepsilon) - 1(1 - \eta))) + \delta^2(1(1 - \eta)) + V^\geq_\delta(t+3)\right] > 0$. Therefore, the effort path is $e(1, 1)$.

Next, we compare $V^t_\delta(\{1 - \varepsilon\}, e(1, 1), \{1\})$ and $V^t_\delta(\{1 - \varepsilon\}, e(1, 1 - \eta), \{1\})$. There is $1(1 - \varepsilon) + \lambda^+(1(1 - \varepsilon) - (1 - \varepsilon)^2) + \delta(1 + \lambda^+(1(1 - \varepsilon))) + \delta^2(1) + V^\geq_\delta(t+3) - \left[1(1 - \varepsilon) + \lambda^+(1(1 - \varepsilon) - (1 - \varepsilon)^2) + \delta(1(1 - \eta) - \lambda^-(1(1 - \varepsilon) - 1(1 - \eta))) + \delta^2(1(1 - \eta)) + V^\geq_\delta(t+3)\right] > 0$. Therefore, we obtain $e(1, 1)$.

Putting all findings together, we conclude that if the DM’s effort is 1 at any $t$, the DM will never choose the lower effort in any subsequent periods.

Now suppose the DM’s effort at period $t$ is $e_t = 1 - \varepsilon$ and the DM’s effort path has the following pattern $(\{e_{t-1}\}, e(e_t, e_{t+1}), \{e_{t+2}\}) = (\{1 - \varepsilon\}, e(1 - \varepsilon, 1 - \varepsilon), \{1 - \varepsilon\})$. Then the DM’s payoff is $V^t_\delta(\{1 - \varepsilon\}, e(1 - \varepsilon, 1 - \varepsilon), \{1 - \varepsilon\}) = (1 - \varepsilon)^2 + \delta((1 - \varepsilon)^2) + \delta^2((1 - \varepsilon)^2) + V^\geq_\delta(t+3)$.

Assume that DM wants to deviate and choose at period $t+1$ the effort $e_{t+1} = 1 - \eta$, where $\eta > \varepsilon$. The DM’s payoff is $V^t_\delta(\{1 - \varepsilon\}, e(1 - \varepsilon, 1 - \eta), \{1 - \varepsilon\}) = (1 - \varepsilon)^2 + \delta((1 - \varepsilon)(1 - \eta) - \lambda^-(1 - \varepsilon)^2 - (1 - \eta)^2) + \delta^2((1 - \varepsilon)(1 - \eta)) + V^\geq_\delta(t+3)$. Comparing these two payoffs we have that $V^t_\delta(\{1 - \varepsilon\}, e(1 - \varepsilon, 1 - \varepsilon), \{1 - \varepsilon\}) > V^t_\delta(\{1 - \varepsilon\}, e(1 - \varepsilon, 1 - \eta), \{1 - \varepsilon\})$.

Putting all findings together, we conclude that if the DM’s effort is $1 - \varepsilon$ at any $t$ the DM will never choose any lower effort in any subsequent periods.
This completes the proof. □

This Lemma 22 states that when the DM has Positive Intertemporal Effort Interdependence (PIEI) and Intertemporal Reference Dependence (IRD) with "$\lambda^+ -$ dominance", then the DM will never choose a lower effort in any subsequent period.

Remark 3: It turns out that neither of these cases considered in this chapter (i.e., the NIEI and IRD case and the PIEI and IRD case) is robust to incorporating diminishing marginal productivity of effort as well as disutility of effort within a period.

4.5 Conclusion

Recall that in Chapter 2, the analysis fully pertained to the Intertemporal Reference Independence (IRI) case. I have found that the DM behaves the same way as in the ZIEI and PIEI cases by choosing the maximum possible effort level. In the NIEI case, the DM alternates between the highest possible effort level and zero effort level.

In Chapter 3, I analysed the Intertemporal Reference Dependence (IRD) case. Unlike Chapter 2, in that chapter the DM compared her consecutive outcomes in adjacent periods, and thus, her payoff at each period would yield either a gain due to an increase in her output or a payoff loss due to a decrease in her outcome, unless her outcomes were exactly the same in consecutive periods. The main findings in that chapter were that (1) in the "$\lambda^- -$ dominance" case, the DM behaves just like the DM in the ZIEI and IRI case choosing an effort path of $(1, 1, 1, ..., 1, 1)$ or slightly differently only in the beginning by choosing an effort path of $(0, 1, 1, ..., 1, 1)$, and (2) in the "$\lambda^+$ -dominance" case, when $\lambda^+$ does not exceed $\lambda^-$ by much, the DM behaves just like the way she did in the $\lambda^- -$ dominant case; but in the case where $\lambda^+$ exceeds $\lambda^-$ significantly, the DM exhibits different effort paths even with an
up-and-down pattern throughout, as long as she will finish with an effort level $e_T = 1$. Thus, overall, when the DM moves from the ZIEI and IRI case to the ZIEI and IRD case, whether or not $\lambda^- > \lambda^+$ starts mattering a lot. The DM with $\lambda^- > \lambda^+$ still behaved very much like the DM with $\lambda^- = \lambda^+ = 0$, while the DM with $\lambda^- \leq \lambda^+$ may start behaving very differently especially as $\lambda^-$ and $\lambda^+$ diverge.

In this chapter I have considered the NIEI and IRD case and the PIEI and IRD case. In the former case, I have obtained the surprising and striking result that the DM’s equilibrium effort path is strongly monotone increasing regardless of $\lambda^- \gtrless \lambda^+$ as long as either $\lambda^-$ and/or $\lambda^+$ is greater than zero. Thus, when the DM moves from the ZIEI and IRD case to the NIEI and IRD case, whether or not $\lambda^- > \lambda^+$ does not matter at all. In addition, for the first time one can observe a strictly monotone-increasing effort path by the DM (and regardless of $\lambda^- \gtrless \lambda^+$). In the PIEI and IRD case, however, there is no full convergence of the effort paths of the $\lambda^-$-dominant and $\lambda^+$-dominant DMs. Nevertheless, there is very strong resemblance between those paths. The $\lambda^-$-dominant DM exhibits an effort path of $(0, 1, 1, ..., 1, 1)$ (i.e., just like that of the DM in the ZIEI and IRI case) while the $\lambda^+$-dominant DM exhibits either an effort path of $(1, 1, 1, ..., 1, 1)$ (i.e., just like that of the DM in the ZIEI and IRI case) or an effort path of $(x, 1, 1, ..., 1, 1)$ where $x > 0$ (i.e., almost like that of the DM in the ZIEI and IRI case).
APPENDIX 1: EXAMPLES $T = 3$ AND $T = 4$

Case: Example $T = 3$

Lemma 23: In the PIEI and $\lambda^+$-dominant DM case, for $T = 3$ and $\delta \in (0, 1)$, the effort path is $e = (1, 1, 1)$ if $\delta < \frac{1 + \sqrt{1 + 4\lambda^+}}{2\lambda^+}$, otherwise the effort path is $e = (1 - \epsilon, 1, 1)$, where $\epsilon \in (0, 1]$.

Proof The proof invokes OSDP. According to Lemma 19, the last period effort is $e_T = 1$. Now suppose that the DM is at period $T - 1$. If the DM’s effort at period $t = 2$ is $e_2 = 1$, then the DM’s payoff is $V(e_2 = 1) = 1e_1 + \lambda^+(1e_1 - e_1) + \delta(1 + \lambda^+(1 - e_1)) = e_1 + \delta(1 + \lambda^+(1 - e_1))$. Suppose that at period $t = 2$ the DM decides to deviate and choose the effort $e_2 = 1 - \epsilon$, where $\epsilon \in (0, 1]$. Then the DM’s payoff is $V(e_2 = 1 - \epsilon) = (1 - \epsilon)e_1 - \lambda^-(1 - (1 - \epsilon)e_1) + \delta(1 - \epsilon + \lambda^+((1 - \epsilon) - (1 - \epsilon)e_1))$. Let’s compare $V(e_2 = 1)$ and $V(e_2 = 1 - \epsilon)$. It is easy to verify that $V(e_2 = 1) - V(e_2 = 1 - \epsilon) = \epsilon[e_1(1 + \lambda^-) + \delta + \delta\lambda^+(1 - e_1)] > 0$. Therefore, the DM’s effort is $e_2 = 1$.

Next, suppose that the DM is at period $t = 1$, and her effort is $e_1 = 1$. The DM’s payoff is $V(e_1 = 1) = 1 + \delta(1) + \delta^2(1)$. If her effort is instead $e_1 = 1 - \epsilon$, then the DM’s payoff is $V(e_1 = 1 - \epsilon) = (1 - \epsilon)e_1 - \lambda^-(e_1 - (1 - \epsilon)e_1) + \delta(1 - \epsilon + \lambda^+((1 - \epsilon) - (1 - \epsilon)e_1))$. Let’s compare $V(e_1 = 1)$ and $V(e_1 = 1 - \epsilon)$. Then we have $V(e_1 = 1) - V(e_1 = 1 - \epsilon) = \epsilon(1 + \delta) - \delta^2\epsilon\lambda^+$. Therefore, her effort path is $e = (1, 1, 1)$, if $\lambda^+ < \frac{1 + \delta}{\delta^2}$; otherwise the effort path is $e = (1 - \epsilon, 1, 1)$. This completes the proof. $\square$

Case: Example $T = 4$

Now consider the optimal effort path for $T = 4$. According to Lemma 19, the last period effort is $e_4 = 1$. Consider the DM at period $t = 3$. If $e_3 = 1$, the DM’s payoff is $V(e_3 = 1) = 1e_2 + \lambda^+(1e_2 - e_2e_1) + \delta(1 + \lambda^+(1 + \lambda^+(1 - e_2)))$. If she decides to choose an effort $e_3 = 1 - \epsilon$ instead, where $\epsilon \in (0, 1]$, then the DM’s payoff is either $V(e_3 = 1 - \epsilon)^+ = (1 - \epsilon)e_2 + \lambda^+((1 - \epsilon)e_2 -
\( e_2 e_1 + \delta (1 - \varepsilon) (1 + \lambda^+(1 + \lambda^+(1 + e_2))) \) or \( V(e_3 = 1 - \varepsilon)^- = (1 - \varepsilon) e_2 - \lambda^- (e_2 e_1 - (1 - \varepsilon) e_2) + \delta (1 - \varepsilon) (1 + \lambda^+(1 + \lambda^+(1 - e_2))). \)

First, we compare \( V(e_3 = 1) \) and \( V(e_3 = 1 - \varepsilon)^- \). It is easy to verify that \( V(e_3 = 1) > V(e_3 = 1 - \varepsilon)^- \). As a result, we have \( V(e_3 = 1) - V(e_3 = 1 - \varepsilon)^- = 1 e_2 + \lambda^+(1 e_2 - e_2 e_1) + \delta (1 + \lambda^+(1 + \lambda^+(1 - e_2))) - a_2 + \varepsilon e_2 + \lambda^- (e_2 e_1) - \lambda^- e_2 + \lambda^- \varepsilon e_2 - \delta + \delta \varepsilon - \delta \lambda^+ + \delta \lambda^+ \varepsilon + \delta \lambda^+ e_2 - \delta \lambda^+ \varepsilon e_2 = (\lambda^+ - \lambda^-) e_2 (1 - e_1) + a_1 \varepsilon + \lambda^- \varepsilon e_2 + \delta \varepsilon + \delta \lambda^+ e - \delta \lambda^+ \varepsilon e_2 > 0. \)

Next, comparing \( V(e_3 = 1) \) and \( V(e_3 = 1 - \varepsilon)^+ \), we have \( V(e_3 = 1) - V(e_3 = 1 - \varepsilon)^+ = e_2 (1 + \lambda^+) + \varepsilon \delta (1 + \lambda^+(1 + \lambda^+(1 - e_2))) > 0. \) Thus her effort path for the last two periods is \( (e_3, e_4) = (1, 1) \).

Now, suppose the DM is at period \( t = 2 \). Suppose that she decides to choose the effort \( e_2 = 1 \). Then her payoff is \( V(e_2 = 1) = 1 e_1 + \lambda^+(1 e_1 - e_1) + \delta (1 + \lambda^+(1 - e_1)) + \delta^2 (1) = e_1 + \delta (1 + \lambda^+(1 - e_1)) + \delta^2. \) However, if the DM decides to choose \( e_2 = 1 - \varepsilon \) instead, then the DM’s payoff is \( V(e_2 = 1 - \varepsilon) = (1 - \varepsilon) e_1 - \lambda^- (e_1 - (1 - \varepsilon) e_1) + \delta (1 - \varepsilon) (1 + \lambda^+(1 - e_1)) + \delta^2 (1 + \lambda^+(1 - (1 - \varepsilon))). \) Let’s compare these two payoffs. Consequently, we have \( V(e_2 = 1) - V(e_2 = 1 - \varepsilon) = e_1 + \delta (1 + \lambda^+(1 - e_1)) + \delta^2 - \left[ (1 - \varepsilon) e_1 - \lambda^- (e_1 - (1 - \varepsilon) e_1) + \delta (1 - \varepsilon) (1 + \lambda^+(1 - e_1)) + \delta^2 (1 + \lambda^+(1 - (1 - \varepsilon))) \right] = \varepsilon e_1 (1 + \lambda^-) + \delta (1 + \lambda^+(1 - e_1)) + \delta^2 \lambda^+. \) Hence, we have that, if \( e_1 (1 + \lambda^-) + \delta (1 + \lambda^+(1 - e_1)) - \delta^2 \lambda^+ > 0, \) the effort is \( e_2 = 1; \) otherwise \( e_2 = 1 - \varepsilon. \)

Next, suppose that the DM is at period \( t = 1 \). We have to consider two subcases. The first one is \( e_1 (1 + \lambda^-) + \delta (1 + \lambda^+(1 - e_1)) - \delta^2 \lambda^+ > 0; \) that is the effort path of the last three periods is \( (e_2, e_3, e_4) = (1, 1, 1) \). Then, if \( e_1 = 1, \) the DM’s payoff at period \( t = 1 \) is \( V(e_1 = 1) = 1 + \delta (1) + \delta^2 (1) + \delta^3 (1) \). Suppose the DM decides to deviate and choose the effort \( e_1 = 1 - \varepsilon. \) Then, the DM’s payoff is \( V(e_1 = 1 - \varepsilon) = (1 - \varepsilon) + \delta (1 - \varepsilon) + \delta^2 (1 + \lambda^+(1 - (1 - \varepsilon))) + \delta^3 (1). \) When we compare \( V(e_1 = 1) \) and \( V(e_1 = 1 - \varepsilon) \), we obtain \( V(e_1 = 1) - V(e_1 = 1 - \varepsilon) = \varepsilon (1 + \delta - \delta^2 \lambda^+). \) If \( 1 + \delta - \delta^2 \lambda^+ > 0, \) her
Putting all together, we have:

\[ \delta_1 - (V \varepsilon) \]

Now suppose that she decides to take an effort \( e_1 = 1; \) otherwise \( e_1 = 1 - \varepsilon \). To summarize, the DM’s effort paths are:

\[
(e_1, e_2, e_3, e_4) = \begin{cases} 
(1, 1, 1, 1), & \text{if } \begin{cases} 1 + \lambda^- + \delta - \delta^2 \lambda^+ > 0 \\
1 + \delta - \delta^2 \lambda^+ > 0 
\end{cases} \\
(1 - \varepsilon, 1, 1, 1), & \text{if } \begin{cases} (1 - \varepsilon)(1 + \lambda^-) + \delta(1 + \lambda^+ \varepsilon) - \delta^2 \lambda^+ > 0 \\
1 + \delta - \delta^2 \lambda^+ < 0 
\end{cases} 
\end{cases}
\]

Now consider the second case, where \( e_1(1 + \lambda^-) + \delta(1 + \lambda^+(1 - e_1)) - \delta^2 \lambda^+ < 0 \); that is the last three periods’ effort path is \((e_2, e_3, e_4) = (1 - \varepsilon, 1, 1)\).

Assume that the DM’s effort at period \( t = 1 \) is \( e_1 = 1 \). As a result, the DM’s payoff is \( V(e_1 = 1) = 1 + \delta((1 - \varepsilon)1 - \lambda^-((1 - (1 - \varepsilon))) + \delta^2(1(1 - \varepsilon)) + \delta^3(1 + \lambda^+(1 - (1 - \varepsilon))) = 1 + \delta((1 - \varepsilon) - \lambda^- \varepsilon) + \delta^2(1 - \varepsilon) + \delta^3(1 + \lambda^+ \varepsilon). \)

Now suppose that she decides to take an effort \( e_1 = 1 - \varepsilon \) instead. Then, the DM’s payoff is \( V(e_1 = 1 - \varepsilon) = (1 - \varepsilon) + \delta((1 - \varepsilon)^2 - \lambda^-((1 - \varepsilon) - (1 - \varepsilon)^2)) + \delta^2(1(1 - \varepsilon) + \lambda^+(1 - (1 - \varepsilon))^2) + \delta^3(1 + \lambda^+(1 - (1 - \varepsilon))) = (1 - \varepsilon) + \delta(1(1 - \varepsilon) - \lambda^- \varepsilon) + \delta^2(1 - \varepsilon) + \delta^3(1 + \lambda^+ \varepsilon). \)

Let’s compare \( V(e_1 = 1) \) and \( V(e_1 = 1 - \varepsilon) \). The we have \( V(e_1 = 1) - V(e_1 = 1 - \varepsilon) = \varepsilon(1 + \delta(1 - \varepsilon(1 + \lambda^-)) - \delta^2 \lambda^+(1 - \varepsilon)). \) If \( 1 + \delta(1 - \varepsilon(1 + \lambda^-)) - \delta^2 \lambda^+(1 - \varepsilon) > 0 \), her effort is \( e_1 = 1 \); otherwise her effort is \( e_1 = 1 - \varepsilon \).

Putting all together, we have:

\[
(e_1, e_2, e_3, e_4) = \begin{cases} 
(1 - \varepsilon, 1, 1, 1), & \text{if } \begin{cases} 1 + \lambda^- + \delta - \delta^2 \lambda^+ < 0 \\
1 + \delta(1 - \varepsilon(1 + \lambda^-)) - \delta^2 \lambda^+(1 - \varepsilon) > 0 
\end{cases} \\
(1 - \varepsilon, 1 - \varepsilon, 1, 1), & \text{if } \begin{cases} (1 - \varepsilon)(1 + \lambda^-) + \delta(1 + \lambda^+ \varepsilon) - \delta^2 \lambda^+ < 0 \\
1 + \delta(1 - \varepsilon(1 + \lambda^-)) - \delta^2 \lambda^+(1 - \varepsilon) < 0 
\end{cases} 
\end{cases}
\]

Listing all effort paths, we have:
The effort path is:

\[
(e_1, e_2, e_3, e_4) =
\begin{cases}
(1, 1, 1, 1), & \text{if } 1 + \lambda^+ + \delta - \delta^2 \lambda^+ > 0 \\
& 1 + \delta - \delta^2 \lambda^+ > 0 \\
(1 - \varepsilon, 1, 1, 1), & \text{if } (1 - \varepsilon)(1 + \lambda^-) + \delta(1 + \lambda^+ \varepsilon) - \delta^2 \lambda^+ > 0 \\
& 1 + \delta - \delta^2 \lambda^+ < 0 \\
(1, 1 - \varepsilon, 1, 1), & \text{if } 1 + \lambda^- + \delta - \delta^2 \lambda^+ < 0 \\
& 1 + \delta(1 - \varepsilon(1 + \lambda^-)) - \delta^2 \lambda^+ (1 - \varepsilon) > 0 \\
(1 - \varepsilon, 1 - \varepsilon, 1, 1), & \text{if } (1 - \varepsilon)(1 + \lambda^-) + \delta(1 + \lambda^+ \varepsilon) - \delta^2 \lambda^+ < 0 \\
& 1 + \delta(1 - \varepsilon(1 + \lambda^-)) - \delta^2 \lambda^+ (1 - \varepsilon) < 0
\end{cases}
\]

Regardless of the size of \( \lambda^+ \) and \( \lambda^- \), if \( \varepsilon < \delta \), then the effort path \((1, 1 - \varepsilon, 1, 1)\) is self-contradicted. In the case where \( \varepsilon > \delta \) the effort paths \((1, 1 - \varepsilon, 1, 1)\) and \((1 - \varepsilon, 1, 1, 1)\) are self-contradicted. To summarise, if \( \varepsilon < \delta \), the effort path is:

\[
(e_1, e_2, e_3, e_4) =
\begin{cases}
(1, 1, 1, 1), & \text{if } 1 + \lambda^+ + \delta - \delta^2 \lambda^+ > 0 \\
& 1 + \delta - \delta^2 \lambda^+ > 0 \\
(1 - \varepsilon, 1, 1, 1), & \text{if } (1 - \varepsilon)(1 + \lambda^-) + \delta(1 + \lambda^+ \varepsilon) - \delta^2 \lambda^+ > 0 \\
& 1 + \delta - \delta^2 \lambda^+ < 0 \\
(1 - \varepsilon, 1 - \varepsilon, 1, 1), & \text{if } (1 - \varepsilon)(1 + \lambda^-) + \delta(1 + \lambda^+ \varepsilon) - \delta^2 \lambda^+ < 0 \\
& 1 + \delta(1 - \varepsilon(1 + \lambda^-)) - \delta^2 \lambda^+ (1 - \varepsilon) < 0
\end{cases}
\]

In the case where \( \varepsilon > \delta \), the effort path is:

\[
(e_1, e_2, e_3, e_4) =
\begin{cases}
(1, 1, 1, 1), & \text{if } 1 + \lambda^+ + \delta - \delta^2 \lambda^+ > 0 \\
& 1 + \delta - \delta^2 \lambda^+ > 0 \\
(1 - \varepsilon, 1, 1, 1), & \text{if } (1 - \varepsilon)(1 + \lambda^-) + \delta(1 + \lambda^+ \varepsilon) - \delta^2 \lambda^+ < 0 \\
& 1 + \delta(1 - \varepsilon(1 + \lambda^-)) - \delta^2 \lambda^+ (1 - \varepsilon) < 0
\end{cases}
\]
5. CONCLUSION

In this thesis, I have considered a forward-looking and potentially impatient DM who has to choose an effort level in each period - where as usual a higher effort level generates a higher level of outcome - in a fully dynamic history-dependent environment by simultaneously considering past, present and future, the main focus being on the finite-horizon setup.

To that end, my analysis has been based on the following key concepts.

(1) the **Zero Intertemporal Effort Interdependence (ZIEI)**, where the DM’s effort in one period does not have any negative or positive impact on her effort in the ensuing period,

(2) the **Negative Intertemporal Effort Interdependence (NIEI)**, where fatigue due to higher effort of the DM in one period affects her effort level of the next period negatively,

(3) the **Positive Intertemporal Effort Interdependence (PIEI)**, where a higher effort of the DM in one period affects her effort level in the next period positively,

(4) the **Intertemporal Reference Independence (IRI)**, where the DM does not compare her consecutive outcomes in adjacent periods, and consequently the DM does not encounter any loss aversion or any gain fondness between adjacent periods due to fluctuations in the values of her outcomes in consecutive periods,

(5) the **Intertemporal Reference Dependence (IRD)**, where the DM compares her consecutive outcomes in adjacent periods, and thus, apart from her standard direct utility from the outcome of her choice of effort in each
period, the DM also obtains a reference payoff gain from an increase in the outcome and/or suffers a reference payoff loss from a decrease in the outcome, unless she encounters identical outcomes in consecutive periods,

(6) the Intertemporal Gain Fondness, where the gain-fondness coefficient $\lambda^+ > 0$ of the DM determines the extent of the reference utility increase she experiences from her intertemporal outcome increment, and

(7) the Intertemporal Loss Aversion, where the loss-aversion coefficient $\lambda^- > 0$ of the DM determines the extent of the reference utility decrease she experiences from that intertemporal outcome drop.

Chapter 1 of this thesis has included the Introduction with relevant literature.

My main results in the following chapters can be summarized as follows:

In Chapter 2, I have focused on the IRI setup. I have considered the (i) ZIEI and IRI, (ii) NIEI and IRI, and (iii) PIEI and IRI cases. I showed that in the ZIEI and IRI case, the DM always chooses the maximum possible effort level. In the NIEI and IRI case, I showed that the DM starts with the highest possible effort level and then follows up with the lowest possible effort level, and so on, thus following this alternating up-and-down pattern throughout. In the PIEI and IRI case, it turned out that the DM still picks the maximum possible effort level every period as in the ZIEI and IRI case.

In Chapters 3 and 4, I have considered a DM who exhibits IRD. In Chapter 3, I have first considered the ZIEI in conjunction with IRD. In that context, I have first considered the “$\lambda^-$-dominant” case. I have found that the DM behaves just like the DM in the ZIEI and IRI when $\lambda^+$ is relatively low, in that she picks the highest possible effort level every period. When $\lambda^+$ is relatively high and she does not discount the future much, however, the DM starts with a low level of effort at the initial period, but behaves just like a standard DM thereafter, by picking the highest effort level every period from then on. I have then considered the “$\lambda^+$-dominant” case. In this case, my results have
turned out to be the same as those of the $\lambda^+ \leq \lambda^-$ case whenever $\lambda^+$ does not exceed $\lambda^-$ by much. When $\lambda^+$ exceeds $\lambda^-$ significantly, however, a much richer dynamic could arise, and fluctuating or volatile efforts could become standard under some parameters.

In Chapter 4, I have kept considering a DM who has IRD. I have considered the NIEI and PIEI cases along with IRD.

In the NIEI and IRD case, it turns out that regardless of the extent of her intertemporal loss aversion and/or the extent of her intertemporal gain fondness, any DM type’s choice of effort path is the same and strongly monotonic over the entire time horizon. In the PIEI and IRD case, I first consider a $\lambda^-$-dominant DM. I find that the DM behaves like a standard DM, who always chooses the highest effort level every period. Next, I consider a $\lambda^+$-dominant DM. In this case too, the DM chooses a path where her effort level is always at the maximum possible level after the first period, while her first period effort (i) too is at the maximum possible level if her discount factor is not high, and (ii) is less than the maximum possible level when her discount factor is high.

Overall, with IRI, there is already significant variation between the DM’s effort paths depending on whether ZIEI and PIEI (choosing the maximum possible effort level all the way) or NIEI (alternating between the highest and lowest possible levels). With IRD, even within ZIEI there is a significant but even richer variation between the DM’s effort paths depending on whether $\lambda^- > \lambda^+$ or not. A $\lambda^-$-dominant DM always chooses the maximum possible effort level, while $\lambda^+$-dominant DM may either always choose the maximum possible effort level or alternate between the highest and lowest possible levels, among other variations of those paths. With IRD with NIEI, surprisingly the DM’s effort path becomes strictly monotone increasing, regardless of $\lambda^- \gtrless \lambda^+$. With IRD and PIEI, however, there is no full convergence of the effort paths of the $\lambda^-$-dominant and $\lambda^+$-dominant DMs,
although there is very strong resemblance between the effort paths of the of the $\lambda^-$-dominant and $\lambda^+$-dominant DMs.
BIBLIOGRAPHY


