Anisotropy and Failure Modeling for Nonlinear Strain
Paths and Its Application to Rigid Packaging

by

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Submitted in fulfillment of the requirements for the degree of
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Abstract

The aluminum beverage can is a highly engineered, premium product whose evolution continues to meet consumer demands for sustainable products. In addition to their superior sustainability, cans provide a unique means for manufacturers to give their brands an edge through shape, graphics, functionality, and convenience. These creative marketing tools are becoming more commonplace as companies seek new ways to influence consumer purchasing decisions. Distinguishing beverage can characteristics such as shaping successfully draw consumer attention to their products. The shaping process presents challenging forming operations requiring numerical simulations that can provided accurate and reliable failure predictions for highly non-linear forming paths. Today, nonlinear finite element analysis (FEA) has become an integral component in the design process of aluminum beverage cans. The integrated design process has resulted in demands for faster turnaround, improved accuracy, and increased reliability in the results of the simulation. The objective of this thesis is to provide reliable and robust methods of formability assessment which enable development of improved product designs through use of beneficial strain paths. The proposed methods can be widely used for predicting failure for products which undergo nonlinear strain paths during forming including rigid-packaging and automotive forming applications.

Experiment results from uniaxial tensile tests, bi-axial bulge tests, and disk compression tests for both AA3104-Thin and AA3104-Thick materials are presented. The results from the experimental tests are used to determine material coefficients for the Hill48, Barlat Yld2000-2D, and Barlat Yld2004-18P models. Optimization techniques are used to determine the material coefficients for both the Yld2000-2D and Yld2004-18P models. The material coefficients are verified using single element and tensile bar models to compare the predicted r-values and normalized stress ratios with the experimental data. Excellent agreement is achieved between the numerical and experimental data for both the r-values and stress anisotropies for the AA3104-Thin alloy. The sensitivity to material model selection and the technique to determine the material coefficients is discussed.

Earing results are documented for cups drawn from AA3104-Thin and AA3104-Thick materials, both of which exhibit 8 ears after drawing. An analytical model is implemented in EXCEL in order to predict earing based on experimental r-values and directional yield stress values. The model gives reasonable estimates of the earing profile in only seconds of CPU time. The model is also capable of estimating the r-values and directional yield stress values given an earing profile and provides valuable data on the contributions of
the $r$-values and directional yield stress values to the overall earing profile. The $r$-values control the earing tendencies and the stress ratios control the earing magnitude. Finite element simulations are developed to study the influence of material model, friction, hardening model, and method of calculating the stress ratios on the predicted earing profile. It is shown that only the YLD2004-18P model is capable of accurately predicting the earing profile, as both the Hill48 and Barlat YLD2000-2D models are only capable of predicting 4 ears. The models also shows the significance of accurately measuring the $r$-values and method selected for calculating the stress ratios. Excellent agreement with the experimental data for earing is achieved using the AA3104-Thin material data.

Wrinkling during the cup forming process performed in the NUMISHEET2014 benchmark test are investigated in view of anisotropy and hardening and confirm the trends based on manufacturing experience. There is always a trade-off between wrinkling and structural performance. Factors that control or reduce wrinkling may also affect localized thinning or structural performance. Results are presented that demonstrate that wrinkling can be controlled or eliminated through optimized tooling design.

A literature review of the traditional experimental and numerical approaches used to develop the forming limit diagram is presented. Several analytical models are reviewed including the Swift’s diffused necking model, the Hill localized necking model, the Hill93 yield criteria FLD, the Stören and Rice vertex model, the Bressan and Williams shear instability model, and the Marciniak–Kuczynski (M-K) model. The M-K model is used to predict forming limit diagrams for both AA3104-Thin and AA3104-Thick materials using the Hill48 models and the Barlat YI2000-2D model and the results are compared with the experimental FLD.

A review of the stress-based forming limit diagram and the mapping from the traditional strain based FLC is presented. Mapping from the strain-based FLC to the PEPS (Polar Effective Plastic Strain) diagram is also summarized. An EXCEL–based post-processing system for general use is introduced. Mapping from strain-based FLC to PEPS for a general yield function is also derived. In addition, a modification to the current method for PEPS is proposed for application in multi-stage forming processes.

A literature review on coupled fracture models, uncoupled fracture models, and continuum damage models is summarized. The modified Mohr Coulomb and extended Mohr Coulomb models used for predicting both triaxiality and Lode angle dependence on fracture are discussed. The concept for the PEPS diagram is further developed to cover the path-independent fracture which is compatible with the stress-
based fracture polygon. Mapping from principal strain space to stress triaxiality space, principal stress space, and PEPS space is accomplished utilizing a generalized mapping technique.

Mechanical tests are conducted on both AA3104-Thin and AA3104-Thick materials to generate fracture data under different stress triaxiality conditions. Tensile tests are performed on flat sheet samples with a central hole, and notched specimens. Torsion of a double bridge and in-plane shear tests are conducted to generate points near pure shear conditions. The Nakajima test is utilized to produce points in bi-axial tension. The data from the experiments is used to develop the fracture locus in principal strain space for both alloys. A mapping process for a general yield function is used to determine the effect of average $r$-value and selection of yield function on fracture prediction. The fracture spaces are strongly influenced by the average $r$-value obtained experimentally and selection of the yield function requires careful use by the analyst.

Finite element modeling is used to validate the Modified Mohr-Coulomb (MMC) fracture model in polar space. Results for the tensile specimens show good agreement with the range of experimental results. When applying anisotropic material models, the various fracture loci predict different failure times in the simulations. This is believed to be a result of the varying strain ratios, triaxiality factors, and stress ratios resulting from anisotropic behavior. The model of the hydraulic bulge test shows very good agreement with the experimental results. The simulation of the tapered cup forming using AA3104-Thick material data confirms the failure observed during experimental testing for wrinkle evaluation. The results for hole expansion during cup drawing show the strain around the hole is non-uniformly distributed due to the anisotropy of the material. In spite of the significant anisotropic effects, the polar based fracture locus is able to accurately capture both the failure strain and failure location. Finally, the model of a cup draw / reverse redraw / expand forming sequence demonstrates the robustness of the modified PEPS fracture theory for a condition with nonlinear forming paths and accurately predicts the onset of failure.

Finally, concepts and ideas for further research on the prediction of fracture for forming applications with non-linear loading paths and anisotropic materials are presented.
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PART I

EARING AND WRINKLING DURING CUP DRAWING
CHAPTER 1

BEVERAGE CAN MANUFACTURING

1.1 INTRODUCTION

The aluminum beverage can is a highly engineered, premium product whose evolution continues to meet both consumer demands for sustainable products and canmaker and beverage brand demands for a highly marketable product. In the year 2015, nearly 95 billion aluminum beverage cans were manufactured in the United States and approximately 350 billion were produced worldwide. Lightweighting of these aluminum D&I beverage cans has been a continuous process for more than 50 years. Aluminum beverage can "ends" have been made progressively smaller over the years in order to reduce cost. Likewise, cost control efforts have resulted in continuous reduction of the net metal requirements for the can body. To reduce the weight and cost of the "bodies", cans with thinner sidewalls, reduced neck diameters and smaller base diameters have been developed. The reduction in cost has been achieved while maintaining functionality, structural performance, and formability of the can. However, with the reduction in metal usage, the likelihood of producing defects during the forming process increases. These defects can include wrinkling, buckling, and fracture. The accurate prediction of these defects is extremely important in sheet metal forming in order to reduce as much as possible the expensive and time consuming “trial-and-error” procedures, as well as to improve the quality of products.

Nonlinear finite element analysis (FEA) has become an integral component of the design process and provides a scientific approach to the design and analysis of thin sheet metal products. A schematic of typical finite element model input and output data is provided in Figure 1-1. The integrated design process has resulted in demands for faster turnaround, improved accuracy, and increased reliability in the results of the simulation. The models have become increasingly complex as complete, multi-physics systems are
studied. FEA is used extensively in rigid packaging to compare alternative designs and to forecast lightweighting for potential cost reduction. Modeling is routinely used to provide root cause solutions to customer issues. Optimization techniques are used to determine the effects of process parameters on both formability and structural performance. This reduces the trial and error process in designing tooling and costs associated with redesign allowing for improved manufacturability of products and shortens the timeline from concept to production. FEA is also used to study the impact of material characteristics including yield stress, strain hardening, strain hardening rate, and material anisotropy on forming.

![Diagram of finite element model inputs and outputs](image)

*Figure 1-1. Schematic of finite element model inputs and outputs.*

However, for the simulations to be truly useful for the design process, they must be able to accurately predict failure. Manufacturing experience suggests that forming defects during canmaking are influenced by many factors such as mechanical properties of the aluminum sheet, tooling geometry, contact conditions, including the effects of lubrication, and process boundary conditions. The experimental analysis of both tensile and compressive defects in these thin sheet metal products is difficult because the effects of all of the factors contributing to the instabilities are complex and small changes in these factors may produce widely varying results. Therefore, a numerical approach is utilized to separate the effect of each variable on defect formation. Figure 1-2 compares the experimental and finite element results for redraw wrinkling during can forming. Figure 1-3 shows experimental and predicted results for a
compressive (buckling) failure during die necking. Figure 1-4 compares results for a beverage can die expansion process.

Figure 1-2. Redraw wrinkling (a) experiment; (b) finite element prediction.

Figure 1-3. Compressive buckling during die necking (a) experiment; (b) finite element prediction.
Figure 1-4. Tensile failure during die expansion of can (a) experiment; (b) finite element prediction.

The objective of this thesis is to provide reliable and easy-to-use methods of formability and failure assessment which enable development of improved product design through the use of optimum strain paths. The proposed methods can be widely used for predicting failure for the examples which undergo nonlinear strain path including rigid-packaging and automotive forming. The methods proposed show path independence and are similar in shape to traditional strain-based forming limit curves. Evaluation is performed in a post-processing state and can be applied to a large variety of material models.

1.2 BEVERAGE CAN EARLY HISTORY

In the late 18th century, as Napoleon Bonaparte led France to war against its neighbors, feeding the French army posed a serious logistical problem. The government offered a 12,000 franc prize to anyone who could devise a practical means for preserving perishable food products. Motivated by the prize money, Nicolas Appert, the son of an innkeeper and owner of a small but successful confectionery shop in Paris, began a period of experimentation in 1795. Using corked-glass containers reinforced with wire and sealing wax and kept in boiling water for varying lengths of time, he preserved soups, fruits, vegetables, juices, dairy products, marmalades, jellies, and syrups. The experimentation process took nearly 14 years eventually leading to his 1810 publication of the first safe methodology for appertization, now commonly called canning. Appert opened the world's first cannery, as an experimental facility in 1804 and expanded it into a commercial factory in 1812. The first food container, made out of tin-plated iron, was patented by Peter Durand of England in 1810. In 1813, after acquiring the patent from Peter Durand, Bryan Dorkin and John Hall opened the first commercial canning factory in England.
By the late 19th century, cans were instrumental in the mass distribution of food products, but it wasn’t until 1909 that the American Can Company made its first attempt to can beer. Their initial attempt was unsuccessful, and with the onset of Prohibition in the United States, American Can Company would have to wait before trying again. In 1933, after two years of research, American Can developed a steel can that could sustain internal pressurize and had a special coating to prevent the carbonated beer from chemically reacting with the tin. The first steel beverage can was introduced commercially in Richmond, Virginia on January 24, 1935 by the Krueger Brewing Company. The early 3-piece steel cans (Figure 1-5) weighed nearly 4 ounces, and had a flat top that required a “church key” for opening. Within three months, over 80 percent of distributors were handling Krueger’s canned beer, and Krueger’s was capturing market share from the “big three” national brewers–Anheuser-Busch, Pabst and Schlitz. In only a few short months, Krueger needed American Can, the company that perfected the technology, to produce 180,000 cans a day to meet the demand. By the end of 1935, 37 breweries were using beer cans and over 200 million cans were sold.

Figure 1-5. Steel beverage can introduced by Krueger Brewing Company

Schlitz soon joined the canning craze but introduced its Lager Beer in a beer can that was quite different than Krueger's and Pabst's flat tops. Schlitz Lager was introduced in a flat bottom, inverted rib, cone-top steel beer can, manufactured by Continental Can Company (Figure 1-6). It appeared in 1935 and remained in production until about 1960. Canning was suspended during World War II, as the materials used were required for the war effort.
In 1954, the Adolph Coors Co. joined forces with Beatrice Foods Co. to found Aluminum International, Inc. to develop an economic process for manufacturing aluminum cans for their products. Their first 8 oz. cans were marketed in 1958 by Beatrice Food’s Hawaii Brewing Corp. for their Primo beer, and Coors beer followed in 1959 (Figure 1-7). Coors headed up the R&D efforts, and they developed a method of continuously casting aluminum and rolling it to about 1/8” thickness, then blanking circular slugs that were the diameter of the can. These slugs were fed into a customized German-made impact extrusion press (Figure 1-8) that punched out 3600 7-ounce flat-bottom can bodies an hour. This early technique for making D&I cans was slow and plagued with tooling problems, and could only make a 7 oz. can with a base that was thicker than necessary (.03”) to withstand the internal forces.

Figure 1-6. Schlitz cone-top beer can.

Figure 1-7. Early aluminum beverage cans – Primo Beer and Coors.
The Reynolds Metals Company had initiated its own R&D program, and in 1963 pioneered the production method for economically producing 12 oz. aluminum cans that is used today. Coors and Kaiser Aluminum soon followed. Kaiser Aluminum established a container R&D facility in Chicago to develop an aluminum can. They adapted the "draw and iron" process of Swiss inventor Jakob Keller to produce D&I aluminum cans. The first 12 oz. cans were sold by the Hamms Brewery in St. Paul, Minnesota in 1964, and Coca-Cola and Pepsi began using these cans by 1967. In the early 1970s, the Aluminum Company of America (Alcoa) introduced its Featherlite can (Figure 1-9) that was drawn from light gauge H-19 aluminum stock and was 20% lighter than the (then) average aluminum can.
In 1959, Ermal Fraze devised a can-opening method that would come to dominate the canned beverage market. His invention, shown in Figure 1-10, was the "pull-tab". This eliminated the need for a separate opener tool by attaching an aluminum pull-ring lever with a rivet to a pre-scored wedge-shaped tab section of the can top. The ring was riveted to the center of the top, which created an elongated opening large enough that one hole simultaneously served to let the beverage flow out while air flowed in. Fraze successfully pitched the concept and negotiated a deal with the Alcoa. In March of 1962, the Pittsburgh Brewing Co. test marketed the first "tap-top" beer can (Figure 1-12) in Virginia. By 1965, over seventy-five percent of beer brewers in the United States of America had adopted Fraze's lid.
The self-opening can underwent many changes to get from the first finger-cutting zip tops to the modern day stay-on-tabs (SOT). The pull ring followed the zip tab and was an improvement in that it no longer cut fingers. But, both types were removable tabs that wound up littering beaches, injuring consumers, and harming wildlife. The first SOT cans with a non-detachable tab (Figure 1-12) were developed by Reynolds and first introduced to the market by Falls City Brewing in 1975, and are still in use today.

Figure 1-11. Tap-top beer can.

Figure 1-12. Image from US Patent 3,967,752 – easy open wall.
1.3 BEVERAGE CAN MANUFACTURING PROCESS

Can plants use mass quantities of aluminum coils every day to produce beverage cans. Each coil typically weighs about 25,000 pounds and, when rolled out flat, can be anywhere from 20,000 to 30,000 feet long and up to six feet wide. The aluminum coils arrive at the can plant and are loaded one at a time onto an "uncoiler" — a machine that unrolls the strip of aluminum at the beginning of the can making line and feeds it into the lubricator. The lubricator applies a thin film of lubricant on the sheet and feeds it into the cupper. The lubrication helps the aluminum flow smoothly during both the cup making and body making processes.

A large machine called a cupping press starts the can shaping process. The press cuts out circular discs from the aluminum sheet approximately 0.274 mm thick, where a ram and punch drives the blank through a draw ring and forms the blanks into shallow cups. The cupping press blanks out at high speeds up to 14 cups every stroke to produce 2,500 to 3,750 cups per minute. The cups drop from the press onto the cup conveyor. The scrap (or skeleton) aluminum left over from these operations is removed and recycled. A typical blanking pattern is shown in Figure 1-13.

![Figure 1-13. Typical cupping press blanking pattern – 14 out configuration.](image)

From the cupping press, the cups are transferred to a bodymaker that uses a punch mounted on a ram to push the cups through a redraw ring and a series of ironing rings (see Figure 1-14) where the thinwall is approximately 0.097 mm thick and the thickwall at the top of the can is approximately 0.147 mm thick. At the bottom of the stroke, the dome profile is formed in to the can. A commercial bodymaker can run up to 450 strokes per minute. The cans exit the bodymaker and are loaded on a trimmer that trims the open end of the can to a uniform height.
The cans now enter the washing and decorating processes. The cans are washed, rinsed, and dried in preparation for the external decorating and internal coating processes. The cleaned cans proceed to a printer, where six to eight colors of ink may be placed on a can at the same time. The can spins around as the label is applied. Finally, a thin film of lacquer is applied to protect the entire label. Next, the can goes to an oven, where the paint and coating are cured. The cans are then conveyed to the internal coaters that spray the inside surface of the can to keep what is in the can from touching or reacting with the metal and maintain product integrity. An internal coater oven then cures the inside spray to seal the coating on the can.

The cans then proceed to a waxer that applies a thin coat of lubricant to the outside of the open edge of the cans in preparation for necking. The cans are transferred to a series of die necking operations (typically 14 stations), each one gradually reducing the diameter of the open end of the can. A flanger then rolls back the open edge of the can to form a lip, or flange, used to attach the lid during the filling and double seaming processes. Finally, all finished cans are tested for leaks. A light tester can find holes smaller than a human hair. A camera inspection system also checks for any irregularities or contamination in the can. A palletizer places the finished cans on pallets, 389 cans per layer, up to 21 layers high, to prepare for shipment to the fillers. The pallets are shipped to soft drink companies, which will put the soft drinks in the cans. A schematic of the can making process is provided in Figure 1-15.
1.4 CURRENT TRENDS

Aluminum beverage cans are not only considered the most common beverage package, but also the most recycled consumer packaging as well as the most valuable container to be recycled. Since aluminum is a sustainable metal, it can be recycled over and over again, and aluminum companies have discovered that by reusing the recycled aluminum, they save 95% of the energy that it would normally take to make a can.
out of pure extracted aluminum and substantially reduces the environmental footprint through reduced energy consumption and CO₂ emissions. Recycled cans are back on store shelves in as little as 60 days. A chart with aluminum recycling facts is given in Figure 1-16.
In addition to their superior sustainability, cans provide a unique means for manufacturers and fillers to give their brands an edge through shape, graphics, functionality, and convenience. These creative marketing tools are becoming more commonplace as companies seek new ways to influence consumer purchasing decisions. Distinguishing beverage can characteristics, such as shaping, successfully draw consumer attention to their products. Steep competition is driving companies across all beverage sectors toward customized aluminum can shapes and sizes spurring innovation, and significant investment in new technology among the industry of can manufacturers providing inroads to new markets for can manufactures and fillers. Samples of shaped aluminum cans are shown in Figure 1-17. Total necking reductions of over 50 percent and total can expansions exceeding 40 percent have been achieved. These challenging forming operations require numerical simulations that can provide accurate and reliable failure predictions for highly non-linear forming paths.

Figure 1-17. Aluminum can shaping technology.

1.5 OUTLINE OF THE THESIS

The present thesis consists of two parts. Part I focuses on simulation of issues related to cup drawing, while Part II addresses forming limit and fracture limit predictions. A total of ten chapters, each with a bibliography of the references, as well as six appendices for additional background information are included in the thesis. Each chapter describes a specific part of the research. The contents of the chapters are described below.

Chapter 2 documents the experimental results for two commonly used alloys in the beverage can industry. The results from the experimental tests are used to determine material coefficients for typical material
models used for thin sheet metal forming simulations. The generated material coefficients are verified using single element and tensile bar models to compare the predicted r-values and normalized stress values with the experimental data. Sensitivity to material model selection and the techniques to determine the material coefficients are discussed.

Chapter 3 presents earing results for cups drawn using both AA3104-Thin and AA3104-Thick materials. An analytical model is developed and implemented in EXCEL in order to predict earing based on experimental r-values and directional yield stress values, providing reasonable estimates of the earing profile in only seconds of CPU time. The model is also capable of providing reasonable estimates of the r-values and directional yield stress values given an earing profile. Additionally, the contributions of the r-values and directional yield stress values to the overall earing profile are shown. Finite element simulations are developed to study the influence of material model, friction, hardening model, and method of calculating the stress ratios on the predicted earing profile.

Chapter 4 investigates wrinkling during the cup forming process in view of anisotropy and hardening. The results presented confirm the trends based on manufacturing experience. Finite element simulations are developed to study the effects of material model, friction, process alignment, and punch design on the wrinkling behavior. A method for calculating a wrinkling factor to determine the severity of wrinkling is also developed.

Chapter 5 provides a literature review on the experimental and numerical approaches used to develop the traditional FLD. Several common analytical models in strain space used to predict the FLC are summarized and several commonly used methods for generating traditional forming limit diagrams are discussed.

Chapter 6 presents an approach for path independency in forming limits using the stress-based forming limit and Polar EPS Effective Plastic Strain (PEPS). A review of the stress-based forming limit diagram and the mapping from the traditional strain based FLC are presented. The mapping from the strain-based FLC to the PEPS Diagram is also summarized. An EXCEL–based post-processing system for general use is also introduced. Finally, a modification to the current PEPS method is proposed for use in multi-step forming processes.

Chapter 7 provides a literature review on coupled fracture models, uncoupled fracture models, and continuum damage models. The modified Mohr Coulomb and Extended Mohr Coulomb models used for predicting both triaxiality and Lode angle dependence on fracture are discussed. A concept for path-
independent fracture criteria with mapping from the fracture polygon to the PEPS fracture curve is
developed.

Chapter 8 summarizes the material characterization tests performed to calculate the fracture surface for
the two beverage can alloys. All mechanical tests were recorded by 3D ARAMIS for the measurement of
fracture strain. The tests are designed to study the effects of stress state stress triaxiality and Lode angle
parameter on fracture.

Chapter 9 describes the numerical simulations for determination of the fracture strain for several of the
tests used to generate the fracture polygon. Results are presented in the principal strain space, the stress
triaxiality space, the principal stress space, and the polar space.

Chapter 10 summarizes the contribution of the current thesis and provides suggestions for future research.

Appendices are provided as background or supplemental information supporting the development of the
work in this thesis. Appendix A provides a general overview of vectors and tensors. Appendix B reviews
some basic concepts of continuum plasticity required for developing material models. Appendix C gives
a brief introduction on yield functions and an overview of some well-known isotropic and anisotropic yield
functions routinely used in finite element simulations today. Appendix D provides the 1st order and 2nd
order partial derivatives of the Hill 1948 yield function for use mapping from principal strain space to other
spaces. Appendix E reviews the technique used for non-linear least squares curve fitting for stress-strain
data to both Voce and Swift hardening laws. Appendix F summarizes the list of publications for the author
of this thesis.
CHAPTER 2

MATERIAL CHARACTERIZATION TESTS

2.1 INTRODUCTION

The accuracy of numerical simulations of sheet forming processes using finite element analysis is highly dependent on the constitutive model used. Comprehensive knowledge about the strain hardening behavior of sheet metals at large strains is often needed in analyzing strain localization and ductile failures in the manufacturing processes of thin sheet metals. This also requires an accurate material model which can be easily calibrated and characterized from physical tests. Various testing methods for the characterization of the plastic behavior of sheet materials are well established. A schematic diagram of some well-known tests and their corresponding stress states in the $\sigma_1-\sigma_2$ plane is shown in Figure 2-1. The yield locus is divided into four quadrants. The first quadrant represents the biaxial tensile stress state. Here the hydraulic bulge, disk compression or biaxial tensile test can be applied. Shear deformation with one tensile and one compression stress component is located in the second and fourth quadrants. Shear and plane torsion tests are suitable for these quadrants. The third quadrant corresponds to biaxial compression where uniaxial and biaxial compression tests can be conducted but present many difficulties for thin sheet materials due to their tendency to buckle. All of these tests are also applicable to measure a flow curve, however with different levels of experimental effort. The simple tension test can be found on the positive half of the axes and is considered as a standard test for many applications due to its simplicity. While uniaxial tension testing is a well-established and widely used technique for characterizing the plastic strain hardening of a sheet metal, the data obtained from such a test in terms of an effective true stress–strain curve is strictly valid only up to the peak load or the onset strain level of diffused necking when the stress state within the growing neck increasingly deviates from the uniform uniaxial tension.
The materials considered in this thesis are AA3104-H19 with a sheet thickness of 0.274 mm and AA3104-H28 with a sheet thickness of 0.475 mm. These are common alloys used in the packaging industry to manufacture aluminum drawn and ironed beverage cans and bottles. Tensile tests were performed on dogbone specimens to determine the directional yield stresses and r-values at every 15° from the rolling direction for the AA3104 samples. Biaxial bulges tests were used to obtain material stress-strain response at the highest possible strain levels and disk compression test were used to determine the biaxial r-value. The data from the characterizations tests are used to generate the constitutive model parameters for the Hill (1948), Barlat et al. (2003), often called Yld2000-2D, and Barlat et al. (2005), called Yld2004-18P, material models for the two alloys. In addition, several methods to determine the stress ratios used in the calibration of the material constants are discussed.

*Figure 2-1. Common testing methods for sheet metal characterization.*

### 2.2 DOGBONE SPECIMENS - DIMENSIONS AND TEST CONDITIONS

Dogbone shaped specimens were cut from flat sheet samples according to the dimensions in Figure 2-2 along different orientations: 0, 15, 30, 45, 60, 75 and 90 degrees from the rolling direction. The loading velocity in the test is set such that the initial strain rate is 0.001 /sec. Axial and transverse strains are measured by an advanced visual 3D Aramis system. In addition, the load is recorded by a load cell.
Figure 2-2. Tensile specimen dimensions.

2.2.1 LOAD DISPLACEMENT CURVES – AA3104-THIN

Load displacement curves are measured from the experiments and compared in Figure 2-3 for each test in the different loading directions. The true stress-true strain curves are then computed from the load stroke curves as illustrated in Figure 2-4. It is observed that anisotropy in the yield stresses is strong for this alloy. A non-linear least squares curve fitting routine (see Appendix E) was used to fit the data to both Swift and Voce hardening models.

2.2.2 R-VALUES – AA3104-THIN

The axial and transverse strain are recorded during loading of the dogbone specimens and compared in Figure 2-5 for the different loading directions. The comparison shows the strong dependence of the orthotropic straining on the loading direction of the AA3104-Thin alloy. The r-values are measured for each test prior to the maximum load based on the curves in Figure 2-5 and are summarized in Table 2-1 and Figure 2-6. The mean values are computed and utilized as the final r-values in the numerical analyses of sheet metal forming of AA3104-Thin material.

2.2.3 LOAD DISPLACEMENT CURVES – AA3104-THICK

Load displacement curves are measured from the experiments and compared in Figure 2-7 for tests in the different loading directions. The true stress-true strain curves are then computed from the load stroke curves as illustrated in Figure 2-8. It is observed that anisotropy in the yield stresses is strong for this alloy.
2.2.4 R-VALUES – AA3104-THICK

The axial strain and transverse strain are recorded during loading of the dogbone specimens and compared in Figure 2-9 for the different loading directions. The comparison shows the strong dependence of the orthotropic straining on the loading direction of AA3104-Thick. The r-values are measured for each test prior to the maximum load based on the curves in Figure 2-9 and are summarized in Table 2-2 and Figure 2-10. The mean values are computed and utilized as the final r-values in the numerical analyses of sheet metal forming of the AA3104-Thick material.

Figure 2-3. Load displacement curves along different loading directions for AA3104-Thin.
Figure 2-4. True stress-true strain curves along different loading directions for AA3104-Thin.

Figure 2-5. Axial strain vs transverse strain of the dogbone specimens along different loading directions for AA3104-Thin.
Figure 2-6. r-values along different loading directions for AA3104-Thin.

<table>
<thead>
<tr>
<th>Loading direction</th>
<th>0</th>
<th>15</th>
<th>30</th>
<th>45</th>
<th>60</th>
<th>75</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>0.42918</td>
<td>0.36569</td>
<td>0.59411</td>
<td>0.81914</td>
<td>0.84495</td>
<td>1.03471</td>
<td>1.07956</td>
</tr>
<tr>
<td>#2</td>
<td>0.38914</td>
<td>0.45622</td>
<td>0.6114</td>
<td>0.83201</td>
<td>0.95297</td>
<td>1.01284</td>
<td>1.0816</td>
</tr>
<tr>
<td>#3</td>
<td>0.39336</td>
<td>0.38188</td>
<td>0.56016</td>
<td>0.91384</td>
<td>0.94541</td>
<td>1.13465</td>
<td>1.16591</td>
</tr>
<tr>
<td>#4</td>
<td>0.39328</td>
<td>0.42286</td>
<td>0.44728</td>
<td>0.9872</td>
<td>0.96413</td>
<td>1.05914</td>
<td>1.17184</td>
</tr>
<tr>
<td>#5</td>
<td>0.41006</td>
<td>0.4392</td>
<td>0.57302</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean value</td>
<td>0.403</td>
<td>0.41317</td>
<td>0.55719</td>
<td>0.88805</td>
<td>0.92687</td>
<td>1.06034</td>
<td>1.12473</td>
</tr>
</tbody>
</table>
Figure 2-7. Load displacement curves along different loading directions for AA3104-Thick.

Figure 2-8. True stress-true strain curves along different loading directions for AA3104-Thick.
Figure 2-9. Axial strain vs transverse strain of the dogbone specimens along different loading directions for AA3104-Thick.

Figure 2-10. r-values along different loading directions for AA3104-Thick.
Table 2-2. Summary of r-values along different loading directions – AA3104-THICK

<table>
<thead>
<tr>
<th>Loading direction</th>
<th>0 (#2)</th>
<th>15 (#3)</th>
<th>30 (#2)</th>
<th>45 (#2)</th>
<th>60 (#2)</th>
<th>75 (#1)</th>
<th>90 (#2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>r-value</td>
<td>0.3941</td>
<td>0.4598</td>
<td>0.6875</td>
<td>0.8646</td>
<td>0.8992</td>
<td>1.0467</td>
<td>1.0040</td>
</tr>
</tbody>
</table>

2.3 HYDRAULIC BULGE TEST

The bulge test is a commonly used experiment to establish the material stress-strain response at the highest possible strain levels (Young et al., 1981). It consists of a metal sheet placed in a die with a circular opening. It is clamped in place and deformed with hydraulic pressure as shown in Figure 2-11.

![Figure 2-11. Schematic of hydraulic bulge test.](image)

The determination of the biaxial stress-strain behavior using the experimental data acquired by hydraulic bulge test takes into account the membrane theory. The ratio between bulge diameter and sheet thickness must be higher than 50, so that it remains in the applicability domain of the membrane theory and therefore, bending effects may be neglected. The stress through the thickness at the pole on the force equilibrium is also neglected and thus it is possible to establish a relationship between stresses, hydraulic pressure, sample geometry and thickness by the following expression:

$$\frac{\sigma_1}{R_1} + \frac{\sigma_2}{R_2} = \frac{p}{t}$$  \hspace{1cm} (2-1)

where $\sigma_1$ and $\sigma_2$ are the principal stresses in the plane of the sheet, and $R_1$ and $R_2$ are the corresponding radii of curvature. The variable $p$ relates to hydraulic bulge pressure and $t$ to sheet metal thickness.
The principal stresses, $\sigma_1, \sigma_2$, can be assumed equivalent under isotropic assumptions defining equibiaxial stress state at the pole ($\sigma_1 = \sigma_2 = \sigma_b$). The same applies to the radius of curvature where $R_1 = R_2 = R$. Therefore, Equation 2-1 can be simplified and flow stress can be determined by:

$$\sigma_b = \frac{pR}{2t} \quad (2-2)$$

For the acquisition of radius of curvature and thickness at the pole a mechanical system is used as shown in Figure 2-12, thus allowing the determination of these variables during the test. The calculation of radius of curvature makes use of a simple geometric construction given by:

$$R = \frac{a^2 + h^2}{2h} \quad (2-3)$$

where $a$ is the radial dimension defined by spherometer and $h$ is the difference between the spherometer support and displacement transducer as presented in Figure 2-12. A correction is performed for half the thickness of sheet, since the calculation is done for the external surface of the cap. The current thickness ($t$) of the sample can be obtained through Equation 2-4, knowing the initial thickness ($t_0$) and the current thickness strain $\varepsilon_t$.

$$t = t_0 \exp(-\varepsilon_t) \quad (2-4)$$

Considering incompressibility of the material, the thickness strain can be obtained as follows:

$$\varepsilon_t = -(\varepsilon_1 + \varepsilon_2) \quad (2-5)$$

As for stresses and radius of curvature, the strains at the plane are also considered the same and therefore the strain in thickness direction is given by

$$\varepsilon_t = -2\varepsilon \quad (2-6)$$

where $\varepsilon$ is the membrane strain.

The determination of membrane strain is performed by measuring the expansion of a circle with an initial diameter of $D_0$. This measurement is performed by an extensometer, which follows the deformation of sheet metal sample during the bulge test. Since the diameter of the circle increases to diameter $D$, the current thickness strain can be obtained by:
\[ \varepsilon_t = -2 \ln \left( \frac{D}{D_0} \right) \]  

(2-7)

Triplicate hydraulic bulge tests were conducted at standard test conditions (constant true strain rate = 0.005/second and constant true strain rate = 0.060/second with the extensometer oriented at forty-five degrees to the sheet rolling direction) for each sample. The experimental true stress – true strain data from the hydraulic bulge test for AA3104-Thin material for both strain rates are shown in Figure 2-13. The experimental true stress – true strain data from the hydraulic bulge test for AA3104-Thick material for both strain rates are shown in Figure 2-14. Both show very good repeatability. The average maximum values of true stress and true strain measured in each set of triplicate tests are listed in Table 2-3. Also shown in Table 2-3 are the optimized constants obtained from curve fitting to the Swift and Voce equations which can be used to describe the true stress versus true strain behaviors of the materials. A plot of the equal biaxial \( \sigma vs \varepsilon \) data as described by the average Voce and Swift constants for the AA3104-Thin and AA3104-Thick materials for the two strain rates is shown in Figure 2-15 and Figure 2-16, respectively.

<table>
<thead>
<tr>
<th>Material</th>
<th>Strain Rate</th>
<th>Maximum Values</th>
<th>Swift</th>
<th>Voce</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>in/in/sec</td>
<td>( \sigma_{\text{MAX}} ) MPa</td>
<td>( \varepsilon_{\text{MAX}} )</td>
<td>K MPa</td>
</tr>
<tr>
<td>AA3104 Thin</td>
<td>0.005</td>
<td>355.9</td>
<td>0.2417</td>
<td>382.4</td>
</tr>
<tr>
<td></td>
<td>0.060</td>
<td>361.3</td>
<td>0.2412</td>
<td>390.5</td>
</tr>
<tr>
<td>AA3104 Thick</td>
<td>0.005</td>
<td>283.0</td>
<td>0.3590</td>
<td>291.6</td>
</tr>
<tr>
<td></td>
<td>0.056</td>
<td>286.6</td>
<td>0.3728</td>
<td>296.2</td>
</tr>
</tbody>
</table>
Figure 2-12. Schematic of hydraulic bulge test measurement system.
Figure 2-13. True stress – true strain data from the hydraulic bulge test for AA3104-Thin: a) Strain rate = 0.005; b) strain rate = 0.060.
Figure 2-14. True stress – true strain data from the hydraulic bulge test for AA3104-Thick: a) Strain rate = 0.005; b) strain rate = 0.060.
**Figure 2-15.** Equal Biaxial true stress – true strain curves for AA3104-Thin.

**Figure 2-16.** Equal Biaxial true stress – true strain curves for AA3104-Thick.
2.4 DISK COMPRESSION TEST

Disk compression tests were conducted using 12.7 mm diameter specimens in order to determine the biaxial r-value. The specimens were loaded in compression using Teflon sheets as lubrication between the faces of the specimens and points of contact with the upper and lower platens of the test fixture. The diameters both parallel and perpendicular to the rolling direction of the sheet and thickness were measured for each specimen prior to and after deformation. Due to the anisotropy of the material, the strains in RD and TD are different. The biaxial r-value is defined as the ratio of the transverse strain, $\varepsilon_{TD}$, to the rolling direction strain, $\varepsilon_{RD}$.

\[
\begin{align*}
\varepsilon_{TD} & = \ln \left( \frac{D_{TD}}{D_0} \right) \\
\varepsilon_{RD} & = \ln \left( \frac{D_{RD}}{D_0} \right)
\end{align*}
\]  

(2-8)

\[
r_{bx} = \frac{\varepsilon_{TD}}{\varepsilon_{RD}}
\]  

(2-9)

The results of the test are summarized in Table 2-4.

Regression analyses of the $\varepsilon_{TD}$ versus $\varepsilon_{RD}$ data were conducted to determine a biaxial plastic strain ratio ($r_{bx}$) for each material. The $r_{bx}$ value determined is summarized in Table 2-4.

![Illustration of disk compression test](image)

*Figure 2-17. Illustration of disk compression test.*
Table 2-4. Disk Compression Test Data – AA3104 Alloys

<table>
<thead>
<tr>
<th>Sample ID</th>
<th>$r_{dx}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AA3104-Thin</td>
<td>0.783</td>
</tr>
<tr>
<td>AA3104-Thick</td>
<td>0.757</td>
</tr>
</tbody>
</table>

The results of the uni-axial tensile tests, bi-axial bulge tests and disk compression tests are summarized in Table 2-5 (AA3104-Thin) and Table 2-6 (AA3104-Thick). The data is used to calculate the coefficients for the anisotropic material models.
Table 2-5. Uniaxial Tensile Test Data – AA3104-Thin Material

<table>
<thead>
<tr>
<th>Test Direction</th>
<th>( K ) MPa</th>
<th>( \varepsilon_0 )</th>
<th>( n )</th>
<th>( A ) MPa</th>
<th>( B ) MPa</th>
<th>( C )</th>
<th>( r )-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>331.461</td>
<td>0.00018</td>
<td>0.05013</td>
<td>293.095</td>
<td>54.965</td>
<td>44.919</td>
<td>0.403</td>
</tr>
<tr>
<td>15</td>
<td>333.035</td>
<td>0.00005</td>
<td>0.04627</td>
<td>295.970</td>
<td>55.233</td>
<td>52.000</td>
<td>0.413</td>
</tr>
<tr>
<td>30</td>
<td>334.550</td>
<td>0.00003</td>
<td>0.04633</td>
<td>296.330</td>
<td>55.340</td>
<td>54.456</td>
<td>0.557</td>
</tr>
<tr>
<td>45</td>
<td>339.400</td>
<td>0.00000</td>
<td>0.05137</td>
<td>290.569</td>
<td>65.802</td>
<td>90.089</td>
<td>0.888</td>
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<tr>
<td>60</td>
<td>346.760</td>
<td>0.00000</td>
<td>0.05007</td>
<td>298.199</td>
<td>70.999</td>
<td>95.833</td>
<td>0.927</td>
</tr>
<tr>
<td>75</td>
<td>351.858</td>
<td>0.00000</td>
<td>0.04996</td>
<td>302.680</td>
<td>71.970</td>
<td>94.020</td>
<td>1.060</td>
</tr>
<tr>
<td>90</td>
<td>347.434</td>
<td>0.00004</td>
<td>0.04938</td>
<td>304.780</td>
<td>59.382</td>
<td>55.140</td>
<td>1.125</td>
</tr>
<tr>
<td>bi-axial</td>
<td>386.430</td>
<td>0.00000</td>
<td>0.04892</td>
<td>353.800</td>
<td>67.050</td>
<td>27.285</td>
<td>0.783</td>
</tr>
</tbody>
</table>

Table 2-6. Uniaxial Tensile Test Data – AA3104-Thick Material

<table>
<thead>
<tr>
<th>Test Direction</th>
<th>( K ) MPa</th>
<th>( \varepsilon_0 )</th>
<th>( n )</th>
<th>( A ) MPa</th>
<th>( B ) MPa</th>
<th>( C )</th>
<th>( r )-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>305.700</td>
<td>0.00860</td>
<td>0.07300</td>
<td>254.900</td>
<td>38.000</td>
<td>34.500</td>
<td>0.394</td>
</tr>
<tr>
<td>15</td>
<td>305.800</td>
<td>0.00970</td>
<td>0.07400</td>
<td>256.300</td>
<td>38.600</td>
<td>31.000</td>
<td>0.460</td>
</tr>
<tr>
<td>30</td>
<td>312.200</td>
<td>0.01500</td>
<td>0.08400</td>
<td>258.700</td>
<td>38.300</td>
<td>26.200</td>
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<td>45</td>
<td>312.300</td>
<td>0.01800</td>
<td>0.08600</td>
<td>259.200</td>
<td>37.800</td>
<td>23.700</td>
<td>0.865</td>
</tr>
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<td>0.11400</td>
<td>271.800</td>
<td>47.100</td>
<td>16.100</td>
<td>0.899</td>
</tr>
<tr>
<td>75</td>
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<td>0.03700</td>
<td>0.12500</td>
<td>274.300</td>
<td>47.300</td>
<td>15.300</td>
<td>1.047</td>
</tr>
<tr>
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<td>358.400</td>
<td>0.04000</td>
<td>0.14000</td>
<td>281.400</td>
<td>52.600</td>
<td>14.500</td>
<td>1.004</td>
</tr>
<tr>
<td>bi-axial</td>
<td>303.010</td>
<td>0.03535</td>
<td>0.06820</td>
<td>283.207</td>
<td>40.493</td>
<td>7.733</td>
<td>0.757</td>
</tr>
</tbody>
</table>
2.5 MATERIAL MODEL COEFFICIENTS

The AA3104 materials are assumed to be elastic-plastic with the isotropic elastic behavior being defined by Young’s modulus and Poisson’s ratio as defined in Table 2-7. The elasto-plastic constitutive model adopted considers anisotropic plastic behavior, being defined by: (i) an associated flow rule, (ii) a yield criterion, and (iii) a hardening law. The yield surface in stress space separates stress states producing elastic and elasto-plastic deformation and is a generalization of the tensile yielding behaviour to multiaxial stress states. Plastic anisotropy is the result of the distortion of the yield surface shape due to the material microstructural state. Regardless of the shape of the yield surface, strain hardening can be isotropic or anisotropic. Only isotropic work hardening behavior will be discussed and is modeled by the Swift law being expressed by:

\[ \sigma = K (\varepsilon_p + \bar{\varepsilon}_p)^n \]  \hspace{1cm} (2-10)

where \( \sigma \) is the flow stress and \( \bar{\varepsilon}_p \) is the equivalent plastic strain. The material parameters \( K \), \( n \), and \( \varepsilon_0 \) are evaluated using only the uniaxial tensile test results. The material mechanical behavior follows Hooke’s law in the elastic regime, being described by the Young modulus and the Poisson ratio. The material data used in the calculation of the material coefficients for each of the yield criterion are given in Table 2-8 and Table 2-9. The stress ratios for the AA3104-Thin material were calculated based on an equivalent strain energy of 10. The stress ratios for the AA3104-Thick were based on an equivalent strain energy of 5.

The yield criterion expresses the relationship between the stress components in the transition from the elastic to the plastic regime, for which evolution is dictated by Equation 2-10. Three yield functions are adopted in this study: (i) Hill 1948 anisotropic, (ii) Barlat Yld2000-2D, and (iii) Barlat Yld2004-18P.

<table>
<thead>
<tr>
<th>Sample</th>
<th>Thickness (mm)</th>
<th>Density (g/cm³)</th>
<th>Young’s Modulus (MPa)</th>
<th>Poisson’s Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>AA3104-Thin</td>
<td>0.274</td>
<td>2.715</td>
<td>68950</td>
<td>0.33</td>
</tr>
<tr>
<td>AA3104-Thick</td>
<td>0.475</td>
<td>2.715</td>
<td>68950</td>
<td>0.33</td>
</tr>
</tbody>
</table>

Table 2-7. Elastic Mechanical Properties
Table 2-8. Summary of AA3104-Thin Uniaxial Test Data for Calculating Material Coefficients

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( K )</th>
<th>( \varepsilon_0 )</th>
<th>( n )</th>
<th>( \varepsilon_{equiv} )</th>
<th>( \sigma_\theta )</th>
<th>( \sigma_\theta / \sigma_0 )</th>
<th>( r )-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>331.46</td>
<td>0.00018</td>
<td>0.05013</td>
<td>0.03731</td>
<td>281.15</td>
<td>1.0000</td>
<td>0.403</td>
</tr>
<tr>
<td>15</td>
<td>333.04</td>
<td>0.00005</td>
<td>0.04627</td>
<td>0.03660</td>
<td>285.79</td>
<td>1.0165</td>
<td>0.413</td>
</tr>
<tr>
<td>30</td>
<td>334.55</td>
<td>0.00003</td>
<td>0.04633</td>
<td>0.03645</td>
<td>286.97</td>
<td>1.0207</td>
<td>0.557</td>
</tr>
<tr>
<td>45</td>
<td>339.40</td>
<td>0.00000</td>
<td>0.05137</td>
<td>0.03671</td>
<td>286.41</td>
<td>1.0187</td>
<td>0.888</td>
</tr>
<tr>
<td>60</td>
<td>346.76</td>
<td>0.00000</td>
<td>0.05007</td>
<td>0.03578</td>
<td>293.50</td>
<td>1.0439</td>
<td>0.927</td>
</tr>
<tr>
<td>75</td>
<td>351.86</td>
<td>0.00000</td>
<td>0.04996</td>
<td>0.03527</td>
<td>297.71</td>
<td>1.0589</td>
<td>1.060</td>
</tr>
<tr>
<td>90</td>
<td>347.43</td>
<td>0.00004</td>
<td>0.04938</td>
<td>0.03560</td>
<td>294.69</td>
<td>1.0482</td>
<td>1.125</td>
</tr>
<tr>
<td>bi-axial</td>
<td>386.43</td>
<td>0.00000</td>
<td>0.04892</td>
<td>0.03212</td>
<td>326.60</td>
<td>1.1617</td>
<td>0.783</td>
</tr>
</tbody>
</table>

Table 2-9. Summary of AA3104-Thick Uniaxial Test Data for Calculating Material Coefficients

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( K )</th>
<th>( \varepsilon_0 )</th>
<th>( n )</th>
<th>( \varepsilon_{equiv} )</th>
<th>( \sigma_\theta )</th>
<th>( \sigma_\theta / \sigma_0 )</th>
<th>( r )-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>305.70</td>
<td>0.00860</td>
<td>0.07300</td>
<td>0.04256</td>
<td>246.06</td>
<td>1.0000</td>
<td>0.394</td>
</tr>
<tr>
<td>15</td>
<td>305.80</td>
<td>0.00970</td>
<td>0.07400</td>
<td>0.04256</td>
<td>245.80</td>
<td>0.9994</td>
<td>0.460</td>
</tr>
<tr>
<td>30</td>
<td>312.20</td>
<td>0.01500</td>
<td>0.08400</td>
<td>0.04253</td>
<td>245.62</td>
<td>0.9985</td>
<td>0.688</td>
</tr>
<tr>
<td>45</td>
<td>312.30</td>
<td>0.01800</td>
<td>0.08600</td>
<td>0.04248</td>
<td>245.35</td>
<td>0.9988</td>
<td>0.865</td>
</tr>
<tr>
<td>60</td>
<td>334.30</td>
<td>0.03100</td>
<td>0.11400</td>
<td>0.04202</td>
<td>248.07</td>
<td>1.0083</td>
<td>0.899</td>
</tr>
<tr>
<td>75</td>
<td>342.30</td>
<td>0.03700</td>
<td>0.12500</td>
<td>0.04181</td>
<td>249.16</td>
<td>1.0129</td>
<td>1.047</td>
</tr>
<tr>
<td>90</td>
<td>358.40</td>
<td>0.04000</td>
<td>0.14000</td>
<td>0.04140</td>
<td>252.27</td>
<td>1.0230</td>
<td>1.004</td>
</tr>
<tr>
<td>bi-axial</td>
<td>303.01</td>
<td>0.03535</td>
<td>0.06820</td>
<td>0.04026</td>
<td>254.08</td>
<td>1.0503</td>
<td>0.757</td>
</tr>
</tbody>
</table>
2.5.1 *HILL’S (1948) YIELD FUNCTION*

Hill (1948) proposed an anisotropic yield criterion, considering that the material has an anisotropic behavior along three orthogonal symmetry planes (orthotropic behavior). The yield criterion is expressed by the quadratic function:

\[
\phi(\sigma) = F(\sigma_{yy} - \sigma_{zz})^2 + G(\sigma_{zz} - \sigma_{xx})^2 + H(\sigma_{xx} - \sigma_{yy})^2 + 2L\sigma_{yy}^2 + 2M\sigma_{zz}^2 + 2N\sigma_{xy}^2 = 2\bar{\sigma}^2
\] (2-11)

where \(\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yx}, \sigma_{zx}\) are the components of the Cauchy stress tensor and \(F, G, H, L, M, N\) are material constants that describe the current state of anisotropy. These can be derived if three tensile yield stresses with respect to the anisotropy axes and three yield stresses for pure shear on each of the orthogonal planes of anisotropy are measured. Additionally, it can easily be noticed that for \(L = M = N = 3F = 3G = 3H\), Equation 2-11 becomes equal to the von Mises criterion for isotropic materials.

For a plane stress state, Hill’s formulation can be expressed as

\[
\phi(\sigma) = (G + H)\sigma_{xx}^2 + (F + H)\sigma_{yy}^2 - 2H\sigma_{xx}\sigma_{yy} + 2N\sigma_{xy}^2 = 2\bar{\sigma}^2
\] (2-12)

In the case of sheet metals, the orthotropic frame axis \(x\) is usually parallel to the rolling direction, \(y\) is parallel to the transverse direction, and \(z\) is collinear with the normal direction. The direction yield stress can be determined using

\[
\sigma_\theta = \frac{1}{\sqrt{2}} \left[ (G + H)\cos^4 \theta + (F + H)\sin^4 \theta - 2H\sin^2 \theta \cos^2 \theta + 2N\sin^2 \theta \cos^2 \theta \right] \] (2-13)

while the uniaxial anisotropy at the same direction \(\theta\) predicted by the yield criterion is defined by:

\[
r_\theta = \frac{H + (2N - F - G - 4H)\sin^2 \theta \cos^2 \theta}{F\sin^2 \theta + G\cos^2 \theta} \] (2-14)

The procedure used to identify the anisotropy parameters based on the experimental yield stress and the experimental coefficients of plastic anisotropy (r-values) is described in detail in Appendix C and are presented in Table 2-10. Although this model has been widely used in industry, mainly because of its simplicity and efficiency, it is known to inaccurately describe the behavior of materials with low r-values typically found in many aluminum alloys (Barlat, Maeda, Chung 1997).
Table 2-10. Material Parameters for the Hill48 Yield Criteria

<table>
<thead>
<tr>
<th>Material</th>
<th>F</th>
<th>G</th>
<th>H</th>
<th>N</th>
</tr>
</thead>
<tbody>
<tr>
<td>AA3104-Thin</td>
<td>R</td>
<td>0.5107</td>
<td>1.4255</td>
<td>0.5745</td>
</tr>
<tr>
<td>AA3104-Thin</td>
<td>S</td>
<td>0.6513</td>
<td>0.8308</td>
<td>1.1692</td>
</tr>
<tr>
<td>AA3104-Thick</td>
<td>R</td>
<td>0.5630</td>
<td>1.4347</td>
<td>0.5653</td>
</tr>
<tr>
<td>AA3104-Thick</td>
<td>S</td>
<td>0.8620</td>
<td>0.9510</td>
<td>1.0490</td>
</tr>
</tbody>
</table>

R = r-value based; S = stress based

Single element and uniaxial tensile bar finite element models were used to compare the predicted r-values and normalized stress values with the experimental data used to generate the constitutive model coefficients. The r-value based coefficients are used in the simulation. The results are summarized in Figure 2-18 and Figure 2-19. Since the r-values were used to generate the coefficients, very good agreement is achieved for r-value predictions between the numerical and experimental data. However, significant differences are observed between the predicted and experimental normalized yield stress values. There are also significant differences in the predicted yield surfaces in the bi-axial and plane strain directions as well as the yield stresses at 90°.
Figure 2-18. Comparison of experimental and predicted (single element and tensile bar) r-values, normalized stress, and yield surfaces using Hill48 for the AA3104-Thin material.
Figure 2-19. Comparison of experimental and predicted (single element and tensile bar) r-values, normalized stress, and yield surfaces using Hill48 for the AA3104-Thick material.
2.5.2 Barlat et al. (2003): YLD2000-2D

Yld2000-2D yield function was proposed to overcome the limitations associated with the previously suggested formulation, Barlat et al. (1997), called YLD96, which could not ensure convexity and required quite complex finite element implementation procedures. Yld2000-2D consists of a non-quadratic plane stress yield function (2D mean plane stress) with eight material parameters (four stress-ratios and four r-values), based on an expansion of the yield criteria introduced by Hershey (1954) and Hosford (1972) for isotropic materials.

\[ \varphi = \varphi' + \varphi'' = 2\bar{\sigma}^a \] (2-15)

where exponent “a” is a material coefficient and is assumed to be 8 in this thesis and

\[ \varphi' = \left| X_1 - X_2 \right|^a \] (2-16)

\[ \varphi'' = \left[ 2X_1 + X_2 \right]^a + \left[ 2X_1 + X_2 \right]^a \] (2-17)

The yield surface is convex when \( a \geq 1 \). The exponent “a” in Equation 2-15 is mainly associated with the crystal structure. A higher “a” value has the effect of increasing the curvature of the rounded vertices of the yield surface. In Equation 2-16 and Equation 2-17, \( X_j \) and \( X_k' \) are the principal values of \( \mathbf{X}' \) and \( \mathbf{X}'' \) (two linear transformations on the stress deviator) defined as

\[
\begin{bmatrix}
X_{xx}' \\
X_{yy}' \\
X_{xy}'
\end{bmatrix} =
\begin{bmatrix}
C_{11} & 0 & 0 \\
0 & C_{22} & 0 \\
0 & 0 & C_{33}
\end{bmatrix}
\begin{bmatrix}
s_{xx} \\
s_{yy} \\
s_{xy}
\end{bmatrix};
\begin{bmatrix}
X_{xx}' \\
X_{yy}' \\
X_{xy}'
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & 0 \\
C_{21} & C_{22} & 0 \\
0 & 0 & C_{33}
\end{bmatrix}
\begin{bmatrix}
s_{xx} \\
s_{yy} \\
s_{xy}
\end{bmatrix}
\] (2-18)

where subscripts x and y represent the rolling and transverse directions of the sheet, respectively. The transformation can also apply on the stress tensor; i.e.,

\[
\mathbf{X}' = \mathbf{C}' \mathbf{s} = \mathbf{C}' \mathbf{T} \mathbf{\sigma} = \mathbf{L}' \mathbf{\sigma}
\] (2-19)

\[
\mathbf{X}'' = \mathbf{C}'' \mathbf{s} = \mathbf{C}'' \mathbf{T} \mathbf{\sigma} = \mathbf{L}'' \mathbf{\sigma}
\] (2-20)

with
\[
T = \begin{bmatrix}
  2/3 & -1/3 & 0 \\
  -1/3 & 2/3 & 0 \\
  0 & 0 & 1
\end{bmatrix}
\] (2-21)

The transformation operators \( \mathbf{L'} \) and \( \mathbf{L''} \) contain three and five anisotropy parameters, respectively, which are denoted as \( \alpha_1 \) to \( \alpha_8 \). When \( m \) is equal to 2 and all the \( \alpha_1 \) to \( \alpha_8 \) are equal to 1, the yield function reduces to von Mises. The anisotropy parameters were determined using the Newton-Raphson iterative method as explained in Barlat et al. (2003), and are listed in Table 2-11. Single element and uniaxial tensile bar finite element models are used to compare the predicted \( r \)-values and normalized stress values with the experimental data. The results are summarized in Figure 2-20 and 2-21. Since the \( r \)-values and stress ratios at \( 0^\circ, 45^\circ, \) and \( 90^\circ \) are used to generate the coefficients, very good agreement is achieved between the numerical and experimental data. The stress ratio for the AA3104-Thin material exhibited more variation at the other orientations. The von Mises and Yld2000-2D yield loci are compared in Figure 2-20 and Figure 2-21 in the plane stress space, in the absence of shear stress.

<table>
<thead>
<tr>
<th>Material</th>
<th>(\alpha_1)</th>
<th>(\alpha_2)</th>
<th>(\alpha_3)</th>
<th>(\alpha_4)</th>
<th>(\alpha_5)</th>
<th>(\alpha_6)</th>
<th>(\alpha_7)</th>
<th>(\alpha_8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AA3104-Thin</td>
<td>0.59438</td>
<td>1.17732</td>
<td>0.81818</td>
<td>0.89239</td>
<td>0.96676</td>
<td>0.62717</td>
<td>0.94672</td>
<td>1.15199</td>
</tr>
<tr>
<td>AA3104-Thick</td>
<td>0.76472</td>
<td>1.10411</td>
<td>0.93280</td>
<td>0.95516</td>
<td>1.01038</td>
<td>0.84837</td>
<td>0.98083</td>
<td>1.08349</td>
</tr>
</tbody>
</table>
Figure 2-20. Comparison of experimental and predicted (single element and tensile bar) $r$-values, normalized stress, and yield surfaces using Barlat Yld2000-2D for the AA3104-Thin material.
Figure 2-21. Comparison of experimental and predicted (single element and tensile bar) r-values, normalized stress, and yield surfaces using Barlat Yld2000-2D for the AA3104-Thick material.
2.5.3 BARLAT ET AL. (2005): YLD2004-18P

For cubic metals, there are usually enough potentially active slip systems to accommodate any shape change. Compressive and tensile yield strengths are virtually identical and yielding is not influenced by the hydrostatic pressure. The yield surface of such materials is usually represented adequately by an even function of the principal values $S_\alpha$ of the stress deviator $s$ suggested by Hosford (1972), i.e.,

$$\phi = |S_1 - S_2| + |S_2 - S_3| + |S_3 - S_1| = 2\bar{\sigma}^a$$

(2-22)

The exponent "$a$" is connected to the crystal structure of the material, i.e., 6 for BCC and 8 for FCC. Extensions of Equation 2-22 for the case of planar anisotropy are briefly summarized for a general stress state. The formulation is based on two linear transformations of the stress deviator. The two linear transformations can be expressed as

$$\tilde{s}' = C' s = C' T \sigma = L' \sigma$$

(2-23)

$$\tilde{s}'' = C'' s = C'' T \sigma = L'' \sigma$$

(2-24)

where $T$ is a matrix that transforms the Cauchy stress tensor $\sigma$ to its deviator $s$. $\tilde{s}'$ and $\tilde{s}''$ are the linearly transformed stress deviators and $C'$ and $C''$ (or $L'$ and $L''$) are the matrices containing the anisotropy coefficients.

For a full stress state, the linear transformations can be expressed in the most general form using the following matrices

$$C' = \begin{bmatrix} 0 & -c'_{12} & -c'_{13} & 0 & 0 & 0 \\ -c'_{21} & 0 & -c'_{23} & 0 & 0 & 0 \\ -c'_{31} & -c'_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c'_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c'_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c'_{66} \end{bmatrix}, \quad C'' = \begin{bmatrix} 0 & -c''_{12} & -c''_{13} & 0 & 0 & 0 \\ -c''_{21} & 0 & -c''_{23} & 0 & 0 & 0 \\ -c''_{31} & -c''_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c''_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c''_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c''_{66} \end{bmatrix}$$

(2-25)

In order to accurately describe both yielding and plastic flow behavior of sheet metals, the coefficients of the above anisotropic yield function commonly need to be optimized explicitly or iteratively from experimental tensile, shear or bi-axial yield stresses and Lankford coefficients. An optimization routine was developed using the LS-OPT code to minimize the residuals between the experimental and predicted $r$-values and normalized stress values by adjusting the weighting functions using Equation 2-26.
The results of the optimization process for the two materials are shown in Table 2-12. Single element and uniaxial tensile bar finite element models are used to compare the predicted r-values and normalized stress values with the experimental data. The results are summarized in Figure 2-22 and Figure 2-23. Since the r-values and stress ratios at every 15° were used to generate the coefficients, very good agreement was achieved between the numerical and experimental data for both the r-values and stress ratios. The von Mises and Yld2004-18P yield loci are compared in Figure 2-22 and Figure 2-23 in the plane stress space, in the absence of shear stress.

Table 2-12. Material Parameters for the YLD2004-18P Yield Criteria

<table>
<thead>
<tr>
<th></th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\alpha_3$</th>
<th>$\alpha_4$</th>
<th>$\alpha_5$</th>
<th>$\alpha_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>AA3104-Thin</strong></td>
<td>1.13021</td>
<td>0.84619</td>
<td>1.11118</td>
<td>0.89335</td>
<td>-0.49552</td>
<td>0.68614</td>
</tr>
<tr>
<td>$\alpha_7$</td>
<td>1.47199</td>
<td>0.96980</td>
<td>1.22709</td>
<td>0.36757</td>
<td>-0.08264</td>
<td>0.52427</td>
</tr>
<tr>
<td>$\alpha_{13}$</td>
<td>0.68771</td>
<td>1.19374</td>
<td>0.98813</td>
<td>0.42780</td>
<td>0.99205</td>
<td>0.63589</td>
</tr>
<tr>
<td><strong>AA3104-Thick</strong></td>
<td>1.00737</td>
<td>0.91603</td>
<td>0.14547</td>
<td>0.68242</td>
<td>0.35206</td>
<td>0.36242</td>
</tr>
<tr>
<td>$\alpha_7$</td>
<td>1.24969</td>
<td>0.94496</td>
<td>0.83955</td>
<td>0.91798</td>
<td>-0.32196</td>
<td>0.92036</td>
</tr>
<tr>
<td>$\alpha_{13}$</td>
<td>1.27087</td>
<td>1.17314</td>
<td>0.73380</td>
<td>0.70479</td>
<td>1.05172</td>
<td>1.01529</td>
</tr>
</tbody>
</table>
Figure 2-22. Comparison of experimental and predicted (single element and tensile bar) r-values, normalized stress, and yield surfaces using Barlat Yld2004-18P for the AA3104-Thin material.
Figure 2-23. Comparison of experimental and predicted (single element and tensile bar) r-values, normalized stress, and yield surfaces using Barlat Yld2004-18P for the AA3104-Thick material.
2.6 STRESS RATIO SENSITIVITY

Several methods are available for determining the stress ratios for use in calculating the material coefficients. The simplest is the initial yield stress (Figure 2-24) which can be determined for the Swift model using

\[
\sigma = K(\varepsilon_o + \varepsilon_p)^n \Rightarrow \varepsilon_p = 0 \implies K(\varepsilon_o)^n
\]

and for the Voce model using

\[
\sigma = A - B \exp(-C \varepsilon_p) \Rightarrow \varepsilon_p = 0 \implies \sigma = A - B
\]

A second method is to use the minimum fracture strain from all of the uniaxial tensile test in all orientations. This is shown pictorially in Figure 2-25. The fracture strain, \(\varepsilon_p^{\text{fract}}\), can then be using in the Swift of Voce equations to calculate the stress value.

\[
\sigma = K(\varepsilon_o + \varepsilon_p^{\text{fract}})^n
\]

\[
\sigma = A - B \exp(-C \varepsilon_p^{\text{fract}})
\]

A third method is to use the minimum plastic work to failure. The plastic strain that produces equivalent plastic work or strain energy as the minimum is determined at each test orientation. Figure 2-26 shows an example of this procedure.

An additional method that uses equivalent plastic work can be used to evaluate the sensitivity of the stress ratio curves from the selected method. Using the data from Table 2-10 and Table 2-11 and both the Swift model and Voce model, the stress ratio curves as a function of plastic work are shown in Figure 2-27 (AA3104-Thin) and Figure 2-28 (AA3104-Thick). For each of the materials the trends in the stress ratio plots are very similar, however, the magnitudes are different and may have an impact on the predicted earing during cup drawing. In addition, the comparison between the AA3104-Thin and AA3104-Thick materials is noticeably different. Table 2-13 provides the material coefficients for the AA3104-Thin material for the YLD2004-18P models for a selected set of stress ratios. These will be utilized in Chapter 3 to study the impact on the earing profile.
Figure 2-24. Stress ratios based on initial yield stress.

Figure 2-25. Stress ratios based on minimum fracture strain.

Figure 2-26. Stress ratios based on minimum fracture strain with minimum strain energy.
Figure 2-27. Stress ratio plots for AA3104-Thin as a function of plastic strain, strain energy, and hardening law.
Figure 2-28. Stress ratio plots for AA3104-Thick as a function of plastic strain, strain energy, and hardening law.
Table 2-13. YLD2004-18P Coefficients using various stress ratio methods.

<table>
<thead>
<tr>
<th>Material</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\alpha_3$</th>
<th>$\alpha_4$</th>
<th>$\alpha_5$</th>
<th>$\alpha_6$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Swift Model</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>S.E. = 0.5</td>
<td>0.95470</td>
<td>0.67420</td>
<td>1.21350</td>
<td>0.86169</td>
<td>-0.51431</td>
<td>0.66937</td>
</tr>
<tr>
<td>S.E. = 40.0 (Thin)</td>
<td>1.24772</td>
<td>1.01250</td>
<td>1.27001</td>
<td>0.53173</td>
<td>0.27997</td>
<td>0.45928</td>
</tr>
<tr>
<td>S.E. = 40.0</td>
<td>0.77344</td>
<td>1.30855</td>
<td>1.02790</td>
<td>0.66165</td>
<td>0.95728</td>
<td>0.61831</td>
</tr>
<tr>
<td>S.E. = 10.0</td>
<td>0.37012</td>
<td>-0.09090</td>
<td>0.53456</td>
<td>0.68517</td>
<td>1.19503</td>
<td>0.99717</td>
</tr>
<tr>
<td>S.E. = 5.0</td>
<td>0.44326</td>
<td>0.99295</td>
<td>0.63559</td>
<td>1.16168</td>
<td>0.86029</td>
<td>1.10068</td>
</tr>
<tr>
<td>S.E. = 10.0</td>
<td>0.30228</td>
<td>-0.09919</td>
<td>0.49473</td>
<td>0.71547</td>
<td>1.16053</td>
<td>0.96127</td>
</tr>
<tr>
<td>S.E. = 5.0</td>
<td>0.90139</td>
<td>-0.49401</td>
<td>0.73748</td>
<td>1.46714</td>
<td>0.96868</td>
<td>1.22394</td>
</tr>
<tr>
<td>S.E. = 7.5</td>
<td>0.40659</td>
<td>1.01316</td>
<td>0.66877</td>
<td>1.12666</td>
<td>0.88561</td>
<td>1.03450</td>
</tr>
<tr>
<td>S.E. = 7.5</td>
<td>0.09013</td>
<td>-0.49401</td>
<td>0.73748</td>
<td>1.48514</td>
<td>0.93975</td>
<td>1.18082</td>
</tr>
<tr>
<td>S.E. = 10.0</td>
<td>1.05520</td>
<td>0.90338</td>
<td>0.98765</td>
<td>0.94521</td>
<td>-0.42226</td>
<td>0.77822</td>
</tr>
<tr>
<td>S.E. = 7.5</td>
<td>1.26763</td>
<td>1.02907</td>
<td>1.05605</td>
<td>0.45899</td>
<td>0.44443</td>
<td>0.49021</td>
</tr>
<tr>
<td>S.E. = 10.0</td>
<td>0.21885</td>
<td>0.97917</td>
<td>0.43138</td>
<td>0.67911</td>
<td>0.85578</td>
<td>0.74746</td>
</tr>
<tr>
<td>S.E. = 7.5</td>
<td>0.68061</td>
<td>1.16984</td>
<td>0.91321</td>
<td>0.37882</td>
<td>1.00241</td>
<td>0.73277</td>
</tr>
</tbody>
</table>

Material Characterization Tests

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2.7 SUMMARY

Various testing methods for the characterization of the plastic behavior of sheet metals are well established. The experiment results from uniaxial tensile tests, bi-axial bulge tests, and disk compression tests for both AA3104-Thin and AA3104-Thick materials are documented. The results from the experimental tests were used to determine material coefficients for the Hill 1948, Barlat Yld2000-2D, and Barlat Yld2004-18P models. Numerical optimization using LS-OPT was used to determine the coefficients for both the Yld2000-2D and Yld2004-18P models. The material coefficients were verified using single element and tensile bar models to compare the predicted r-values and normalized stress values with the experimental data. Excellent agreement was achieved between the experimental and numerical directional r-values and stress ratios using the Barlat Yld2004-18P model. A high degree of anisotropic hardening was observed in the stress ratio plots for both materials. For materials that exhibit this behavior, the material model selection and the technique to determine the material coefficients is critical for accurate forming simulation predictions. The material coefficient data for each of the material models will be used in the following chapters to evaluate earing during conventional cup drawing and wrinkling during a tapered cup drawing process. In addition, sensitivity to material model selection and the technique to determine the material coefficients will be discussed.
2.8 REFERENCES

CHAPTER 3

EARING IN CUP DRAWING

3.1 LITERATURE REVIEW – EARING

Today, the worldwide production of aluminum beverage cans is over 350 billion units per year. These aluminum beverage cans are generally produced through the mechanical cold forming process of drawing and ironing that starts from large coils of cold-rolled sheet. The average sheet thickness is about 0.274 mm and is typically an Al-alloy with about 1% manganese and 1% magnesium, which is referred to as alloy AA3104. In order to meet the strength requirements for beverage cans, the rigid container sheet (RCS) bodystock AA3104 is used in the highly strain hardened, cold rolled H19 temper. The process of manufacturing cans is described in detail by Hosford and Duncan (1994).

At a can manufacturing plant, the coil of Al-sheet is fed continuously from an uncoiler and lubricator into a cupping press, referred to as the cupper, which blanks out up to 14 cups every stroke running at 150 strokes per minute (spm). Immediately after blanking, the cupping press ram drives the blank through the draw ring to form a shallow cup, in what is called the cup drawing operation. Figure 3-1 shows the production cup of a typical aluminum bottle. It is seen that the leading edge of this cup is not perfectly flat, but shows a number of undulations at the top of the cup, known as ‘ears’ or ‘peaks’, balanced by an equal number of low points, called ‘troughs’ or ‘valleys’. The earing profile that develops is a result of the significant plastic anisotropy in the sheet which can be attributed to the presence of preferred crystallographic orientation, or texture, acquired during the thermo-mechanical production of the sheet. Excessive earing during deep drawing of a textured sheet can cause major problems in the production of aluminum containers or beverage cans - Zaidi and Sheppard (1985); Courbon (2003). Earing is highly undesirable since it requires extra metal to be trimmed from the top of the can, leading to loss of material.
More severely, too pronounced earing is detrimental for cup and can handling and affects the process of can-making negatively, by stretching and clipping off ears, leading to machine down time and, hence, reduction of line-efficiency. The characteristic earing profiles of deep drawn cups form because the sheet texture gives rise to different radial elongations in different directions of the blank. Engler (2012) states that in a rolled sheet with a pronounced deformation texture, the ears are usually observed at the four positions ±45° to the rolling direction around the drawn cup, which is characteristic of heavily cold rolled sheet. The height of the ears, and hence the percentage of earing, generally increases with the amount of cold rolling applied. In the soft annealed, re-crystallized state, ears are often found at angles of 0° and 90° to the rolling direction.

In efforts to understand and predict earing, with the ultimate purpose to control or design around them, there have been three major approaches: continuum plasticity, crystal plasticity, and analytical models. The phenomenological description of plastic deformation in metals is the most commonly used strategy in the numerical simulation of forming processes. The main concept to describe the sheet orthotropic behavior is the yield surface, used to describe yielding and the plastic flow of the material. Because of this dual role of the yield surface, particular care and accuracy for its modeling is required. Also, due to the complexity of the underlying mechanism of plastic flow and the increasingly advanced alloying technologies, the yield surface modeling has become more complex, relying on an increasing number of material parameters. Generally, r-value and stress directionalities are the key input parameters for phenomenological constitutive models. These anisotropies are directly related to earing of a drawn cup. For example, Hill’s 1948 yield function (Hill, 1948) accepts either r-values or yield stresses along 0°, 45°, and 90° as anisotropy parameters, while Yld91 (Barlat et al., 1991a) uses only the yield stress values for the balanced biaxial value as well as along the three major directions (0°, 45°, 90°). Yld2000-2D model (Barlat et al., 2003) accommodates both r-value and stress directionalities for the three uniaxial and balanced biaxial directions. Yld2004-18P model (Barlat et al., 2005) utilizes r-value and yield stress data every 15° from the rolling direction as well as a biaxial datum. Thus, based on the combination of these directionalities, Yld2004-18P model is able to predict more than four ears in cup drawing as shown in Yoon et al. (2006). Characteristics for linear transformation yield functions are well summarized in Barlat et al. (2007). It has been shown that a good prediction of these material directionalities controls the overall accuracy of the earing profile. Recently, there have been many advanced approaches developed to describe plastic anisotropy. The strain rate potential is another concept that can describe plastic anisotropy (Barlat et al., 1993; Yoon et al., 1995; Chung et al., 1996; Kim et al., 2008a,b; Rabahallah et al., 2009; Van Houtte et al., 2009; Cazacu et al., 2010). Non-associated flow plasticity was also implemented
to the finite element method to predict plastic anisotropy (Civitanic et al., 2008; Taherizadeh et al., 2009). Experimentally, it was shown that the yield surface shape can evolve in complex ways (Kuroda and Tvergaard, 2001; Kuwabara, 2007). For the corresponding modeling, it has been acknowledged that more advanced models should capture the distortion of the yield surface (Wu et al., 2005; Holmedal et al., 2008; Aretz, 2008; Korkolis and Kyriakides, 2008; Stoughton and Yoon, 2009). Yield criteria to describe plastic anisotropy for complex hcp materials were proposed by Cazacu et al. (2006) and Plunkett et al. (2008). Anisotropic hardening behaviors for the hcp materials were also investigated (Plunkett et al., 2007; Nixon et al., 2009). A review on hardening models has been made by Chaboche (2008). Rousselier et al. (2009) also predicted a complicated earing profile of AA 2090-T3 with a reduced polycrystal approach. Yoon et al. (2010) studied the effects of the evolution in anisotropy and the directionality in hardening on the predictions of the earing profile and investigated a new methodology that incorporates multiple hardening curves corresponding to uniaxial tension along several orientations with respect to the rolling direction, and to biaxial tension. Improved accuracy in the prediction of the non-uniformity of the cup height profile was achieved using distortional hardening.

In crystal plasticity, grains in polycrystalline materials usually have preferential crystallographic orientations, for which the material is referred to have a ‘texture’. Especially for sheets, the texture is developed during the rolling and annealing processes, resulting in planar anisotropy. In order to take better account of the influence of texture on earing, several models based on the plastic slip of a single crystal or of a polycrystal have been proposed. Tucker’s (1961) approach, based on the Schmid law for FCC single crystal, is well known and affords relatively reasonable agreement with the results of experiments. The changes of texture during cold rolling and annealing processes were experimentally observed by X-ray diffraction by Kao (1985). The texture of metal sheets experimentally measured is represented by three-dimensional crystallite orientation distribution functions (ODF). Kanetake et al. (1983, 1985) calculated earing in cup drawing based on ODF for steel, aluminum, Al-Mg alloy and copper sheets. Houtte et al. (1987) used yield curves derived from ODF for aluminum alloys. Even though their calculations are close to the results in earing shape, there is a significant difference in earing height. Mathur et al. (1991), Kalindindi et al. (1992), and Becker et al. (1993), attempted to use a model based on the Taylor (1938), and Bishop and Hill (1951a, 1951b) (TBH) polycrystal theory, combining with finite-element simulation. However, these calculations are very CPU intensive. Some efforts were based on the mixture of the continuum and crystal plasticity approaches: the continuum mechanics of textured polycrystals (CMTP), in which the effects of typical texture components appearing in aluminum and its alloy sheets on earing were analyzed quantitatively (Chan, 1995; Hu et al., 1998). To model earing in cup
drawing, some utilized the visco-plastic self-consistent (VPSC) polycrystal plasticity scheme developed by Tomé and coworkers (Lebensohn and Tomé, 1993; Kocks et al., 1998). In this approach, the polycrystal structure was represented by a set of individual grains and each grain is treated as an ellipsoidal viscoplastic inclusion that is embedded in and interacts with a homogeneous effective medium having average polycrystal properties. Earing patterns were calculated for magnesium alloys (Walde and Riedel, 2007). Engler and Kalz (2004) have devised a polycrystal-plasticity model to tackle earing with the viscoplastic self-consistent (VPSC) code put forward by Lebensohn and Tomé (1993). The boundary conditions for the VPSC computations were derived from detailed FEM analysis of the stresses and strains operating during cup deep drawing. The resulting load history was cast into an idealized set of boundary conditions that are input in the polycrystal-plasticity model. The VPSC model then yields the information on the variations in radial strain under different in-plane angles and, therefore, on the resulting earing profile. The novel method to determine earing profiles from sheet texture was validated with a number of examples covering a wide range of sheet textures typically encountered in commercial aluminum sheet products. To render finite element models based on crystal plasticity more flexible in the treatment of large polycrystalline entities, the texture component crystal plasticity finite element method (TCCP-FEM) was also developed by introducing an effective way of describing the texture of macroscopic samples at each integration point. Simulations were conducted for brass, copper, aluminum, aluminum alloy, low carbon steel and stainless steel sheets (Raabe et al., 2005; Zhao et al., 2004; Tikhovskiy et al., 2007, 2008). Also, some works considered the evolution of texture as well as the initial texture (Inal et al., 2000; Walde and Riedel, 2007).

Compared to the above mentioned methods, there have been few studies on prediction of the earing profile based on analytical approaches for a single step cup drawing. Hosford and Caddell (1983) and Chung et al. (1996) provided a quantitative trend between the r-value anisotropy and the earing profile in a mild steel and an aluminum alloy, respectively. Using a different approach, Barlat et al. (1991b) attempted to correlate the stress anisotropy (not r-values) to the earing trends by applying the stress condition at the rim. Recently, an analytical approach considering the r-value directionality as a main contributor to the earing profile was derived by Yoon et al. (2006). The method provides a simple tool for the prediction of the earing profile using, as input, basic information including the r-value directionality, the initial blank size and the cup radius. However, the method did not consider the stress directionality. Yoon et al. (2008) simply combined Yoon et al. (2006) and Barlat et al. (1991b) to consider both r-value and yield stress directionalities on earing prediction. Mulder and Nagy (2009) further improved Yoon et al. (2008) considering non-uniform strain in the flange and the process effects. Yoon et al. (2011)
proposed an analytical approach with arbitrary anisotropic yield function to determine the earing profile in the drawing and ironing processes. It indicated that the distribution of earing has strong dependency on the \( r \)-value and yield stress directionalities. Further works conducted by Chung et al. (2011) proved that the earing profile could be determined analytically with only the anisotropic properties of yield stress and the \( r \)-value.

Various efforts have been made in order to reduce earing, including methods to control the anisotropy in the sheet manufacturing processes, such as in rolling and annealing processes (Kanetake et al., 1983; Kao, 1985; Yu et al., 1993; Jahazi and Goudarzi, 1997; Saha et al., 2007). Also, Thiruvarudchelvan and Loh (1994) added an extra annealing process before the drawing process to minimize earing, while Gavas and Izciler (2006) and Ku et al., (2007) modified the tool geometry and/or the blank holding system. The blank holding force effect on earing was also studied by Demirci et al. (2008). An alternative approach to reduce the earing problem is to find an initial blank shape, referred to as the optimal blank shape, which leads to an ear-free product with uniform flange. Roughly speaking, at the positions of the blank where ears will develop during deep drawing, material is removed from the blank, whereas there is extra material added to the blank which is able to fill the troughs. An experimental trial-and-error process to determine the best blank shape is very expensive and time-consuming. Therefore, numerical simulations present an attractive and effective alternative for the design of such “convoluted cut-edges”. In particular when a fast blank optimization is required for rapid reaction to a customer’s request, a fast and reliable numerical blank design tool is necessary. In the literature there are a number of approaches to determine the optimum blank shape upon sheet forming; detailed literature surveys on the methods have been given by Pegada et al. (2002) and Wang et al. (2009). Nowadays, most approaches are based on finite-element analysis (e.g. Chung et al., 1997; Dick et al., 2007). Barros et al. (2013) reported the accurate prediction of the earing during cup drawing depends not only on the correct modeling of the materials mechanical behavior, but also on the accuracy of the global process modeling including the blank holder, the yield criterion selected, the work hardening law, and the strategy used to identify the materials parameters.

Neto et al. (2016) studied the influence of elastic deformation of the tooling during cup drawing and concluded the accuracy of the sheet metal forming simulation can be improved by considering the elastic response of the forming tools in the numerical model.

Other authors proposed approaches within optimum blank geometry to minimize earing based on numerical simulations (Toh and Kobayashi, 1985; Barlat et al., 1994; Chen and Sowerby, 1996; Chung et al., 1997; Pegada et al., 2002; Kishor and Kumar, 2002; Lin and Kwan, 2009).
This chapter documents the experimental earing data generated using both AA3104-Thin and AA3104-Thick alloys. The data is used to verify both analytical and finite element simulations of the cup drawing process. The analytical approach directly utilizes measured tensile test data without involving any yield functions resulting in computational cost that are significantly reduced compared to finite element simulations. The model can also be utilized to approximate the r-values and directional yield stress data from measurement of the earing profile. Finally, detailed finite element analyses are performed to evaluate the choice of material model, friction, and method to generate material coefficients.

3.2 EXPERIMENTAL RESULTS

The cup drawing operation involves three axisymmetric forming tools (die, blank-holder and punch), which are schematically illustrated in Figure 3-2, as well as their main dimensions in millimetres provided in Table 3-1. The circular blank of an AA3104 aluminium alloy with a diameter of 76.12 mm and initial thicknesses of 0.274 mm and 0.475 mm are used in the experiments.

The deep drawing of the cylindrical aluminum cups is carried out at room temperature in a hydraulic press with maximum capacity of 5 tons. The press is equipped with a monitoring system (Keithley), which enables the acquisition of the punch force evolution as a function of its stroke and the blank-holder force.

Figure 3-1. Typical cup used for producing an aluminum bottle showing earing.

Earing in Cup Drawing
The experimental tests are run in triplicate for each process condition in order to investigate the reproducibility. The drawing process is performed with a constant punch travel speed of 140 mm/s and a constant blank-holder force of 8.9 kN. The cup is drawn completely considering a punch travel of 25 mm, while the gap between the blank-holder and the die is not restricted by any rigid stop or standoff during the drawing process. All forming tools are made of AISI A2 steel with 58–60 of Rockwell C hardness and surface roughness about 2–4 µm (finish working surfaces). An oil lubricant is applied to both sides of the blank to reduce the frictional forces between the forming tools and the blank sheet during the experiments.

![Schematic of tooling for cup drawing.](image)

**Figure 3-2. Schematic of tooling for cup drawing.**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>D3</th>
<th>D4</th>
<th>D6</th>
<th>R2</th>
<th>R3</th>
<th>T1</th>
</tr>
</thead>
<tbody>
<tr>
<td>AA3104-Thin</td>
<td>45.72</td>
<td>46.74</td>
<td>76.12</td>
<td>1.016</td>
<td>1.905</td>
<td>0.274</td>
</tr>
<tr>
<td>AA3104-Thick</td>
<td>45.72</td>
<td>47.75</td>
<td>76.12</td>
<td>1.016</td>
<td>1.905</td>
<td>0.475</td>
</tr>
</tbody>
</table>
Figure 3-3. Drawn cups for AA3104-Thin and AA3104-Thick alloys.

Figure 3-4. Experimental punch for curve for AA3104-Thin material.
3.3 PREDICTION OF EARING PROFILE – ANALYTICAL MODEL (Yoon et al, 2011)

As shown in Figure 3-5a, when the material is isotropic, the isotropic circumferential strain for the cup wall is defined as:

\[ \varepsilon_{\theta}^{iso} = \left( \frac{R_c}{R} \right) \quad \text{for} \quad R_c \leq R \leq R_b \]  

(3-1)

In Equation 3-1, the effect of anisotropy on the circumferential strain is not included. The anisotropic contribution is derived from the yield stress directionality in the rest of this section. The stress components with respect to the orthotropic material coordinates are related to those in cylindrical coordinates by:

\[
\begin{align*}
\sigma_{xx} &= \sigma_{rr} \cos^2 \theta + \sigma_{\theta\theta} \sin^2 \theta - \sigma_{r\theta} \sin 2\theta \\
\sigma_{yy} &= \sigma_{rr} \sin^2 \theta + \sigma_{\theta\theta} \cos^2 \theta + \sigma_{r\theta} \sin 2\theta \\
\sigma_{xy} &= (\sigma_{rr} - \sigma_{\theta\theta}) \sin \theta \cos \theta + \sigma_{r\theta} \cos 2\theta 
\end{align*}
\]

(3-2)

At the flange, it is assumed that \( \sigma_{r\theta} = 0 \). Thus, the stress components in the flange become:

\[
\begin{align*}
\sigma_{xx} &= \sigma_{rr} \cos^2 \theta + \sigma_{\theta\theta} \sin^2 \theta \\
\sigma_{yy} &= \sigma_{rr} \sin^2 \theta + \sigma_{\theta\theta} \cos^2 \theta \\
\sigma_{xy} &= (\sigma_{rr} - \sigma_{\theta\theta}) \sin \theta \cos \theta 
\end{align*}
\]

(3-3)

Defining a yield function of \( F(\sigma_{xx}(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta}), \sigma_{yy}(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta}), \sigma_{xy}(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta})) = c \), assuming yielding occurs completely around the flange, and using the normality flow rule leads to:

\[
\frac{\partial F}{\partial \sigma_{rr}} d\sigma_{rr} + \frac{\partial F}{\partial \sigma_{\theta\theta}} d\sigma_{\theta\theta} + \frac{\partial F}{\partial \theta} d\theta = 0
\]

(3-4a)

or

\[
\frac{\partial F}{\partial \sigma_{rr}} d\sigma_{rr} + \frac{\partial F}{\partial \sigma_{\theta\theta}} d\sigma_{\theta\theta} = - \frac{\partial F}{\partial \theta} d\theta
\]

(3-4b)

where (using the relationship of \( \dot{\varepsilon}_{\theta\theta} = \dot{\lambda} \frac{\partial F}{\partial \sigma_{\theta\theta}} \))
\[
\frac{d\hat{F}}{d\theta} \, d\theta = \left( \frac{\partial F}{\partial \sigma_{xx}} + \frac{\partial F}{\partial \sigma_{yy}} + \frac{\partial F}{\partial \sigma_{xy}} \right) \frac{d\theta}{\lambda} = \left( \hat{\epsilon}_{xx} + \hat{\epsilon}_{yy} + 2\hat{\epsilon}_{xy} \right) \frac{d\theta}{\lambda} \]  

(3-4c)

\[
\frac{d\hat{F}}{d\sigma_{rr}} = \left( \frac{\partial F}{\partial \sigma_{xx}} + \frac{\partial F}{\partial \sigma_{yy}} + \frac{\partial F}{\partial \sigma_{xy}} \right) \frac{d\sigma_{rr}}{\lambda} = \left( \hat{\epsilon}_{xx} + \hat{\epsilon}_{yy} + 2\hat{\epsilon}_{xy} \right) \frac{d\sigma_{rr}}{\lambda} \]  

\[
\frac{d\hat{F}}{d\sigma_{\theta\theta}} = \left( \frac{\partial F}{\partial \sigma_{xx}} + \frac{\partial F}{\partial \sigma_{yy}} + \frac{\partial F}{\partial \sigma_{xy}} \right) \frac{d\sigma_{\theta\theta}}{\lambda} = \left( \hat{\epsilon}_{xx} + \hat{\epsilon}_{yy} + 2\hat{\epsilon}_{xy} \right) \frac{d\sigma_{\theta\theta}}{\lambda} \]  

Substituting Equation 3-4c into Equation 3-4b together with the strain transformations of

\[
\begin{align*}
\hat{\epsilon}_{rr} &= \hat{\epsilon}_{xx} \cos^2 \theta + \hat{\epsilon}_{yy} \sin^2 \theta + 2\hat{\epsilon}_{xy} \sin \theta \cos \theta \\
\hat{\epsilon}_{\theta\theta} &= \hat{\epsilon}_{xx} \sin^2 \theta + \hat{\epsilon}_{yy} \cos^2 \theta - 2\hat{\epsilon}_{xy} \sin \theta \cos \theta \\
\hat{\epsilon}_{\theta\theta} &= (\hat{\epsilon}_{yy} - \hat{\epsilon}_{xx}) \sin \theta \cos \theta + 2\hat{\epsilon}_{xy} \left( \sin^2 \theta - \cos^2 \theta \right)
\end{align*}
\]  

(3-5)

Equation 3-4b becomes

\[
\frac{2\hat{\epsilon}_{\theta\theta}(\sigma_{rr} - \sigma_{\theta\theta})}{\hat{\epsilon}_{\theta\theta} \sigma_{rr} + \hat{\epsilon}_{\theta\theta} \sigma_{\theta\theta}} = -1
\]  

(3-6)

Using the assumption of the simple shear condition \( \hat{\epsilon}_{\theta\theta} = -\hat{\epsilon}_{rr} \) (no thickness change) with the existence of \( \hat{\epsilon}_{\theta\theta} \), Equation 3-6 reduces to

\[
\frac{d(\sigma_{rr} - \sigma_{\theta\theta})}{\sigma_{rr} - \sigma_{\theta\theta}} = 2\frac{\hat{\epsilon}_{\theta\theta}}{\hat{\epsilon}_{\theta\theta}} \, d\theta = 2\Gamma \, d\theta
\]  

(3-7)

where the parameter \( \Gamma \) is that introduced by Barlat and Richmond (1987) to characterize the shear deformation occurring during uniaxial tension (or compression) in a given sheet direction. Equation 3-7 relates the variation of circumferential stress in various directions to the parameter \( \Gamma \).

In order to solve Equation 3-7 explicitly, the deformation in the flange is assumed along a direction of \( \theta \). As shown in Figure 3-5a, it is assumed that the radial tensile traction can be modeled from the yield stress directionality (and the compressive deformation is formulated from the r-value directionality in an uncoupled way). Then, the tensile and compressive contributions are superimposed. By assuming that the differences of the radial tensile stresses are attributed from yield stress directionality, the stress state at the innermost flange is applied for the entire flange area to model yield stress anisotropy. Then, \( \sigma_{rr} - \sigma_{\theta\theta} = \sigma_{(r)}^y \) by ignoring \( \sigma_{(r)} \). Then, Equation 3-7 further reduces to
\[
\frac{d\left(\sigma'_{(\theta)}\right)}{\sigma'_{(\theta)}} = 2\Gamma d\theta
\]  

(3-8a)

If the compressive contribution is mainly considered from yield stress directionality, Equation 3-7 reaches the following relationship (Barlat et al., 1991b):

\[
\frac{d\left(\sigma'_{(\theta+90)}\right)}{\sigma'_{(\theta+90)}} = 2\Gamma d\theta
\]  

(3-8b)

For the kinematics relation in Equation 3-8a, if \(v_r\), \(v_{\theta}\) and \(v_z\) represent components of the velocity vector at any point of the flange with respect to cylindrical coordinates (Figure 3-5a), the strain rate components can, in principle, be calculated and related to the stress by the constitutive law. However, the stress state at any point of the flange is unknown and non-uniform. Therefore, two additional assumptions are necessary to calculate the earing profile. Although \(v_{\theta}\) and \(v_z\) are not equal to zero, they are negligible when compared to the radial velocity. Additionally, it is assumed that the earing tendency is mainly imposed by the radial velocity of points located at the rim of the flange. Except at the very end of the drawing operation, these points are always part of the flange. Using the relationship between the velocity field and strain rates for each point on the flange, it is possible to show that (Barlat et al., 1991b)

\[
\frac{dv_r}{d\theta} = \frac{2\dot{\varepsilon}_{r\theta}}{\dot{\varepsilon}_{\theta\theta}} = 2\Gamma d\theta
\]  

(3-8c)

Combination of Equation 3-8c and Equation 3-8a gives

\[
\frac{dv_r(\theta)}{v_r(\theta)} = \frac{d\sigma_r'}{\sigma_r'}
\]  

(3-9a)

or

\[
d\left(\ln v_r(\theta)\right) = d\left(\ln \sigma_r'\right)
\]  

(3-9b)

Therefore, it appears that the curve that represents the normalized variation of the radial velocity of the rim of the flange as a function of angular position is the same as the one representing the normalized stresses variation. Integration of Equation 3-9b leads to

\[
\ln v_r(\theta) + C_1 = \ln \sigma_r' + C_2
\]  

(3-10a)
or

\[ C_1 - C_2 = \ln \sigma_\gamma - \ln \nu_r(\theta) \]  \hspace{1cm} (3-10b)

Boundary conditions are imposed to remove the constants C1 - C2 as

\[ C_1 - C_2 = \ln \sigma_{\text{ref}} - \ln \nu_{\text{ref}} \]  \hspace{1cm} (3-11)

By substituting Equation 3-11 into Equation 3-10b, the following relationship can be obtained

\[ \ln \frac{\nu_{\text{ref}}}{\nu_r(\theta)} = \ln \frac{\sigma_{\text{ref}}}{\sigma_\gamma} \]  \hspace{1cm} (3-12)

Next, consider the green part of Figure 3-6 for the elongation from the stress directionality. As a first approximation, the total cup height after elongation can be obtained by multiplying the punch speed \( v_p(=v_{\text{ref}}) \) by the time \( t_f \) that the outer edge of the flange needs to travel from the original position \( r = R_b \) to the punch radius. This leads to the following relationship between the decrement of blank radius and the time increment necessary for the decrement:

\[ h^{\text{ANI}}(\theta) = v_p t_f = v_p \left( \frac{R_b - R_c}{\nu_r(\theta)} \right) = \frac{v_{\text{ref}}}{\nu_r(\theta)} (R_b - R_c) \]  \hspace{1cm} (3-13)

In Equation 3-13 and later, superscript “ANI” means the contribution from stress directionality. Then, the definition of the radial strain is given as follows:

\[ \varepsilon_r^{\text{ANI}} = \ln \frac{h^{\text{ANI}}(\theta)}{R_b - R_c} = \ln \frac{v_{\text{ref}}}{\nu_r(\theta)} \]  \hspace{1cm} (3-14a)

From Equation 3-12

\[ \varepsilon_r^{\text{ANI}} = \ln \frac{\sigma_{\text{ref}}}{\sigma_\gamma} \]  \hspace{1cm} (3-14b)

By applying the simple shear condition, the circumferential and radial strains from the contribution of the yield stress can be written as

\[ \varepsilon_\theta^{\text{ANI}} = -\varepsilon_r^{\text{ANI}} = \ln \frac{\sigma_\gamma}{\sigma_{\text{ref}}} \]  \hspace{1cm} (3-15)
Equation 3-15 shows the contribution of stress anisotropy on the circumferential strain. There is no thickness change for Equation 3-15 due to the simple shear assumption. The procedure from Equation 3-2 through Equation 3-15 can be simply verified from a moment equilibrium by assuming no thickness change as

$$\sigma_{(\theta)}^t \cdot L_{(\theta)} = \sigma_{ref} \cdot L_{(ref)}$$  \hspace{1cm} (3-16a)$$

Then,

$$\varepsilon_{r}^{ANI} = \ln \left( \frac{L_{(\theta)}}{L_{(ref)}} \right) = \frac{\sigma_{ref}}{\sigma_{(\theta)}}$$  \hspace{1cm} (3-16b)$$

Equation 3-16b is the same with Equation 3-14b. The average and directional circumferential strains are defined in Equation 3-1 and Equation 3-15. By merging the two equations, the total circumferential strain can be defined as

$$\varepsilon_{\theta} = \varepsilon_{\theta}^{ISO} + \beta \varepsilon_{\theta}^{ANI} = \ln \left( \frac{R_{L}}{R} \right) + \beta \ln \left( \frac{\sigma_{(\theta)}}{\sigma_{ref}} \right) = \ln \left( \frac{R_{L}}{R} \left( \frac{\sigma_{(\theta)}}{\sigma_{ref}} \right)^{\beta} \right)$$  \hspace{1cm} (3-17)$$

In Equation 3-17, a deceleration factor, $\beta$, is introduced. When the stress mode at the inner most flange is applied for the entire flange (as shown in Figure 3-CC), $\beta = 1.0$. If the linear distribution of the radial tensile stress is assumed, $\beta = 0.5$. The recommended value of $\beta$ is $0.5 \leq \beta \leq 1$. For the isotropic materials, Equation 3-17 reduces to Equation 3-1.
3.3.1 HEIGHT CONSIDERING YIELD STRESS AND R-VALUE DIRECTIONALITIES

The normality rule for a circumferential strain is combined with Equation 3-17 as

$$\varepsilon_\theta = \lambda \frac{\partial F}{\partial \sigma_\theta} = \ln \left( \frac{R_c}{R} \left( \frac{\sigma_{(0)}}{\sigma_{\text{ref}}} \right)^\beta \right)$$

(3-18)

where $F$ is a yield function. Then, from Equation 3-18, the plastic parameter $\lambda$ is defined as

$$\lambda = \ln \left( \frac{R_c}{R} \left( \frac{\sigma_{(0)}}{\sigma_{\text{ref}}} \right)^\beta \right) \left/ \frac{\partial F}{\partial \sigma_\theta} \right.$$  

(3-19)

By using the normality rule for $\varepsilon_r$ together with Equation 3-19, the total radial strain becomes

$$\varepsilon_r = \lambda \frac{\partial F}{\partial \sigma_r} = \frac{\partial \sigma_r}{\partial F} \ln \left( \frac{R_c}{R} \left( \frac{\sigma_{(0)}}{\sigma_{\text{ref}}} \right)^\beta \right)$$

(3-20)

The quantity $\left( \frac{\partial F}{\partial \sigma_r} / \frac{\partial F}{\partial \sigma_\theta} \right)$ has the following ratio by using the normality rule:

$$\left( \frac{\partial F}{\partial \sigma_r} / \frac{\partial F}{\partial \sigma_\theta} \right) = \left( \lambda \frac{\partial F}{\partial \sigma_r} / \lambda \frac{\partial F}{\partial \sigma_\theta} \right) = \left( \frac{\varepsilon_r}{\varepsilon_\theta} \right)$$

(3-21)

The ratio varies in the flange. However, in order to derive an explicit formula, it is assumed that the deformation at the rim dominates the earing. As shown in Figure 3-4, the rim behavior in the direction defined by $\theta$ is controlled by the property of the material in compression in the direction defined by $\theta + 90$. Assuming that, for a given direction, uniaxial tension and compression lead to identical $r$-values, these can be expressed as a function of the strains at the rim:

$$r_{\theta, 90} = \frac{\varepsilon_r}{\varepsilon_\theta} = -\frac{\varepsilon_r}{\varepsilon_r + \varepsilon_\theta}$$

(3-22)

Here, the subscripts $r$, $\theta$, $t$ correspond to the radial, circumferential, and thickness directions, respectively. It is assumed that the $r$-values are constant and the strains in Equation 3-22 are plastic strains. From Equation 3-21, the following relationship is obtained:
\[ \varepsilon_\theta : \varepsilon_r = -(r_{\theta+90} + 1) : r_{\theta+90} \]  
\[ \varepsilon_r : \varepsilon_\theta = r_{\theta+90} : 1 \]  
(3-23a)
(3-23b)

Then
\[ \varepsilon_\theta : \varepsilon_r : \varepsilon_\theta \bigg|_{a} = -(r_{\theta+90} + 1) : r_{\theta+90} : 1 \]  
(3-23c)

In order to model r-value anisotropy, the quantity of \( \left( \frac{\partial F}{\partial \sigma_r} / \frac{\partial F}{\partial \sigma_\theta} \right) \) can be defined as a constant term by using the relationship of Equation 23-a at the rim, i.e.,
\[ \left( \frac{\partial F}{\partial \sigma_r} / \frac{\partial F}{\partial \sigma_\theta} \right)_{at R=R_c} = \frac{\varepsilon_r}{\varepsilon_\theta} \left|_{at R=R_c} = - \frac{r_{\theta+90}}{1+r_{\theta+90}} \right. \]  
(3-24)

Then, substituting Equation 3-24 into Equation 3-20, the total radial strain becomes
\[ \varepsilon_r = \left( \frac{r_{\theta+90}}{1+r_{\theta+90}} \right) \ln \left( \frac{R_c}{R} \right) \left( \frac{\sigma^\theta}{\sigma_{\text{ref}}} \right)^\beta \]  
\[ \left( \frac{1}{1+r_{\theta+90}} \right) \]  
(3-25)

If \( \sigma^\theta / \sigma_{\text{ref}} = 1 \), Equation 3-25 reduces to Yoon et al. (2006).

The thickness strain can be also derived from the relationship of Equation (3-23b) (the incompressibility condition) as
\[ \varepsilon_t = - \left( \frac{\varepsilon_r}{r_{\theta+90}} \right) \ln \left( \frac{R_c}{R} \right) \left( \frac{\sigma^\theta}{\sigma_{\text{ref}}} \right)^\beta \]  
(3-26)

Equation 3-26 is valid only for the region where \( \varepsilon_t > 0 \).

Finally, the total height of a cup can be obtained from the logarithmic integral of Equation 3-25 as follows:
\[ H^\text{out}(\theta) = t_0 + r_c \int_{R_c}^{R_c} \exp(\varepsilon_t) dR \]  
(3-27a)

where
In Equation 3-27b, the mathematical relationship of \( \exp(\ln(a)) = a \) is applied. The integrated form of Equation 3-27a can be expressed as

\[
H^{\text{cup}}(\theta) = t_0 + r_c + \frac{R_b}{A_{\theta+90} + 1} \left( (d)^{A_{\theta+90}} - \frac{1}{d} \right) (B_\theta)^{A_{\theta+90}} \tag{3-28a}
\]

where

\[
A_{\theta+90} = \frac{r_{\theta+90}}{1 + r_{\theta+90}}
\]

\[
B_\theta = \left( \frac{\sigma_{\text{ref}}}{\sigma_{(\theta)}} \right)^\beta
\]

\[
d = \frac{R_b}{R_c}
\]

\[
0.5 \leq \beta \leq 1
\]

\[
\sigma_{\text{ref}} = \left( \int_0^{2\pi} \sigma_{(\theta)} d\theta \right) / 2\pi
\]

for the quarter symmetry with every 15° of data

\[
\sigma_{\text{ref}} = \left(1 / 12\right) \left( \sigma_0 + 2 \left( \sigma_{15} + \sigma_{90} + \sigma_{45} + \sigma_{60} + \sigma_{75} + \sigma_{90} \right) \right) \tag{3-28c}
\]

It is further interesting to derive the specific contributions from r-value and yield stress directionalities to the cup height. For this purpose, an isotropic contribution can be derived with \( r_{\theta+90} = 1 \) and \( \sigma_{\text{ref}} / \sigma_{(\theta)} = 1 \) as

\[
H^{\text{iso}} = t_0 + r_c + \frac{R_b}{1.5} \left( (d)^{0.5} - \frac{1}{d} \right) \tag{3-29}
\]

A correction factor can be applied to the isotropic cup height using output from a finite element simulation and can be added to the overall cup height calculations. The cup height from r-value directionality is obtained by excluding yield stress contribution with the assumption of \( \sigma_{\text{ref}} / \sigma_{(\theta)} = 1 \), i.e.,
\[ H^{\text{value}}(\theta) = t_0 + r_e + \frac{R_b}{A_{\theta=90}} \left( (d)^{\theta=90} - \frac{1}{d} \right) \]  \hspace{1cm} (3-30)

In a similar approach, the cup height from yield stress directionality is obtained by using \( r_{\theta=90} = 1 \), i.e.,

\[ H^{\text{stress}}(\theta) = t_0 + r_e + \frac{R_b}{1.5} \left( (d)^{0.5} - \frac{1}{d} \right)(B_y)^{0.5} \]  \hspace{1cm} (3-31)

Then, the cup height contributions from r-value and yield stress directionalities can be derived as

\[ \Delta H^{\text{value}}(\theta) = H^{\text{value}}(\theta) - H^{\text{iso}} \]  \hspace{1cm} (3-32a)

and

\[ \Delta H^{\text{stress}}(\theta) = H^{\text{stress}}(\theta) - H^{\text{iso}} \]  \hspace{1cm} (3-32b)

Equation 3-32 is useful to understand the explicit contributions from r-value and yield stress directionalities to earing.
Figure 3-5. Deformation of an element on the flange: (a) stress state at various locations on the flange; (b) stress states on the yield surface.
Figure 3-6. Initial blank and drawn cup: the deformation zone for analytical solution is denoted by green line.

3.3.2 PREDICTION OF R-VALUES AND YIELD STRESS DIRECTIONALITIES

Considering that compressive and shearing modes are independent, Equation 3-28a can be utilized to determine \( r_{\theta,90} \) and \( \sigma_{\theta} \) as a function of \( H_b \) and \( R_b \). The following process can be used:

Assuming \( \beta = 0 \), then Equation 3-28a becomes

\[
H^{(a)}(\theta) = t_o + r_c + \frac{R_b}{A_{\theta,90}} + 1 \left( (d)^{(\theta,90)} - \frac{1}{d} \right)
\]  

(3-33)

Rearranging the terms gives

\[
\frac{H^{(a)}(\theta) - (t_o + r_c)}{R_b} = \frac{(d)^{(\theta,90)} - (1/d)}{A_{\theta,90} + 1}
\]  

(3-34)

which can be solved iteratively for \( A_{\theta,90} \). Using Equation 3-28b, \( r_{\theta,90} \) becomes
\[ r_{\theta+90} = \frac{A_{\theta+90}}{1-A_{\theta+90}} \]  \hspace{1cm} (3-35)

Next, assuming \( \beta = 1 \), Equation 3-28 becomes

\[ H^{\text{cap}}(\theta) = t_0 + r_c + \frac{R_b}{A_{\theta+90} + 1} \left( \frac{\sigma_{\text{ref}}}{\sigma_{\theta}} \right)^{A_{\theta+90}} \]  \hspace{1cm} (3-36)

In Equation 3-36, all variables are known except for \( \left( \frac{\sigma_{\text{ref}}}{\sigma_{\theta}} \right) \).

\[ \left( \frac{\sigma_{\text{ref}}}{\sigma_{\theta}} \right) = \left[ \left( H^{\text{cap}}(\theta) - t_0 - r_c \right) \left( \frac{A_{\theta+90} + 1}{R_b} \right) \right] \left( \frac{\sigma^{A_{\theta+90}}}{\left( 1 \right)} \right) \]  \hspace{1cm} (3-37)

Equation 3-37 can be used to solve for the direction yield stress values \( \sigma_{\theta} \).

### 3.3.3 VERIFICATION - APPLICATION TO CUP DRAWING – AA2090-T3

For the verification purpose, a cup drawing example considering the AA2090-T3 sheets is used. The tool dimensions used for the cup drawing process using the AA2090-T3 sheet are provided in Table 3-2. The yield stress and r-value data for the material are given in Table 3-3.

| Table 3-2. Dimensions of tools, cup, and the blank (Yoon et al., 2006) |
|---|---|---|---|---|---|---|---|
| \( t_0 \) (mm) | \( r_c \) (mm) | \( D_p \) (mm) | \( D_d \) (mm) | \( D_b \) (mm) | \( R_c \) (mm) | \( R_b \) (mm) | \( d = R_b / R_c \) |
| 1.60 | 12.70 | 97.46 | 101.48 | 158.76 | 50.74 | 79.380 | 1.5350 |

| Table 3-3. r-values and Normalized yield stresses for AA2090-T3 (Yoon et al., 2006) |
|---|---|---|---|---|---|---|
| \( r_{00} \) | \( r_{15} \) | \( r_{30} \) | \( r_{45} \) | \( r_{60} \) | \( r_{75} \) | \( r_{90} \) |
| 0.2115 | 0.3269 | 0.6923 | 1.5769 | 1.0385 | 0.5384 | 0.6923 |
| \( \sigma_0 / \sigma_0 \) | \( \sigma_{15} / \sigma_0 \) | \( \sigma_{30} / \sigma_0 \) | \( \sigma_{45} / \sigma_0 \) | \( \sigma_{60} / \sigma_0 \) | \( \sigma_{75} / \sigma_0 \) | \( \sigma_{90} / \sigma_0 \) |
| 0.0000 | 0.9605 | 0.9102 | 0.8114 | 0.8096 | 0.8815 | 0.9102 |
The equations from Section 3.3.1 are easily implemented into an Excel spreadsheet. The earing profile can be predicted using the analytical Equation 3-28a, which involves the r values and yield stresses as well as the geometrical features of the cup drawing test. The predicted earing profile is shown in Figure 3-7. The contributions from the r-value and yield stress effects are plotted in Figure 3-8.

![Cup Height Profile](image)

**Figure 3-7.** Analytical earing profile for AA2090-T3.

![Cup Height Contribution](image)

**Figure 3-8.** Contribution of r-value and yield stress anisotropy to earing for AA2090-T3.

### 3.3.4 ANALYTICAL EARING PREDICTION – AA3104-THIN

Dimensions required to calculate the cup height are listed in Table 3-4. The normalized yield stress and r-value of AA3104-Thin material measured every 15° from the rolling direction are also shown in Table 3-5. The total cup height calculated from Equation 3-28a based on the material data listed in Table 3-5 are compared with the experiment result in Figure 3-9 and they show good agreement in the earing profile.
However, the result obtained shows a major discrepancy in height, which was reasonably corrected after imposing a height correction factor from the finite element calculations for an isotropic cup. The contributions of the r-value and yield stress anisotropy are provided in Figure 3-10, which confirms that earing profile is mainly contributed by the r-value anisotropy.

| Table 3-4. Dimensions of tools, cup, and the blank – IT3 Cup |
|-----------------|----------------|-----------------|----------------|-----------------|-----------------|-----------------|
| $t_0$ (mm) | $r_c$ (mm) | $D_p$ (mm) | $D_d$ (mm) | $D_b$ (mm) | $R_c$ (mm) | $R_b$ (mm) | $d = R_b / R_c$ |
| 0.274 | 2.286 | 45.72 | 46.736 | 76.124 | 22.86 | 38.062 | 1.665 |

| Table 3-5. r-values and Normalized yield stresses for AA3104-Thin |
|------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $r_{00}$ | $r_{15}$ | $r_{30}$ | $r_{45}$ | $r_{60}$ | $r_{75}$ | $r_{90}$ |
| 0.403 | 0.413 | 0.557 | 0.888 | 0.927 | 1.060 | 1.125 |
| $\sigma_0 / \sigma_0$ | $\sigma_{15} / \sigma_0$ | $\sigma_{30} / \sigma_0$ | $\sigma_{45} / \sigma_0$ | $\sigma_{60} / \sigma_0$ | $\sigma_{75} / \sigma_0$ | $\sigma_{90} / \sigma_0$ |
| 1.0000 | 1.0165 | 1.0207 | 1.0187 | 1.0439 | 1.0589 | 1.0482 |

**Figure 3-9.** Earing profile for AA3104-Thin using analytical model.
3.3.5 ANALYTICAL EARING PREDICTION – AA3104-THICK

Dimensions required to calculate the cup height for the AA3104-Thick material are listed in Table 3-6. The normalized yield stress and r-value measured every 15° from the rolling direction are also shown in Table 3-7. The total cup height calculated from Equation 3-28a based on the material data listed in Table 3-7 are compared with the experiment result in Figure 3-11. For this case, the earing prediction is unacceptable showing ears at both 15° and 45°. There appears to be no contribution from the yield stress anisotropy, as expected, from the experimental yield stress data. Figure 3-12 shows little to no contribution from the directional yield stress to the earing profile.

![Figure 3-10. Contribution of r-value and yield stress anisotropy to earing for AA3104-Thin.](image)

| Table 3-6. Dimensions of tools, cup, and the blank – IT3 Cup – AA3104-Thick |
|---|---|---|---|---|---|---|---|
| $t_0$ (mm) | $r_c$ (mm) | $D_p$ (mm) | $D_d$ (mm) | $D_b$ (mm) | $R_c$ (mm) | $R_b$ (mm) | $d = R_b / R_c$ |
| 0.475 | 2.286 | 45.72 | 47.244 | 76.124 | 22.86 | 38.062 | 1.665 |

| Table 3-7. r-values and Normalized yield stresses for AA3104-Thick |
|---|---|---|---|---|---|---|---|---|
| $r_{00}$ | $r_{15}$ | $r_{30}$ | $r_{45}$ | $r_{60}$ | $r_{75}$ | $r_{90}$ |
| 0.394 | 0.460 | 0.688 | 0.865 | 0.899 | 1.047 | 1.004 |
| $\sigma_0 / \sigma_0$ | $\sigma_{15} / \sigma_0$ | $\sigma_{30} / \sigma_0$ | $\sigma_{45} / \sigma_0$ | $\sigma_{60} / \sigma_0$ | $\sigma_{75} / \sigma_0$ | $\sigma_{90} / \sigma_0$ |
| 1.0000 | 0.9994 | 0.9985 | 0.9988 | 1.0083 | 1.0129 | 1.0230 |
3.4 PREDICTION OF EARING PROFILE – FINITE ELEMENT MODEL

The numerical simulations of the cup drawing process is performed using LS-DYNA, an advanced multi-physics simulation software package developed to solve highly nonlinear transient dynamic finite element analysis using explicit time integration. The deformable sheet, or blank, is modeled using 13680 shell elements (type=16) with full iterative plasticity. The element uses the Bathe-Dvorkin transverse shear treatment to eliminate w-mode hourglassing. Other modes of hourglassing are eliminated in the shell by virtue of the selective reduced (S/R) integration. The S/R integration here means that full integration (4 in-plane integration points) is used except for purposes of calculating transverse shear. To eliminate transverse shear locking, only 1 in-plane integration point is considered in calculating transverse shear.
The blank zone in which earing is measured is discretized with a fine structured mesh to accurately reproduce the profile. The forming tools are considered rigid bodies with its surface modelled by discrete shell elements. The friction between the blank and the forming tools is described by the classical Coulomb’s law. The friction coefficient between sheet and tools is assumed to be constant and taken as $\mu=0.03$. The blank-holder force of 8.9 kN is applied through the use of non-linear springs attached from a rigid plate to the lower pressure pad. The cup is drawn completely by displacement of 22 mm of the upper die. The punch is considered completely rigid in the model.

The progression of earing in the finite element simulation is depicted in Figure 3-13. The cup profile showing the thickness fringe in the fully drawn condition in provided in Figure 3-14 and shows the maximum thickening at approximately $\pm75^\circ$ from the rolling direction. A comparison of the experimental and predicted punch forming loads in shown in Figure 3-15 and shows excellent agreement.
Figure 3-13. Earing progression for AA3104-Thin using YLD2004-18P model.
Figure 3-14. Cup profile showing thickness fringe for AA3104-Thin using YLD2004-18P model.

Figure 3-15. Comparison of experimental and prediction punch force using for AA3104-Thin.
3.4.1 EFFECT OF MATERIAL MODEL - AA3104-THIN

Three material models are examined for their impact on earing in this section: Hill48S (stress-based), Barlat Yld2000-2D, and Barlat Yld2004-18P models. The material coefficients are listed in Table 2-10 (Hill48S), Table 2-11 (YLD2000-2D), and Table 2-12 (YLD2004-18P). The results are shown in Figure 3-16.

As can be seen in the figure, the earing profiles predicted from the three material models are noticeably different. The amplitude of the earing profile based on Hill’s 1948 yield function is much larger than the experimental profile. This is consistent with the amplitude of the normalized yield stress anisotropy of Figure 2-18, which is also overestimated, since it was discussed in the work of Yoon et al. (2011) that the stress anisotropy dominates the earing profile amplitude while the r-value anisotropy dominates the earing tendency. The Yld2000-2D model shows reasonable agreement with experimental data in the earing amplitude, but only predicts 4 ears. The YLD2004-18P model shows excellent agreement in both the earing amplitude and earing profile. As can be observed in the figure, the small ears around 0° and 180° are successfully predicted.

![Figure 3-16. Predicted earing for AA3104-Thin using different material models.](image-url)
3.4.2 EFFECT OF FRICTION - AA3104-THIN
The impact of friction on earing in the cup drawing process is shown in Figure 3-17. It can be seen that as the coefficient of friction increases, the depth of the valley at 90° deceases. This is a result of the increased contact pressure at that orientation due to localized thickening from the drawing process. Also, the magnitude of the ear at 90° decreases with increased friction as a result of increased resistance to metal flow.

![Figure 3-17](image)

*Figure 3-17. Predicted earing for various coefficients of friction.*

3.4.3 EFFECT OF HARDENING LAW - AA3104-THIN
The effect of the hardening law used with YLD2004-18P is shown in Figure 3-18, where both the Swift and Voce hardening models are compared. The Figure shows a decrease in the earing amplitude at 45° using the Voce model. This is most likely due to the difference in the directional yield stress values calculated for the two models.
3.4.4 EFFECT OF STRESS RATIO - AA3104-THIN

The impact of the method used to calculate the stress ratios with YLD2004-18P is shown in Figure 3-19. Using the Swift model, stress ratios were calculated based on equivalent strain energies of 0.5, 10, and 40. A strain energy of 0.5 is nearly equivalent to using the initial yield stress method. The strain energy of 10 was based on the strain at failure method. The earing tendencies are the same for each of the models. The earing magnitudes are slightly different at both 45° and 90°. All of the predictions are considered quite acceptable. An equivalent strain energy level of 40 has the highest strain hardening and produces results closest to the experimental data.

Figure 3-18. Comparison of earing profile for AA3104-Thin using Swift and Voce models.
3.4.5 EFFECT OF MATERIAL MODEL - AA3104-THICK

Three material models are examined for their impact on earing in this section: Hill48S, Barlat Yld2000-2D, and Barlat Yld2004-18P models. The material coefficients are listed in Table 2-10 (Hill48S), Table 2-11 (Yld2000-2D), and Table 2-12 (Yld2004-18P). The results are shown in Figure 3-20.

As can be seen in the figure, the earing profiles predicted from the three material models are noticeably different. As shown in the figure, the amplitude of the earing profile based on Hill’s 1948 yield function is much larger than the experimental profile. The Yld2000-2D model underestimates the earing magnitude and only predicts 4 ears. The Yld2004-18P model also underestimates the earing amplitude. It also predicts 8 ears, but in the wrong orientation. The data suggests that the directional yield stress data from the experiments will not predict the amplitude of the earing profile. Since the r-values for the AA3104-Thick are similar to the r-values for the AA3104-Thin material, a model was built using the AA3104-Thick r-values and the AA3104-Thin stress ratios. The results are plotted in Figure 3-21 which suggests the r-values are accurate (correct trend), but the stress ratios may be inaccurate (earing amplitude).
**Figure 3-20.** Predicted earing for AA3104-Thick using different material models.

**Figure 3-21.** Comparison of earing profile for AA3104-Thick using Swift models with adjusted stress ratio.
3.5 SUMMARY

Earing results have been documented for cups drawn from AA3104-Thin and AA3104-Thick materials. Both materials display 8 ears after drawing, dominated by the 45° earing with small ears at both 0-180° and 90-270°. The experimental data suggests that an advanced material model like YLD2004-18P is required to accurately predict the earing profile. An analytical model was developed and implemented in EXCEL in order to predict earing based on experimental r-values and directional yield stress values. The model gives reasonable estimates of the earing profile in only seconds of CPU time, making it very useful for sensitivity studies. In addition, the model is capable of estimating the r-values and directional yield stress values given an earing profile. The model also provides valuable data on the level of contributions of the r-values and directional yield stress values to the overall earing profile. Results show the r-values control the earing tendencies and the stress ratios control the earing magnitude. A major drawback of the current derivation is that it does not account for the effects of the blank holding force and friction, however, the current derivation still provides a first approximation of the anisotropic effects.

Finite element simulations were also developed to study the influence of material model, friction, hardening model, and method of calculating the stress ratios on the predicted earing profile. It was shown that only the YLD2004-18P model was capable of accurately predicting the earing profile, as both the Hill48 and Barlat YLD2000-2D models are only capable of predicting 4 ears. Increased localized blank holder pressure due to non-uniform thickening during the drawing process was also observed. An anisotropic friction model that accounts for the contact conditions in the rolling versus the transverse sheet texture may provide improved predictions. The models also showed the significance of accurately measuring the r-values and method selected for calculating the stress ratios (due anisotropic hardening). Excellent agreement with the experimental data on the cup earing was achieved using the AA3104-Thin material data.
3.6 REFERENCES


4.1 LITERATURE REVIEW – WRINKLING

Wrinkling is a type of localized buckling of sheet metal that results from an instability under compressive stresses. The initiation and growth of wrinkles are influenced by many factors such as mechanical properties of the sheet material including the yield stress, strain hardening, strain hardening rate, anisotropy, current stress state, tooling geometry, contact conditions including friction, and other process boundary conditions (Kim et al., 2003a). The accurate prediction of wrinkling is extremely important in sheet metal forming processes in order to reduce the expensive and time consuming “trial-and-error” procedures, as well as improving the quality of products. In most problems, a non-linear elastic material model is used for analysis, and complex contact conditions are not incorporated. In sheet metal forming wrinkling problems, however, an elastic-plastic material model must be used since most wrinkling occurrences in these processes take place in the plastic region, and complex contact conditions should be considered.

Initial efforts to study wrinkling focused on analytical methods and treated the phenomenon as a bifurcation problem. The buckling of a column or a compressed circular plate, and the wrinkling of a deep drawn cup are typical examples of bifurcation problems. The development of a general theory of stability of structural systems initiated with the fundamental work of Koiter (1945). The concept of Koiter’s work is known as the perturbation method. Shanley (1947) first showed theoretically that the buckling load of a centrally compressed short column coincides with the tangent modulus. Geckeler (1952) produced a mathematical analysis and two useful expressions for the condition where no blankholder was used, namely the critical stress at which buckling occurs, and the number of waves or lobes into which the flange
buckles. Senior (1956) derived flange wrinkling criterion from energy considerations and a one-dimensional model that considered the effect of a spring-loaded blank holder at a constant force. The lower and upper limits of the critical buckling stress and wave number were given for the cases without a binder force. This one-dimensional model is limited to the cases where the flange width is small compared with the blank radius.

Hill’s bifurcation and uniqueness theory (Hill, 1958) initiated the general analytical study of plastic wrinkling. Based on Hill’s theory, Hutchinson (1974) presented a bifurcation theory for structures in the plastic region using the Donnell-Mushtari-Vlasov (DMV) theory. Hutchinson and Neale (1985) used this theory to study the bifurcation phenomenon of doubly curved sheet metal by using shallow shell theory. However, the investigation was limited to regions of the sheet which were free of any surface contact, and therefore, the effect of the binder was not included. Triantafyllidis and Needleman (1980) studied an annular plate by modeling the binder as an elastic foundation and investigated the effect of binder stiffness on the critical buckling stress and the wave number. Their results were found to compare favorably with some previous empirical models for the cases of no binder constraint, but no comparison was given in the cases with normal constraint. Based on a two-dimensional buckling model of an elastic-plastic annular plate resting on an elastic foundation, Yu and Johnson (1982) used the energy method to determine the critical conditions and also quantitatively investigated the effects of a binder on the critical stress and wave number for isotropic, rigid-plastic materials.

The implementation of the above studies into real forming analyses was hampered by the lack of clear definition on the relationship between the normal stiffness of the elastic foundation and the parameters used in forming process, i.e. binder force/pressure. A similar energy method was utilized in Yossifon and Tirosh (1984, 1985) to predict critical fluid pressure for preventing plastic buckling during hydroforming processes. Triantafyllidis (1985) numerically studied the puckering instability problem in the hemispherical cup test based on a proposed bifurcation criterion using the phenomenological corner theory instead of J2 theory. The effect of geometry and material properties on the onset of non-axisymmetric plastic instability was also investigated. Fatnassi et al. (1985) carried out theoretical investigations to predict the non-axisymmetric buckling in the throat of circular elastic-plastic tubes subjected to a nosising operation along a frictionless conical die. The buckling point and associated modes are determined by Hill’s bifurcation theory in conjunction with a non-axisymmetric buckling mode. Tuğcu (1991a) used a bifurcation-of-equilibrium approach to investigate the buckling of simply supported elastic-plastic plates with infinite curvatures under biaxial loading, including small amounts of shear.
stresses. Tuğcu (1991b) also used the wrinkling theory to study the discrepancy between the predictions of flow and deformation theories to the wrinkling behavior of a flat plate. Wang et al. (1994) used a similar approach to study wall wrinkling for an anisotropic shell with compound curvatures and applied the criterion to axisymmetric shrink flanging.

Cao and Boyce (1997) developed a buckling criterion for a rectangular plate subjected to edge compression and lateral constraints imposed by rigid binders using a combination of finite element analysis and energy conservation law. The wrinkling criterion defines the buckling mode/wavelength and the buckling stress as a function of applied lateral constraint for the cases of both elastic and elastic-plastic plates. Cao and Wang (2000) developed an analytical model for the onset of flange wrinkling under normal constraints and transverse tension. A wrinkling limit diagram for anisotropic sheet metals subjected to biaxial plane-stress was proposed by Kim and Son (2000) using bifurcation analysis for thin shells based on the DMV theory. Chu and Xu (2001) analyzed the onset of flange wrinkling of a deep drawn cup as an elastoplastic bifurcation problem. The flange was modeled as an elastoplastic annular plate subject to axisymmetric radial tension along its inner edge. A closed-form solution for the critical drawing stress was developed based on an assumed nonlinear plastic stress field and the deformation theory of plasticity. The effects of flange width, drawing ratios, material properties, strain hardening on the onset of wrinkling were investigated. Shafaat et al. (2011) analyzed the onset of wall wrinkling for galvanized steel sheets and investigated the effect of different yield criteria. They found that when the Hill’s (1948) yield function was used, the analytical critical cup height was in good agreement with the experimental results. The effect of plastic anisotropy and lubrication on the wrinkling behaviour in the spherical cup deep drawing process was investigated by Anarestani et al. (2011) using the bifurcation theory incorporated in an incremental finite element analysis. The results show that the plastic buckling tendency decreases with increasing plastic anisotropy.

It is difficult to analytically solve wrinkling initiation and growth while considering all the contributing factors because the effects of the features are very complex and studies of wrinkling behavior show a wide scattering of data for small deviations in factors, as is common in instability phenomena, thus limiting the analytical approaches to relatively simple problems. Owing to these difficulties, the study of wrinkling have been carried out on a case by case basis for a given process, and a generalized wrinkling criterion, which can be used effectively for various processes, has not been proposed. Today, wrinkling is mainly studied using the finite element method, which provides detailed information concerning wrinkling behaviour of sheet metals in complex industrial problems.
Unlike the analytical approach, numerical simulations using the Finite Element Method (FEM) can deal with a variety of more complicated forming conditions involving friction effects, tooling effects, and process effects. Two types of wrinkling analyses are performed with the finite element method: bifurcation analysis of a structure without imperfections and non-bifurcation analysis, which assumes an initial imperfection or perturbation force due to load eccentricities. Because the finite element analyses of sheet metal forming processes involve strong nonlinearities in geometry, material, and contact, convergence problems using implicit solution techniques are frequently observed.

Non-bifurcation analyses sometimes lead to reasonable results since all real structures have inherent imperfections, such as material non-uniformity or geometric irregularities. Thus, most wrinkling analyses have been carried out using non-bifurcation analysis (Chan, 1993; Cao and Boyce, 1995). However, the results obtained by non-bifurcation analysis are sensitive to the amplitude of the initial imperfections, which are chosen arbitrarily. A kind of non-bifurcation procedure can be also obtained, with good results, using explicit solution algorithms in the FEM (Dick, (2002), Dick and Yoon, 2008)), thus benefiting from intrinsic round-up errors associated with this procedure as defined in Magalhaes Correia and Ferron (2004). These numerical errors can act as the imperfections in the conventional non-bifurcation procedure. Additionally, Kawka et al. (2001) observed high mesh sensitivity in explicit analyses, typical on non-bifurcation procedures. Neto et al. (2015) reported the punch force evolution and the shape of the formed wrinkles in tapered cup drawing are very sensitive to the mesh size in the blank, and the selection of the appropriate yield criterion to model the material anisotropy allows for accurate prediction of the shape, frequency, and amplitude of the wrinkles.

Bifurcation analysis, on the other hand, assumes no induced imperfections at the model level (Wang and Lee (1993) and Wang et al. (1994)), thus benefiting from no need for the introduction of artificial empirical imperfections. In order to capture wrinkling initiation, energy or geometry-based wrinkling indicators are needed to locate the zones in the model that are prone to the appearance of wrinkling. This procedure has been used in conjunction with implicit solution techniques in FEM. In the bifurcation problem, the stiffness matrix of linearized finite element equation becomes singular at a bifurcation point and the solution procedure by the Newton–Raphson method cannot be carried out to completion. Riks (1979) proposed the continuation method by which the post-bifurcation analysis can be carried out along the secondary solution path and implemented in the buckling analysis of elastic shell structure. The bifurcation point and the post-bifurcation behavior can be analyzed by the continuation method. This approach allows users to analyze the wrinkling behavior of sheet metal in a more exact and rigorous way,

In this work, non-bifurcation analyses based on the explicit finite element method are carried out for simplification purposes. Imperfections are naturally imposed with material anisotropy. Effects of material characteristics including the yield stress, strain hardening, strain hardening rate, and anisotropy on wrinkle formation are investigated. The formability is also evaluated by monitoring both the wrinkling magnitude and forming loads during the cup drawing process.

### 4.2 CUP WRINKLING - EXPERIMENT

Following the beverage can cup drawing operation, the cups are transferred and fed into a bodymaker. The cup is held in place by a redraw sleeve in front of a moving punch, which first pushes the cup through the redraw ring to reduce its diameter to the punch diameter whilst retaining the sheet thickness. The punch then forces the redrawn cup through a series of three ironing rings each with decreasing diameter. This reduces the thickness of the metal (wall ironing) to about one-third the original thickness. At the end of the stroke the can base is formed when the punch pushes the drawn and ironed can against the doming die. The entire punch stroke of the body-maker – the so called “redraw-and-ironing” operation – is accomplished at up to 450 strokes per minute.

Advanced beverage can manufacturing technology and cost control efforts have resulted in a consistent reduction of the net metal cost and can weight and which has led to an increase in the likelihood of wrinkling during can production. A simplified cup drawing experiment was developed to emulate the wrinkling that occurs during the cup redrawing process in can manufacturing and was used as a benchmark for Numisheet2014 – see Dick et al. (2013), Dick and Yoon (2015). The setup can easily be used to study the influence of materials, process operating conditions, including lubrication, and tooling geometry on wrinkling. The objective of this study is to investigate the effects of the material model, tooling alignment, and tooling geometry on the cup wrinkling (puckering) behavior.
The cup drawing operation involves three axisymmetric forming tools (die, blank-holder and punch), which are schematically illustrated in Figure 4-1, as well as their key dimensions in millimetres shown in Table 4-1. The circular blank of the AA3104 aluminium alloys with 64.77 mm diameter is utilized. The deep drawing of the cylindrical aluminium cup is carried out at room temperature in a hydraulic press with maximum capacity of 5 tonnes. The press is equipped with a monitoring system (Keithley), which enables the acquisition of the punch force evolution as a function of its stroke and the blank-holder force. The experimental tests are run in groups of 5 for each process condition in order to investigate the reproducibility. The drawing process is performed with a constant punch travel speed of 140 mm/s and a constant blank-holder force of 8.9 kN. The cup is drawn completely using a punch travel of 20 mm. The gap between the blank-holder and the die is not restricted by any rigid stop or standoff during the drawing process. All forming tools are made of AISI A2 steel with 58–60 of Rockwell C hardness and surface roughness about 2–4 μm (finish working surfaces). An oil lubricant is applied to both sides of the blank to reduce the frictional forces between the forming tools and the blank sheet during the experiments.

Figure 4-1. Schematic of the forming tools used in the drawing process with detailed view of the punch geometry.
The wrinkling behaviour during the drawing process is defined in this study by the wrinkle amplitude and frequency developed along the unsupported side-wall area of the cup. The wrinkle contours are measured in the outer surface of the drawn cup using a coordinate measuring machine (CMM). The geometrical measurements are performed at several circumferential paths (layers) with increments of 0.5 mm along the z-direction (negative) from the top surface of the drawn cup. The radial distance of the points is calculated with respect to the cup axis, which is evaluated along the circumferential direction specified by the rolling direction (see Figure 4-3). The processed CMM scan data provides the amplitude and frequency of the wrinkle pattern. Only the results for the scan at z=4.5 mm are reported. The final geometry of three typical cylindrical aluminium drawn cups from the AA3104-Thin material is presented in Figure 4-4, highlighting the development of wrinkles in the conical side-wall of the drawn cup. The typical wrinkle frequency in the experimental cups is characterized by 12 waves.
Figure 4-3. Coordinate system of the cup used for wrinkle description.
Figure 4-4. Wrinkle profile for three typical cups using AA3104-Thin material.
4.3 CUP WRINKLING - FINITE ELEMENT MODEL

The numerical simulations of the cup drawing process are performed using LS-DYNA, an advanced multiphysics simulation software package developed to solve highly nonlinear transient dynamic finite element analysis using explicit time integration. The deformable sheet, or blank, is modeled using 57540 shell elements (type=10 - Belytschko-Wong-Chiang formulation) with full iterative plasticity. The blank zone in which wrinkling is expected to occur is discretized with a fine structured mesh to accurately reproduce the wrinkling pattern. The forming tools are considered as rigid bodies with their surface modelled by discrete shell elements. The friction between the blank and the forming tools is described by the classical Coulomb’s law. The friction coefficient between sheet and tools is assumed to be constant and taken as $\mu=0.03$. The blank-holder force of 8.9 kN is applied through the use of non-linear springs attached from a rigid plate to the lower pressure pad. The cup is drawn completely by displacement of 20 mm of the upper die. In this thesis, the finite element model does not contain any initial imperfection to induce wrinkling. The wrinkling behaviour of the sheet metal forming process is analysed using the non-bifurcation analysis and compared with experimental measurements. The initiation of the wrinkles arises as an artefact of the explicit time integration and round-off errors present in the finite element simulation. The influence of the material constitutive model, friction conditions, tool alignment and tool geometry are analysed.

4.3.1 EFFECT OF MATERIAL MODEL - AA3104-THIN

Three material models are examined for their impact of cup wrinkling in this section: Hill48S (stress-based), Barlat’s Yld2000-2D, and Barlat’s Yld2004-18P models. The stress based Hill48 model was used as the $r$-value based model failed due to excessive thinning during the drawing operation as a results of the significant differences in directional yield stresses determined by the material input. The material coefficients are listed in Table 2-10 (Hill48S), Table 2-11 (Yld2000-2D), and Table 2-12 (YLD2004-18P). The strain hardening of the material is described using the Swift equation:

$$\sigma = K(\varepsilon_p + \varepsilon_0)^n$$  \hspace{1cm} (4-1)

where $K=331.46$ MPa, $\varepsilon_0=0.00018$, and $n=0.05013$.

The results of the simulations showing the wrinkle patterns are provided in Figure 4-5. The Hill48S model predicts 10 waves while the YLD2000-2D and YLD2004-18P models predict 12 waves. Based on these results, and for numerical efficiency reasons, additional studies in this chapter on wrinkling will utilize only the YLD2000-2D material model.
4.3.2 EFFECT OF FRICTION - AA3104-THIN

The impact of increased friction is shown in Figure 4-6. The coefficient of friction in the model was increased to \( \mu=0.06 \). The results show a decrease in the overall amplitude of the wrinkles as the increased friction provides higher restraining forces during the drawing operation. In addition, the amplitude of the wrinkles is further reduced in the transverse direction due to higher contact pressures as a result of more thickening from anisotropic effects.

4.3.3 EFFECT OF TOOL AND BLANK ALIGNMENT - AA3104-THIN

The sensitivity of the wrinkling profile as a result of alignment is evaluated using the data provided in Table 4-2. For condition V03, the sheet is shifted 0.013 mm in the diagonal direction, the punch is offset -0.005 mm in the rolling direction, and the blankholder tool is rotated 0.1 degrees about a vector (1,-1) in the x-y plane. The results are shown in Figure 4-7 and show that full depth wrinkles are not formed between 90° and 120° and between 240° and 270°. These types of features are typical in the experimental cups. For condition V06, only the blank is shifted -0.025 mm in the rolling direction. The results are plotted in Figure 4-8 and show a decrease in the number of wrinkles from 12 to 11.

<table>
<thead>
<tr>
<th></th>
<th>Sheet Offset</th>
<th>Punch Offset</th>
<th>Non-Parallel BH</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Magnitude</td>
<td>Direction</td>
<td>Magnitude</td>
</tr>
<tr>
<td>V03</td>
<td>0.0005</td>
<td>45°</td>
<td>0.0002</td>
</tr>
<tr>
<td>V06</td>
<td>0.0010</td>
<td>180°</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

4.3.4 EFFECT OF PUNCH DESIGN - AA3104-THIN

In order to investigate the impact of the punch design on wrinkling, a \( 2^3 \) factorial numerical design of experiments was generated by modifying the punch geometric variables D2, H1, and R2. The dimensions for the three variables are given in Table 4-3. In order to evaluate the severity of wrinkling, a wrinkle factor is calculated using

\[
WF = \sqrt{\sum_{\theta=1}^{360} (r_{ij} - r_i)^2}
\]  

(4-2)
where $WF$ is the wrinkle factor, $r_\theta$ is the radial dimension to the wrinkled profile, and $r_t$ is the theoretical or average radius. If $WF$ equals 0, no wrinkling occurs.

<table>
<thead>
<tr>
<th>Model ID</th>
<th>$D2$</th>
<th>$H1$</th>
<th>$R2$</th>
<th>$H1'$</th>
<th>Wrinkle Factor</th>
</tr>
</thead>
<tbody>
<tr>
<td>DOX1</td>
<td>31.750</td>
<td>5.207</td>
<td>3.810</td>
<td>9.017</td>
<td>11.57</td>
</tr>
<tr>
<td>DOX2</td>
<td>33.782</td>
<td>5.207</td>
<td>3.810</td>
<td>9.017</td>
<td>8.74</td>
</tr>
<tr>
<td>DOX3</td>
<td>31.750</td>
<td>3.429</td>
<td>3.810</td>
<td>7.239</td>
<td>1.28</td>
</tr>
<tr>
<td>DOX4</td>
<td>33.782</td>
<td>3.429</td>
<td>3.810</td>
<td>7.239</td>
<td>1.81</td>
</tr>
<tr>
<td>DOX5</td>
<td>31.750</td>
<td>3.937</td>
<td>5.080</td>
<td>9.017</td>
<td>8.48</td>
</tr>
<tr>
<td>DOX6</td>
<td>33.782</td>
<td>3.937</td>
<td>5.080</td>
<td>9.017</td>
<td>6.65</td>
</tr>
<tr>
<td>DOX7</td>
<td>31.750</td>
<td>2.159</td>
<td>5.080</td>
<td>7.239</td>
<td>0.70</td>
</tr>
<tr>
<td>DOX8</td>
<td>33.782</td>
<td>2.159</td>
<td>5.080</td>
<td>7.239</td>
<td>0.84</td>
</tr>
</tbody>
</table>

The first case is the reference simulation compatible with the experimental cup measured at the axial position of $z=-4.5$ mm shown in Figure 4-4. The wrinkling profiles for the DOX study using the Yld2000-2D model are provided in Figure 4-9. Analysis of the data shows that as the $D2$ increases, $H1$ decreases, and $R2$ increases, the tendency to wrinkle decreases. The experimental result using the DOX8 tooling data is shown in Figure 4-10 and indicates essentially no wrinkling.

### 4.3.5 EFFECT OF MATERIAL MODEL - AA3104-THICK

Three material models are once again examined for their impact of cup wrinkling using the AA3104-Thick material ($t=0.475$ mm): Hill48S, Barlat’s Yld2000-2D, and Barlat’s Yld2004-18P models. The material coefficients are listed in Table 2-10 (Hill48S), Table 2-11 (YLD2000-2D), and Table 2-12 (YLD2004-18P). The tooling dimensions used in the finite element model are provided in Table 4-1. The results of the simulation are shown in Figure 4-11. Here the amplitude of the wrinkles are significantly reduced compared with the cups having 0.274 mm thickness. The Hill48S model and Barlat’s YLD2000-2D models show both low amplitude and low frequency wrinkling. An image of the experimental cup is provided in Figure 4-12 and shows no wrinkling.
Figure 4-5. Wrinkling profile for AA3104-Thin using Hill48S model.

Figure 4-6. Effect of friction on wrinkling profile for AA3104-Thin using YLD2000-2D model.
Figure 4-7. Effect of tool alignment on wrinkling profile for AA3104-Thin using YLD2000-2D model.

Figure 4-8. Effect of blank offset on wrinkling profile for AA3104-Thin using YLD2000-2D model.
Figure 4-9. Effect of punch geometry on wrinkling profile for AA3104-Thin using YLD2000-2D model.

Figure 4-10. Image of drawn cup using AA3104-Thin material for condition DOX8.
Figure 4-11. Wrinkling profiles for AA3104-Thick using Hill485, YLD2000-2D, and YLD2004-18P models.

Figure 4-12. Image of drawn cup using AA3104-Thick material.
4.4 SUMMARY

Wrinkling during the tapered cup forming process performed in the NUMISHEET2014 benchmark test was investigated in view of anisotropy and friction. A numerical technique was developed to quantify the severity wrinkling resulting from the drawing process. This allows for optimization of both tooling design and material parameters (yield stress, strain hardening, strain hardening rate, etc.) to minimize or eliminate the wrinkling. The results presented confirm the trends based on manufacturing experience. The study on material characteristics indicates that anisotropy promotes wrinkling and increased friction decreases the amplitude of wrinkling due to increased restraining forces during drawing. The sensitivity study on punch profile demonstrates that the wrinkling can be controlled or eliminated through tooling design. Reducing the profile height, increasing the nose radius and profile radius, and increasing the base diameter tend to minimize the wrinkling behavior as these directional changes all reduce the unsupported cross-sectional length of metal in the profile region of the cup. However, there is a trade-off between wrinkling and structural performance of the final design. Factors that control or reduce wrinkling may also affect localized thinning or structural integrity. An acceptable solution must balance both the wrinkling and structural performance requirements. The effect of anisotropic friction between the tooling and the sheet should be investigated as the anisotropic thickening causes high localized contact pressures which locally reducing the amplitude of wrinkling. The Barlat Yld2000-2D and Barlat Yld2004-18P models accurately predict the magnitude and frequency of the wrinkling profile, although the pattern was slightly out of phase with the experimental results. Factors such as tooling alignment explain the minor differences observed between experimental results, including wrinkle pattern frequency and localized wrinkles that are not fully developed.

The influence of tooling alignment on both the magnitude and frequency of wrinkles should be further studied experimentally as it is very challenging to ensure that all installed tooling are completely concentric. Finally, even though a very fine mesh was used in this study, mesh refinement should be further studied as a finer mesh will allow for a better approximation of the wrinkled shapes and will impact the wrinkle magnitude. However, the overall wrinkling trends should remain the same.
4.5 REFERENCES


PART II

FORMING LIMITS

AND

FRACTURE LIMITS
CHAPTER 5

STRAIN BASED FORMING LIMIT DIAGRAMS

5.1 INTRODUCTION

The success of sheet forming operations can be limited by several phenomena, such as wrinkling, plastic flow localization, and fracture. For a given forming operation, the sheet may undergo deformation up to a given strain prior to failure by one of these limiting phenomena. Accurate and reliable prediction of these failure modes is critical for successfully designing forming operations, particularly operations requiring multiple forming steps. With the introduction of high strength, thin gauge materials, the likelihood of necking or fracture during the forming processes increases. One tool that has been used by engineers to help predict the onset of necking and localization during sheet metal forming operations is the Forming Limit Diagram (FLD). The concept of the FLD was first introduced by Keeler and Backofen (1964) and Goodwin (1968). This diagram is usually plotted on the axes representing the major (\(\varepsilon_1\)) and the minor (\(\varepsilon_2\)) strains in the plane of a sheet and corresponds to the maximum admissible local strains achievable just prior to the occurrence of visible defects in the sheet metal like necking and fracture (Figure 5-1). The experimental technique to determine the FLD involves subjecting specimens of the considered sheet metal to different in-plane strain states by simple tensile testing or stretching over a hemispherical punch. The limiting strains in the FLD corresponding to necking describe a curve called the forming limit curve (FLC). The state of strain at the onset of localized necking is called the forming limit strain. The FLC is typically divided into two branches: the “right branch”, which is valid for positive major and minor strains (stretching), and the “left branch”, which is applicable for positive major and negative minor strains (drawing).
This chapter provides a literature review on the experimental and numerical approaches used to develop the traditional FLD. Several common analytical models in strain space used to predict the FLC are summarized.

Figure 5-1. Forming limit diagram.

5.2 BACKGROUND

The concept of the forming limit diagram (FLD) was first introduced in the 1960’s by Keeler and Backofen (1964) and today represents a generally accepted measure of sheet metal formability. They performed tests using several materials including steel, copper, brass and aluminum by securely clamping and then stretching the sheet with a solid hemispherical punch. Their work determined strains on the right side of the FLD. The experimental technique of Keeler was further developed by Goodwin (1968) to include the strains on the left side of the FLD and produced a successful FLD for a mild steel used for a stamping process with the earliest FLD named as the Keeler-Goodwin diagram. Through their efforts the FLC concept and the circle grid analysis technique became accepted in tool and die shops as diagnostic techniques. The FLD can also be constructed by experiments such as hemispherical punch-stretch tests developed by Marciniak (1967), also known as the Marciniak cup, displayed in Figure 5-2. Hecker (1975) proposed an experimental technique which enabled generation of the entire diagram using fewer tests by utilizing a hemispherical punch with different widths of the sheet samples and different types of lubricants. Although, Hecker’s model was a significant improvement to FLD construction, the methodology employed involved some complexity and variability in adjusting the lubricity. Unfortunately, the experimental approach is very time consuming and needs specialized laboratory equipment. In addition, experimental studies by Nakazima (1968), shown in Figure 5-3, and Graf and Hosford (1990), as
well as numerical studies by Barata da Rocha et al. (1984), Gotoh (1978), Hiwatashi et al., (1998), Wu et al. (1998), and Kuroda and Tvergaard (2000), show that strain path changes significantly affect the shape and location of the FLC. The characterization of forming limits in strain space is therefore a practical challenge for complex forming processes due to this sensitivity and is limited to sheet metals undergoing proportional loading. Experimental work by Ishigaki (1977) at Toyota Motors Company demonstrated the concept of using the dynamic strain path for improving the formability. As can be seen in Figure 5-4, Toyota engineers achieved a remarkable improvement in the formability, by taking advantage of the dynamic nature of the strain FLC.

![Figure 5-2. Schematic diagram showing tooling set-up for the Marciniak cup test.](image1)

![Figure 5-3. Typical Nakajima test specimens and associated FLC.](image2)
Figure 5-4. Nonlinear strain path concept developed by Toyota and applied to tryout of a quarter panel stamped from a deep draw quality steel. Forming limit curves are experimental.

Due to the time and resources required to develop an FLD experimentally, many analytical models have been proposed such as Hill’s (1952) localized necking model, Swift’s (1952) diffused necking model, the Marciniak–Kuczynski model (1967), known as the M-K model, the Vertex theory by Stören and Rice (1975) and Zhu et al. (2001), and the modified maximum force criterion (MMFC) by Hora (2006). For a review of these existing strain-based theoretical models and their corresponding stress-based forms, the reader is referred to Stoughton and Zhu (2004). All of these models were derived from the necking and stability analysis in the plane of a sheet. Bressan and Williams (1983) proposed an alternative approach based on stability analysis through the thickness. Alsos (2008) combined this criterion with Hill’s instability criterion known as the BWH criterion and used it for numerical analysis of sheet metal instability. The range of validity of all these models is restricted to the strain ratio $-1/2 \leq \beta \leq 1$, corresponding to stress states between the uniaxial and biaxial tension.

Recently, there have been several important developments in forming limits and fracture. With the advancements in photogrammetric equipment, it is possible to record the entire experiment and identify the onset of necking directly. Thus, the so called time dependent evaluation methods have been proposed by Volk and Hora (2010), and Merklein et al., (2010). With advancement of the digital image correlation
(DIC) analysis methods, accurate measurement of strain near necking zones is possible today. Thus, experimental construction of the FLD using the DIC method has become increasingly accurate. Wang et al. (2014) discussed in detail several DIC procedures to generate the FLD, which can be affected by many factors, such as strain rate (the forming speed), strain hardening properties, normal and planar anisotropy of the sheet metals (Hariharen et al., 2014), thickness of the sheet, through thickness stress (Assempour et al., 2010; Hashemi et al., 2014) and lubrication conditions.

In the area of modeling, the enhanced modified maximum force criterion (eMMFC) by Hora and Tong (2006), the fracture limit based prediction by Isik et al. (2014), and the numerical stability approach in the CrachFEM code by Hooputra et al. (2004) are examples of new developments. Specifically, Volk and Suh (2013) proposed a phenomenological approach, the so-called Generalized Forming Limit Concept (GFLC) to predict the localized necking on arbitrary deformation history with an unlimited number of non-linear strain increments. The GFLC consists of the conventional FLC and an acceptable number of experiments with bi-linear deformation history. Every deformation state is built up of two linear strain increments that can be transformed to a pure linear strain path with the same used formability of the material.
Figure 5-5. Forming Limit Diagram (FLD): major strain ($\varepsilon_1$) vs. minor strain ($\varepsilon_2$) plot for AA6111-T4: (a) biaxial pre-strain paths; (b) uniaxial and plane strain pre-strain paths; pre-strain given in parallel to rolling direction (RD); (c) uniaxial and plane strain pre-strain paths; pre-strain given in parallel to transverse direction (TD) - (Graf and Hosford – 1994)
5.3 ANALYTICAL MODELS FOR FLD

In sheet forming analysis a distinction is made between *diffuse* and *localized* necking. In sheet specimens, localized necking occurs after diffuse necking. In industrial stampings, the maximum allowable strain is determined by localized rather than by diffuse necking. According to Hosford and Caddell (1993), *diffuse* necking is accompanied by the contraction strains in both the width and the thickness directions of the sheet. The full neck develops gradually and considerable extension is still possible after the onset of diffuse necking. Finally, a condition will be reached where a sharp *localized* neck can form. Its width is of the order of the sheet thickness. In a localized neck the strain along the necking band is zero and, assuming volume constancy, the thickness strain is exclusively provided by the remaining in-plane strain. Figure 5-6 illustrates both diffused and localized necking.

![Figure 5-6. Tensile bar exhibiting both diffuse and local necking.](image)

Theoretical methods for predicting the FLD are generally divided into three different approaches to describe the condition in which the failure of necking in sheet metals occurs. The first approach treats the onset of failure as a condition that leads to plastic instability. Swift’s (1952) diffused necking model predicts instability in a plane stress loading condition when the load in a principal direction reaches to a critical magnitude. Hill (1952) suggested a postulate that the onset of a localized necking through the thickness results from the force equilibrium equation when a hardening induced by deformation with increase in stress, is equal to the geometrical softening due to reduction in thickness. This leads to a strain situation with zero extension which governs only negative minor strains, so merely the left side of FLD can
be predicted. Keeler and Backofen (1964) experimental observations clearly showed that the sheet fails by a process of strain localization in a narrow neck in the same manner as for the negative minor strains.

The second approach is based on the work done by Storen and Rice (1975). They initially incorporated J2 deformation theory of plasticity into a classical bifurcation analysis to achieve the limit strains in the whole region of strain paths by using a simplified constitutive model of a pointed vertex on the subsequent yield loci. In this method, the development of a yield vertex causes bifurcation from a state of uniform deformation. In the S-R analysis, the work-hardening exponent $n$, is decisive of the FLD, and the predicted FLD does not always fit the measured ones in great variety.

The third approach that was suggested by Marciniak and Kuczynski (1967), known as the M-K method, was an attempt to present a mathematical model for solving the mentioned problem. The RHS of the FLC for a material was determined by assuming that there is a pre-existing imperfection perpendicular to the major stress axis in an infinite sheet to explain the development of localized necking during biaxial loading. Sowerby and Duncan (1971) used Hill’s (1948) yield criterion to calculate the FLD and showed that the right hand side FLD increases with an increase in the strain hardening exponent $n$ value and strongly depends on the normal anisotropic index $r$-value. Comparison of the numerical and experimental FLD showed that the Hill (1948) yield function over-predicted the FLC, especially for aluminum materials. Hutchinson and Neale (1978), by giving the idea of existence of an initial nonuniformity, predicted the entire FLD by introducing an imperfection at an angle to the major stress axis. The angle selected is the one that minimizes the limit strain. Huang (2000) reported the results obtained by the M-K method are extremely dependent on the initial size of imperfection and the predicted limit strains near the balanced biaxial stretching are extremely large compared to the experimental values. Friedman and Pan (2000) determined the effect of various yield criteria on the right-hand side of the forming limit diagram. Results indicate that the effects of plastic anisotropy on the limit strains in biaxial stretching operations are significantly dependent on selection of yield criterion for a given $r$-value. They introduced an angle that is specific for each loading path and is defined by the angle between plane strain and the specific point on the yield surface determined from the loading path to characterize the influence of the shape of the yield locus on the FLC. Cao et. al. (2000) incorporated the anisotropic yield criterion of Karafillis and Boyce (1993) in the M–K model to predict localized necking in sheet metal alloys for linear and non-linear strain paths. The calculated FLD’s exhibited good agreement with the experimental results of Graf and Hosford (1993, 1994). They also reported the high stress exponents in the K-B yield criterion have the effect of lowering the forming limit prediction to the desired level and therefore improve the accuracy of the
prediction of the FLD. Yao and Cao (2002) developed the FLC’s, using M–K model considering the effects of pre-strains and kinematic hardening. The exponent of the yield function used in their work is assumed to decrease with increasing pre-strain. Yoshida and Suziki (2008) analyzed the forming limits of a sheet metal subjected to linear and combined loading paths, using the M–K model and found that the stress based FLC is independent of strain path only if the work hardening behavior is not affected by the strain path change. Aretz (2010) extended Hills (1952) localized necking model to predict a forming limit curve for negative minor strains and its predictions are closer to that of the M–K model and concluded that the extended model is an efficient alternative to the M–K model in the negative minor strain regime. The implementation of different yield criteria in the M–K model has been investigated by several authors including Banabic (1999), Banabic and Dannenmann (2001), Banabic et al. (2004), Wang et al. (2006), Zhang et al. (2011), and Ghazanfari et al. (2012). Butuc et al. (2003, 2006, 2011) investigated the influence of different hardening laws, yields functions, strain rate sensitivity and also strain path changes on stress-based forming limits. Refer to Ganjiani and Assempour (2007) for a more detailed review of developments based on the M-K theory.

5.3.1 NADDRG MODEL

The North American deep Drawing Research Group (NADDRG) proposed an empirical equation to calculate an FLC which simplifies the theoretical prediction. In the introduced model, the FLC is composed of two lines which intersect at the point defined as FLC0 in the plane strain region of the FLC.

The slopes of the line are approximately 20 and 45 located on the right and left hand sides of the forming limit diagram. The proposed equation to calculate the limit strain by Keeler and Brazier (1975) is:

\[
\varepsilon_{10} = \frac{(23.3 + 14.13t)n}{0.21}
\] (5-1)

where \(t\) is the sheet thickness in millimeters and \(n\) is the strain hardening exponent in the Hollomon power law equation. FLC0 is the most important parameter in the FLC and is defined as the major principal strain at the onset of necking in the plane strain condition and can be determined experimentally from a tensile test specimen having a geometry of large sample width compared to its length. According to Holmberg et. al. (2004), the forming limit at the plane strain condition is of particular interest for the reasons listed below:
• It is, normally, the mode of deformation where the material can resist the lowest amount of strain, i.e. it is the minimum point of the FLC.

• It is often a critical strain condition for real stamped parts.

• The plane strain condition is an unstable condition with rapid localization.

• The forming limit in plane strain is often used when ranking the formability of different materials.

• The plane strain formability limit is a good measure to be combined with theoretical models of FLCs.

5.3.2 SWIFT’S DIFFUSED NECKING MODEL

For sheet metal forming, Swift (1952) proposed a diffuse necking criteria, which states that a diffuse neck starts when the load reaches a maximum along both principal directions.

\[
\begin{align*}
\dot{\sigma}_1 &= \sigma_1 \dot{\varepsilon}_1 \\
\dot{\sigma}_2 &= \sigma_2 \dot{\varepsilon}_2
\end{align*}
\]  

(5-2)

Swift exploited the Considère criterion to study the diffuse necking and proposed a general diffuse necking condition:

\[
\frac{d\sigma_e}{\sigma_e d\varepsilon_e} = \frac{\sigma_1 \left( \frac{\partial f}{\partial \sigma_1} \right)^2 + \sigma_2 \left( \frac{\partial f}{\partial \sigma_2} \right)^2}{\frac{\partial f}{\partial \sigma_p} \left( \frac{\partial f}{\partial \sigma_1} \frac{\sigma_1}{\sigma_1} + \frac{\partial f}{\partial \sigma_2} \frac{\sigma_2}{\sigma_2} \right)}
\]

(5-3)

Using a Hollomon stress-strain relationship

\[
\bar{\varepsilon}_p = \frac{\frac{\partial f}{\partial \sigma_p} \left( \frac{\sigma_1}{\sigma_1} \frac{\partial f}{\partial \sigma_1} + \frac{\sigma_2}{\sigma_2} \frac{\partial f}{\partial \sigma_2} \right)^n}{\sigma_1 \left( \frac{\partial f}{\partial \sigma_1} \right)^2 + \sigma_2 \left( \frac{\partial f}{\partial \sigma_2} \right)^2}
\]

(5-4)

The principal strains can be determined as follows:
\[ \varepsilon_1 = \frac{\sigma_1 \left( \frac{\partial f}{\partial \sigma_1} \right)^2 + \sigma_2 \left( \frac{\partial f}{\partial \sigma_2} \right)^2}{\sigma_1 \left( \frac{\partial f}{\partial \sigma_1} \right)^2 + \sigma_2 \left( \frac{\partial f}{\partial \sigma_2} \right)^2} n \]  
(5-5)

\[ \varepsilon_2 = \frac{\sigma_2 \left( \frac{\partial f}{\partial \sigma_2} \right)^2 + \sigma_1 \left( \frac{\partial f}{\partial \sigma_1} \right)^2}{\sigma_1 \left( \frac{\partial f}{\partial \sigma_1} \right)^2 + \sigma_2 \left( \frac{\partial f}{\partial \sigma_2} \right)^2} n \]  
(5-6)

For Hill’s 1948 anisotropic yield function, the derivatives can be calculated as

\[ \frac{\partial f}{\partial \sigma_1} = \frac{1}{2\sqrt{\sigma}} \left( 2\sigma_1 - \frac{2R_{00}}{1 + R_{00}} \sigma_2 \right) = \frac{1}{2\sqrt{\sigma}} \left( \frac{2\sigma_1}{1 + R_{00}} \right) (1 + R_{00} - R_{00} \alpha) \]
\[ \frac{\partial f}{\partial \sigma_2} = \frac{1}{2\sqrt{\sigma}} \left( \frac{2R_{00} (1 + R_{90})}{1 + R_{00}} \sigma_2 - \frac{2R_{00}}{1 + R_{00}} \sigma_1 \right) = \frac{1}{2\sqrt{\sigma}} \left( \frac{2\sigma_1}{1 + R_{00}} \right) \left( \frac{R_{00} (1 + R_{90})}{R_{90}} \alpha - R_{00} \right) \]  
(5-7)

where \( \alpha = \sigma_2 / \sigma_1 \). Solving for the principal strains gives:

\[ \varepsilon_1 = \frac{(1 + R_{00} - \alpha R_{00}) \left( 1 + R_{00} + \frac{2 R_{00} (1 + R_{90})}{R_{90}} - 2 \alpha R_{00} \right)}{(1 + R_{00} - \alpha R_{00})^2 + \alpha \left( \frac{R_{00} (1 + R_{90})}{R_{90}} - R_{00} \right)^2} n \]  
(5-8)

\[ \varepsilon_2 = \frac{\left( \frac{R_{00} (1 + R_{90})}{R_{90}} - R_{00} \right) \left( 1 + R_{00} + \frac{2 R_{00} (1 + R_{90})}{R_{90}} - 2 \alpha R_{00} \right)}{(1 + R_{00} - \alpha R_{00})^2 + \alpha \left( \frac{R_{00} (1 + R_{90})}{R_{90}} - R_{00} \right)^2} n \]  
(5-9)

For Hill’s normal anisotropy \( R_{00} = R_{45} = R_{90} = R \), these equations reduce to:

\[ \varepsilon_1 = \frac{(1 + R - \alpha R) \left( 1 + R + \alpha^2 (1 + R) - 2 \alpha R \right)}{(1 + R - \alpha R)^2 + \alpha (\alpha + \alpha R - R)^2} n \]  
(5-10)

\[ \varepsilon_2 = \frac{\left( \alpha + \alpha R - R \right) \left( 1 + R + \alpha^2 (1 + R) - 2 \alpha R \right)}{(1 + R - \alpha R)^2 + \alpha (\alpha + \alpha R - R)^2} n \]  
(5-11)
For an isotropic von Mises material \( R_{00} = R_{45} = R_{90} = 1 \), the equations further reduce to:

\[
\varepsilon_1 = \frac{2n(2-\alpha)(1-\alpha + \alpha^2)}{(2-\alpha)^2 + \alpha (2\alpha - 1)^2} = \frac{2n(2-\alpha)(1-\alpha + \alpha^2)}{4 - 3\alpha - 3\alpha^2 + 4\alpha^3}
\]  

(5-12)

\[
\varepsilon_2 = \frac{2n(2\alpha - 1)(1-\alpha + \alpha^2)}{(2-\alpha)^2 + \alpha (2\alpha - 1)^2} = \frac{2n(2\alpha - 1)(1-\alpha + \alpha^2)}{4 - 3\alpha - 3\alpha^2 + 4\alpha^3}
\]

(5-13)

It is generally considered that the Swift criterion underestimates the material formability, in particular on the negative minor strain regime in FLD, by assuming diffused necking. Hill (1952) postulated that the localization necking is triggered when the increase of the stress by strain hardening is balanced by the geometrical softening raised by the thickness reduction.

### 5.3.3 Hill’s Localized Necking Model

Hill (1952) discussed the localized necking in thin sheet by considering the discontinuity of stress and strain, and laid the foundation of localized instability analysis. His theory is based on the characteristics of governing equations in plane stress. Localized necking is possible only when the equations are hyperbolic, when the characteristic directions are those of zero extension. Therefore, local instability is possible only when the incremental strain ratio is negative, i.e. corresponding to the strain state in the left hand side of FLD. Depending on the material and the type of loading, this type of instability can occur before or after the maximum load point. The change in cross-section dimensions associated with localized necking is negligible. In the case of uniaxial tension, the localized necking develops along a direction, which is inclined with respect to the loading direction. Hill assumed that the necking direction is coincident with the direction of zero-elongation and thus the straining in the necking region is due only to the sheet thinning.

Hill’s general expression for localized necking condition is given as:

\[
\frac{d\sigma}{\sigma \varepsilon} = \left( \frac{\partial f}{\partial \sigma_1} \right) + \left( \frac{\partial f}{\partial \sigma_2} \right)
\]

(5-14)

Assuming that the stress-strain relation can be expressed using the Hollomon equation and using the Hill’48 yield criterion with the instability condition gives
The major and minor strains can be determined as

\[ \varepsilon_1 = \left( \frac{\partial f}{\partial \sigma_1} + \frac{\partial f}{\partial \sigma_2} \right) n \]  \hspace{1cm} (5-16)

\[ \varepsilon_2 = \left( \frac{\partial f}{\partial \sigma_1} + \frac{\partial f}{\partial \sigma_2} \right) n \]  \hspace{1cm} (5-17)

Substituting from Equations 5-7 gives

\[ \varepsilon_1 = \frac{R_{90} \left( 1 + R_{oo} - R_{oo} \alpha \right)}{R_{oo} \alpha + R_{90}} n \]  \hspace{1cm} (5-18)

\[ \varepsilon_2 = \frac{R_{oo} \left( \alpha + R_{90} \alpha - R_{90} \right)}{R_{oo} \alpha + R_{90}} n \]  \hspace{1cm} (5-19)

For normal anisotropy, the equations reduce to

\[ \varepsilon_1 = \frac{1 + R - R \alpha}{1 + \alpha} n \]  \hspace{1cm} (5-20)

\[ \varepsilon_2 = \frac{\alpha + R \alpha - R}{1 + \alpha} n \]  \hspace{1cm} (5-21)

And for an isotropic von Mises material the equations become

\[ \varepsilon_1 = \frac{2 - \alpha}{1 + \alpha} n \]  \hspace{1cm} (5-22)
\[ e_2 = \frac{(2\alpha - 1)}{(1+\alpha)} n \]  
(5-23)

It can be seen from the above equations that

\[ e_1 + e_2 = n \]  
(5-24)

The limit strains for the LHS ( \( \beta \leq 0 \) ) of the FLD can be obtained for various stress ratios.

5.3.4 FLD USING HILL’93 YIELD CRITERION

Hill (1993) proposed a new yield criterion for orthotropic sheet metal. The general expression of the Hill’s 1993 yield criterion is

\[ \frac{\sigma_1^2 - c \sigma_{12} + \sigma_2^2}{\sigma_0^2} + \frac{(p + q - \frac{p\sigma_1 + q\sigma_2}{\sigma_b})}{\sigma_0 \sigma_{90}} = 1 \]  
(5-25)

where \( c, p, \) and \( q \) are non-dimensional parameters and are determined by

\[ \frac{c}{\sigma_0 \sigma_{90}} = \frac{1}{\sigma_0^2} + \frac{1}{\sigma_{90}^2} + \frac{1}{\sigma_b^2} \]  
(5-26)

\[ \left( \frac{1}{\sigma_0^2} + \frac{1}{\sigma_{90}^2} - \frac{1}{\sigma_b^2} \right) p = \frac{2r_0 (\sigma_b - \sigma_{90})}{(1 + r_0) \sigma_0^2} - \frac{2r_{90} \sigma_b}{(1 + r_{90}) \sigma_{90}^2} + \frac{c}{\sigma_0} \]  
(5-27)

\[ \left( \frac{1}{\sigma_0^2} + \frac{1}{\sigma_{90}^2} - \frac{1}{\sigma_b^2} \right) q = \frac{2r_0 (\sigma_b - \sigma_0)}{(1 + r_0) \sigma_0^2} - \frac{2r_0 \sigma_b}{(1 + r_0) \sigma_{90}^2} + \frac{c}{\sigma_{90}} \]  
(5-28)

In the above equations, \( \sigma_b \) is the yield stress in biaxial tension, \( \sigma_0 \) and \( \sigma_{90} \) are the yield stresses in uniaxial tension at 0° and 90° to the rolling direction, respectively, and \( r_0 \) and \( r_{90} \) are the r-values at 0° and 90° to the rolling direction, respectively. The plastic function \( f \) is

\[ 2f = \frac{1}{\sigma_0^2} \left[ \sigma_1^2 - \left( 2 - \frac{\sigma_0^2}{\sigma_b^2} \right) \sigma_1 \sigma_2 + \sigma_2^2 + \left( p + q - \frac{p\sigma_1 + q\sigma_2}{\sigma_b} \right) \sigma_1 \sigma_2 \right] - 1 \]  
(5-29)

The derivatives of the function \( f \) are

\[ \frac{\partial f}{\partial \sigma_1} = \xi = \left[ \frac{2}{\sigma_0^2} + \frac{\alpha}{\sigma_0 \sigma_{90}} \right] (p + q - c - t(2p + q\alpha)) \]  
(5-30)
\[
\frac{\partial f}{\partial \sigma_2} = \omega = \sigma_1 \left[ \frac{2\alpha}{\sigma_0^2} + \frac{1}{\sigma_0 \sigma_90} \left( p + q - c - t \left( p + 2qa \right) \right) \right]
\]

\[
\frac{\partial f}{\partial \sigma} = \eta = \sqrt{\frac{1}{\sigma_0^2} - \frac{ca}{\sigma_0 \sigma_90} + \frac{\alpha^2}{\sigma_0^2} + \left( p + q - pt - qt \alpha \right)}
\]

(5-31)

(5-32)

where \( t = \sigma_1 / \sigma_9 \). For local necking \( (\varepsilon_2 \leq 0) \), and applying the Levi-Mises equation for a material and a Swift hardening law

\[
\varepsilon_1 = \frac{\partial f / \partial \sigma_1}{\partial f / \partial \sigma_1 + \partial f / \partial \sigma_2} n - \frac{\partial f / \partial \sigma_1}{\partial f / \partial \sigma} \varepsilon_0
\]

(5-33)

In reduced form

\[
\varepsilon_1 = \frac{\xi}{\xi + \omega} n - \frac{\xi}{\eta} \varepsilon_0
\]

(5-34)

\[
\varepsilon_2 = \frac{\omega}{\xi + \omega} n - \frac{\omega}{\eta} \varepsilon_0
\]

(5-35)

For local necking \( (\varepsilon_2 \geq 0) \), the expressions for the limit strains are

\[
\varepsilon_1 = \frac{\sigma_1 \left( \frac{\partial f}{\partial \sigma_1} \right)^2 + \sigma_2 \left( \frac{\partial f}{\partial \sigma_2} \right)^2}{\sigma_1 \left( \frac{\partial f}{\partial \sigma_1} \right)^2 + \sigma_2 \left( \frac{\partial f}{\partial \sigma_2} \right)^2} n - \frac{\partial f / \partial \sigma_1}{\sigma_0 \sigma_90} \varepsilon_0
\]

(5-36)

\[
\varepsilon_2 = \frac{\sigma_1 \left( \frac{\partial f}{\partial \sigma_1} \right)^2 + \sigma_2 \left( \frac{\partial f}{\partial \sigma_2} \right)^2}{\sigma_1 \left( \frac{\partial f}{\partial \sigma_1} \right)^2 + \sigma_2 \left( \frac{\partial f}{\partial \sigma_2} \right)^2} n - \frac{\partial f / \partial \sigma_2}{\sigma_0 \sigma_90} \varepsilon_0
\]

(5-37)

In abbreviated form, the equations become

\[
\varepsilon_1 = \frac{\xi^2 + \alpha \xi \omega}{\xi^2 + \alpha \omega^2} n - \frac{\xi}{\eta} \varepsilon_0
\]

(5-38)

\[
\varepsilon_1 = \frac{\alpha \omega^2 + \xi \omega}{\xi^2 + \alpha \omega^2} n - \frac{\xi}{\eta} \varepsilon_0
\]

(5-39)
5.3.5 STOREN AND RICE’S VERTEX THEORY

In order to create a bifurcation that allows a neck to form in plane strain, Storen and Rice (1975) introduce a vertex on the yield surface whose discontinuity grows until a plane strain plastic strain increment is possible. They used deformation theory to simulate the formation of this vertex, and therefore, their analysis is restricted to conditions of proportional loading. They derived the following relation for the major strain at the instant of instability using a Von Mises yield function and a power law stress-strain relation to define the incremental stress rates which leads to an FLC constructed as (using $\beta = \dot{\varepsilon}_2 / \dot{\varepsilon}_1$)

\[
\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 3\beta^2 + n(2 + \beta)^2 \\ 2(2 + \beta)(1 + \beta + \beta^2) \end{pmatrix} \quad 0 \leq \beta \leq 1
\]

\[
\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 3 + n(1 + 2\beta)^2 \\ (1 + 2\beta)(1 + \beta + \beta^2) \end{pmatrix} \quad -1 \leq \beta \leq 0
\]

5.3.6 BRESSAN-WILLIAMS-HILL (BWH) MODEL

In plasticity, the main mechanism of deformation comes from slip arising from shear on certain preferred combinations of crystallographic planes. Experimental results have shown that failure planes in sheet metal lie close to the direction of maximum shear stress (Bressan and Williams, 1983). It can be reasonably assumed that the instability may take place before any visual signs of local necking. Based on this observation, a shear stress based instability criterion may well be useful in estimating the point of local necking. Bressan and Williams (1983), the BW criterion, proposed a simple expression for failure that has been applied with good results (Alsos et al., 2008). The basis for the BW expression follows three basic assumptions. First of all, the shear instability is initiated in the direction through the thickness at which the material element experiences no change of length. This indicates a critical through thickness shear direction. Secondly, the instability is triggered by a local shear stress which exceeds a critical value. This means that the initiation of local necking is described as a material property. Finally, elastic strains are neglected. This is reasonable since the elastic strains are small compared to the plastic strains at the localized necking state.
From Figure 5-7, and using the assumptions above, a mathematical formulation for the BW criterion can be developed. The inclined plane through the element thickness at which shear instability occurs (indicated by the plane normal $x_n$) forms an angle $\pi/2 - \theta$ to the shell plane. The material experiences zero elongation in this direction, indicating that $\dot{\varepsilon}_i = 0$.

![Figure 5-7. Local shear instability in a material element. No elongation takes place in the $x_i$ direction.](image)

Using Mohr's circle (Figure 5-8a) and the strain transformation given in Equation 5-42 gives

\[
\begin{align*}
\dot{\varepsilon}_x' &= \frac{\varepsilon_1 + \varepsilon_3}{2} + \frac{\varepsilon_1 - \varepsilon_3}{2} \cos(2\theta) + \varepsilon_{xy} \sin(2\theta) \\
\dot{\varepsilon}_y' &= \frac{\varepsilon_1 + \varepsilon_3}{2} - \frac{\varepsilon_1 - \varepsilon_3}{2} \cos(2\theta) - \varepsilon_{xy} \sin(2\theta) \\
\varepsilon_{xy}' &= -\frac{\varepsilon_1 - \varepsilon_3}{2} \sin(2\theta) + \varepsilon_{xy} \cos(2\theta)
\end{align*}
\] (5-42)

The relation between the angle of the inclined plane and the principal strain rates can be determined

\[
\dot{\varepsilon}_i = \dot{\varepsilon}_{xy} = \frac{\dot{\varepsilon}_1 + \dot{\varepsilon}_3}{2} + \frac{\dot{\varepsilon}_1 - \dot{\varepsilon}_3}{2} \cos(2\theta) = 0
\] (5-43)

where $\dot{\varepsilon}_{xy} = 0$ and $\cos(2\theta) = \cos(2(\pi/2 - \theta))$. Using plastic incompressibility, $\dot{\varepsilon}_3 = -(1 + \beta)$, leads to

\[
\cos(2\theta) = -\frac{\dot{\varepsilon}_1 + \dot{\varepsilon}_3}{\dot{\varepsilon}_1 - \dot{\varepsilon}_3} = \frac{\beta}{2 + \beta}
\] (5-44)

It can be easily shown that

\[
\sin(2\theta) = \frac{2\sqrt{1 + \beta}}{2 + \beta}
\] (5-45)
Using Mohr’s circle for plane stress or the stress transformation equations, the corresponding stress state can be obtained

\[
\tau_{\alpha} = \frac{\sigma_1}{2} \sin(2\theta) = \frac{\sigma_1 \sqrt{1+\beta}}{2+\beta}
\]

(5-46)

The BW criterion can be calibrated by considering that the Hill expression and the BW criterion will be the equal at the plane strain condition, \( \beta = 0 \).

\[
\tau_{\alpha} = \frac{K}{\sqrt{3}} \left( \frac{2}{\sqrt{3}} \hat{\varepsilon}_1 \right)^a
\]

(5-47)

where \( \hat{\varepsilon}_1 \) is equal to the uniform elongation in plane strain.

The BW criterion was initially intended for the positive quadrant of the FLD, but the mathematical expression is also valid for negative values. However, as the strain rate ratio becomes negative, the validity of the BW criterion becomes questionable. Hence, in order to cover the full range of \( \beta \), the Hill and BW criteria have been combined into one criterion, from now on referred to as the BWH criterion. Formulated in terms of the strain rate ratio, \( \beta \), the criterion reads
The equivalent stress can be calculated using

\[ \sigma_{eq} = \frac{\sqrt{3}}{2 + \beta} \sqrt{1 + \beta + \beta^2} \]  

The equivalent strain (based on the power law equation) can be calculated using

\[ \varepsilon_{eq} = \left( \frac{\sigma_{eq}}{K} \right)^{\frac{1}{n}} \]  

and the major principal strain can be determined as

\[ \varepsilon_1 = \frac{\sqrt{3}}{2} \frac{\varepsilon_{eq}}{\sqrt{1 + \beta + \beta^2}} \]  

The Bressan-Williams-Hill shear instability criterion can provide a consistent approach to the onset of local necking in sheet metal forming and is easily implemented into a finite element code. The simple nature of the criterion makes it very CPU efficient and attractive in analysis of structures where material and failure data are limited and may therefore be a cost effective and consistent alternative to more complex failure criteria.

**5.3.7 MARCINIAK-KUCZYNSKI (M-K) MODEL**

The model developed by Marciniak and Kuczynski (M–K criterion) is based on a thickness imperfection approach. This model has been one of the most widely applied models for FLD prediction. The failure strain depends mainly on the growth of an initial inhomogeneity presenting as a narrow band in material. Generally, accuracy of the predicted FLD can be significantly influenced by the shape of employed yield surface, hardening rule and strain rate sensitivity. For steel sheet materials, various yield functions and different hardening laws have been considered in the M–K criterion. The quadratic Hill’s 48 yield function has been successfully applied for steel sheets over a long period and is still widely used.

The modified M-K theory is used to calculate the forming limits when there is an initial angle between the groove and the major principal stress direction under each loading condition (see Figure 5-9). Under each
loading condition, the minimum value of the calculated forming limit is regarded as the forming limit for angles varying from $0^\circ$ to $90^\circ$.

For area $a$,

\[
\sigma_{nn}^a = \sigma_1^a \cos^2 \phi + \sigma_2^a \sin^2 \phi \tag{5-52}
\]

\[
\sigma_{nt}^a = (\sigma_2^a - \sigma_1^a) \sin \phi \cos \phi \tag{5-53}
\]

\[
\sigma_{tt}^a = \sigma_1^a \sin^2 \phi + \sigma_2^a \cos^2 \phi \tag{5-54}
\]

For area $b$,

\[
\sigma_{nn}^b = \sigma_{11}^b \cos^2 \phi + \sigma_{22}^b \sin^2 \phi + \sigma_{12}^b \sin 2\phi \tag{5-55}
\]

\[
\sigma_{nt}^b = \sigma_{22}^b \cos^2 \phi + (\sigma_{22}^b - \sigma_{11}^b) \sin \phi \cos \phi \tag{5-56}
\]

\[
\sigma_{tt}^b = \sigma_{11}^b \sin^2 \phi + \sigma_{22}^b \cos^2 \phi + \sigma_{12}^b \sin 2\phi \tag{5-57}
\]

where $n$ and $t$ are the normal and tangential directions of the groove, respectively. According to the force equilibrium of the section between $a$ and $b$, we have

\[
\begin{align*}
F_{nn}^a &= F_{nn}^b \\
F_{nt}^a &= F_{nt}^b \\
\end{align*}
\tag{5-58}
\]

which gives

\[
\begin{align*}
\sigma_{nn}^a t^a &= \sigma_{nn}^b t^b \\
\sigma_{nt}^a t^a &= \sigma_{nt}^b t^b \\
\end{align*}
\tag{5-59}
\]

and

\[
\frac{\sigma_{nn}^a}{\sigma_{nt}^a} = \frac{\sigma_{nn}^b}{\sigma_{nt}^b} \tag{5-60}
\]

\[
\begin{align*}
t^a &= e^{\dot{c}^a t_0^a} \\
t^b &= e^{\dot{c}^b t_0^b} \\
\end{align*}
\tag{5-61}
\]
where \( t_a^o \) and \( t_b^o \) are the initial thickness of the normal and groove areas of the sheet before deformation, \( t^o \) and \( t^b \) are the true thickness of the sheet in the normal and groove areas during deformation, and \( \varepsilon_3 \) is the strain in the thickness direction.

Set \( f_0 = t^b / t^o \) where \( f_0 \) is the initial thickness imperfection.

\[
\frac{\sigma_1^o \cos^2 \phi + \sigma_2^o \sin^2 \phi}{(\sigma_{11}^o - \sigma_{12}^o)\sin\phi\cos\phi} = \frac{\sigma_1^b \cos^2 \phi + \sigma_2^b \sin^2 \phi + \sigma_{12}^b \sin 2\phi}{\sigma_{11}^b \cos^2 \phi + (\sigma_{11}^o - \sigma_{12}^o)\sin\phi\cos\phi}
\]

(5-62)

The angle of the groove, \( \phi \), is updated during the deformation using

\[
\tan(\phi + d\phi) = \tan(\frac{1 + d\varepsilon_1^o}{1 + d\varepsilon_2^o})
\]

(5-63)

where \( d\varepsilon_1^o \) and \( d\varepsilon_2^o \) are the principal strain increments in the plane of region \( a \).

The compatibility equation between the areas \( a \) and \( b \) is

\[
d\varepsilon_{in}^a = d\varepsilon_{in}^b
\]

(5-64)

where

\[
\begin{align*}
d\varepsilon_{in}^a &= d\varepsilon_1^a \sin^2 \phi + d\varepsilon_2^a \cos^2 \phi \\
d\varepsilon_{in}^b &= d\varepsilon_1^b \sin^2 \phi + d\varepsilon_2^b \cos^2 \phi - 2d\varepsilon_{12}^b \sin \phi \cos \phi
\end{align*}
\]

(5-65)

There are many methods to determine the stability point. Generally, the instability point is determined when the ratio of the strain increments outside the groove to those inside the groove are smaller than a certain value, typically 0.01.

The experimental technique to determine the FLD involved subjecting specimens of the AA3104-Thin and AA3104-Thick sheet to different in-plane strain states, by simple tensile testing or stretching over a hemispherical punch (Nakajima test). By varying the specimen width, different deep draw and stretch forming conditions occur on the sheet metal surface (from biaxial tension to plane strain to uniaxial tension). The tests on the material samples followed the ISO 12004-2:2008 standard and used a 101.6 mm ball traveling at 1.5mm/s with strains being measured using an Aramis 5M system.
The predicted forming limit diagrams for the AA3104-Thin material using the M-K theory with Swift’s hardening law for isotropic (von Mises), Hill48 (r-value based and stress ratio based), and Yl2000-2D (Swift and Voce hardening) material models are shown in Figure 5-10. The FLD predictions using the isotropic and Hill48 models significantly over-predict formability in the bi-axial direction, whereas the Yld2000-2D model underestimates the formability. Reasonable estimates are achieved in the plane strain direction. In uniaxial tension, the isotropic and Hill48 models over predict the formability.

The predicted forming limit diagrams for the AA3104-Thick material using the M-K theory with Swift’s hardening law for isotropic (von Mises), Hill48 (r-value based and stress ratio based), and Yl2000-2D (Swift and Voce hardening) material models are shown in Figure 5-11. Similar to the AA3104-Thin models, the FLD predictions using the isotropic and Hill48 models significantly over-predict formability in the bi-axial direction, whereas the Yld2000-2D model underestimates the formability. Reasonable estimates are achieved in the plane strain direction. In uniaxial tension, the isotropic and Hill48 models over predict the formability.

![Figure 5-9. Initial defect used in the M-K model.](image)
Figure 5-10. AA3104-Thin FLD using M-K model compared to experimental FLD.

Figure 5-11. AA3104-Thick FLD using M-K model compared to experimental FLD.
5.4 SUMMARY

This chapter provided a literature review of the traditional experimental and numerical approaches used to develop the forming limit diagrams. Several analytical models have been reviewed including Swift’s diffused necking model, Hill’s localized necking model, Hill’s (1993) yield criteria FLD, Stören and Rice’s vertex model, Bressan and Williams’s shear instability model, and Marciniak–Kuczynski’s model. All of these models were derived from the necking and stability analysis in the plane of a sheet. The M-K model was used to predict forming limit diagrams for AA3104-Thin and AA3104-Thick materials using Hill48 and Yld2000-2D models. The results were compared with the experimental FLD and show the importance of the yield function and hardening law on the prediction of the FLD particularly in the bi-axial region of the curve. Most of the models significantly over-predict the formability in the bi-axial direction.
5.5 REFERENCES


6.1 STRESS-BASED FLC BACKGROUND

In industrial sheet metal forming applications, complex work pieces and finished parts are usually manufactured in multi-step, discontinuous processes resulting in non-linear deformation paths. In the design process, most engineers and designers routinely use strain-based forming limit diagrams to assess formability. While this approach may provide general trends, it is only valid when deformation occurs along linear strain paths, i.e., \( \beta = \frac{d\varepsilon_2}{d\varepsilon_1} \) is constant. Under non-proportional loading conditions, the FLD cannot be applied to predict whether the forming process will be successful.

For computational purposes, one of the most promising solutions for dealing with strain path effects on the FLC is to use a stress-based approach as proposed by Kleemola and Pelkkikangas (1977), Arrieux et al. (1982), Stoughton (2000), and Stoughton and Yoon (2005). Kuwabara et al. (2003) measured the stress state near the forming limit of a tube deformed using internal pressure and end feed under proportional and non-proportional conditions. Paul (2015) provided a methodical comparison among all common available strain path independent strain/stress based limiting criteria and concluded that the selection of both the yield criterion and the strain-hardening law have substantial influence on the results. These authors confirmed that the forming limit as characterized by the state of stress is insensitive to the loading history. These authors have also shown that the FLC in stress space is path-independent and should be suitable for the analysis of any forming problem. Once path-independence of the FLC is established, either experimentally or analytically, then the limits to formability can be predicted accurately using a combination of the FLC and finite element simulation, not only for proportional loading but also in cases where a sheet element has a complex strain history. Despite the usefulness of the FLC concept, awareness of the path-independence of the FLC is not widespread.
One of the concerns about the stress-based FLC is attributed to the reduction of the slope of the true stress–strain relation. Due to this effect, larger changes in strain occur at stress levels close to the necking limit compared to stress levels further below the limit stress. To remedy this difficulty, Stoughton and Yoon (2012) proposed the PEPS diagram (PEPSD) as one of the metrics to assess formability. The new path in the PEPS diagram is determined based on the magnitude of effective plastic strain radius and the direction of the strain increment in the conventional strain diagram. It has been shown that the PEPS Diagram has a one-to-one mathematical correspondence to the stress-Based FLC. It has also been demonstrated from experimental data that the forming limit curve in the PEPS diagram is insensitive to changes in strain path. As can be seen in Figure 6-1, the FLCs described in the strain space with different pre-strains shown in Hosford (1993) are mapped to a single curve in the stress space and Polar EPS (PEPS) space and verifies the path independence in both the stress-based FLC and PEPS diagram. In addition, the PEPS diagram does not depend on the stress–strain relation. In one sense, that means they are less complex to use. This independence with respect to the stress–strain relation has an even bigger advantage in that the forming limit criterion might be extendable to material models in which the stress–strain relation is not monotonic. Path independent necking models can be a powerful tool to design nonlinear paths to maximize the formability.

The PEPS diagram can be mapped from a conventional forming limit curve, which can be obtained from experiments or M-K theory. The mapping procedure from a conventional FLD to the PEPS diagram involves a yield function. Stoughton and Yoon (2012) considered only Hill’s normal anisotropy. In this thesis, a general mapping theory including non-quadratic yield functions has been developed. Both Hill’s normal anisotropy and Yld2000-2D are employed to demonstrate a general mapping procedure. Dick et al. (2015) implemented the path independent Polar EPS Diagram and Stress-Based FLC into a User Material interface provided by commercial software.

Approaches considering path independency in forming limits based on the stress-based forming limit and Polar EPS Effective Plastic Strain (PEPS) diagram which appear to be an effective solution for nonlinear effects are discussed. A review of the stress-based forming limit diagram and the mapping from the traditional strain based FLC are presented. The mapping from the strain-based FLC to the PEPSD is also summarized. Finally, an EXCEL–based post-processing system for general use is introduced. The related theory has been implemented into a user material model in the commercial software LS-DYNA.
Figure 6-1. Experimental forming limit curves for linear strain paths and for a bilinear strain path after 0.07 strain in equal biaxial tension in (a) strain and (b) stress spaces. The green dashed lines with arrows in both figures show the corresponding strain and calculated stress increments due to pre-strain and three blue dashed lines show selected strain and corresponding calculated stress increments to the final point on the strain FLC. Note that the overlay of the two experimental stress FLCs is proof that stress-based FLCs are independent of the loading history.

6.2 REVIEW OF STRESS-BASED FLC THEORY
A representation of the forming limit behavior for proportional loading in strain space, i.e., the locus of principal strains, is specified as follows.

\[ \text{strain-FLC} = \begin{bmatrix} \varepsilon_{1}^{FLC} \\ \varepsilon_{2}^{FLC} \end{bmatrix} = \begin{bmatrix} 1 \\ \beta \end{bmatrix} \]  

where \( \beta = \left( \frac{\dot{e}_{2}^{FLC}}{\dot{e}_{1}^{FLC}} \right) \), a parameter in the range \( \beta = [-1,1] \) that defines the ratio of the plastic principal in-plane strain rates. For a point on the strain FLC for a linear strain path, \( \beta \) is a constant given by the ratio of the plastic strains at the point. A serious limitation of the strain-based FLC is that it applies only to cases of proportional loading, and will lead to a false assessment when the strain-path is highly non-linear.

A solution to this issue is to use the stress-based FLC, which has been shown to be independent of loading history. This section reviews how to derive the stress-based forming limit criterion from the strain-based FLC based on Hill’s (1948) model for a metal with normal anisotropy. The derived equations will be used to explain the history dependent variables accounting for nonlinearity later.

The minor principal stress, \( \sigma_{2}^{FLC} \), is proportional to the major principal stress, \( \sigma_{1}^{FLC} \), by a parameter \( \alpha = [-1,1] \), i.e.,

\[ \sigma_{2}^{FLC} = \alpha \sigma_{1}^{FLC} \]  

Note that with the specified range for the \( \alpha \) parameter, the magnitude of the minor principal stress is always less than or equal to the major stress. The major principal stress can be used as a normalizing factor in the following derivation without concern about singularities in the calculations.

The Hill (1948) yield function for normal anisotropy under plane stress is defined in terms of the principal stress as

\[ \bar{\sigma}(\sigma_{1}^{FLC}, \sigma_{2}^{FLC}) = \sqrt{\left(\sigma_{1}^{FLC}\right)^{2} + \left(\sigma_{2}^{FLC}\right)^{2} - \frac{2\bar{r}}{1+\bar{r}} \sigma_{1}^{FLC} \sigma_{2}^{FLC}} \]  

where \( \bar{r} \) is the averaged \( r \)-value. The ratio of the major and minor principal stresses defines the parameter \( \alpha \) as implied by Equation 6-2. Then, Equation 6-3 can be written as

\[ \bar{\sigma} = \sigma_{1}^{FLC} \sqrt{1 + \alpha^{2} - \frac{2\bar{r}}{1+\bar{r}} \alpha} \]  

(6-4)
or
\[
\frac{\sigma}{\sigma_{1}^{RC}} = \sqrt{1 + \alpha^2 - \frac{2\bar{r}}{1+\bar{r}}} \alpha = \sigma_{\alpha}(1, \alpha) \quad (6-5)
\]

By using the associated flow rule with Equation 6-4, the major and minor plastic strains under linear strain paths can be defined as
\[
\dot{e}_1^{RC} = \dot{e}_p = \frac{\partial \dot{\sigma}}{\partial \sigma_1^{RC}} = \dot{e}_p \left( \frac{\sigma_1^{RC}}{\sigma} \right) \left( 1 - \frac{\bar{r}}{1+\bar{r}} \alpha \right) \quad (6-6)
\]
\[
\dot{e}_2^{RC} = \dot{e}_p = \frac{\partial \dot{\sigma}}{\partial \sigma_2^{RC}} = \dot{e}_p \left( \frac{\sigma_1^{RC}}{\sigma} \right) \left( \alpha - \frac{\bar{r}}{1+\bar{r}} \right) \quad (6-7)
\]

Then, the principal strain ratio, \( \beta \) is defined as a function of the principal stress ratio \( \alpha \) as
\[
\beta = \frac{\dot{e}_1^{RC}}{\dot{e}_1^{RC}} = \frac{\partial \dot{\sigma}}{\partial \sigma_1^{RC}} = \frac{\alpha + \bar{r} \alpha - \bar{r}}{1-\bar{r} \alpha + \bar{r}} \quad (6-8)
\]

Alternatively,
\[
\alpha = \frac{\beta + \bar{r} + \bar{r} \beta}{1 + \bar{r} + \bar{r} \beta} \quad (6-9)
\]

On the other hand, the effective plastic strain at necking can be also defined by adding Equation 6-6 and Equation 6-7 and rearranging terms as
\[
\dot{e}_p^{RC} = \frac{\dot{e}_1^{RC} + \dot{e}_2^{RC}}{\dot{\sigma}} = \frac{(1+\beta)\dot{e}_1^{RC}}{\dot{\sigma}_1^{RC} + \dot{\sigma}_2^{RC}} \left( \frac{1+\bar{r}}{1+\alpha} \right) \quad (6-10)
\]

By inserting Equation 6-9 into Equation 6-10, the strain-rate potential for Hill's (1948) normal anisotropic yield function can be derived as
\[
\dot{e}_p^{RC} = \frac{1+\bar{r}}{\sqrt{1+2\bar{r}}} \sqrt{\left( \dot{e}_1^{RC} \right)^2 + \left( \dot{e}_2^{RC} \right)^2 + \frac{2\bar{r}}{1+\bar{r}} \dot{e}_1^{RC} \dot{e}_2^{RC}} \quad (6-11)
\]
or

\[ \dot{\varepsilon}^{\text{FLC}}_p = \int_0^t \dot{\varepsilon}^{\text{FLC}}_p \, dt \quad (6-12) \]

Using a hardening law and Equation 6-5,

\[ \bar{\sigma} = \sigma^{\text{FLC}}_1 \bar{\sigma}_\sigma(1, \alpha) = h(\varepsilon^{\text{FLC}}_p) \quad (6-13) \]

where

\[ h(\varepsilon^{\text{FLC}}_p) = K(\varepsilon \sigma + \varepsilon^{\text{FLC}}_p) \quad \text{or} \quad h(\varepsilon^{\text{FLC}}_p) = A - B \exp(-C \varepsilon^{\text{FLC}}_p) \quad (6-14) \]

We can finally calculate the major principal stress; i.e.,

\[ \sigma^{\text{FLC}}_1 = \frac{h(\varepsilon^{\text{FLC}}_p)}{\bar{\sigma}_\sigma(1, \alpha)} = \sqrt{1 + \alpha^2 - \frac{2\alpha}{1 + \alpha}} \quad (6-15) \]

and the minor principal stress using Equation 6-2.

The equations for calculating the effective plastic strain and principal stresses for an arbitrary bilinear plastic strain path is shown in Figure 6-2 for a special case of Hill’s (1948) normal anisotropy and generalizes the mapping procedure from the principal plastic strain-space to the principal stress space for arbitrary bilinear plastic strain increments to an intermediate strain state, \((e_x, e_y) = (e_x, e_y)\), ending at a final plastic strain state, \((e_x + \Delta e_x, e_y + \Delta e_y)\). An example of this transformation for two of the FLCs shown for AA2008-T4 is shown in Figure 6-2. In the figure, the black curve is the FLC for the condition of zero pre-strain. The FLC for a bilinear path with an equal-biaxial plastic pre-strain \((e_x, e_y) = (0.07, 0.07)\) is shown in red. There is no significant difference in the calculated stress FLCs for linear and nonlinear strain paths seen in Figure 6-7. Since these FLCs are determined from experiment, the analysis leads to the conclusion that stress-based forming limits are insensitive to the deformation history, and depend only on the stress state. This surprising result of a path independent stress-based forming limit is theoretically validated by review of bifurcation analyses that were originally used to explain the strain FLC, as explained in Stoughton and Zhu (2004). It is shown in this reference that the bifurcation models lead to an instability condition that can be conveniently expressed explicitly in terms of the current state of the true stress, without explicit dependence on any history variables. This result occurs prior to imposing any assumption on the
strain path that may have been involved to reach the critical stress condition at which the instability will occur, which of course is necessary to derive a strain forming limit criterion for a given strain path.

Normal Anisotropic Hill Model

\[
\begin{align*}
\varepsilon_1(t) &= \varepsilon_A + t\Delta\varepsilon_1 \\
\varepsilon_2(t) &= \varepsilon_B + t\Delta\varepsilon_2
\end{align*}
\]

\[
\bar{\varepsilon}_p(t) = t \frac{1 + \bar{\rho}}{\sqrt{1 + 2\bar{\rho}}} \sqrt{\Delta\varepsilon_1^2 + \Delta\varepsilon_2^2 + \frac{2\bar{\rho}}{1 + \bar{\rho}} \Delta\varepsilon_1 \Delta\varepsilon_2 + \frac{1 + \bar{\rho}}{\sqrt{1 + 2\bar{\rho}}} \sqrt{\Delta\varepsilon_A^2 + \Delta\varepsilon_B^2 + \frac{2\bar{\rho}}{1 + \bar{\rho}} \varepsilon_A \varepsilon_B} \]
\]

\[
\beta(t) = \frac{\Delta\varepsilon_2}{\Delta\varepsilon_1} \quad \alpha(t) = \frac{\beta(t) + \bar{\rho} + \bar{\rho} \beta(t)}{1 + \bar{\rho} + \bar{\rho} \beta(t)}
\]

\[
\sigma_1(t) = \frac{K(\varepsilon_1 + \bar{\varepsilon}_p(t))^n}{\sqrt{1 + \alpha(t)^2 - \frac{2\bar{\rho}}{1 + \bar{\rho}} \alpha(t)}}
\]

\[
\sigma_2(t) = \alpha(t) \sigma_1(t)
\]

Figure 6-2. Mapping procedure from plastic strain FLC to Stress FLC for a bi-linear strain path from \((\varepsilon_A, \varepsilon_B)\) with final increment \((\Delta\varepsilon_A, \Delta\varepsilon_B)\).

Figure 6-3 generalizes the mapping procedure from the principal plastic strain-space to the principal stress space for arbitrary plastic strain histories, \((\varepsilon_1, \varepsilon_2) = (\varepsilon_1(t), \varepsilon_2(t))\). The general equations to derive the effective plastic strain and principal stresses for a non-quadratic yield function or other yield functions with planar anisotropy are described in Stoughton and Yoon (2005).
6.3 POLAR EFFECTIVE PLASTIC STRAIN (PEPS) DIAGRAM

To understand the history dependent variables for non-linear deformation, it is helpful to more carefully review the mapping procedure from plastic strain space to stress space as illustrated in Figure 6-7. It is useful to point out that that the accounting of nonlinear paths comes into play at only two places. The first is in the definition of $\beta$, which for non-linear strain paths is defined in terms of the ratio of the current strain rates. Nonlinear paths also play a role in the definition of the effective plastic strain $\bar{\varepsilon}_p$, which is defined by the time integral of a function of the strain rates. This integration is complicated by the fact that $\beta$ is also changing in time. No other relation depends explicitly on deformation history. Since the stress tensor components are then defined explicitly in terms of the effective plastic strain and $\alpha$, or indirectly in terms of the effective plastic strain and $\beta$, the forming limit for linear and nonlinear deformations can also be characterized as a simple limit on the accumulated effective plastic strain, $\bar{\varepsilon}_p$, as a function of $\beta$, or as a function of $\alpha$.

One of the concerns about the stress FLC is the reduction of the slope of the true stress-strain relation. Due to this effect, larger changes in strain occur at stress levels close to the necking limit compared to stress levels further below the limit stress. This makes it difficult to visually see or quantify the margin of safety in the stress diagram without a magnifying glass or overlay of the contours of equivalent strain in the stress FLC. To remedy this difficulty, we will consider using the effective plastic strain as one of the
metrics to assess formability. Although effective plastic strain is described as a type of strain, it is not directly linked to the principal or tensor components of the strain tensor. It is however, uniquely linked to the stress tensor through the yield function and stress-strain relation, and therefore falls under the category of a stress metric.

The idea of an FLC based on the variables \((\overline{\varepsilon}_p, \alpha)\) was proposed by Yoshida et. al. (2007) and the idea of an FLC based on the variables \((\overline{\varepsilon}_p, \beta)\) was proposed by Zeng et al. (2008) and by Yoshida et al. (2007). These ideas are illustrated in Figure 6-4. The alternate diagrams are mathematically equivalent to the stress FLC for models with positive work hardening, as is evident in the one-to-one relationship between the stress state and the values of the alternate variables due to the monotonic hardening relationship, but these proposals have several important practical advantages: First they scale with the magnitude of strain, so it is easier to visualize safety margins for conditions that are near to the necking limit. Second, they do not depend on the stress-strain relation at all. In one sense, that means they are less complex to use; But this independence with respect to the stress-strain relation has an even bigger advantage in that the forming limit criterion might be extendable to material models in which the stress-strain relation is not monotonic. Stoughton and Yoon (2012) proposed a mathematically equivalent solution to using the \((\overline{\varepsilon}_p, \beta)\) variables as proposed by Zeng et al. (2008), which may be more appealing to industrial engineering applications. We propose to plot the data in a polar diagram of the \(\overline{\varepsilon}_p\) variable with the angle defined as the arctangent of the ratio of the principal strain rates,

\[
\theta = \tan^{-1}(\beta) = \tan^{-1}\left(\frac{\dot{\varepsilon}_2}{\dot{\varepsilon}_1}\right)
\]

In a Cartesian equivalent system, the variables of the proposed diagram become

\[
(y, x) = (\overline{\varepsilon}_p \cos(\theta), \overline{\varepsilon}_p \sin(\theta))
\]

where

\[
\begin{bmatrix}
\overline{\varepsilon}_p(t) \cos(\theta) \\
\overline{\varepsilon}_p(t) \sin(\theta)
\end{bmatrix} = \frac{1}{\sqrt{1 + \beta(t)^2}} 
\begin{bmatrix}
\overline{\varepsilon}_p(t) \\
\overline{\varepsilon}_p(t) \beta(t)
\end{bmatrix}
\]

(6-16)
It is important to emphasize, to avoid confusion, that there is no physical significance to the meaning of the variables \((y, x)\) in Equation 6-17. This transformation is only useful to employ standard 2D plotting software that may only accept Cartesian coordinate data. Equation 6-17 is necessary to enable use of this software to plot data in what effectively becomes a polar diagram. The physically meaningful variables with respect to formability are the angle \(\theta\) and the effective plastic strain, which is the radial variable in the polar diagram. The direction \(\theta\) reflects the direction of the current in-plane principal plastic strain rates, in a Cartesian coordinate system superimposed on the polar diagram.

The schematic diagram to implement the Polar EPS (Polar Effective Plastic Strain) diagram is shown in Figure 6-5 for a bilinear path. The new path is determined based on the magnitude of effective plastic strain radius and the direction of the strain increment in the conventional strain diagram used to define the direction in the new diagram by a line on the new EPS path projecting back to the origin. The advantages of the PEPS diagram are no significant/noticeable path dependence, no dependence on the stress-strain relation, and the shape is similar to the Strain FLC for the as-received making it an intuitive tool to use for those familiar with strain-based FLD’s. One limitation of the concept is the effective strain depends on the constitutive law used.
Figure 6.5. Graphical illustrations of Polar EPS Diagram (a) bi-linear path (b) arbitrary path.

6.4 EXCEL-BASED OR POST PROCESSING-BASED SYSTEM

In order to directly process the results from any FE software in a post-processing way, the following methodology has been newly developed. It has been implemented into both an Excel-Based System and a FORTRAN code for the convenience of use at ATC. This approach is independent from constitutive models. Any FE outcome can be utilized to trace the deformation under Polar EPS and stress-based FLC.

Let’s consider that stress and strain increments: \( \Delta \sigma = \sigma_{n+1} - \sigma_n \) and \( \Delta \varepsilon = \varepsilon_{n+1} - \varepsilon_n \).

Considering that strain from a commercial code is total strain, it is necessary to extract only the plastic part of the total strain. By using the constitutive relationship of

\[
\Delta \varepsilon = C \Delta \varepsilon^p = C (\Delta \varepsilon - \Delta \varepsilon^p)
\]

(6-19)
The plastic strain can be calculated as

\[ \Delta \varepsilon^p = \Delta \varepsilon - C^{-1} \Delta \sigma \] (6-20)

where

\[
C = \frac{E}{1 - \nu^2} \begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & (1-\nu) \frac{1}{2}
\end{bmatrix} = \begin{bmatrix}
a_1 & a_2 & 0 \\
a_2 & a_1 & 0 \\
0 & 0 & G
\end{bmatrix}
\] (6-21)

and

\[
C^{-1} = \begin{bmatrix}
a_1 & -a_2 & 0 \\
-a_2 & a_1 & 0 \\
0 & 0 & \frac{1}{G}
\end{bmatrix}
\] (6-22)

where \( d = a_1 * a_1 - a_2 * a_2 \).

By using Equation 6-22, the explicit form of Equation 6-20, the plastic strain increment can be expressed as

\[ \Delta \varepsilon^p_{xx} = \Delta \varepsilon_{xx} - \frac{a_1}{d} \Delta \sigma_{xx} - \frac{a_2}{d} \Delta \sigma_{yy} \] (6-23)

\[ \Delta \varepsilon^p_{yy} = \Delta \varepsilon_{yy} - \frac{a_2}{d} \Delta \sigma_{xx} + \frac{a_1}{d} \Delta \sigma_{yy} \] (6-24)

\[ \Delta \varepsilon^p_{xy} = \Delta \varepsilon_{xy} - \frac{1}{G} \Delta \sigma_{yy} \] (6-25)

By using Mohr’s Circle formula, the principal plastic strains are defined as

\[ \Delta \varepsilon^p_{1,2} = \frac{\Delta \varepsilon^p_{xx} + \Delta \varepsilon^p_{yy}}{2} \pm \sqrt{\left( \frac{\Delta \varepsilon^p_{xx} - \Delta \varepsilon^p_{yy}}{2} \right)^2 + \left( \Delta \varepsilon^p_{xy} \right)^2} \] (6-26)
Recalling the relationship of $\beta = \frac{\Delta \varepsilon_2}{\Delta \varepsilon_1}$, the Polar EPS space can be expressed as

$$
\begin{bmatrix}
\bar{\varepsilon}_p(t) \cos(\theta)
\\bar{\varepsilon}_p(t) \sin(\theta)
\end{bmatrix} = \frac{1}{\sqrt{1 + \beta(t)^2}}
\begin{bmatrix}
\bar{\varepsilon}_p(t)
\bar{\varepsilon}_p(t) \beta(t)
\end{bmatrix}
$$

(6-27)

where $\bar{\varepsilon}_p(t)$ is equivalent plastic strain, which can be taken from the output of a commercial finite element code.

Stresses used in the stress-based FLC can be calculated from Mohr-Circle directly as

$$
\Delta \sigma_{1,2} = \frac{\Delta \sigma_{xx} + \Delta \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\Delta \sigma_{xx} - \Delta \sigma_{yy}}{2}\right)^2 + \left(\Delta \sigma_{xy}\right)^2}
$$

(6-28)

The important thing is to exclude the unloading part and to project the stresses back to the yield surface before unloading. It can be implemented with the following formula as

$$
\Delta \sigma_p^\sigma = \frac{h(\bar{\varepsilon}_p)}{\sigma(\bar{\varepsilon})}
$$

(6-29)

In Equation 6-29, $\Delta \sigma_p^\sigma$ represent the projected stress. Also, $h(\bar{\varepsilon}_p)$ and $\sigma(\bar{\varepsilon})$ denotes the effective stresses from a hardening and a yield function, respectively.

In order to verify the implementation, two methodologies have been demonstrated. Figure 6-6 shows Polar EPS and stress-based FLC by using Excel-based post processing. In the figure, a conventional forming limit predicts failure, while path-independent Polar EPS and stress-based FLC predict safe forming. Figure 6-6 also shows FORTRAN-based post processing using the large results of a square cup simulation using commercial software Autoform. In this case, the conventional forming limit predicts safe forming, while polar EPS predicts failure. FORTRAN-based post-processing is beneficial when a large scale of data is processed.
6.5 MAPPING FROM STRAIN-BASED FLC TO STRESS-BASED FLC

The discussion above assumed that deformation occurs along linear strain paths, i.e., $\rho = \frac{d e_1}{d e_1}$ is constant. In practice, particularly for multi-step forming, this is not the case. Moreover, it was shown by Stoughton and Yoon (2005) that non-linear strain paths have an influence on the FLC. Graf and Hosford (1993) showed that the FLC strongly depends on the strain path for steel and aluminum alloy sheets, respectively. The characterization of forming limits in strain space is therefore a practical challenge for complex forming processes due to this sensitivity.

In order to compute the stress-based FLC, a representation of the forming limit behavior for proportional loading in strain space, i.e., the locus of principal strains, is specified as follows:

$$strain - FLC = \left[ \begin{array}{c} e_{1}^{FLC} \\ e_{2}^{FLC} \end{array} \right] = e_{1}^{FLC} \left( \rho, \theta \right) \left[ \begin{array}{c} 1 \\ \rho \end{array} \right]$$  \hspace{1cm} (6-30)

where $\rho$ is a parameter in the range $\rho = [-1, +1]$ that defines the ratio of the principal strains on the FLC.

A serious limitation of the strain-based FLD is that it applies only to cases of proportional loading, and will lead to a false assessment of formability in a multi-stage cup drawing operation, where the strain-path is
highly non-linear. The solution to the latter issue is to use the stress-based FLD, which has been shown to be independent of loading history. This section describes how to derive the stress-based forming limit criterion from the strain-based FLC’s defined in the above strain-based forming limit criterion.

The minor principal stress, \( \sigma_{2}^{FLC} \), is proportional to the major stress by a parameter \( \alpha = [-1,+1] \), i.e., \( \sigma_{2}^{FLC} = \alpha \sigma_{1}^{FLC} \). Note that with the specified range for the \( \alpha \) parameter, the magnitude of the minor principal stress is always less than or equal to the major stress, so that we can use the major principal stress as a normalizing factor in the following derivation without concern for singularities in the calculations. The stress tensor components at the stress state \( (\sigma_{1}^{FLC}, \sigma_{2}^{FLC} (=\alpha \sigma_{1}^{FLC}), \theta) \) on the FLC are given by

\[
\begin{bmatrix}
\sigma_{11}^{FLC} \\
\sigma_{22}^{FLC} \\
\sigma_{12}^{FLC}
\end{bmatrix} = \begin{bmatrix}
\sigma_{1}^{FLC} \cos^2(\theta) + \sigma_{2}^{FLC} \sin^2(\theta) \\
\sigma_{1}^{FLC} \sin^2(\theta) + \sigma_{2}^{FLC} \cos^2(\theta) \\
(\sigma_{1}^{FLC} - \sigma_{2}^{FLC}) \sin(\theta) \cos(\theta)
\end{bmatrix} \begin{bmatrix}
\lambda_{11} \\
\lambda_{22} \\
\lambda_{12}
\end{bmatrix}
\]

(6.31)

where

\[
\begin{bmatrix}
\lambda_{11} \\
\lambda_{22} \\
\lambda_{12}
\end{bmatrix} = \begin{bmatrix}
\cos^2(\theta) + \alpha \sin^2(\theta) \\
\sin^2(\theta) + \alpha \cos^2(\theta) \\
(1-\alpha) \sin(\theta) \cos(\theta)
\end{bmatrix}
\]

(6.32)

Specifically when \( \theta = 0 \) (a normal anisotropy), Equation 6.32 reduces to

\[
\begin{bmatrix}
\lambda_{11} \\
\lambda_{22} \\
\lambda_{12}
\end{bmatrix} = \begin{bmatrix}
1 \\
\alpha \\
0
\end{bmatrix}
\]

(6.33)

Using Equation 6.33 with the selected material model, we can define the yield function to within a constant \( \sigma_{1}^{FLC} \),

\[
\bar{\sigma}(\sigma_{11}^{FLC}, \sigma_{22}^{FLC}, \sigma_{12}^{FLC}) = \sigma_{1}^{FLC} \bar{\sigma}(\lambda_{11}, \lambda_{22}, \lambda_{12})
\]

(6.34)

The gradient of the plastic potential at this stress state is also uniquely defined in terms of the \( \lambda_{ij} \) coefficients and the material parameters, independent of the magnitude of \( \sigma_{1}^{FLC} \). Under proportional loading, the plastic strain tensor components are proportional to the gradients of the plastic potential,
\[
\begin{bmatrix}
    e_{11}^{\text{FLC}} \\
    e_{22}^{\text{FLC}} \\
    2e_{12}^{\text{FLC}}
\end{bmatrix}
= \tilde{e}_p^{\text{FLC}} \begin{bmatrix}
    p_{11} \\
    p_{22} \\
    p_{12}
\end{bmatrix}
= \tilde{e}_p^{\text{FLC}} \begin{bmatrix}
    \frac{\partial \sigma}{\partial \sigma_{11}^{\text{FLC}}} \\
    \frac{\partial \sigma}{\partial \sigma_{22}^{\text{FLC}}} \\
    \frac{\partial \sigma}{\partial \sigma_{12}^{\text{FLC}}}
\end{bmatrix}^T
\]  

(6-35)

and the principal strains on the forming limit are given by

\[
\begin{bmatrix}
    e_1^{\text{FLC}} \\
    e_2^{\text{FLC}}
\end{bmatrix}
= \frac{1}{2} \begin{bmatrix}
    e_{11}^{\text{FLC}} + e_{22}^{\text{FLC}} + \sqrt{(e_{11}^{\text{FLC}} - e_{22}^{\text{FLC}})^2 + (2e_{12}^{\text{FLC}})^2} \\
    e_{11}^{\text{FLC}} + e_{22}^{\text{FLC}} - \sqrt{(e_{11}^{\text{FLC}} - e_{22}^{\text{FLC}})^2 + (2e_{12}^{\text{FLC}})^2}
\end{bmatrix}
\]

(6-36)

Substitution of Equation 6-35 into Equation 6-36 gives,

\[
\begin{bmatrix}
    e_1^{\text{FLC}} \\
    e_2^{\text{FLC}}
\end{bmatrix}
= \frac{\tilde{e}_p^{\text{FLC}}}{2} \begin{bmatrix}
    p_{11} + p_{22} + \sqrt{(p_{11} - p_{22})^2 + p_{12}^2} \\
    p_{11} + p_{22} - \sqrt{(p_{11} - p_{22})^2 + p_{12}^2}
\end{bmatrix}
\]

(6-37)

Note that the principal strains at this stress state are oriented at an angle \(\theta' \neq \theta\) for anisotropic materials given by

\[
\tan(2\theta') = \frac{2e_{12}^{\text{FLC}}}{e_{11}^{\text{FLC}} - e_{22}^{\text{FLC}}} = \frac{p_{12}}{p_{11} - p_{22}}
\]

(6-38)

meaning that the orientation of the principal strains is, in general, not parallel to the orientation of the principal stresses for an anisotropic material. From Equation 6-37, the ratio of principal strains is defined uniquely in terms of the gradients of the plastic potential,

\[
\rho = \frac{e_2^{\text{FLC}}}{e_1^{\text{FLC}}} = \frac{p_{11} + p_{22} - \sqrt{(p_{11} - p_{22})^2 + p_{12}^2}}{p_{11} + p_{22} + \sqrt{(p_{11} - p_{22})^2 + p_{12}^2}}
\]

(6-39)

Using the values for \(\theta'\) and \(\rho\) from Equation 6-38 and Equation 6-39, respectively, the strain tensor components on the forming limit are now fully specified for this stress state using

\[
\begin{bmatrix}
    e_{11}^{\text{FLC}} \\
    e_{22}^{\text{FLC}} \\
    2e_{12}^{\text{FLC}}
\end{bmatrix}
= e_1^{\text{FLC}}(\rho, \theta') \begin{bmatrix}
    \cos^2(\theta') + \rho \sin^2(\theta') \\
    \sin^2(\theta') + \rho \cos^2(\theta') \\
    2(1 - \rho) \sin(\theta') \cos(\theta')
\end{bmatrix}
\]

(6-40)
where \( e_1^{FLC} (\rho, \theta') \) is defined by Equation 6-30. Adding together the two components of the principal strains from Equation 6-37 and solving for the effective plastic strain leads to the following equation, whose value can now be calculated explicitly,

\[
\bar{\varepsilon}_p^{FLC} = \frac{e_1^{FLC} + e_2^{FLC}}{p_{11} + p_{22}} = \frac{(1+\rho)e_1^{FLC} (\rho, \theta')}{p_{11} + p_{22}} = \frac{2e_1^{FLC} (\rho, \theta')}{p_{11} + p_{22} \sqrt{(p_{11} - p_{22})^2 + p_{12}^2}}
\]  

(6-41)

Using the power law or Voce law and inverting Equation 6-34, we can finally calculate the major principal stress,

\[
\sigma_1^{FLC} (\alpha, \theta) = \frac{K(\varepsilon_b + \bar{\varepsilon}_p^{FLC})}{\alpha} \sigma \left( \lambda_{11}, \lambda_{22}, \lambda_{12} \right)
\]  

(6-42)

or

\[
\sigma_1^{FLC} (\alpha, \theta) = \frac{A - B \exp(-C \bar{\varepsilon}_p^{FLC})}{\alpha} \sigma \left( \lambda_{11}, \lambda_{22}, \lambda_{12} \right)
\]  

(6-43)

and the minor principal stress \( \sigma_2^{FLC} = \alpha \sigma_1^{FLC} \). This procedure is followed for all values of \( \alpha = [-1, +1] \), defining the locus of points \( (\sigma_1^{FLC}, \sigma_2^{FLC}) \) on the stress-based FLC for major stress at an angle \( \theta \) to the rolling direction of the sheet. Each point on this curve, characterized by the function \( \sigma_1^{FLC} (\alpha, \theta) \), has a corresponding point on the experimental strain-based FLC given by the function \( e_1^{FLC} (\rho, \theta') \). The only difference is that the stress-based curves are applicable for arbitrary loading history, while the strain-based curves apply only for proportional loading.

**6.6 MAPPING FROM STRAIN-BASED FLC TO POLAR EPS FOR A GENERAL YIELD FUNCTION**

The stress component in the principal space is defined as

\[
\left[ \sigma \right] = \left[ \begin{array}{c} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{array} \right] = \left[ \begin{array}{c} \sigma_1 \\ \sigma_2 \end{array} \right] = \sigma_1 \left[ \begin{array}{c} 1 \\ \alpha \end{array} \right]
\]  

(6-44)

Then, the strain ratio can be connected by the stress-ratio with the flow rule as
\[
\beta = \frac{\varepsilon_p^o}{\varepsilon_{p1}^o} = \frac{\partial \sigma}{\partial \sigma_1} \frac{\partial \sigma}{\partial \sigma_2}
\]  
(6-45)

Equation 6-45 can be converted to the following non-linear equation for any yield function as

\[
F = \frac{\partial \sigma}{\partial \sigma_1 (1)} - \beta \frac{\partial \sigma}{\partial \sigma_2 (\alpha)} = 0
\]  
(6-46)

Equation 6-46 can be solved by the N-R method by linearizing Equation 6-45 as

\[
F + \frac{\partial F}{\partial \alpha} \Delta \alpha = 0
\]  
(6-47)

Then, \( \Delta \alpha \) can be derived as

\[
\Delta \alpha = -\frac{F}{\frac{\partial F}{\partial \alpha}} = -\frac{\beta \frac{\partial \sigma}{\partial \sigma_1 (1)} - \frac{\partial \sigma}{\partial \sigma_2 (\alpha)}}{\beta \frac{\partial^2 \sigma}{\partial \sigma_1 (1) \partial \sigma_2 (\alpha)} - \frac{\partial^2 \sigma}{\partial \sigma_2 (\alpha) \partial \sigma_2 (\alpha)}}
\]  
(6-48)

where

\[
\alpha_{i+1} = \alpha_i + \Delta \alpha
\]  
(6-49)

The initial guess for \( \alpha_0 \) is generated using the von Mises yield function, i.e.,

\[
\alpha_0 = \frac{2\beta + 1}{2 + \beta}
\]  
(6-50)

Once \( \alpha \) is obtained, the effective plastic strain (strain-rate potential) can be obtained as

\[
\varepsilon_p = \frac{\varepsilon_{p1}^o + \varepsilon_{p2}^o}{p_{11} + p_{22}} = \frac{(1+\beta)\varepsilon_{p1}^o}{\frac{\partial \sigma}{\partial \sigma_1 (1)} + \frac{\partial \sigma}{\partial \sigma_2 (\alpha)}}
\]  
(6-51)

Finally, the Polar EPS diagram can be expressed in the following

\[
\begin{bmatrix}
\varepsilon_p(t) \cos(\theta) \\
\varepsilon_p(t) \sin(\theta)
\end{bmatrix} = \frac{1}{\sqrt{1 + \beta(t)^2}} \begin{bmatrix}
\varepsilon_p(t) \\
\varepsilon_p(t) \beta(t)
\end{bmatrix}
\]  
(6-52)
See Appendix D for a more detailed example for calculating \( \alpha \) using Hill’s 1948 anisotropic yield function.

6.7 MAPPING FROM STRAIN BASED-FLD TO STRESS-BASED FLD and PEPS - AA6111-T4 EXAMPLE

As discussed previously, the major limitation of the traditional strain-based FLD is that the curves are not static, rather they are inherently dynamic for sheet metal forming problems. This result suggests that the strain-based FLD for the as-received condition provides little to no clue about the margin of safety for a forming process that is non-proportional. The forming limit data for AA6111-T4 from Graf and Hosford, (1994), shown in Figure 5-5, is used to produce the stress-based forming limit curves and polar effective plastic strain curves.

The stress-based forming limit curves are provided in Figure 6-7. These curves indicated that the curves are nearly path independent for the pre-straining paths and loading conditions conducted in the experiment. Additional experimental data will help improve the understanding of load path dependency. The corresponding PEPS curves and shown in Figure 6-8. These curves show some path dependency, particularly for high bi-axial pre-straining.
Figure 6-7. Stress Based Forming Limit Diagram (SBFLD): major stress ($\sigma_1$) vs. minor stress ($\sigma_2$) plot for AA6111-T4: (a) biaxial pre-strain paths; (b) uniaxial and plane strain pre-strain paths; pre-strain given in parallel to rolling direction (RD); (c) uniaxial and plane strain pre-strain paths; pre-strain given in parallel to transverse direction (TD) - [Graf and Hosford – 1994].
Figure 6-8. Polar Effective Plastic Strain Diagram (PEPSD): $\epsilon_p \cos(\theta)$ vs. $\epsilon_p \sin(\theta)$ plot for AA6111-T4: (a) biaxial pre-strain paths; (b) uniaxial and plane strain pre-strain paths; pre-strain given in parallel to rolling direction (RD); (c) uniaxial and plane strain pre-strain paths; pre-strain given in parallel to transverse direction (TD) - (Graf and Hosford – 1994).
6.8 PROPOSED MODIFICATION TO POLAR EPS

A potential shortcoming of the original polar effective plastic strain diagram is illustrated in Figure 6-9. The diagram represents an initial loading condition in uniaxial compression. This is observed on the flange of a circular blank during cup drawing, or the leading edge of a cylindrical can during die necking. A second operation is employed to expand the geometry resulting from the first forming operation. In this process, the leading edge of the cylinder will be in uniaxial tension. In the current method, once the second forming operation begins, the vector would rotate through the PEPS forming limit curve and indicate necking or failure. This is typically not observed during the forming and shaping of aluminum beverage containers. The proposed modification calculates the direction of the vector in the same way as the original method, but uses the incremental plastic strain to calculate the magnitude of the new vector, and the ending point of the previous vector as its starting point. For proportional loading, the two processes are equivalent. This technique is outlined below.

As before, the principal plastic strain increments are defined using Equation 6-20, and Equation 6-22 through Equation 6-25

$$\Delta \varepsilon_{1,2} = \frac{\Delta \varepsilon_{xx}^p + \Delta \varepsilon_{yy}^p}{2} \pm \sqrt{\left(\frac{\Delta \varepsilon_{xx}^p - \Delta \varepsilon_{yy}^p}{2}\right)^2 + (\Delta \varepsilon_{xy}^p)^2}$$  \hspace{1cm} (6-53)

Recalling the relationship of $\beta = \frac{\Delta \varepsilon_2}{\Delta \varepsilon_1}$, the incremental change to the Polar EPS space can be expressed as

$$\begin{bmatrix} \Delta \vec{\varepsilon}_p(t) \cos(\theta) \\ \Delta \vec{\varepsilon}_p(t) \sin(\theta) \end{bmatrix} = \frac{1}{\sqrt{1 + \beta(t)^2}} \begin{bmatrix} \Delta \varepsilon_{p2}(t) \\ \Delta \varepsilon_{p1}(t) \beta(t) \end{bmatrix}$$  \hspace{1cm} (6-54)

where $\Delta \vec{\varepsilon}_p(t)$ is equivalent plastic strain increment, which can be taken from the output of a commercial finite element code. The position of the polar plastic strain is updated using

$$\begin{bmatrix} \vec{\varepsilon}_p(t) \cos(\theta) \\ \vec{\varepsilon}_p(t) \sin(\theta) \end{bmatrix} = \begin{bmatrix} \vec{\varepsilon}_p(t) \cos(\theta) \end{bmatrix}_{i-1} + \begin{bmatrix} \Delta \vec{\varepsilon}_p(t) \cos(\theta) \end{bmatrix}$$  \hspace{1cm} (6-55)

$$\begin{bmatrix} \vec{\varepsilon}_p(t) \sin(\theta) \end{bmatrix} = \begin{bmatrix} \vec{\varepsilon}_p(t) \sin(\theta) \end{bmatrix}_{i-1} + \begin{bmatrix} \Delta \vec{\varepsilon}_p(t) \sin(\theta) \end{bmatrix}$$

This new method will be utilized in Section 9.7 in the analysis of the expansion of an aluminum cup produced using a draw / reverse redraw operation.
6.9 SUMMARY

A review of the stress-based forming limit diagram and the mapping from the traditional strain based FLC is presented. The mapping from the strain-based FLC to the PEPSD is also summarized. An EXCEL–based post-processing system for general use in mapping from the strain-based system to a stress-based system and polar-based system for a path independent criteria is introduced. This technique is easily implemented in User MATerial (UMAT) functions in commercial codes. A mapping from strain-based FLC to Polar EPS for a general yield function is also derived. An example for AA6111-T4 is provided that demonstrates the nearly path independence of both the stress-based and polar-based diagrams. The polar-based system shows some path dependence for high bi-axial pre-straining. Finally, the framework for a modification to the current method for PEPS is proposed for application in multi-stage, discontinuous forming processes.

Figure 6-9. Proposed modification to Polar EPS method.
6.10 REFERENCES

CHAPTER 7

DUCTILE FRACTURE MODELS

7.1 LITERATURE REVIEW – FRACTURE

Failure and fracture predictions are the most important issues in analysis and design of metal forming processes. Most fracture theories have been proposed as a combination of brittle and ductile fracture models as shown in Figure 7-1a. However, it can be noted that shear deformation is also a major source of fracture by studying the Mohr-Coulomb fracture theory and the work of Wierzbicki et al. (2005). It has been proven that the criterion based on only brittle and ductile fractures is not successful for the prediction of fracture modes combining shear deformation, in spite of its generality. Subsequently, it has been proposed to construct an additional fracture plane which considers shear deformation as shown in Figure 7-1b. Also, it has been demonstrated that fracture is directly related to triaxiality. Stoughton (2008) further showed that during the investigation of the hemming process, the mechanisms for fracture and cracking, with or without necking, or necking without fracture or cracking, appear to depend explicitly on the interaction of the applied stress, including inhomogeneous distributions, and the nature of the metal.

![Figure 7-1a. Johnson-Cook failure model.](image1)

![Figure 7-1b. Failure model with shear plane.](image2)
Recently, advanced aluminum alloys, magnesium alloys, and high strength steels have been introduced in the automotive industry to satisfy the increasing requirement for high fuel efficiency and improved safety. These advanced metals, however, fail in ductile fracture with little necking compared with conventional alloys. In addition, ductile fracture is also observed in shear and compression at low or negative stress triaxiality in bulk metal forming by Bao and Wierzbicki (2004), Børvik et al. (2010), and Khan and Liu, (2012a, 2012b). Thus, advanced ductile fracture criteria are more suitable to predict failure of these advanced metals in wide loading conditions from tension, shear, and compression.

From the microscopic point of view, ductile fracture of metals can also be explained with the integral processes of void nucleation, void growth, void coalescence, and propagation of micro-cavities or cracks. These ductile fracture criteria can be classified into three branches: coupled fracture criteria, continuum damage criteria, and uncoupled fracture criteria. Coupled fracture criteria assume that strength of metals is affected by accumulated damage induced by nucleation, and growth and coalescence of voids, while damage predicted by uncoupled fracture criteria has no effect on the load capability of metals before final fracture.

### 7.1.1 COUPLED Fracture MODELS

One of the most popular coupled ductile fracture criteria is the Gurson–Tvergaard–Needleman (GTN) ductile fracture criterion. The original formulation of this model proposed by Gurson (1977) assumed that the degradation of the load carrying capacity, and ultimately the fracture of ductile metals, are caused by the evolution of voids. Gurson’s original model takes into account only the growth of pre-existing voids, without assuming any generative mechanisms. In order to overcome this limitation, Tvergaard (1982) and Tvergaard and Needleman (1984) proposed mathematical descriptions of the void nucleation, void growth, and void coalescence. The accumulated damage in the model is represented by void volume fraction. The void volume fraction is coupled by the constitutive equation to induce softening effects. Gologanu et al.(1997) extended Gurson’s approach, initially limited to spherical or cylindrical voids, to spheroidal ones. Their model, referred to as the GLD model, was obtained through limit-analysis of a spheroidal RVE containing a prolate or oblate spheroidal confocal void and subjected to conditions of homogeneous boundary strain rate. It consisted of four elements: a macroscopic yield criterion depending upon porosity and void shape, a macroscopic flow rule obeying the normality property, and evolution equations for the porosity and the void shape parameter. Wen et al. (2005) proposed a modified Gurson model to account for the softening effect due to void growth. They concluded that excessive growth of voids may lead to macroscale softening behavior of the microvoided material. Siruguet and Leblond
(2004) developed a model for porous ductile materials accounting for both void shape effects and influence of inclusions. Barsoum and Faleskog (2007a) performed experiments on a double notched tube (DNT) specimen subjected to combined tension–torsion load and showed that ductile failure not only depends on stress triaxiality, but is also strongly affected by the type of deviatoric stress state that prevails, which can be quantified by a stress invariant that discriminates between axisymmetric stressing and shear dominated stressing, e.g., the Lode parameter. As shown by Nahshon and Hutchinson (2008), additional modifications representing shear softening are necessary to obtain reasonable predictions of strain localization at low stress triaxialities. The Nahshon-Hutchinson model can predict failure under shear-dominated loading, such as during the cutting of sheet metal (Nahshon and Xue, 2009). However, Nielsen and Tvergaard (2009) found that this modification is inadequate in the case of high stress triaxialities, compromising the predictive capabilities of the original Gurson model for loading conditions where void growth is the main damage mechanism. Consequently, Nielsen and Tvergaard (2010) proposed a slight modification of the Nahshon-Hutchinson model, making the damage accumulation under shear-dominated stress states active only for low stress triaxialities. Jackiewicz (2011) extended the Gurson model with a new void coalescence criterion based on a simple assumption that a singular value of the effective stress triggers the coalescence of microvoids in materials. Xue et al. (2013) applied a two damage parameter extension to the GTN model to simulate the tension–torsion tests of two steels, Weldox 420 and Weldox 960, to investigate the Lode dependence of ductile fracture. The simulations captured the fact that the fracture strain does not decrease monotonically with increasing stress triaxiality.

### 7.1.2 CONTINUUM DAMAGE MODELS

Another popular coupled ductile fracture criterion is the continuum damage mechanics (CDM) initially introduced by Kachanov (1958) and further improved by Lemaitre (1985, 1996), Chaboche (1981, 1988a, 1988b), Saanouni and Chaboche (2003), Brünig (2003a, 2003b, 2006), and others. CDM considers the mechanics of material damage and its mechanical effects within the framework of continuum mechanics. CDM introduces a continuous damage variable by establishing an additional damage evolution equation for representing the local distribution of micro-defects. Brünig and Gerke (2011) implemented a generalized and extended version of Brünig’s damage model to simulate damage evolution in ductile metals under dynamic loading.
7.1.3 **UNCOUPLIED FRACTURE MODELS**

Uncoupled ductile fracture criteria were developed based on microscopic mechanisms, various hypotheses or experimental observations of ductile fracture. McClintock (1968) analytically investigated the growth of a cylindrical void while Rice and Tracey (1969) analyzed void growth using a single spherical void in an infinite solid under remote loading. Both their results stated that void growth was mainly controlled by the stress triaxiality. LeRoy et al. (1981) considered nucleation, shape change and coalescence of voids in the Rice–Tracey model. Johnson and Cook (1983) proposed an empirical model based on these findings and incorporated the effects of strain rate and temperature. Wilkins et al. (1980) proposed a weighting function depending on the asymmetry of the deviatoric principal stresses in addition to hydrostatic pressure. It was assumed that the effect of hydrostatic pressure and stress asymmetry on ductility were separable. Numerous phenomenological ductile fracture criteria (Freudenthal, 1950; Cockcroft and Latham, 1968; Brozzo et al., 1972; Oh et al., 1979; Oyane et al., 1980; Clift et al., 1990; Ko et al., 2007) were developed and widely employed to solve engineering problems (ex. compressive upsetting tests, axisymmetric extrusion, strip compression and tension, drawing and hub-hole expanding) due to their simplicity and few fracture parameters to be evaluated experimentally. Attempts to define a more general fracture criterion have led to the introduction of the third invariant of the stress tensor in the weighting function (e.g. Wierzbicki and Xue, 2005). Bai and Wierzbicki (2010) transposed the classical Mohr–Coulomb fracture criterion into the space of stress triaxiality, Lode angle and equivalent plastic strain, defining the so-called Modified Mohr–Coulomb (MMC) fracture criterion.

Bao and Wierzbicki (2004) used carefully designed fracture tests (see Figure 7-2) together with finite element simulation and divided the fracture mechanisms of metallic materials into three different branches with shear modes for negative stress triaxialities, void-growth-dominated modes for large positive triaxialities, and mixed modes for lower positive stress triaxialities. It was found that the ductile fracture strain is not necessarily a monotonic decreasing function of stress triaxiality. This finding was further developed by Wierzbicki and Xue (2009), Xue (2005), Bai and Wierzbicki (2008) to incorporate the effect of Lode angle (related to the third deviatoric stress invariant J3) to ductile fracture. Stoughton and Yoon (2011) also made use of their data to show that a Mohr–Coulomb criterion can describe the onset of fracture in stress space. Another criterion in stress space was proposed by Khan and Liu (2012) after supplementing Bao and Wierzbicki’s (2004) data with additional experimental data points. Bai and Wierzbicki (2010) postulated a Mohr–Coulomb fracture model in stress space to come up with the corresponding damage indicator model in the combined strain–stress space of equivalent plastic strain,
stress triaxiality and Lode angle parameter. A micro-mechanism inspired ductile fracture model was proposed by Lou and Huh (2013). Their model had also been partially validated using the data reported for aluminum 2024-T351 by Bao and Wierzbicki (2004).

Experimental results on advanced high strength steels by Li et al. (2010), Luo and Wierzbicki (2010) validated the applicability of the modified Mohr–Coulomb model. The investigation by Gao et al. (2009) on the ductile fracture of aluminum 5083 alloy showed the strong stress state effects on the plastic response and the ductile fracture behavior. These effects can be described by using the stress triaxiality and the Lode angle (which is related to the third invariant of the stress deviator). In particular, it is found that stress triaxiality has a relatively small effect on plasticity but a significant effect on ductile failure strain. On the other hand, the effect of the Lode angle on ductile fracture is negligible but its effect on plasticity is significant. By taking the advantages of both uncoupled models (ease of formulation and implementation) and coupled models (integration of damage to material behavior), Lian et al. (2013) have provided a hybrid damage mechanics model which explicitly considers the local state of stress by relying on all three invariants of the stress tensor. This model relies on the Bai and Wierzbicki (2010) uncoupled damage model, but its damage evolution law has been modified into a modified BW (MBW) model. The novelty in the model is that it distinguishes between damage and fracture, and in addition to the damage evolution and fracture, it quantitatively describes the damage initiation, which corresponds to a critical phase of material microstructure. Instead of using Lode angle (parameter), Lou et al. (2012), Lou and Huh (2013) used the Lode parameter and proposed a criterion by directly incorporating the existence of cut-off value of stress triaxiality. Khan and Liu (2012a) established a phenomenological isotropic fracture criterion based on the magnitude of stress vector (MSV) and the first invariant of stress tensor, which has been recently extended to study the effect of strain rate and temperature Khan and Liu (2012b). Voyiadjis et al. (2012) proposed a damage model which incorporates the effect of all three stress invariants directly and investigated the effects of large amplitude cyclic loading.

However, little is known on the effect of non-monotonic or non-proportional loading paths on the onset of ductile fracture. Bao and Treitler (2004) performed reverse loading experiments on notched axisymmetric bar AA2024-T351 specimens with pre-compression followed by tension all the way to fracture and observed a substantial increase in ductility due to pre-compression and developed a criterion to predict crack formation under complex reverse loadings. Bai (2008) discussed the phenomenological modification of the accumulation rule of damage indicator models to account for the effect of non-proportional loading histories on the onset of ductile fracture. He also showed important effects of non-
linear loading paths on ductile fracture during the crushing of prismatic columns. Benzerga et al. (2012) performed two-dimensional axisymmetric computations on a unit cell containing a periodic distribution of initially spherical voids to demonstrate that the strain to fracture is strongly path-dependent and cannot be represented as a function of the stress state only. The effect of the loading path on ductile failure has been investigated in detail in the context of sheet metal forming (e.g. Kleemola and Pelkkikangas 1977), Wagoner and Laukonis (1983), Hosford and Caddell (2014). It is now well established that the failure strains, i.e. the strains at the onset of localized necking, are strongly loading path dependent. More recent studies on bi-axially loaded aluminum tubes Korkolis et al. (2010), Korkolis and Kyriakides (2009), Korkolis and Kyriakides (2011a), and Korkolis and Kyriakides (2011b) confirm a loading path dependency of the forming liming diagram (FLD) in both strain and stress space. Furthermore, they confirmed the non-quadratic form of the yield surface of aluminum. The results of Korkolis and Kyriakides (2009) provided insight into the loading path dependency of ductile fracture. In a first approximation, the stress state in their radial loading experiments may be considered as constant up to the point of fracture initiation (rupture). In their experiments with corner loading paths, the specimens were pre-loaded under uniaxial tension, before continuously increasing the stress triaxiality throughout plastic loading up to that of selected radial loading experiments.
Figure 7-2. Specimens used by Bao and Wierzbicki to calibrate the fracture locus for a wide range of stress triaxiality.

7.2 MODIFIED MOHR COULOMB MODEL

Bai and Wierzbicki (2010) developed a novel weighting function by transforming the classical stress-based Mohr–Coulomb failure criterion into the space of stress triaxiality, Lode parameter, and equivalent plastic strain. The resulting phenomenological ductile fracture model is called the ‘Modified Mohr–Coulomb’ (MMC) model, which can predict both triaxiality and Lode angle dependence. The model is an extension of the maximum shear stress fracture criterion and is thus able to predict shear fracture which has been shown to be a major failure mode of sheet metal under various loading conditions. Moreover, the MMC criterion has only three free parameters to be calibrated and is thus an ideal choice for industrial applications based on Luo and Wierzbicki (2009), and Li et al. (2010).

Three invariants of the stress tensor \([\sigma]\) are used in the development of the model. The invariants are described with the principal stresses denoted, \(\sigma_1, \sigma_2, \sigma_3\). A variant of the first invariant to the stress tensor is as follows,
\[ p = -\sigma_m = -\frac{1}{3} \text{tr}(\sigma) = -\frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \]  \hspace{1cm} (7-1)

Variants of the second and third invariants of the deviatoric stress tensor \([S]\) can be written as

\[ q = \bar{\sigma} = \sqrt{\frac{3}{2}} [S] : [S] = \sqrt{\frac{1}{2}} \left[ (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] \]  \hspace{1cm} (7-2)

and

\[ r = \left( q \frac{[S] : [S]}{[S]} \right)^{\frac{1}{3}} = \left( \frac{27}{2} \text{det}([S]) \right)^{\frac{1}{3}} = \left( \frac{27}{2} (\sigma_1 - \sigma_m)(\sigma_2 - \sigma_m)(\sigma_3 - \sigma_m) \right)^{\frac{1}{3}} \]  \hspace{1cm} (7-3)

where \([S] = [\sigma] + p[I]\)  \hspace{1cm} (7-4)

and \([I]\) is the identity tensor and \(s_1 \geq s_2 \geq s_3\) is assumed. The stress triaxiality factor, \(\eta\), and Lode parameter, \(L\), are defined as

\[ \eta = -\frac{p}{q} = \frac{\sigma_m}{\bar{\sigma}} \]  \hspace{1cm} (7-5)

\[ L = \frac{2\sigma_2 - \sigma_1 - \sigma_3}{\sigma_1 - \sigma_3} \]  \hspace{1cm} (7-6)

Under the assumption of plane stress, the relationship between the stress triaxiality and the strain ratio \(\alpha\) for an isotropic material is:

\[ \eta = \frac{1 + \alpha}{\sqrt{3} \sqrt{1 + \alpha + \alpha^2}} \]  \hspace{1cm} (7-7)

The Lode angle is related to the normalized third invariant \(\xi\) by

\[ \xi = \left( \frac{r}{q} \right)^3 = \cos(3\theta) \]  \hspace{1cm} (7-8)
The normalized third deviatoric stress invariant can be expressed in term of the Lode angle shown in Malvern (1969), Xu and Liu (1995). Since the range of the Lode angle is $0 \leq \theta \leq \pi/3$, the range of $\xi$ is $-1 \leq \xi \leq 1$. The geometrical representation of the Lode angle is shown in Figure 7-3.

![Diagram of stress invariants and coordinate systems](image)

**Figure 7-3. Three types of coordinate systems in the space of principal stresses.**

The Lode angle can be normalized by the Lode angle parameter

$$\bar{\theta} = 1 - \frac{6\theta}{\pi} = 1 - \frac{2}{\pi}\arccos \xi$$

(7-9)

Such that the range of $\bar{\theta}$ is $-1 \leq \bar{\theta} \leq 1$. Various stress states encountered in “classical” specimens used for plasticity and fracture testing can be uniquely characterized by the defined set of parameters $(\eta, \bar{\theta})$, as shown in Table 7-1.
Table 7-1. Ten types of classical specimens for plasticity and fracture calibration

<table>
<thead>
<tr>
<th>No.</th>
<th>Specimen Type</th>
<th>Stress Triaxiality, $\eta$</th>
<th>Lode Angle Parameter, $\bar{\Theta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Smooth round bars, tension</td>
<td>$\frac{1}{3}$</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>Notched round bars, tension</td>
<td>$\frac{1}{3} + \sqrt{2}\ln\left(1 + \frac{a}{2R}\right)$</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>Plastic plane strain, tension</td>
<td>$\frac{\sqrt{3}}{3}$</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>Flat grooved plates, tension Bai et. al. (2006b)</td>
<td>$\frac{\sqrt{3}}{3}\left[1 + \sqrt{2}\ln\left(1 + \frac{t}{4R}\right)\right]$</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>Torsion or shear</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>Cylinders, compression</td>
<td>$-\frac{1}{3}$</td>
<td>-1</td>
</tr>
<tr>
<td>7</td>
<td>Equi-biaxial plane stress, tension</td>
<td>$\frac{2}{3}$</td>
<td>-1</td>
</tr>
<tr>
<td>8</td>
<td>Equi-biaxial plane stress, compression</td>
<td>$-\frac{2}{3}$</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>Plastic plane strain, compression</td>
<td>$-\frac{\sqrt{3}}{3}$</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>Notched round bars, compression</td>
<td>$-\left[\frac{1}{3} + \sqrt{2}\ln\left(1 + \frac{a}{2R}\right)\right]$</td>
<td>-1</td>
</tr>
</tbody>
</table>

In the expressions of $\eta$, $R$ is the radius of a notch or groove, $a$ is the radius of a round bar at the notch, $t$ is the thickness of a flat grooved plate at the groove.

It was shown by Wierzbicki and Xue (2005), and Bai and Wierzbicki (2008) that the plane stress condition, $\sigma_3 = 0$, uniquely relates the parameters $\eta$ and $\xi$ or $\bar{\Theta}$ through

$$
\xi = \cos(3\bar{\Theta}) = \cos\left(\frac{\pi}{2}(1 - \bar{\Theta})\right) = \sin\left(\frac{\pi}{2}\bar{\Theta}\right) = -\frac{27}{2} \eta \left(\eta^2 - \frac{1}{3}\right)
$$

(7-10)

A plot of Equation 7-10 is shown in Figure 7-4. The function has three roots corresponding to pure shear $(\eta = 0, \bar{\Theta} = 0)$. 
Figure 7-4. Conceptual representation of the initial stress states on the plane of $\eta$ and $\theta$.

The modified Mohr-Coulomb model is uniquely defined by five parameters, including three material parameters $C_1, C_2, C_3$ and takes the following form:

$$
\varepsilon_f(\eta, \theta) = \left[ \frac{A}{C_2} \left( C_3 + \frac{\sqrt{3}}{2} \sqrt{1 - C_1} \left( \sec \left( \frac{\pi \theta}{6} \right) - 1 \right) \right) \right] \times 
\left[ \sqrt{1 + \frac{C_2^2}{3}} \cos \left( \frac{\pi \theta}{6} \right) + C_1 \left( \eta + \frac{1}{3} \sin \left( \frac{\pi \theta}{6} \right) \right) \right]^{-1/n}
$$

(7-11)

where $A$ and $n$ are the strength coefficient and the exponent of the power law hardening rule, respectively. For plane stress conditions, the equation reduces to

$$
\varepsilon_f(\eta) = \left[ \frac{A}{C_2} f_3 \left( \frac{1 + C_1^2}{3} f_1 + C_1 \left( \eta + \frac{2}{3} f_2 \right) \right) \right]^{-1/n}
$$

(7-12)

where

$$
f_1 = \cos \left[ \frac{1}{3} \arcsin \left( \frac{27}{2} \eta \left( \eta^2 - \frac{1}{3} \right) \right) \right]
$$

(7-13)

$$
f_2 = \sin \left[ \frac{1}{3} \arcsin \left( \frac{27}{2} \eta \left( \eta^2 - \frac{1}{3} \right) \right) \right]
$$

(7-14)
The definition of the equivalent strain for material exhibiting plastic isotropy is

\[
d\bar{\varepsilon}_p = \frac{2d\varepsilon_1}{\sqrt{3}}\sqrt{1 + \alpha + \alpha^2}
\]  

(7-17)

The principal stresses can be calculated using the following

\[
\begin{align*}
\sigma_1 &= \sigma_m + s_1 = \sigma_m + \frac{(3-L)\bar{\sigma}}{3\sqrt{L^2 + 3}} = \left(\eta + \frac{(3-L)}{3\sqrt{L^2 + 3}}\right)\bar{\sigma} \\
\sigma_2 &= \sigma_m + s_2 = \sigma_m + \frac{(2L)\bar{\sigma}}{3\sqrt{L^2 + 3}} = \left(\eta + \frac{2L}{3\sqrt{L^2 + 3}}\right)\bar{\sigma} \\
\sigma_3 &= \sigma_m + s_3 = \sigma_m - \frac{(3+L)\bar{\sigma}}{3\sqrt{L^2 + 3}} = \left(\eta - \frac{3+L}{3\sqrt{L^2 + 3}}\right)\bar{\sigma}
\end{align*}
\]  

(7-18)

The MMC fracture locus shown in Figure 7-5 consists of four branches in the space of equivalent strain to fracture and the stress triaxiality \((\bar{\varepsilon}_f, \eta)\). The first branch corresponds to the stress states between equi-biaxial tension \((\eta = 2/3)\) and plane strain \((\eta = 1/\sqrt{3})\). The second branch covers the range from plane strain \((\eta = 1/\sqrt{3})\) to uniaxial tension \((\eta = 1/3)\). The third branch extends from uniaxial tension \((\eta = 1/3)\) to pure shear \((\eta = 0)\). The fourth and last branch is applicable to the stress states between pure shear \((\eta = 0)\) and uniaxial compression \((\eta = -1/3)\). A fifth branch ranging from uniaxial compression \((\eta = -1/3)\) to balanced biaxial compression \((\eta = -2/3)\) will be discussed in the next section. The four branches of the MMC fracture locus can be transformed to the space of principal strains \((\varepsilon_1, \varepsilon_2)\) with the resulting locus of fracture points referred to as the Fracture Forming Limit Diagram (FFLD), as shown in Figure 7-6. The first branch corresponds to the stress states between equi-biaxial tension \((\alpha = 1)\) and plane strain \((\alpha = 0)\). The second branch covers the range from plane strain \((\alpha = 0)\) to uniaxial tension \((\alpha = -1/2)\). The third branch extends from uniaxial tension \((\alpha = -1/2)\) to pure shear \((\alpha = -1)\). The fourth
and last branch is applicable to the stress states between pure shear \((\alpha = -1)\) and uniaxial compression \((\alpha = -2)\).

**Figure 7-5.** The 2D MMC plane stress fracture locus.

**Figure 7-6.** The 2D MMC plane stress fracture locus represented in the space of principal true strains.

### 7.3 EXTENDED MODIFIED MOHR COULOMB MODEL

Lou et al. (2014) proposed a macroscopic ductile fracture criterion based on micro-mechanism analysis of nucleation, growth, and shear coalescence of voids from experimental observation of fracture surfaces. The proposed ductile fracture model contains a changeable cut-off value for the stress triaxiality to represent effects of micro-structure, the Lode parameter, temperature, and strain rate on ductility of metals. The proposed model can provide a satisfactory prediction of ductile fracture for metals from compressive upsetting tests to plane strain tension with slanted fracture surfaces. The Lode dependent form of the proposed ductile fracture criterion is
The proposed ductile fracture criterion with the suggested Lode dependent cut-off value for the stress triaxiality in Equation 7-16 is applied to construct the fracture locus of AA 2024-T351 since these experimental results cover wide loading conditions from compression, shear, and tension. The experimental data points are taken from Bai and Wierzbicki (2010), and Khan and Liu (2012a). Bao and Wierzbicki (2004) carried out the first 15 tests while Bai and Wierzbicki (2010) simulated these tests and reported the equivalent plastic strains to fracture with the corresponding stress triaxiality and Lode angle parameter. Khan and Liu (2012a) carried out three additional tests of AA2024-T351 since the experimental data of Bao and Wierzbicki (2004) in some stress triaxiality is insufficient or missing. Three principal stresses to fracture for these tests are also summarized in Table 7-2. The Hollomon power strain hardening rule is

$$\sigma = 740 \times (\bar{\sigma}^p)^{0.15}$$

for AA2024-T351 and is summarized in Table 7-2.

**Table 7-2. Experimental results of fracture stresses and strains for AA2024-T351.**

<table>
<thead>
<tr>
<th>Test #</th>
<th>$\sigma_1$ [MPa]</th>
<th>$\sigma_2$ [MPa]</th>
<th>$\sigma_3$ [MPa]</th>
<th>$\sigma_m$ [MPa]</th>
<th>$\sigma_f$</th>
<th>$\eta_f$</th>
<th>$\eta$</th>
<th>$\dot{\eta}$</th>
<th>$\dot{\gamma}$</th>
<th>$L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>707.9</td>
<td>45.3</td>
<td>45.0</td>
<td>266.0</td>
<td>662.8</td>
<td>0.469</td>
<td>0.401</td>
<td>0.999</td>
<td>-0.99</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>793.5</td>
<td>180.0</td>
<td>180.0</td>
<td>394.4</td>
<td>613.7</td>
<td>0.283</td>
<td>0.626</td>
<td>0.999</td>
<td>-0.99</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>902.3</td>
<td>336.5</td>
<td>524.9</td>
<td>566.0</td>
<td>0.167</td>
<td>0.927</td>
<td>0.998</td>
<td>-0.99</td>
<td>-0.822</td>
<td>0.79</td>
</tr>
<tr>
<td>4</td>
<td>699.6</td>
<td>338.2</td>
<td>23.0</td>
<td>353.6</td>
<td>586.4</td>
<td>0.210</td>
<td>0.603</td>
<td>0.075</td>
<td>-0.06</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>71.0</td>
<td>0</td>
<td>-620.4</td>
<td>-183.1</td>
<td>658.8</td>
<td>0.451</td>
<td>-0.278</td>
<td>-0.082</td>
<td>0.79</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>122.5</td>
<td>-0.8</td>
<td>-572.1</td>
<td>-150.1</td>
<td>641.9</td>
<td>0.380</td>
<td>-0.234</td>
<td>-0.681</td>
<td>0.64</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>122.4</td>
<td>-0.3</td>
<td>-565.6</td>
<td>-147.8</td>
<td>635.6</td>
<td>0.356</td>
<td>-0.223</td>
<td>-0.679</td>
<td>0.64</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>131.9</td>
<td>-0.2</td>
<td>-555.1</td>
<td>-141.1</td>
<td>631.4</td>
<td>0.341</td>
<td>-0.224</td>
<td>-0.652</td>
<td>0.61</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>116.3</td>
<td>-2.8</td>
<td>-627.5</td>
<td>-171.3</td>
<td>692.0</td>
<td>0.622</td>
<td>-0.248</td>
<td>-0.714</td>
<td>0.68</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>349.6</td>
<td>0</td>
<td>-327.8</td>
<td>7.3</td>
<td>586.7</td>
<td>0.211</td>
<td>0.012</td>
<td>0.036</td>
<td>0.03</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>451.2</td>
<td>-0.1</td>
<td>-237.8</td>
<td>71.1</td>
<td>606.2</td>
<td>0.261</td>
<td>0.117</td>
<td>0.338</td>
<td>0.31</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>628.2</td>
<td>12.5</td>
<td>-0.3</td>
<td>213.5</td>
<td>622.2</td>
<td>0.310</td>
<td>0.343</td>
<td>0.966</td>
<td>-0.95</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>680.5</td>
<td>32.4</td>
<td>-0.5</td>
<td>237.5</td>
<td>665.1</td>
<td>0.480</td>
<td>0.357</td>
<td>0.918</td>
<td>-0.90</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>640.6</td>
<td>27.7</td>
<td>0.6</td>
<td>223.0</td>
<td>626.9</td>
<td>0.326</td>
<td>0.356</td>
<td>0.929</td>
<td>-0.91</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>657.7</td>
<td>22.6</td>
<td>22.3</td>
<td>234.2</td>
<td>635.3</td>
<td>0.355</td>
<td>0.369</td>
<td>0.999</td>
<td>-0.99</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>355.2</td>
<td>0</td>
<td>-355.2</td>
<td>0</td>
<td>615.3</td>
<td>0.288</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>506.8</td>
<td>0</td>
<td>-160.4</td>
<td>115.5</td>
<td>603.2</td>
<td>0.253</td>
<td>0.191</td>
<td>0.536</td>
<td>-0.51</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>0</td>
<td>-226.7</td>
<td>-715.7</td>
<td>-314.1</td>
<td>633.6</td>
<td>0.348</td>
<td>-0.496</td>
<td>-0.398</td>
<td>0.36</td>
<td></td>
</tr>
</tbody>
</table>

The proposed ductile fracture criterion with the suggested Lode dependent cut-off value for the stress triaxiality in Equation 7-19 is applied to construct the fracture locus of AA 2024-T351 in Table 7-2 for its accuracy verification. To apply the proposed criterion in Equation 7-19, the constant offset of $C$ is first set to 1/3.
Then three material constants need to be calculated by fracture experiments. The material constants were optimized to be $C_1 = 4.0983, C_2 = 0.4316, C_3 = 0.3914$ by the least square error method with 18 experimental data points of AA2024-T351 in Table 7-2. Using the calculated material constants, the fracture locus is constructed and compared with 18 experimental data points in the space of $(\eta, \bar{\sigma}^p)$ as depicted in Figure 7-7. The fracture locus can be transformed to the space of principal strains $(\varepsilon_1, \varepsilon_2)$ with the resulting locus shown in Figure 7-8.

Figure 7-7. Branches of the fracture locus for plane stress in the space of $(\eta, \bar{\sigma}^p)$. 
7.4 MAPPING FROM STRESS FRACTURE TO POLAR EPS FRACTURE

The stress components are given in the principal space:

$$
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_{12}
\end{bmatrix} =
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{12}
\end{bmatrix} =
\begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3
\end{bmatrix} =
\begin{bmatrix}
1 \\
\alpha \\
0
\end{bmatrix}
$$

(7-20)

Then, using Equation 7-20 with the selected material model, we can define the yield function to within a constant $\sigma_1$,

$$
\bar{\sigma}(\sigma_{11}, \sigma_{22}, \sigma_{12}) = \sigma_1 \bar{\sigma}(\lambda_{11}, \lambda_{22}, \lambda_{12}) = \sigma_1 \bar{\sigma}(1, \alpha, 0)
$$

(7-21)

Equation 7-21 can be now connected to a hardening model with Power or Voce model as

$$
\sigma_1 \bar{\sigma}(1, \alpha, 0) = K \left( \epsilon_0 + \epsilon_\sigma \right)^n
$$

(7-22)

or

$$
\sigma_1 \bar{\sigma}(1, \alpha, 0) = A - B \exp(-C\epsilon_\sigma)
$$

(7-23)
From Equation 7-22, the effective plastic strain can be obtained as

$$\bar{\varepsilon}_p = \left( \frac{\sigma_1 \bar{\sigma}(1,\alpha,0)}{K} - \varepsilon_0 \right)^{1/n}$$

(7-24)

or from Equation 7-23,

$$\bar{\varepsilon}_p = -\ln \left( \frac{A - \sigma_1 \bar{\sigma}(1,\alpha,0)}{B} \right) / C$$

(7-25)

Then, the principal components of plastic strain are expressed as

$$\begin{bmatrix} \varepsilon_1^p \\ \varepsilon_2^p \end{bmatrix} = \frac{\bar{\varepsilon}_p}{2} \begin{bmatrix} p_{11} + p_{22} + \sqrt{(p_{11} - p_{22})^2 + p_{12}^2} \\ p_{11} + p_{22} - \sqrt{(p_{11} - p_{22})^2 + p_{12}^2} \end{bmatrix}$$

(7-26)

where

$$\begin{bmatrix} p_{11} \\ p_{22} \\ p_{12} \end{bmatrix} = \left[ \frac{\partial \bar{\sigma}}{\partial \sigma_{11}}, \frac{\partial \bar{\sigma}}{\partial \sigma_{22}}, \frac{\partial \bar{\sigma}}{\partial \sigma_{12}} \right]^T$$

(7-27)

Recalling the relationship of $\beta = \frac{\varepsilon_2^p}{\varepsilon_1^p}$, the polar EPS space can be expressed as

$$\begin{bmatrix} \bar{\varepsilon}_p(t) \cos(\theta) \\ \bar{\varepsilon}_p(t) \sin(\theta) \end{bmatrix} = \frac{1}{\sqrt{1 + \beta(t)^2}} \begin{bmatrix} \bar{\varepsilon}_p(t) \\ \bar{\varepsilon}_p(t) \beta(t) \end{bmatrix}$$

(7-28)

### 7.5 MAPPING FROM STRAIN-BASED SYSTEM FOR A GENERAL YIELD FUNCTION

The stress component in the principal space is defined in Equation 7-20. The strain ratio can be determined by the stress-ratio with the flow rule as

$$\beta = \frac{d \varepsilon_2^p}{d \varepsilon_1^p} = \frac{d \lambda}{d \varepsilon_1^p} \frac{\partial \bar{\sigma}}{\partial \sigma_{22}} = \frac{d \lambda}{d \varepsilon_1^p} \frac{\partial \bar{\sigma}}{\partial \sigma_{12}}$$

(7-29)
Equation 7-29 can be converted to the following non-linear equation for any yield function as

\[
F = \frac{\partial \sigma}{\partial \sigma_1(1)} - \beta \frac{\partial \sigma}{\partial \sigma_2(\alpha)} = 0
\quad (7-30)
\]

Equation 7-30 can be solved by the N-R method by linearizing Equation 7-29 as

\[
F + \frac{\partial F}{\partial \alpha} \Delta \alpha = 0
\quad (7-31)
\]

Then, \( \Delta \alpha \) can be derived as

\[
\Delta \alpha = -\frac{F}{\frac{\partial F}{\partial \alpha}} = -\frac{\beta \frac{\partial \sigma}{\partial \sigma_1(1)} - \frac{\partial \sigma}{\partial \sigma_2(\alpha)}}{\beta \frac{\partial^2 \sigma}{\partial \sigma_1(1) \partial \sigma_2(\alpha)} - \frac{\partial^2 \sigma}{\partial \sigma_2(\alpha) \partial \sigma_2(\alpha)}}
\quad (7-32)
\]

where

\[
\alpha_{i+1} = \alpha_i + \Delta \alpha
\quad (7-33)
\]

Once \( \alpha \) is obtained, the effective plastic strain (strain-rate potential) can be obtained as

\[
\bar{\varepsilon}_p = \varepsilon_1^p + \varepsilon_2^p = \frac{(1 + \beta) \varepsilon_1^p}{p_{11} + p_{22} + \frac{\partial \sigma}{\partial \sigma_1(1)} + \frac{\partial \sigma}{\partial \sigma_2(\alpha)}}
\quad (7-34)
\]

The effective stress can be calculated from the hardening law (Swift) as

\[
\bar{\sigma} = K \left( \varepsilon_0 + \bar{\varepsilon}_p \right)^\sigma
\quad (7-35)
\]

The mean stress is determined using

\[
p = -\sigma_m = -\frac{1}{3} \text{tr}(\sigma) = -\frac{1}{3} (\sigma_1 + \sigma_2 + \sigma_3)
\quad (7-36)
\]

The stress triaxiality factor, \( \eta \), and Lode parameter, \( L \), are defined as

\[
\eta = \frac{\sigma_m}{\bar{\sigma}}
\quad (7-37)
\]
\[
L = \frac{2\sigma_2 - \sigma_1 - \sigma_3}{\sigma_1 - \sigma_3}
\] 

Finally, the Polar EPS diagram can be expressed using Equation 7-28 (assuming proportional loading).

Using the above equations, mapping from principal strain space to effective plastic strain versus triaxiality space, principal stress space, and polar effective plastic strain space can be accomplished.

### 7.6 SUMMARY

The Mohr-Coulomb and extended Mohr-Coulomb fracture models are briefly reviewed. The concept for the PEPS diagram is being further developed to cover the path-independent fracture which is compatible with the stress-based fracture polygon by Stoughton and Yoon (2011). With this approach, the strain path can be utilized to describe the post-necking behavior. For this purpose, the mapping from the fracture polygon to the PEPS fracture curve is required. A general mapping method for this purpose is also developed in this chapter. The generalized mapping procedure has been implemented an EXCEL and allows for mapping the fracture locus from principal strain space to principal stress space, stress triaxiality space, and PEPS space for fracture prediction.
7.7 REFERENCES


growth and coalescence of microvoids under low, medium and high stress triaxiality loadings. Engng
high strain rates and temperatures. In: Proceedings of the 7th International Symposium on Ballistics,
Hague, pp. 541–547.
SSSR. Otdelenie Technicheskikh Nauk 8, 26–31 (in Russian).
Plast. 35, 1–12.
Khan, A.S., Liu, H., (2012b). Strain rate and temperature dependent fracture criteria for isotropic and
of steel copper and brass. Sheet Metal Ind. 54 (6), 591–592.
2059–2080.
107 (1), 83–89.
Lian J, Sharaf M, Archie F, Münstermann S., (2013). A hybrid approach for modelling of plasticity and
sheets using a phenomenological fracture model. International Journal of Solids and Structures 47,
3084–3102.
Englewood Cliffs, New Jersey.
371.


CHAPTER 8

MATERIAL CHARACTERIZATION TESTS - FRACTURE

8.1 INTRODUCTION

The materials considered in this study are AA3104-H19 having a sheet thickness of 0.274 mm and AA3104-H28 having a sheet thickness of 0.475 mm. These are common alloys used in the packaging industry to manufacture aluminum drawn and ironed beverage cans and bottles. Tensile tests were performed on specimens with a central hole, and notched specimens. Torsion of a double bridge tests and in-plane shear tests were conducted to generate the points near pure shear conditions. The Nakajima test was utilized to produce points in bi-axial tension. The data from the experiments is used to develop the fracture locus in principal strain space. A typical fracture surface is shown in Figure 8-1.
8.2 SPECIMENS WITH CENTRAL HOLE - DIMENSIONS AND TEST CONDITIONS

Specimens with a central hole in Figure 8-2 are prepared along different directions: 0, 15, 30, 45, 60, 75 and 90 measured from the rolling direction. The specimens are loaded at a velocity of 0.3 mm / min such that the straining is very slow. Patterns are prepared on the specimen surface and loading processes are recorded by a 3D GOM ARAMIS system for measurement of the fracture strain. Axial elongation is also extracted from the DIC analysis using the initial gauge length of 20 mm. In addition, the load is recorded by a load cell.

![Specimen Dimensions](image)

*Figure 8-2. Specimen dimensions for specimens with a central hole*

The deformation of specimens is analyzed by the DIC method. Figure 8-3 illustrates a typical distribution of the von Mises equivalent strain one step before the onset of ductile fracture computed by the DIC method. It is obvious that the maximum deformation takes place at the inner edge of the central hole. Thus, the equivalent strain at the inner edge of the central hole before ductile fracture is viewed as the equivalent strain to fracture. It must be emphasized that this strain consists of both elastic and plastic strains. However, the elastic strain is negligible compared to the large plastic strain at fracture. A typical evolution of the equivalent strain at the inner edge of the hole is compared with the load-stroke curve in Figure 8-4 for test #1 along the rolling direction. The equivalent strain at the reduction of load is taken as the equivalent strain to fracture. The graphs showing the evolution of the equivalent strain at the inner edge of the hole with the comparison to the load-stroke curves for each testing orientation are provided in Figure 8-5 through Figure 8-11. The fracture strokes of each test for the AA3104-Thin samples are summarized in Table 8-1 and compared in Figure 8-12. The fracture strains measured by the DIC method are summarized in Table 8-2 and compared in Figure 8-13. It is observed that the effect of the loading direction does not show identical results for both the fracture strokes and the fracture strains even though
the fracture strokes and fracture strains have similar tendencies according to loading directions. This is a result of the anisotropic hardening of the metal.

The graphs for the AA3104-Thick specimens showing the evolution of the equivalent strain at the inner edge of the hole with the comparison to the load-stroke curves for each testing orientation are provided in Figure 8-14 through Figure 8-20. The fracture strokes of each test for the AA3104-Thin samples are summarized in Table 8-3 and compared in Figure 8-21. The fracture strains measured by the DIC method are summarized in Table 8-4 and compared in Figure 8-22. Once again, it is observed that the effect of the loading direction does not show equivalent results for both the fracture strokes and the strains even though the fracture strokes and strains have similar tendencies according to loading directions. This is a result of the anisotropic hardening of the metal.

![Graph](image.png)

*Figure 8-3. Distribution of the von Mises equivalent strain before fracture for test #1 of specimens with a central hole.*
Figure 8-4. Evolution of equivalent strain and the load-stroke curve for test #1 of specimens with a central hole.
Figure 8-5. AA3104-Thin Specimens with a Central Hole - Loading direction: 00.
Figure 8-6. AA3104-Thin Specimens with a Central Hole - Loading direction: 15.
Figure 8-7. AA3104-Thin Specimens with a Central Hole - Loading direction: 30.
Figure 8-8. AA3104-Thin Specimens with a Central Hole - Loading direction: 45.
Figure 8-9. AA3104-Thin Specimens with a Central Hole - Loading direction: 60.
Figure 8-10. AA3104-Thin Specimens with a Central Hole - Loading direction: 75.
Figure 8-11. AA3104-Thin Specimens with a Central Hole - Loading direction: 90.
### Table 8-1. Fracture strokes of specimens with a central hole – AA3104-Thin

<table>
<thead>
<tr>
<th>Loading direction</th>
<th>0</th>
<th>15</th>
<th>30</th>
<th>45</th>
<th>60</th>
<th>75</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>0.2989</td>
<td>0.3829</td>
<td>0.397</td>
<td>0.3748</td>
<td>0.3685</td>
<td>--</td>
<td>0.3997</td>
</tr>
<tr>
<td>#2</td>
<td>0.3136</td>
<td>0.3858</td>
<td>0.389</td>
<td>0.3817</td>
<td>0.3655</td>
<td>0.3976</td>
<td>0.3787</td>
</tr>
<tr>
<td>#3</td>
<td>0.3133</td>
<td>0.3739</td>
<td>0.3736</td>
<td>0.3565</td>
<td>0.3721</td>
<td>0.367</td>
<td>0.3718</td>
</tr>
<tr>
<td>#4</td>
<td>0.3136</td>
<td>0.3955</td>
<td>0.3895</td>
<td>0.3643</td>
<td>0.3613</td>
<td>0.4012</td>
<td></td>
</tr>
<tr>
<td>#5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.3487</td>
<td></td>
</tr>
<tr>
<td>Mean value</td>
<td>0.30985</td>
<td>0.38453</td>
<td>0.38725</td>
<td>0.36933</td>
<td>0.36685</td>
<td>0.37862</td>
<td>0.3834</td>
</tr>
</tbody>
</table>

### Table 8-2. Fracture strains of specimens with a central hole – AA3104-Thin

<table>
<thead>
<tr>
<th>Loading direction</th>
<th>0</th>
<th>15</th>
<th>30</th>
<th>45</th>
<th>60</th>
<th>75</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>0.15619</td>
<td>0.17213</td>
<td>0.17991</td>
<td>0.16309</td>
<td>0.18216</td>
<td>0.18084</td>
<td>0.18533</td>
</tr>
<tr>
<td>#2</td>
<td>0.13845</td>
<td>0.15527</td>
<td>0.17577</td>
<td>0.1724</td>
<td>0.18562</td>
<td>0.18277</td>
<td>0.18289</td>
</tr>
<tr>
<td>#3</td>
<td>0.1623</td>
<td>0.16373</td>
<td>0.15666</td>
<td>--</td>
<td>0.19517</td>
<td>0.14063</td>
<td>0.18743</td>
</tr>
<tr>
<td>#4</td>
<td>0.16124</td>
<td>0.18634</td>
<td>0.17394</td>
<td>0.17857</td>
<td>0.17445</td>
<td>0.19227</td>
<td></td>
</tr>
<tr>
<td>#5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.1739</td>
<td></td>
</tr>
<tr>
<td>Mean value</td>
<td>0.15454</td>
<td>0.16937</td>
<td>0.17157</td>
<td>0.17135</td>
<td>0.18435</td>
<td>0.17408</td>
<td>0.18522</td>
</tr>
</tbody>
</table>
Figure 8-12. Effect of loading directions on fracture strokes for specimens with a central hole.

Figure 8-13. Effect of loading directions on fracture strains for specimens with a central hole.
Figure 8-14. AA3104-Thick Specimen with a Central Hole - Loading direction: 00.
Figure 8-15. AA3104-Thick Specimen with a Central Hole - Loading direction: 15.
Figure 8-16. AA3104-Thick Specimens with a Central Hole - Loading direction: 30.
Figure 8-17. AA3104-Thick Specimen with a Central Hole - Loading direction: 45.
Figure 8-18. AA3104-Thick Specimen with a Central Hole - Loading direction: 60.
Figure 8-19. AA3104-Thick Specimen with a Central Hole - Loading direction: 75.
Figure 8-20. AA3104-Thick Specimen with a Central Hole - Loading direction: 90.
Table 8-3. Fracture strokes of specimens with a central hole – AA3104-Thick

<table>
<thead>
<tr>
<th>Loading direction</th>
<th>0</th>
<th>15</th>
<th>30</th>
<th>45</th>
<th>60</th>
<th>75</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>0.40985</td>
<td>0.41469</td>
<td>0.46344</td>
<td>0.45641</td>
<td>0.47438</td>
<td>0.44858</td>
<td>0.48839</td>
</tr>
<tr>
<td>#2</td>
<td>0.40446</td>
<td>0.42093</td>
<td>0.46518</td>
<td>0.47084</td>
<td>0.47663</td>
<td>0.50802</td>
<td>0.45904</td>
</tr>
<tr>
<td>#3</td>
<td>0.38956</td>
<td>0.41384</td>
<td>0.44213</td>
<td>0.44783</td>
<td>0.48319</td>
<td>0.49014</td>
<td>0.4858</td>
</tr>
<tr>
<td>#4</td>
<td>0.40936</td>
<td>0.39401</td>
<td>0.45617</td>
<td></td>
<td>0.49094</td>
<td>0.5061</td>
<td></td>
</tr>
<tr>
<td>Mean value</td>
<td>0.4033</td>
<td>0.4165</td>
<td>0.4412</td>
<td>0.4578</td>
<td>0.4781</td>
<td>0.4844</td>
<td>0.4848</td>
</tr>
</tbody>
</table>

Table 8-4. Fracture strains of specimens with a central hole – AA3104-Thick

<table>
<thead>
<tr>
<th>Loading direction</th>
<th>0</th>
<th>15</th>
<th>30</th>
<th>45</th>
<th>60</th>
<th>75</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>0.33541</td>
<td>0.31739</td>
<td>0.35739</td>
<td>0.30796</td>
<td>0.34953</td>
<td>0.28273</td>
<td>0.32734</td>
</tr>
<tr>
<td>#2</td>
<td>0.28521</td>
<td>--</td>
<td>0.3547</td>
<td>0.36127</td>
<td>0.32575</td>
<td>0.35831</td>
<td>0.32612</td>
</tr>
<tr>
<td>#3</td>
<td>0.26942</td>
<td>0.29427</td>
<td>0.30778</td>
<td>0.335</td>
<td>0.36116</td>
<td>0.35396</td>
<td>0.35946</td>
</tr>
<tr>
<td>#4</td>
<td>0.31152</td>
<td>0.3216</td>
<td>0.27513</td>
<td>0.3508</td>
<td></td>
<td>0.32442</td>
<td>0.33617</td>
</tr>
<tr>
<td>Mean value</td>
<td>0.3004</td>
<td>0.3111</td>
<td>0.3238</td>
<td>0.3388</td>
<td>0.3455</td>
<td>0.3299</td>
<td>0.3373</td>
</tr>
</tbody>
</table>
Figure 8-21. Effect of loading directions on fracture strokes for specimens with a central hole.

Figure 8-22. Effect of loading directions on fracture strains for specimens with a central hole.
8.3 NOTCHED SPECIMENS - DIMENSIONS AND TEST CONDITIONS

Notched specimens are prepared according to the dimension in Figure 8-23 along different orientations: rolling, diagonal and transverse directions. The specimens are loaded at a velocity of 0.3 mm / min such that the strain-rate is small. Patterns are prepared on the specimen surface and loading processes are recorded by a 3D GOM ARAMIS system for the later measurement of fracture strain. Axial elongation is also extracted from DIC analysis where the initial gauge length is 20 mm. In addition, the load is recorded by a load cell.

![Figure 8-23. Dimensions for notched specimens.](image)

The deformation of specimens is analyzed by the DIC method. Figure 8-24 illustrates a typical distribution of the von Mises equivalent strain one step before the onset of ductile fracture computed by the DIC method. The equivalent strain at the center of the notched gauge before ductile fracture is viewed as the equivalent strain to fracture. It should be stressed that this strain consists of both elastic strain and plastic strain. But the elastic strain is negligible compared to the big plastic strain at fracture. The evolution of the equivalent strain at the center is compared with the load-stroke curve in Figure 8-25 for a typical specimen along the rolling direction. The equivalent strain at the drop of load is taken as the equivalent strain to fracture.

The graphs for the AA3104-Thin specimens showing the evolution of the equivalent strain at the middle of the notch with comparison to the load-stroke curves for each testing orientation are provided in Figure 8-26 (RD), Figure 8-27 (DD), and Figure 8-28 (TD). The fracture strokes of each test for the AA3104-Thin samples are summarized in Table 8-5 and compared in Figure 8-29. The fracture strains measured by the DIC method are summarized in Table 8-6 and compared in Figure 8-30.
The graphs for the AA3104-Thick specimens showing the evolution of the equivalent strain at the middle of the notch with comparison to the load-stroke curves for each testing orientation are provided in Figure 8-31 (RD), Figure 8-32 (DD), and Figure 8-33 (TD). The fracture strokes of each test for the AA3104-Thin samples are summarized in Table 8-7 and compared in Figure 8-34. The fracture strains measured by the DIC method are summarized in Table 8-8 and compared in Figure 8-35.

For both sets of specimens, it is observed that the effect of the loading direction shows different results for both the fracture strokes and strains even though the fracture strokes and strains have similar tendencies according to loading directions. This is once again a result of the anisotropic hardening of the metal.

Figure 8-24. Distribution of the von Mises equivalent strain before fracture for notched specimens.
Figure 8-25. Evolution of equivalent strain and the load-stroke curve for test #1 of notched specimens.
Figure 8-26. AA3104-Thin Specimen with a Notched R5 - Loading direction: 00.
Figure 8-27. AA3104-Thin Specimen with a Notched R5 - Loading direction: 45.
Figure 8-28. AA3104-Thin Specimen with a Notched R5 - Loading direction: 90.
### Table 8-5. Fracture strokes of notched specimens – AA3104-Thin

<table>
<thead>
<tr>
<th>Loading direction</th>
<th>0</th>
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<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>0.3279</td>
<td>0.3276</td>
<td>0.2814</td>
</tr>
<tr>
<td>#2</td>
<td>0.324</td>
<td>0.3282</td>
<td>0.3093</td>
</tr>
<tr>
<td>#3</td>
<td>0.3219</td>
<td>0.3225</td>
<td>0.3039</td>
</tr>
<tr>
<td>#4</td>
<td></td>
<td>0.3495</td>
<td>0.2979</td>
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<tr>
<td>#5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean value</td>
<td>0.3246</td>
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<td>0.29813</td>
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</table>

### Table 8-6. Fracture strains of notched specimens – AA3104-Thin

<table>
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<tr>
<th>Loading direction</th>
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<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>0.09346</td>
<td>0.08648</td>
<td>0.0668</td>
</tr>
<tr>
<td>#2</td>
<td>0.10357</td>
<td>0.09667</td>
<td>0.0699</td>
</tr>
<tr>
<td>#3</td>
<td>0.10136</td>
<td>0.09192</td>
<td>0.07922</td>
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<tr>
<td>#4</td>
<td>0.10043</td>
<td>0.10213</td>
<td>0.07286</td>
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<tr>
<td>#5</td>
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<tr>
<td>Mean value</td>
<td>0.0997</td>
<td>0.0943</td>
<td>0.07219</td>
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</tbody>
</table>
Figure 8-29. Effect of loading directions on fracture strokes for notched specimens.

Figure 8-30. Effect of loading directions on fracture strains for notched specimens.
Figure 8-31. AA3104-Thick Specimen with a Notched R5 - Loading direction: 00.
Figure 8-32. AA3104-Thick Specimen with a Notched R5 - Loading direction: 45.
Figure 8-33. AA3104-Thick Specimen with a Notched R5 - Loading direction: 90.
### Table 8-7. Fracture strokes of notched specimens – AA3104-Thick

<table>
<thead>
<tr>
<th>Loading direction</th>
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<td>#1</td>
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<td>#2</td>
<td>0.37941</td>
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<td>0.3309</td>
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<tr>
<td>#3</td>
<td>0.38786</td>
<td>0.39184</td>
<td>0.32104</td>
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<tr>
<td>#4</td>
<td>0.37441</td>
<td>--</td>
<td>0.345</td>
</tr>
</tbody>
</table>

Mean value: 0.3801, 0.3966, 0.3342

### Table 8-8. Fracture strains of notched specimens – AA3104-Thick

<table>
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</thead>
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<td>0.20979</td>
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<td>0.1797</td>
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<tr>
<td>#2</td>
<td>0.21068</td>
<td>0.22295</td>
<td>0.18</td>
</tr>
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<td>0.21748</td>
<td>0.2238</td>
<td>0.16842</td>
</tr>
<tr>
<td>#4</td>
<td>0.20601</td>
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</tr>
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</table>

Mean value: 0.2110, 0.2345, 0.1760
Figure 8-34. Effect of loading directions on fracture strokes for AA3104-Thick notched specimens.

Figure 8-35. Effect of loading directions on fracture strains for AA3104-Thick notched specimens.
8.4 TORSION OF DOUBLE BRIDGE SPECIMENS - DIMENSIONS AND TEST CONDITIONS

Torsion tests for sheet materials are still rarely applied. The initial work for the in-plane torsion test was done by Marciniak (1961), who proposed this test in order to investigate cyclic hardening of copper. A round sheet specimen is clamped concentrically in the center and on its outer rim. By a relative rotation between inner and outer fixtures, shear deformation takes place in the annular area in-between. Compared to other shear tests, the loading is applied as a moment instead of a force couple. Therefore, no additional unwanted reaction moment is created which has to be compensated by the clamps. Further developments by Tekkaya et al. (1982) allow the usage of the in-plane torsion test for the flow curve determination. The in-plane torsion test is free of edge effects due to the absence of any edges. Thus, a high deformation can be achieved. Shouler and Allwood (2010) introduced a shear test specimen that has double test zones which can effectively avoid the torque caused by a load imbalance on each test zone. This test can be conducted like a tensile test and by accordingly arranged cutouts, shear and tension loading is applied on parts of the specimen. Brosius et al. (2011) presented a modified geometry with two slits in the round sheet specimen, creating two shear zones where the deformation is localized. By twisting the clamps, both bridges are deformed in the same orientation. This modified specimen geometry exhibits completely different characteristics than the original in-plane torsion test without slits. While the original in-plane torsion test is able to generate a shear stress state without edge effects, the twin bridge specimen is suitable to determine shear curves for anisotropic materials. Yin et al. (2012) presented a method based on the modified geometry to identify kinematic hardening parameters using an inverse approach without stress measurements.

A conventional shear fracture specimen (Figure 8-43) was initially attempted with the AA 3104-Thin material. However, due to the thin gauge, out of plane buckling occurred and thus, was not able to achieve fracture. Due to this limitation, a new shear method based on torsion was used for the AA 3104-Thin material. A double-bridged torsion specimen and torsion experiment equipment are illustrated in Figure 8-36. Torsion specimens are prepared along different directions: rolling, diagonal, and transverse directions. The in-plane torsion tests were conducted to characterize fracture behavior in shear. Patterns are prepared on the specimen surface and deformation procedures are recorded for the further analysis of plastic deformation until final ductile fracture.
The deformation of the specimens is analyzed by the DIC method. Figure 8-37 illustrates the distribution of the von Mises equivalent strain one step before the onset of ductile fracture computed by the DIC method. Ductile fracture initiates at the edge of the bridge. However, the stress state at the edge is far away from pure shear but closer to uniaxial tension. Accordingly, the equivalent strain at the center of the bridge before ductile fracture is viewed as the equivalent strain to fracture in shear. It should be stressed that this strain consists of both elastic strain and plastic strain. But the elastic strain is negligible compared to the large plastic strain at fracture. The evolution of the strain components at the bridge center is compared in Figure 8-38 for test #3 along the diagonal direction. The equivalent strain before onset of ductile fracture is taken as the equivalent strain to fracture. Results for the three orientations (RD, DD, TD) are shown in Figure 8-39, Figure 8-40, and Figure 8-41.
Figure 8-37. Distribution of the von Mises equivalent strain before fracture for AA3104-Thin torsion specimens.

Figure 8-38. Evolution of strain components at the bridge center for test #3 of AA3104-Thin torsion specimens.
Figure 8-39. AA3104-Thin Specimen Torsion Test - Loading direction: 00.
**Figure 8-40.** AA3104-Thin Specimens Torsion Test - Loading direction: 45.
Figure 8.41. AA3104-Thin Specimen Torsion Test - Loading direction: 90.
The fracture strains measured by DIC method are summarized in Table 8-9 and compared in Figure 8-42. Anisotropy is also observed to be very strong in shear. Four tests were carried out for each direction, but deformation of only one bridge of the in-plane torsion specimens was recorded. Once ductile fracture initiated at the other bridge whose deformation was not recorded for DIC analysis, there will be no fracture for the recorded bridge. Therefore, only one fracture strain was measured by DIC along the RD and the DD, however, only two fracture strains were computed for the TD, even though four torsion tests were conducted for each direction.

Table 8-9. Fracture strains of in-plane torsion specimens – AA3104-Thin

<table>
<thead>
<tr>
<th>Loading direction</th>
<th>0</th>
<th>45</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>--</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>#2</td>
<td>--</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>#3</td>
<td>0.30363</td>
<td>0.32968</td>
<td>0.39025</td>
</tr>
<tr>
<td>#4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>#5</td>
<td></td>
<td></td>
<td>0.36805</td>
</tr>
<tr>
<td>Mean value</td>
<td>0.30363</td>
<td>0.32968</td>
<td>0.36915</td>
</tr>
</tbody>
</table>

Figure 8-42. Fracture strain of AA3104-Thin torsion tests along different loading directions.
8.5 IN-PLANE SHEAR SPECIMENS - DIMENSIONS AND TEST CONDITIONS

A simple in-plane shear specimen in Figure 8-43 is employed to study ductile fracture behavior in shear. The specimens are prepared along different directions: rolling, diagonal, and transverse directions. Patterns are prepared on the specimen surface and deformation procedures are recorded for the further analysis of plastic deformation until final ductile fracture.

![Figure 8-43. Dimensions for notched specimens.](image)

The deformation of the specimens is analysis by the DIC method. Figure 8-44 illustrates the distribution of the von Mises equivalent strain one step before the onset of ductile fracture computed by the DIC method. Ductile fracture probably initiates at the edge of the shear gauge. However, the stress state at the edge is far away from pure shear but closer to uniaxial tension. Accordingly, the equivalent strain at the center of the gauge before ductile fracture is viewed as the equivalent strain to fracture in shear. It should be stressed that this strain consists of both elastic strain and plastic strain. But the elastic strain is negligible compared to the big plastic strain at fracture. The evolution of the strain at the gauge center is compared with load-stroke curves in Figure 8-45 for test #3 along RD. The equivalent strain before onset of ductile fracture is taken as the equivalent strain to fracture. The results in the RD, the DD, and the TD directions are shown in Figure 8-46, Figure 8-47, and Figure 8-48, respectively.
Figure 8-44. Distribution of the von Mises equivalent strain before fracture for AA3104-Thick in-plane shear specimens.

Figure 8-45. Evolution of equivalent strain and the load-stroke curve for test #3 of AA3104-Thick in-plane shear specimens along RD.
Figure 8-46. AA3104-Thick Specimen In-Plane Shear Test - Loading direction: 00.
Figure 8-47. AA3104-Thick Specimen In-Plane Shear Test - Loading direction: 45.
Figure 8-48. AA3104-Thick Specimens In-Plane Shear Test - Loading direction: 90.
Effect of the loading directions for the fracture strokes in shear is illustrated in Figure 8-49 and summarized in Table 8-10. The fracture strains measured by the DIC method are summarized in Table 8-11 and compared in Figure 8-50. Anisotropy is also observed to be very strong in shear. Four tests were carried out for each direction, but deformation of only one bridge of the in-plane shear specimens was recorded. One ductile fracture initiated at the other bridge whose deformation was not recorded for DIC analysis, there will be no fracture for the recorded bridge. Therefore, only one fracture strain was measured by DIC for along RD and DD, and two fracture strains were computed for TD, even though four torsion tests were conducted for each direction.

Table 8-10. Fracture strokes of AA3104-Thick in-plane shear specimens

<table>
<thead>
<tr>
<th>Loading direction</th>
<th>0</th>
<th>45</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>1.27371</td>
<td>0.85288</td>
<td>0.93773</td>
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<tr>
<td>#2</td>
<td>1.27403</td>
<td>0.8312</td>
<td>1.00282</td>
</tr>
<tr>
<td>#3</td>
<td>1.26473</td>
<td>0.84676</td>
<td>0.96991</td>
</tr>
<tr>
<td>Mean value</td>
<td>1.2708</td>
<td>0.8436</td>
<td>0.9702</td>
</tr>
</tbody>
</table>

Table 8-11. Fracture strains of AA3104-Thick in-plane shear specimens

<table>
<thead>
<tr>
<th>Loading direction</th>
<th>0</th>
<th>45</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1</td>
<td>0.50239</td>
<td>0.56</td>
<td></td>
</tr>
<tr>
<td>#2</td>
<td>0.65</td>
<td>0.53122</td>
<td>0.54843</td>
</tr>
<tr>
<td>#3</td>
<td>0.66198</td>
<td>0.51</td>
<td>0.565</td>
</tr>
<tr>
<td>Mean value</td>
<td>0.6560</td>
<td>0.5145</td>
<td>0.5578</td>
</tr>
</tbody>
</table>
Figure 8-49. Effect of loading directions on fracture strokes for the AA3104-Thick in-plane shear specimens.

Figure 8-50. Fracture strain of AA3104-Thick in-plane shear tests along different loading directions.
8.6 NAKAJIMA TEST

The Nakajima test was carried out on both the AA3104-Thin and AA3104-Thick samples to measure fracture strain near the balanced biaxial tension condition. The contacting surfaces of the specimens and the punch were lubricated by soft plastic and cream so that ductile fracture initiates at the dome of the specimens. Three tests were done on both sets of samples and all the fracture paths were observed along the transverse directions as depicted in Figure 8-51.

The loading processes of the Nakajima tests were recorded by the 3D ARAMIS system for the measurement of the fracture strain. The distribution of the von Mises equivalent plastic strain for the AA3104-Thin specimen is depicted in Figure 5-52. The evolution of strain is compared with the load stroke curve in Figure 8-53. The strain at the drop of load is viewed as the fracture strain. Fracture strains are measured by DIC for three Nakajima tests: 0.2560, 0.2822 and 0.2870. The average fracture strain of these three tests is calculated as 0.2751.

The distribution of the von Mises equivalent plastic strain for the AA3104-Thick specimen is depicted in Figure 8-54. The evolution of strain is compared with the load stroke curve in Figure 8-55. The strain at the drop of load is viewed as the fracture strain. Fracture strains are measured by DIC for three Nakajima tests: 0.5081, 0.4860 and 0.4999. The average fracture strain of these three tests is calculated as 0.4980.

Figure 8-51. Nakajima tests results showing fracture orientation for AA3104-Thin and AA3104-Thick.
Figure 8-52. Distribution of the von Mises equivalent strain before fracture for AA3104-Thin Nakajima tests.
Figure 8-53. Evolution of equivalent strain and the load-stroke curve for test #3 for AA3104-Thin Nakajima specimens.
Figure 8-54. Distribution of the von Mises equivalent strain before fracture for AA3104-Thick Nakajima tests.
**Figure 8-55.** Evolution of equivalent strain and the load-stroke curve for test #3 for AA3104-Thick Nakajima specimens.
8.7 FRACTURE LOCUS – AA3104-THIN

The fracture locus of AA3104-Thin material is constructed by a newly proposed ductile fracture criterion below:

\[
\left( \frac{2}{\sqrt{l^2 + 3}} \right)^{C_1} \left( \frac{f(\eta, L, C)}{f(1/3, -1, C)} \right)^{C_1} \frac{\varepsilon_1^p}{C_3} = C_3 \tag{8-1}
\]

with

\[
f(\eta, L, C) = \eta + C_4 \frac{3 - L}{3\sqrt{l^2 + 3}} + C \tag{8-2}
\]

In the modified Lode dependent cutoff value for the stress triaxiality in Equation 8-2, effect of the Lode parameter is modeled by the material constant \( C_4 \) introduced in Equation 8-2. Then the shear ductile fracture criterion in Equation 8-1 is enhanced by the modified Lode dependent cutoff value for the stress triaxiality and denoted as DF2015 – see Lou et al. (2016). In the DF2015 criterion, there are five material constants of \( C_1, C_2, C_3, C_4 \) and \( C \). \( C_1 \) controls shear coalescence of void, \( C_2 \) adjusts the void growth, \( C_3 \) is equal to the fracture strain at uniaxial tension, \( C_4 \) represents the Lode-governed torsion of voids and adjusts the Lode dependence of the cutoff value for the stress triaxiality, and \( C \) models the height of the cutoff value for the stress triaxiality. In the DF2015 model, the addition of the biaxial fracture strain is considered compared to the DF2014 model which accounts for tension, shear, and plane strain fracture strains. The cut-off value \( C \) shown in the model is the same as Lou et al. (2014).

The fracture loci are illustrated in Figure 8-56 along the rolling direction, Figure 8-57 along the diagonal direction, and Figure 8-58 along the transverse direction in the space of stress triaxiality, Lode parameter, and the equivalent strain to fracture. Figure 8-59 and Figure 8-60 compare the fracture loci under plane stress of different loading directions in the space of stress triaxiality and the fracture strain, and in the space of the principal strains. It is obvious that anisotropy in ductile fracture is strong and must be considered in modeling of ductile fracture for metal forming.
Figure 8-56. Fracture locus for AA3104-Thin along the rolling direction in the space of the stress triaxiality, Lode parameter and equivalent strain to fracture.

Figure 8-57. Fracture locus for AA3104-Thin along the diagonal direction in the space of the stress triaxiality, Lode parameter and equivalent strain to fracture.
Figure 8-58. Fracture locus for AA3104-Thin along the transverse direction in the space of the stress triaxiality, Lode parameter and equivalent strain to fracture.

Figure 8-59. Comparison of plane stress fracture loci for AA3104-Thin under different loading directions in the space of stress triaxiality and equivalent strain.
Figure 8-60. Comparison of plane stress fracture loci for AA3104-Thin under different loading directions in principal strain space.

8.8 FRACTURE LOCUS – AA3104-THICK

The fracture locus of AA3104-Thick material is constructed by a newly proposed ductile fracture criterion using Equation 8-1 and Equation 8-2. The fracture loci are illustrated in Figure 8-61 along the rolling direction, Figure 8-62 along the diagonal direction, and Figure 8-63 along the transverse direction in the space of stress triaxiality, Lode parameter, and the equivalent strain to fracture. Figure 8-64 and Figure 8-65 compare the fracture loci under plane stress of different loading directions in the space of stress triaxiality and the fracture strain, and in the space of the principal strains. It is obvious that anisotropy in ductile fracture is strong and must be considered in modeling of ductile fracture for metal forming.
Figure 8-61. Fracture locus for AA3104-Thick along the rolling direction in the space of the stress triaxiality, Lode parameter and equivalent strain to fracture.

Figure 8-62. Fracture locus for AA3104-Thick along the diagonal direction in the space of the stress triaxiality, Lode parameter and equivalent strain to fracture.
Figure 8-63. Fracture locus for AA3104-Thick along the transverse direction in the space of the stress triaxiality, Lode parameter and equivalent strain to fracture.

Figure 5-64. Comparison of plane stress fracture loci for AA3104-Thick under different loading directions in the space of stress triaxiality and equivalent strain.
8.9 MAPPING FROM STRAIN-BASED SYSTEM FOR A GENERAL YIELD FUNCTION

The generalized from equations developed in Section 7.5 are used in this section to study the effects of average r-value and choice of yield function of the calculated fracture loci. The data from the strain-based fracture surface are used to determine the strain rate ratio ($\beta$), the stress ratio ($\alpha$) using the selected yield function, the effective plastic strain ($\varepsilon_p$), and the effective stress ($\bar{\sigma}$) from the hardening law. Furthermore, the major ($\sigma_1$) and minor ($\sigma_2$) stresses and the hydrostatic/mean stress ($\sigma_m$) can be determined by application of yield criterion and strain-hardening law of the material. Finally, the stress triaxiality factor ($\eta$) and the Lode parameter ($\ell$) can be calculated from the principal stresses. The Lode angle ($\vartheta$) and Lode angel parameter ($\bar{\ell}$) can be determined using the Lode parameter.

The theoretical fracture loci spaces computed for different transversely isotropic materials ($r_0 = r_{65} = r_{90} = \bar{r}$) are presented in Figure 8-66. From these curves it can be seen that given the strain based fracture curve, the mapping to the other fracture spaces is significantly affected by the average r-bar value. The bi-axial and plane strain stresses at failure increase with increasing r-values. This also impacts the calculated triaxiality values and shifts the curves to the right. In the polar space, for positive strain ratios, the formability increases with increased r-bar values. The shapes of the fracture loci are

Figure 8-65. Comparison of plane stress fracture loci for AA3104-Thick under different loading directions in principal strain space.
sensitive to $T$ and the predicted success of the forming operation is dependent on a careful selection of the material's characteristics.

The theoretical fracture loci spaces computed for different material models (von Mises, Hill48R, Hill48S, Yld2000-2D) are presented in Figure 8-67. From these curves it can be seen that given the strain based fracture curve, the mapping to the other fracture spaces is strongly influenced by the selection of the yield function. In the triaxiality space, the predicted failure in bi-axial tension and uniaxial tension is noticeably different for the different yield function. In stress space, the stress-based Hill (1948) model and Yld2000-2D model show increased formability in bi-axial tension. A similar trend is shown in the polar space. The shapes of the fracture loci are very sensitive to yield function selection. Additional experimental data can help better define the appropriate yield criteria for application.
Figure 8-66. Fracture loci for AA3104-Thin material showing effect of r-bar.
Figure 8-67. Fracture loci for AA3104-Thin material showing effect of yield function.
8.10 SUMMARY

Mechanical tests were conducted on both AA3104-Thin and AA3104-Thick materials to generate fracture data under different stress triaxiality conditions. These two alloys are commonly used in the packaging industry to manufacture aluminum drawn and ironed beverage cans and bottles. Tensile tests were performed on specimens with a central hole, and notched specimens. Torsion of a double bridge and in-plane shear tests were conducted to generate points near pure shear conditions. Nakajima tests were utilized to produce points in bi-axial tension. The data from the experiments was used to develop the fracture locus in principal strain space for both alloys. A mapping process for a general yield function was utilized to determine the effect of average r-value and selection of the yield function for stress triaxiality space, principal stress state, and polar effective plastic strain space for fracture. The fracture spaces are strongly influenced by the average r-value obtained experimentally and selection of the yield function for both the AA3104-Thin and AA3104-Thick materials requires careful use by the analyst. The fracture loci developed using the DF2015 criteria will be utilized in Chapter 9 for the prediction of failures in the finite element simulation of several metal forming operations.
8.11 REFERENCES


CHAPTER 9

FRACUTRE MODELS - FINITE ELEMENT VERIFICATION

9.1 BACKGROUND

This chapter presents several finite element simulations to validate the Polar EPS based fracture model. Fracture loci are shown in the principal strain space, triaxiality versus effective plastic strain space, principal stress space, and polar space. Models of the dogbone tensile test, the tensile specimen with a central hole, and the tensile specimen with a notch are used to help verify the fracture loci developed from the experimental testing performed on these samples. Simulations using the AA3104-Thin data include both the von Mises (isotropic) and the Yld2004-18P model in both the rolling direction (RD) and transverse direction (TD). A model of the hydraulic bulge test using material properties for the AA3104-Thin material and Yld2004-18P is also discussed. Simulation of the tapered cup forming using the AA3104-Thick material data is performed to predict the failure observed during the testing for wrinkling. The experimental data suggests that the plane strain failure is initiated by excessive thinning due to the small nose radius on the punch. Finally, a model of a cup draw / reverse redraw / expansion forming sequence is used to test the robustness of the fracture theory for a condition with nonlinear forming paths. In this example, the leading edge of the sheet experiences uniaxial compression during the drawing operations and uniaxial tension in the expansion process.

9.2 DOGBONE SPECIMEN

This example illustrates the fracture prediction of a plane stress tensile specimen. The Swift equation uses a power law to represent the flow curve.

\[ \sigma = K(\varepsilon_0 + \varepsilon_p)^n \]  

(9-1)
Generally, the strain at maximum load denotes the onset of plastic instability, or necking. For annealed materials the rate of strain hardening immediately after yielding is very high, but steadily decreases as the deformation continues. If the Swift equation is used to describe the strain hardening, the rate of strain hardening can be calculated as:

$$\frac{d\sigma}{d\varepsilon} = nK(\varepsilon_0 + \varepsilon^p)^{n-1} = \frac{n\sigma}{(\varepsilon_0 + \varepsilon^p)}$$

As the deformation continues, the cross-sectional area of the specimen reduces. At some point, the increasing load capability due to strain hardening and the decreasing load capability due to cross-sectional area reduction produce a maxima on the load-displacement curve. Beyond this point deformation is unstable. Once this localized deformation begins the stress in this region will be higher than elsewhere, leading to the formation of localized necking. For strain hardening described by the Swift equation, it can be shown that necking initiates when

$$\varepsilon^p = n\varepsilon_0$$

The maximum load at the onset of necking is determined using:

$$P = A_o Kn^p \exp(-n)$$

where $A_o$ is the original cross-sectional area of the bar.

The geometry of the dogbone tensile specimen is shown in Figure 9-1 with a detailed view of the finite element mesh shown in Figure 9-2. The center of the bar has a width of 6.0 mm and a thickness of 0.274 mm resulting in a cross-sectional area of 1.644 mm². The model uses 13863 deformable shell elements (type=16 in LS-DYNA) for the simulation. The initial model uses an elastic-plastic von Mises material with isotropic hardening. Young's modulus is 68.9 MPa and Poisson's ratio is 0.33. The Swift coefficients are $K=331.46$ MPa, $\varepsilon_0=0.00018$, and $n=0.05013$. Displacement boundary conditions at the end of the specimen are imposed using rigid bodies. The rigid bodies are tied to the ends of the deformable tensile specimen using *CONTACT_TIED_SURFACE_TO_SURFACE_OFFSET. A total displacement of 2.54 mm is achieved in 400 equal size increments. The theoretical peak load is 444.9 N. The deformed shape after the onset of necking and the associated effective plastic strain contour plot for the isotropic model are shown in Figure 9-3. The corresponding load-displacement data is plotted in Figure 9-4. The predicted maximum tensile load is 446.8 N.
Both an elastic-plastic von Mises material with isotropic hardening and Yld2004-18P model are used to simulate the tensile test. For the Yld2004-18P model, the test is simulated in both the rolling direction and transverse direction using the material parameters shown in Table 9-1.

Table 9-1. Material Parameters for the YLD2004-18P Yield Criteria.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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<tr>
<td>$\alpha_2$</td>
<td>0.84619</td>
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<td>$\alpha_3$</td>
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<td>$\alpha_4$</td>
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<td>$\alpha_7$</td>
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<tr>
<td>$\alpha_{18}$</td>
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Figure 9-1. Finite element model for dogbone tensile specimen.

Figure 9-2. Detailed view of finite element model for dogbone tensile specimen.
Figure 9-3. Effective plastic strain plot for dogbone tensile specimen showing necking.

Figure 9-4. Load-displacement curves for dogbone tensile specimen – AA3104-Thin.

The fracture loci for principal strain space, stress triaxiality space, principal stress space, and polar space for the isotropic model are provided in Figure 9-5. The data from Table 9-2 suggests that shell element ID 6389 will be the initial failure location and will fail at a displacement of 1.26 mm. This corresponds to an effective plastic strain of 0.151 and a maximum principal strain of 0.153. The ratio of the plastic strains is $\beta = 0.493$ compared to the theoretical value of 0.5. For the isotropic condition, all of the fracture loci predict the same
time of failure in the analysis. This is a result of a linear loading path to the point of failure and the fact that the fracture loci are mapped assuming von Mises conditions.

Table 9-2. Predicted fracture strokes for AA3104-Thin dogbone tensile specimen.

<table>
<thead>
<tr>
<th></th>
<th>Strain Ratio $\beta$</th>
<th>Strain Space</th>
<th>Triaxiality Space</th>
<th>Stress Space</th>
<th>Polar Space</th>
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<td>1.60 mm</td>
<td>1.60 mm</td>
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<td>1.63 mm</td>
<td>1.63 mm</td>
<td>1.63 mm</td>
</tr>
<tr>
<td><strong>YLD2004-18P RD</strong></td>
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</tr>
<tr>
<td>EL6389</td>
<td>-0.288</td>
<td>1.59 mm</td>
<td>1.60 mm</td>
<td>1.60 mm</td>
<td>1.60 mm</td>
</tr>
<tr>
<td>EL6508</td>
<td>-0.288</td>
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<td>1.63 mm</td>
<td>1.63 mm</td>
<td>1.63 mm</td>
</tr>
<tr>
<td><strong>YLD2004-18P TD</strong></td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
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</tr>
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<td>1.75 mm</td>
<td>1.74 mm</td>
<td>1.75 mm</td>
</tr>
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Figure 9-5. Fracture loci for AA3104-Thin material. Dogbone tensile specimen using von Mises model.
Figure 9-6. Fracture loci for AA3104-Thin material. Dogbone tensile specimen using YLD2004-18P (RD) model.
Figure 9-7. Fracture loci for AA3104-Thin material. Dogbone tensile specimen using YLD2004-18P (TD) model.
9.3 TENSILE SPECIMEN WITH CENTRAL HOLE

The geometry of the tensile specimen with a central hole is shown in Figure 9-8 with a detailed view of the finite element mesh shown in Figure 9-9. The center of the bar has a width of 20.0 mm and a thickness of 0.274 mm. The hole has a diameter of 4.0 mm resulting in a cross-sectional area of 4.384 mm². The model uses 9,790 deformable shell elements (type=16 in LS-DYNA) for the simulation. The von Mises material with isotropic hardening is modeled using *MAT_PIECEWISE_LINEAR_PLASTICITY. Young's modulus is 68.9 MPa and Poisson's ratio is 0.33. The Swift coefficients are $K=331.46$ MPa, $\epsilon_o=0.00018$, and $n=0.05013$. Displacement boundary conditions at the end of the specimen are imposed using rigid bodies. The rigid bodies are tied to the ends of the deformable tensile specimen using *CONTACT_TIED_SURFACE_TO_SURFACE_OFFSET. A total displacement of 0.508 mm is achieved in 400 equal size increments. The deformed shape after the onset of necking and the associated effective plastic strain contour plot is shown in Figure 9-10 with a detailed view showing the necking provided in Figure 9-11. The corresponding load-displacement data is plotted in Figure 9-12. The predicted maximum tensile load for the von Mises model is 908.7 N. For the Yld2004-18P model, the test is simulated in both the rolling direction and transverse direction using the material parameters provided in Table 9-1.

The fracture loci for principal strain space, stress triaxiality space, principal stress space, and polar space for the isotropic model are provided in Figure 9-13. The data from the polar space suggests that shell element ID 3489 will be the initial failure location and will fail at a displacement of 0.30 mm. This corresponds to an effective plastic strain of 0.135 and a maximum principal strain of 0.139. The ratio of the plastic strains is $\beta=-0.446$. For the isotropic condition, all of the fracture loci predict the approximately equivalent stoke length to failure.

The fracture loci for principal strain space, stress triaxiality space, principal stress space, and polar space for the Yld2004-18P model in the rolling direction are provided in Figure 9-14. The data from the polar space suggests that shell element ID 3489 will be the initial failure location and will fail at a displacement of 0.271 mm. This corresponds to an effective plastic strain of 0.103 and a maximum principal strain of 0.107. The ratio of the plastic strains is $\beta=-0.269$. The experimental results produced failure at 0.310 mm stroke length and an effective plastic strain of 0.154.

The fracture loci for principal strain space, stress triaxiality space, principal stress space, and polar space for the Yld2004-18P model in the transverse direction are provided in Figure 9-15. The data from the polar space suggests that shell element ID 3490 will be the initial failure location and will fail at a
displacement of 0.347 mm. This corresponds to an effective plastic strain of 0.152 and a maximum principal strain of 0.149. The ratio of the plastic strains is $\beta = -0.445$. The experimental results produced failure at 0.383 mm stroke length and an effective plastic strain of 0.185.

Table 9-3. Predicted fracture strokes for AA3104-Thin tensile specimen with central hole

<table>
<thead>
<tr>
<th></th>
<th>Strain Ratio $\beta$</th>
<th>Strain Space</th>
<th>Triaxiality Space</th>
<th>Stress Space</th>
<th>Polar Space</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Isotropic</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>EL3489</td>
<td>-0.446</td>
<td>0.296 mm</td>
<td>0.300 mm</td>
<td>0.305 mm</td>
<td>0.300 mm</td>
</tr>
<tr>
<td>EL3490</td>
<td>-0.320</td>
<td>0.301 mm</td>
<td>0.305 mm</td>
<td>0.305 mm</td>
<td>0.306 mm</td>
</tr>
<tr>
<td>EL3521</td>
<td>-0.439</td>
<td>0.304 mm</td>
<td>0.307 mm</td>
<td>0.312 mm</td>
<td>0.307 mm</td>
</tr>
<tr>
<td><strong>YLD2004-18P RD</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>EL3489</td>
<td>-0.269</td>
<td>0.263 mm</td>
<td>0.300 mm</td>
<td>0.343 mm</td>
<td>0.271 mm</td>
</tr>
<tr>
<td>EL3490</td>
<td>-0.249</td>
<td>0.276 mm</td>
<td>0.299 mm</td>
<td>0.378 mm</td>
<td>0.284 mm</td>
</tr>
<tr>
<td>EL3521</td>
<td>-0.264</td>
<td>0.267 mm</td>
<td>0.301 mm</td>
<td>0.368 mm</td>
<td>0.274 mm</td>
</tr>
<tr>
<td><strong>YLD2004-18P TD</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>EL3489</td>
<td>-0.509</td>
<td>0.361 mm</td>
<td>0.319 mm</td>
<td>0.314 mm</td>
<td>0.356 mm</td>
</tr>
<tr>
<td>EL3490</td>
<td>-0.445</td>
<td>0.343 mm</td>
<td>0.307 mm</td>
<td>0.295 mm</td>
<td>0.347 mm</td>
</tr>
<tr>
<td>EL3521</td>
<td>-0.501</td>
<td>0.376 mm</td>
<td>0.324 mm</td>
<td>0.318 mm</td>
<td>0.371 mm</td>
</tr>
</tbody>
</table>

Figure 9-8. Finite element model for tensile specimen with central hole.
Figure 9-9. Detailed view of finite element mesh for tensile specimen with central hole.

Figure 9-10. Effective plastic strain plot for tensile specimen with central hole.

Figure 9-11. Detailed plot showing effective plastic strain for tensile specimen with central hole.
Figure 9-12. Load-displacement curves for tensile specimen with central hole.
Figure 9-13. Fracture loci for AA3104-Thin material. Tensile specimen with central hole using von Mises isotropic model.
Figure 9-14. Fracture loci for AA3104-Thin material. Tensile specimen with central hole using YLD2004-18P (RD) model.
Figure 9-15. Fracture loci for AA3104-Thin material. Tensile specimen with central hole using YLD2004-18P (TD) model.
9.4 TENSILE SPECIMEN WITH NOTCH

The geometry of the tensile specimen with a notch is shown in Figure 9-16 with a detailed view of the finite element mesh shown in Figure 9-17. The center of the bar has a width of 20.0 mm and a thickness of 0.274 mm. The semi-circular notch has a radius of 5.0 mm resulting in a cross-sectional area of 2.74 mm$^2$. The model uses 5471 deformable plane stress shell elements (type=16 in LS-DYNA) for the simulation. The von Mises material with isotropic hardening is modeled using *MAT_PIECEWISE_LINEAR_PLASTICITY. Young's modulus is 68.9 MPa and Poisson's ratio is 0.33. The Swift coefficients are $K=331.46$ MPa, $\epsilon_i=0.00018$, and $n=0.05013$. Displacement boundary conditions at the end of the specimen are imposed using rigid bodies. The rigid bodies are tied to the ends of the deformable tensile specimen using *CONTACT_TIED_SURFACE_TO_SURFACE_OFFSET. A total displacement of 0.508 mm is achieved in 400 equal size increments. The deformed shape after the onset of necking and the associated effective plastic strain contour plot is shown in Figure 9-18 with a detailed view showing the necking provided in Figure 9-19. The corresponding load-displacement data is plotted in Figure 9-20. The predicted maximum tensile load for the von Mises model is 829.9 N. For the Yld2004-18P model, the test is simulated in both the rolling direction and transverse direction using the properties shown in Table 9-1.

The fracture loci for principal strain space, stress triaxiality space, principal stress space, and polar space for the isotropic model are provided in Figure 9-13. The data from the polar space suggests that shell element ID 1108 will be the initial failure location and will fail at a displacement of 0.282 mm. This corresponds to an effective plastic strain of 0.093 and a maximum principal strain of 0.089. The ratio of the plastic strains is $\beta=-0.115$. For the isotropic condition, all of the fracture loci predict approximately equivalent stoke lengths to failure.

The fracture loci for principal strain space, stress triaxiality space, principal stress space, and polar space for the Yld2004-18P model in the rolling direction are provided in Figure 9-14. The data from the polar space suggests that shell element ID 1108 will be the initial failure location and will fail at a displacement of 0.303 mm. This corresponds to an effective plastic strain of 0.0992 and a maximum principal strain of 0.102. The ratio of the plastic strains is $\beta=-0.202$. The experimental results produced failure at 0.325 mm stroke length and an effective plastic strain of 0.0997.

The fracture loci for principal strain space, stress triaxiality space, principal stress space, and polar space for the Yld2004-18P model in the transverse direction are provided in Figure 9-15. The data from the
polar space suggests that shell element ID 1108 will be the initial failure location and will fail at a displacement of 0.289 mm. This corresponds to an effective plastic strain of 0.0756 and a maximum principal strain of 0.0712. The ratio of the plastic strains is $\beta = 0.171$. The experimental results produced failure at 0.298 mm stroke length and an effective plastic strain of 0.0722.

### Table 9-4. Predicted fracture strokes for AA3104-Thin tensile specimen with central hole.

<table>
<thead>
<tr>
<th>Material</th>
<th>Strain Ratio $\beta$</th>
<th>Strain Space</th>
<th>Triaxiality Space</th>
<th>Stress Space</th>
<th>Polar Space</th>
</tr>
</thead>
<tbody>
<tr>
<td>Isotropic</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>EL1045</td>
<td>-0.471</td>
<td>0.280 mm</td>
<td>0.283 mm</td>
<td>0.283 mm</td>
<td>0.283 mm</td>
</tr>
<tr>
<td>EL1077</td>
<td>-0.115</td>
<td>0.280 mm</td>
<td>0.282 mm</td>
<td>0.282 mm</td>
<td>0.282 mm</td>
</tr>
<tr>
<td>EL1108</td>
<td>-0.115</td>
<td>0.280 mm</td>
<td>0.282 mm</td>
<td>0.282 mm</td>
<td>0.282 mm</td>
</tr>
<tr>
<td>YLD2004-18P RD</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>EL1045</td>
<td>-0.276</td>
<td>0.319 mm</td>
<td>0.365 mm</td>
<td>0.350 mm</td>
<td>0.326 mm</td>
</tr>
<tr>
<td>EL1077</td>
<td>-0.202</td>
<td>0.295 mm</td>
<td>0.306 mm</td>
<td>0.316 mm</td>
<td>0.303 mm</td>
</tr>
<tr>
<td>EL1108</td>
<td>-0.202</td>
<td>0.295 mm</td>
<td>0.305 mm</td>
<td>0.314 mm</td>
<td>0.303 mm</td>
</tr>
<tr>
<td>YLD2004-18P TD</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>EL1045</td>
<td>-0.493</td>
<td>0.402 mm</td>
<td>0.382 mm</td>
<td>0.365 mm</td>
<td>0.400 mm</td>
</tr>
<tr>
<td>EL1077</td>
<td>-0.171</td>
<td>0.286 mm</td>
<td>0.311 mm</td>
<td>0.262 mm</td>
<td>0.289 mm</td>
</tr>
<tr>
<td>EL1108</td>
<td>-0.171</td>
<td>0.286 mm</td>
<td>0.310 mm</td>
<td>0.262 mm</td>
<td>0.289 mm</td>
</tr>
</tbody>
</table>

**Figure 9-16.** Finite element model for tensile specimen with notch.
Figure 9-17. Detailed view of finite element mesh for tensile specimen with notch.

Figure 9-18. Effective plastic strain plot for tensile specimen with notch.

Figure 9-19. Detailed plot showing effective plastic strain for tensile specimen with notch.
Figure 9-20. Load-displacement curves for tensile specimen with notch.
Figure 9-21. Fracture loci for AA3104-Thin material. Tensile specimen with notch using von Mises isotropic model.
Figure 9-22. Fracture loci for AA3104-Thin material. Tensile specimen with notch using YLD2004-18P (RD) model.
Figure 9-23. Fracture loci for AA3104-Thin material. Tensile specimen with notch using YLD2004-18P (TD) model.
9.5 HYDRAULIC BULGE TEST

A schematic of the hydraulic bulge test is provided in Figure 9-24 showing a die opening of 150 mm and a die radius of 12.7 mm. The sheet sample used in the experiment is a regular octagon shape having a span of 228.6 mm resulting in the length of the sides being 94.69 mm. The finite element mesh of the deformable sheet is shown in Figure 9-25. The model uses 27,632 deformable plane stress shell elements (type=16) for the simulation. The anisotropic material behavior is defined using the Yld2004-18P model with material parameters given in Table 9-1. Young's modulus is 68.9 MPa and Poisson's ratio is 0.33. The Swift coefficients are $K=331.46$ MPa, $\varepsilon_0=0.00018$, and $n=0.05013$. Rigid bodies are used to define the upper and lower tooling in the model. The lower die is fixed, and the upper die has a displacement boundary condition applied to clamp the sheet. Contact between the deformable sheet and the dies is simulated using *CONTACT_FORMING_ONE_WAY_SURFACE_TO_SURFACE with a coefficient of friction of 0.15. The deformed shape after closing the upper die is provided in Figure 9-26 and shows a similar pattern of wrinkling on the outer edge as seen in the experiment. After closing the dies, pressure is applied to the sheet until failure.

![Figure 9-24. Schematic of the hydraulic bulge test showing critical dimensions.](image)
Figure 9-25. Finite element mesh using in the hydraulic bulge test simulation.

Figure 9-26. Deformed mesh after closing upper die in hydraulic bulge model.
Figure 9-27. Deformed mesh showing localized effective plastic strain at pressure of 1.69 MPa.

Figure 9-28. Detailed view of localized effective plastic strain and element selection for fracture prediction in hydraulic bulge simulation.
A typical failure observed during the hydraulic bulge testing of the AA3104-Thin sheet samples is provided in Figure 9-29. The fracture loci for principal strain space, triaxiality space, principal stress space, and polar space for the hydraulic bulge model are provided in Figure 9-30. The data from the polar space suggests the initial failure occurs at a pressure of 1.55 MPa and a center dome displacement of 22.3 mm.

*Figure 9-29. Typical failure observed in hydraulic bulge experiments for AA3104-Thin.*
Figure 9-30. Fracture loci for AA3104-Thin material. Hydraulic bulge specimen using YLD2004-18P (RD) model.
9.6 TAPERED CUP DRAW – AA3104-THICK

The numerical simulations of the tapered cup drawing process is performed using LS-DYNA. A schematic of the forming tools is provided in Figure 9-31 with the associated tooling dimensions given in Table 9-5. The deformable sheet, or blank, is modeled using 57,540 shell elements (type=10, Belytschko-Wong-Chiang formulation) with full iterative plasticity. Elements are defined in 1 degree segments around the circumference. The forming tools are considered rigid bodies with its surface modelled by discrete shell elements. Contact between the deformable sheet and the dies is simulated using *CONTACT_ONE_WAY_SURFACE_TO_SURFACE. The friction between the blank and the forming tools is described by the classical Coulomb’s law. The friction coefficient between sheet and tools is assumed to be constant and taken as $\mu=0.03$. The blank-holder force of 8.9 kN is applied through the use of non-linear springs attached from a rigid plate to the lower pressure pad. The cup is drawn completely by displacement of 20 mm of the upper die. The punch is considered completely rigid in the model. The material model used is the Barlat Yld2000-2D with the coefficients defined in Table 9-6. Young's modulus is 68.9 MPa and Poisson's ratio is 0.33. The strain hardening of the material is described using the Swift equation:

$$\sigma = K(\varepsilon_0 + \varepsilon^p)^n$$

The Swift coefficients are $K=305.70$ MPa, $\varepsilon_0=0.0086$, and $n=0.0730$.

The deformed mesh at time of failure for tapered cup drawing simulation using AA3104-Thick material is displayed in Figure 9-32 with the deformed mesh showing percent thickness reduction at the time of failure shown in Figure 9-33. The finite element mesh for showing element ID’s of interest is provided in Figure 9-34. These elements are located at a radial position of 15.26 mm, a position contacting the outside nose radius of the punch. A typical failure observed during tapered cup drawing is shown in Figure 9-35. The fracture loci for principal strain space, stress triaxiality space, principal stress space, and polar space for the hydraulic bulge model are provided in Figure 9-36. The data from the polar space suggests the initial failure occurs at a punch stroke of 6.6 mm. The detailed view of fracture loci in the polar space shown in Figure 9-37 shows the initial element locations for failure. The data indicates the failure initiates in the rolling direction which agrees well with the experimental result.
Figure 9-31. Schematic of the forming tools used in the drawing process with detail view of the punch geometry.

Table 9-5. Tools dimensions for cup drawing process.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>D1</th>
<th>D2</th>
<th>D3</th>
<th>D4</th>
<th>D5</th>
<th>D6</th>
<th>R1</th>
<th>R2</th>
<th>R3</th>
<th>T1</th>
<th>H1</th>
<th>H2</th>
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</thead>
<tbody>
<tr>
<td>AA3104-Thick</td>
<td>15.24</td>
<td>31.75</td>
<td>45.72</td>
<td>45.72</td>
<td>47.75</td>
<td>64.77</td>
<td>1.016</td>
<td>3.810</td>
<td>1.905</td>
<td>0.475</td>
<td>3.429</td>
<td>34.29</td>
</tr>
</tbody>
</table>

Table 9-6. Material Parameters for the AA3104-Thick material using YLD2000-2D Yield Criteria.

<table>
<thead>
<tr>
<th>Material</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\alpha_3$</th>
<th>$\alpha_4$</th>
<th>$\alpha_5$</th>
<th>$\alpha_6$</th>
<th>$\alpha_7$</th>
<th>$\alpha_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AA3104-Thick</td>
<td>0.76472</td>
<td>1.10411</td>
<td>0.93280</td>
<td>0.95516</td>
<td>1.01038</td>
<td>0.84837</td>
<td>0.98083</td>
<td>1.08349</td>
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</table>
Figure 9-32. Deformed mesh at time of failure for tapered cup drawing simulation using AA3104-Thick material.

Figure 9-33. Deformed mesh showing percent thickness reduction at time of failure for tapered cup drawing simulation using AA3104-Thick material.
Figure 9-34. Finite element mesh for tapered cup drawing simulation showing element ID’s.

Figure 9-35. Failure observed during tapered cup drawing simulation using AA3104-Thick material.
Figure 9-36. Fracture loci for AA3104-Thick material. Tapered cup drawing using YLD2004-18P (RD) model.
Figure 9-37. Detailed view of fracture loci in polar space for tapered cup drawing using AA3104-Thick material and YLD2004-18P (RD) model showing initial failed elements.
9.7 HOLE EXPANSION DURING CUP DRAWING – AA3104-THIN

The experimental drawing of an AA3104-Thin circular blank with a centralized hole is carried out at room temperature in a hydraulic press with maximum capacity of 5 tons. A schematic of the forming tools is provided in Figure 9-38 with the associated tooling dimensions given in Table 9-7. The circular blank has a diameter of 76.124 mm and a centralized hole with a diameter of 22.860 mm. The drawing process is performed with a constant punch travel speed of 140 mm/s and a constant blank-holder force of 8.9 kN. All forming tools are made of AISI A2 steel with 58–60 Rockwell C hardness having a surface roughness of about 2–4 µm (finished working surfaces). An oil lubricant is applied to both sides of the blank to reduce the frictional forces between the forming tools and the blank sheet during the experiments. The draw depth of the blank is adjusted using stop blocks and shims to control the overall depth and to capture the onset of fracture. A typical forming progression is provided in Figure 9-39 and shows the onset of fracture at 90° to the rolling direction. Additional fractures occur in the RD as a result of increased punch displacement. A detailed view of one sample showing the fracture initiation is provided in Figure 9-40.

The numerical simulation of hole expansion during cup drawing with the AA3104-Thin material is performed using LS-DYNA. The deformable sheet, or blank, is modeled using 27,000 shell elements (type=10, Belytschko-Wong-Chiang formulation) with full iterative plasticity. Shell elements are defined in 1 degree segments around the circumference. The forming tools are considered rigid bodies with their surface modelled by discrete shell elements. Contact between the deformable sheet and the dies is simulated using *CONTACT_ONE_WAY_SURFACE_TO_SURFACE. The friction between the blank and the forming tools is described by the classical Coulomb’s law and the friction coefficient between the sheet and tools is assumed to be constant and is taken as μ=0.03. The blank-holder force of 8.9 kN is applied through the use of non-linear springs attached from a rigid plate to the lower pressure pad. The cup is drawn using a punch displacement of 5.08 mm. The material model used is Yld2004-18P with the coefficients defined in Table 9-1. Young’s modulus is 68.9 MPa and Poisson’s ratio is 0.33. The strain hardening of the material is described using the Swift equation:

\[ \sigma = K(\varepsilon^p + \varepsilon^n) \]

The Swift coefficients are \( K=331.46 \) MPa, \( \varepsilon^p=0.00018 \), and \( n=0.05013 \).

The sequence of deformation from the finite element model for the circular blank with a centralized hole using AA3104-Thin material is displayed in Figure 9-41 and shows failure initiating in the TD direction. A plot of the principal strain ratios at the point of failure initiation is displayed in Figure 9-42 and shows...
highly anisotropic behavior. A fracture locus plot in polar space for several key elements on the edge of the hole is given in Figure 9-43 with a detailed view at the time of fracture displayed in Figure 9-44. The results show that failure initiates with element ID 5476 at $74^\circ$ from the rolling direction. The response is quite linear up to the point of failure as shown in the graph. However, the response is very anisotropic and failure results in the elements closest to the plane strain condition. A comparison of the experimental and numerical simulation data for the hole dimensions, blank dimensions, and draw depth at failure in given in Table 9-8 and shows excellent agreement.
Figure 9-38. Schematic of forming tools for hole expansion during cup drawing.

Table 9-7. Tools dimensions for hole expansion during cup drawing.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>D3</th>
<th>D4</th>
<th>D6</th>
<th>D7</th>
<th>R2</th>
<th>R3</th>
<th>T1</th>
</tr>
</thead>
<tbody>
<tr>
<td>AA3104-Thin</td>
<td>45.72</td>
<td>46.74</td>
<td>76.12</td>
<td>22.86</td>
<td>1.016</td>
<td>1.905</td>
<td>0.274</td>
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</table>
Figure 9-39. Forming progression for blank with centralized circular hole showing onset of fracture.

Figure 9-40. Detailed view of fracture initiation for blank with centralized circular hole.
Figure 9-41. Finite element simulation showing forming progression and fracture initiation location.

Table 9-8. Comparison of FEM and experimental data for hole expansion during cup drawing.

<table>
<thead>
<tr>
<th>Hole Dimensions at Failure (mm)</th>
<th>Blank Dimensions at Failure (mm)</th>
<th>Draw Depth (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>RD</td>
<td>TD</td>
</tr>
<tr>
<td><strong>Experiment</strong></td>
<td>25.552</td>
<td>25.092</td>
</tr>
<tr>
<td><strong>FEM</strong></td>
<td>25.603</td>
<td>25.405</td>
</tr>
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</table>
**Figure 9-42.** Strain ratios for hole elements versus angle from rolling direction.

**Figure 9-43.** Fracture prediction in Polar EPS space for hole expansion during cup drawing.
Figure 9-44. Detailed view of fracture prediction in PEPS space for hole expansion during cup drawing.
9.8 CUP DRAW / REVERSE REDRAW / EXPAND – AA3104-THIN

This example evaluates the ability of the fracture models to predict failure under complex nonlinear strain paths. A beverage can cup draw, reverse redraw, and expansion process is used. During the drawing operation, the leading edge of the sheet is under uniaxial compression. During expansion, the leading edge is subjected to uniaxial tension. The cup forming is a two-stage process of drawing and reverse redrawning which occur sequentially during a single stroke in the cupping press. Reverse redrawning occurs after the 1st operation cup is fully drawn. Figure 9-45 shows a schematic representation of the forming process and tooling used in the cup forming operation. The geometry and dimensions for the draw die, draw punch, and reverse redraw punch are shown in Figure 9-46. The cupping press has a ram forming speed of approximately 400.0 mm/sec. The blank holder force during the cup drawing operation is 21.1 kN and the hold-down force during the reverse redrawning operation is 16.6 kN. The process is well lubricated and the coefficient of friction is assumed to be 0.03. After forming, the cup is placed in a cup holder and a tapered expansion tool is driven inside the cup until failure. The geometry and dimensions for the expansion tools are shown in Figure 9-47. The tooling dimensions for the cup draw / reverse redraw operation are provided in Table 9-9 and the dimensions for the cup expansion operation are given in Table 9-10. The punch speed is 25 mm/sec and the coefficient of friction is 0.03.

The numerical simulation of the cup forming process is performed using LS-DYNA. The round blank has an initial diameter of 162.86 mm and is modeled using 29,820 shell elements (type=16) with full iterative plasticity. The forming tools are considered rigid bodies with its surface modelled by discrete shell elements. Contact between the deformable sheet and the dies is simulated using *CONTACT_FORMING_ONE_WAY_SURFACE_TO_SURFACE. The friction between the blank and the forming tools is described by the classical Coulomb’s law. The friction coefficient between sheet and tools is assumed to be constant and taken as \( \mu = 0.03 \). The blank-holder force and hold-down forces are applied through the use of non-linear springs attached from a rigid plate to the respective tools. The 1st operation cup is drawn completely by displacement of 45 mm of the upper draw die. The 2nd operation reverse redraw cup is drawn completely by displacement of 75 mm of the reverse redraw punch. The draw punch (reverse redraw die) is considered completely rigid in the model. The material model used is the Barlat Yld2004-18P with the material parameters defined in Table 9-1. Young’s modulus is 68.9 MPa and Poisson’s ratio is 0.33. The strain hardening of the material is described using the Swift equation with \( K=331.46 \text{ MPa} \), \( \varepsilon_0=0.00018 \), and \( n=0.05013 \).
Figure 9-45. Schematic of cup draw / reverse draw forming process.
Figure 9-46. Parametric dimensions for cup draw / reverse redraw forming tools.
Figure 9-47. Parametric dimensions for cup expansion tools.

Table 9-9. Dimensions for cup draw / reverse redraw operation (mm)

<table>
<thead>
<tr>
<th></th>
<th>D1</th>
<th>D2</th>
<th>D3</th>
<th>D4</th>
<th>D5</th>
<th>D11</th>
<th>D12</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>107.95</td>
<td>77.242</td>
<td>79.274</td>
<td>108.87</td>
<td>162.86</td>
<td>53.848</td>
<td>76.073</td>
</tr>
<tr>
<td>R1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.134</td>
<td>3.023</td>
<td>2.286</td>
<td>3.175</td>
<td>6.096</td>
<td>3.556</td>
<td>0.274</td>
</tr>
</tbody>
</table>

Table 9-10. Dimensions for cup expansion operation (mm)

<table>
<thead>
<tr>
<th></th>
<th>D1</th>
<th>D2</th>
<th>D3</th>
<th>D4</th>
<th>H1</th>
<th>Beta</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>69.723</td>
<td>85.090</td>
<td>76.708</td>
<td>108.87</td>
<td>58.405</td>
<td>4.7</td>
</tr>
<tr>
<td>R1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3.175</td>
<td>2.540</td>
<td>12.700</td>
<td>3.493</td>
<td>0.635</td>
<td></td>
</tr>
<tr>
<td>R2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R5</td>
<td></td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 9-48. Cup drawing operation. (a) geometry; (b) fringe of effective plastic strain; (c) fringe of shell thickness.
Figure 9-49. Cup reverse redrawing operation. (a) geometry; (b) fringe of effective plastic strain; (c) fringe of shell thickness.
Figure 9-50. Cup expansion operation with expansion tool engaged: (a) fringe of effective plastic strain; (b) fringe of shell thickness.
The cup geometry and fringes of the effective plastic strain and shell thickness after the 1st forming operation are shown in Figure 9-48. Likewise, the cup geometry and fringes of the effective plastic strain and shell thickness after the 2nd forming operation are shown in Figure 9-49. The effect of the material anisotropy are clearly shown through the resulting earing profile and thickness distribution on the leading edge. The effective plastic strain and shell thickness in the expansion operation are provided in Figure 9-50. The fringes clearly show the onset of localization. A photograph of the finished cup and fracture cup after expansion are shown in Figure 9-51.

The fracture locus for in polar space is provided in Figure 9-52. The data from the polar space suggests the initial failure occurs when the leading edge of the beverage cup reaches a diameter of 81 mm. This corresponds to an effective plastic strain of 1.373. The experiment work shows an average diameter at failure of 80 mm (measured on the tool).

![Figure 9-51. Cup expansion operation showing initial cup and fractured cup.](image-url)
Figure 9-52. Fracture locus for cup draw / reverse draw / expansion process with nonlinear strain path.

9.9 SUMMARY

This chapter presents several finite element simulations to validate the Polar EPS fracture model. Results for the tensile specimens show good agreement with the range of experimental results. When using an isotropic model, all of the fracture loci predict failures at very similar times during the simulation. When using anisotropic material models, the various fracture loci predict different failure times in the simulations. This is believed to be a result of the varying strain ratios, stress triaxiality factors, and stress ratios resulting from anisotropic behavior. The model of the hydraulic bulge test showed very good agreement with the experimental results. The simulation of the tapered cup forming using AA3104-Thick material data confirmed the failure observed during experimental testing for wrinkle evaluation. The model of a hole expansion during cylindrical cup drawing using a circular blank demonstrates the robustness of the PEPS theory for a highly anisotropic material and accurately predicts the onset and location of fracture. Finally, the model of a cup draw / reverse redraw / expansion forming sequence demonstrated the robustness of the fracture theory for a condition with nonlinear forming paths and accurately predicted the onset of failure. Based on the results presented, the DF2015 criteria can accurately predict both the stroke and location of non-linear / discontinuous forming operations with anisotropic strain paths and can be applied to general industrial forming operations.
CHAPTER 10

CONCLUDING REMARKS

10.1 CONCLUSION

In this thesis, complete material characterizations including anisotropy and fracture have been conducted for both AA3104-thin and AA3014-thick sheet samples, which are two commonly used alloys in the beverage can industry. Directional tensile testing along every 15 degrees from the rolling as well as biaxial bulge testing are conducted to accurately model anisotropy and plastic deformation. In the fracture characterization tests, a special torsion shear specimen to prevent buckling of the AA3014-thin sheet has been successfully conducted. In addition, the Nakajima test has been carried out for obtaining the biaxial fracture data.

The calibrated anisotropic data have been used for accurately predicting the highly anisotropic response including eight ears and wrinkling during a cup drawing simulation using the Yld2004-18P model. Shear, tension, plane strain and biaxial fracture data are incorporated into an advanced fracture model (DP 2015). A polar EPS forming limit diagram is extended to predict the fracture (called Polar EPS Fracture diagram) when considering nonlinear strain paths. Both necking and post necking can be virtualized through this new Polar EPS diagram. A general method to map from the principal strain space to various stress triaxiality space, principal stress space, and Polar EPS space was proposed by incorporating a non-quadratic yield function. A model of a hole expansion during cup drawing demonstrated the robustness of the PEPS fracture theory for the condition of a highly anisotropic material and accurately predicts the onset and location of failure.
The model of a cup draw / reverse redraw / expansion forming sequence demonstrated the robustness of the fracture theory for a condition with nonlinear forming paths and accurately predicted the onset of failure. Based on the results presented, the DF2015 criteria can accurately predict both the stroke and location of non-linear / discontinuous forming operations with anisotropic strain paths and can be applied to general industrial forming operations.

Therefore, the presented model in this thesis provides scientific basis for modeling anisotropy and fracture for thin sheet metal and can be successfully utilized for the simulation of multi-stage forming processes with enhanced accuracy and numerically efficient computational times.

10.2 FURTHER RESEARCH

Research needs to continue in the development of testing methods and techniques, particularly for thin sheet. The directional tensile data including the normalized yield stress and r-value anisotropies are the required input for the advanced material models such as the Barlat Yld2000-2D and Barlat Yld2004-18P models. Higher accuracy and reliability in the experimental data will produce more accurate material coefficients resulting in improved finite element output. Determination of the stress ratios played a significant role in the prediction of the earing profile. Many techniques can be employed, but a new method considering anisotropic hardening observed in the experimental data may be required. Fracture test developments are also needed for thin sheet metals loaded in shear or uniaxial compression as wrinkling becomes an issue. These fracture points are needed for beverage can forming processes such as multi-stage die necking where fractures can occur after significant diameter reductions.

Research needs to continue on anisotropy in work hardening using the Yld2000-2D and Yld2004-18P advanced yield functions with non-constant stress ratios to account for anisotropic hardening in different directions. The beverage can bodystock alloys are highly anisotropic and accounting for the hardening affects will impact the ability to accurately predict springback and earing during cup drawing. With improved predictions, compensation of the earing characteristic can be achieved using convolute cutedge designs allowing for improved overall sheet utilization and significant cost savings in the industry.

Improvements to the material models for numerical efficiency and ease of use in generating the material coefficients are required. Currently, an optimization routine is used to generate the
coefficients for the Barlat Yld2004-18P model. This is a highly iterative process and very sensitive to the input and weighting factors on both the normalized yield stress data and r-value data. An alternative approach may be the Non-Associated Flow Rule (NAFR) which offers flexibility in terms of selection of combinations of yield and plastic potential functions. The NAFR model may result in reduced computational time, without a sacrifice in accuracy.

A framework is needed for constitutive modeling of plasticity to describe the evolution of anisotropy which varies continuously as a function of plastic strain and the anisotropic-kinematic hardening effect in sheet metals for cyclic forming processes utilized for shaped cans which involve both die necking (uniaxial compression) and die expansion (uniaxial tension). The evolution of anisotropy should also improve the prediction of earing during cup drawing.

Finally, further methodologies considering nonlinear strain paths and triaxiality factors are needed to refine the shape of the fracture loci. This will help improve the overall curve fitting process and should provide improved failure surfaces. Today, most fracture surfaces are generated using the modified Mohr-Coulomb criteria in combination with either von Mises or Hill’s material model. Fracture loci for advanced anisotropic material models combined with the generalized mapping techniques in this thesis are needed for the highly anisotropic high-strength aluminum alloys.
A.1 - BACKGROUND

In mechanics and other fields of physics, quantities are represented by vectors and tensors. Essential manipulations with these quantities will be summarized in this section. For quantitative calculations and programming, components of vectors and tensors are needed, which can be determined in a coordinate system with respect to a vector basis. The three components of a vector can be stored in a column. A vector represents a physical quantity which is characterized by its direction and its magnitude. The length of the vector represents the magnitude, while its direction is denoted with a unit vector along its axis, also called the working line.

Some fundamental properties of vectors include

\[ \mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v} \quad \text{Commutative Law} \]  
\[ u + (\mathbf{v} + \mathbf{w}) = (u + \mathbf{v}) + \mathbf{w} \quad \text{Associative Law} \]  
\[ \alpha (\mathbf{v} + \mathbf{w}) = \alpha \mathbf{v} + \alpha \mathbf{w} \quad \text{Distributive Law} \]  
\[ (\alpha + \beta) \mathbf{v} = \alpha \mathbf{v} + \beta \mathbf{v} \quad \text{Distributive Law} \]  
\[ \alpha (\mathbf{v} + \mathbf{w}) = (\alpha \beta) \mathbf{v} \quad \text{Associative Law} \]

Tensors are simply mathematical objects that can be used to describe physical properties, just like scalars and vectors. In fact, tensors are merely a generalization of scalars and vectors; a scalar is a zero rank tensor, and a vector is a first rank tensor. The rank (or order) of a tensor is defined by the number of directions (and hence the dimensionality of the array) required to describe it. For example, properties
that require one direction (first rank) can be fully described by a 3×1 column vector, and properties that require two directions (second rank tensors), can be described by 9 numbers, as a 3×3 matrix. As such, in general an nth rank tensor can be described by \(3^n\) coefficients.

**Zero Order Tensors (Scalars):** A scalar is a quantity, e.g., a positive or negative number, which does not have an associated direction. For example, time, temperature, pressure, and density are scalar quantities.

**First Order Tensors (Vectors):** A vector is a quantity that has magnitude and one associated direction. For example, a velocity vector is a useful encapsulation of speed (how fast something is moving) with direction (which way it is going). A force vector is a succinct representation of its magnitude (how hard something is being pushed) with its direction (which way it is being pushed).

**Second Order Tensors:** A second order tensor, or dyad, is a quantity that has magnitude and two associated directions. For example, product of inertia is a measure of how far mass is distributed in two directions. Stress is associated with forces and areas (both regarded as vectors).

**Fourth Order Tensors:** A fourth-order tensor relates two second-order tensors and appear as elasticity and compliance tensors.

### A.2 VECTOR DECOMPOSITION

Writing a vector \(\mathbf{a}\) as the sum of its components is called decomposition. Components of a vector are its projection on to the coordinate axes (Figure A-1).

\[
\mathbf{a} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3 = \sum_{i=1}^{3} a_i \mathbf{e}_i
\]  

(A-6)

where \(\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}\) are unit basis vectors of the coordinate system, \(\{a_1, a_2, a_3\}\) are components of \(\mathbf{a}\), and \(\{a_1 \mathbf{e}_1, a_2 \mathbf{e}_2, a_3 \mathbf{e}_3\}\) are component vectors of \(\mathbf{a}\). This can alternatively be denoted by array notation as

\[
\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = (a_1 \ a_2 \ a_3)^T
\]  

(A-7)

Using the Pythagorean Theorem, the norm of a vector \(\mathbf{a}\) in three-dimensional space can be written as

\[
\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}
\]  

(A-8)
A.3 DOT PRODUCT

The dot product of two vectors $\mathbf{a} = [a_1, a_2, a_3, \ldots, a_n]$ and $\mathbf{b} = [b_1, b_2, b_3, \ldots, b_n]$ is defined as

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i = \|\mathbf{a}\|\|\mathbf{b}\| \cos \theta \quad (A-9)$$

And is shown pictorially in Figure A-2.

In any orthonormal coordinate system including Cartesian coordinates, it holds that

$$e_1 \cdot e_1 = 1 \quad e_1 \cdot e_2 = 1 \quad e_1 \cdot e_3 = 1$$
$$e_2 \cdot e_1 = 1 \quad e_2 \cdot e_2 = 1 \quad e_2 \cdot e_3 = 1$$
$$e_3 \cdot e_1 = 1 \quad e_3 \cdot e_2 = 1 \quad e_3 \cdot e_3 = 1$$

Several uses for the dot-product in geometry, statics, and motion analysis, include calculating an angle between two vectors (very useful in geometry), calculating a vector’s magnitude (e.g., distance is the magnitude of a position vector), calculating a unit vector in the direction of another vector, determining when two vectors are perpendicular, determining the component (or measure) of a vector in a certain direction, and changing a vector equation into a scalar equation.
A.4 CROSS PRODUCT

Given two linearly independent vectors \( \mathbf{a} \) and \( \mathbf{b} \), the cross product, \( \mathbf{a} \times \mathbf{b} \), is a vector that is perpendicular to both and therefore normal to the plane containing them (Figure A-3).

\[
\mathbf{a} \times \mathbf{b} = (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3)(b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3)
\]  
(A-11)

\[
\mathbf{a} \times \mathbf{b} = (a_x b_y - a_y b_x)\mathbf{e}_1 + (a_y b_z - a_z b_y)\mathbf{e}_2 + (a_z b_x - a_x b_z)\mathbf{e}_3 = \begin{vmatrix}
\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3
\end{vmatrix}
\]  
(A-12)

The magnitude of the cross product can be interpreted as the positive area of the parallelogram having \( \mathbf{a} \) and \( \mathbf{b} \) as sides. In any orthonormal coordinate system including Cartesian coordinates, it holds that

\[
\begin{align*}
e_1 \times e_1 &= 0 & e_1 \times e_2 &= e_3 & e_1 \times e_3 &= -e_2 \\
e_2 \times e_1 &= -e_3 & e_2 \times e_2 &= 0 & e_2 \times e_3 &= e_1 \\
e_3 \times e_1 &= e_2 & e_3 \times e_2 &= -e_1 & e_3 \times e_3 &= 0
\end{align*}
\]

Several uses for the cross-product in geometry, statics, and motion analysis, include calculating perpendicular vectors, e.g., \( \mathbf{v} = \mathbf{a} \times \mathbf{b} \) is perpendicular to both \( \mathbf{a} \) and \( \mathbf{b} \), determining when two vectors are parallel, e.g., \( \mathbf{a} \times \mathbf{b} = 0 \) when \( \mathbf{a} \) is parallel to \( \mathbf{b} \), calculating the moment of a force or linear momentum, e.g., \( \mathbf{M} = \mathbf{r} \times \mathbf{F} \) and \( \mathbf{H} = \mathbf{r} \times \mathbf{mv} \), calculating velocity/acceleration formulas, e.g., \( \mathbf{v} = \omega \times \mathbf{r} \) and \( \mathbf{a} = \alpha \times \mathbf{r} + \omega \times (\mathbf{\omega} \times \mathbf{r}) \), and calculating the area of a triangle whose sides have length \( |\mathbf{a}| \) and \( |\mathbf{b}| \).
A.5 SCALAR TRIPLE PRODUCT

The scalar triple product of three vectors \( A, B, \) and \( C \) is denoted \([A,B,C]\) and defined by

\[
[A,B,C] = A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}
\]  \hspace{1cm} (A-14)

where \( A \cdot B \) denotes a dot product, \( A \times B \) denotes a cross product, \( |A| \) denotes a determinant, and \( A_i, B_i, \) \( C_i \) are components of the vectors of \( A, B, \) and \( C \) respectively. The scalar triple product can also be written in terms of the permutation symbol \( e_{ijk} \) as

\[
A \cdot (B \times C) = e_{ijk} A_i B_j C_k
\]  \hspace{1cm} (A-15)

where Einstein summation has been used to sum over repeated indices.

The volume of a parallelepiped (Figure A-4) whose sides are given by the vectors \( A, B, \) and \( C \) is given by the absolute value of the scalar triple product

\[
V_{parallelepiped} = |A \cdot (B \times C)|
\]  \hspace{1cm} (A-16)

![Figure A-4. Vector triple product.](image)

A.6 KRONECKER DELTA

In mathematics, the Kronecker delta, named after Leopold Kronecker, is a function of two variables, usually just positive integers. The function is 1 if the variables are equal and 0 otherwise:

\[
\delta_{ij} = \begin{cases} 
0 & \text{if } i \neq j \\
1 & \text{if } i = j 
\end{cases}
\]  \hspace{1cm} (A-17)
where the Kronecker delta $\delta_{ij}$ is a piecewise function of variables $i$ and $j$. The Kronecker delta appears naturally in many areas of mathematics, physics and engineering, as a means of compactly expressing its definition above. In linear algebra, the $n \times n$ identity matrix $I$ has entries equal to the Kronecker delta:

$$I_{ij} = \delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$  \hspace{1cm} (A-18)

A.7 LEVI-CIVITA

The Levi-Civita (epsilon) function is rather more complicated, since it is a function of three free indices and is defined as

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i j k) \text{ is an even permutation of } (123) \\ -1 & \text{if } (ijk) \text{ is an odd permutation of } (123) \\ 0 & \text{otherwise, i.e. if } i = j \text{ or } j = k \text{ or } k = i \end{cases}$$  \hspace{1cm} (A-19)

The total number of possible ways of arranging the digits 1 2 3 is six. A mnemonic device to remember the even and odd permutations of 123 is illustrated in the Figure A-5. Note that even permutations of 123 are obtained by selecting any three consecutive numbers from the sequence 123123 and the odd permutations result by selecting any three consecutive numbers from the sequence 321321.

![Figure A-5. Permutations of 123.](image)

A.8 TENSORS

Tensors typically relate a vector to another vector, or another second rank tensor to a scalar. The dyads are basic tensors which are given in matrix notation as
A 2nd order tensor is written as

\[ S = S_{ij} e_i \otimes e_j \]  \hspace{1cm} (A-21)

In expanded form,

\[ S = S_{11}e_1 \otimes e_1 + S_{12}e_1 \otimes e_2 + S_{13}e_1 \otimes e_3 + S_{21}e_2 \otimes e_1 + S_{22}e_2 \otimes e_2 + S_{23}e_2 \otimes e_3 + S_{31}e_3 \otimes e_1 + S_{32}e_3 \otimes e_2 + S_{33}e_3 \otimes e_3 \]  \hspace{1cm} (A-22)

In matrix form,

\[
[S] = \begin{bmatrix}
S_{11} & S_{12} & S_{13} \\
S_{21} & S_{22} & S_{23} \\
S_{31} & S_{32} & S_{33}
\end{bmatrix}
\]  \hspace{1cm} (A-23)

Other properties of 2nd order dyad operations include:

\[ u \otimes v = u, e_i \otimes v, e_j = (u, v) e_i \otimes e_j \]
\[ (u \otimes v)^T = (u, v) e_j \otimes e_i = (u, v) e_i \otimes e_j \]
\[ (u \otimes v)^T = (v \otimes u) \]
\[ (u \otimes v)w = u (v \cdot w) = (u, w) w, e_j \]
\[ A : B = A_j B_j = tr(A^T B) \]
A.9 INVARIANTS

Invariants of a tensor are scalar functions of the tensor components which remain constant under a basis change. That is to say, the invariant has the same value when computed in two arbitrary bases. A symmetric second order tensor always has three independent invariants.

Examples of invariants are:

1. The three eigenvalues
2. The determinant
3. The trace
4. The inner and outer products

These are not all independent since any of A-4 can be calculated in terms of 1.

The manual way of computing principal stresses is to solve a cubic equation for the three principal values. The equation results from setting the following determinant equal to zero. The λ values, once computed, will equal the principal values of the stress tensor.

\[
\begin{vmatrix}
\sigma_{11} - \lambda & \sigma_{12} & \sigma_{13} \\
\sigma_{12} & \sigma_{22} - \lambda & \sigma_{23} \\
\sigma_{13} & \sigma_{23} & \sigma_{33} - \lambda \\
\end{vmatrix} = 0
\]

(A-25)

\[
(\sigma_{11} - \lambda)[(\sigma_{22} - \lambda)(\sigma_{33} - \lambda) - \sigma_{23}\sigma_{33}] - \sigma_{12}[\sigma_{12}(\sigma_{33} - \lambda) - \sigma_{23}\sigma_{13}] + \sigma_{13}[\sigma_{12}\sigma_{23} - \sigma_{13}(\sigma_{22} - \lambda)] = 0
\]

(A-26)

\[
\lambda^3 - (\sigma_{11} + \sigma_{22} + \sigma_{33})\lambda^2 + (\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{13}^2)\lambda - (\sigma_{11}\sigma_{23}\sigma_{33} - \sigma_{11}\sigma_{23}^2 - \sigma_{12}\sigma_{23}\sigma_{33} + 2\sigma_{12}\sigma_{23}\sigma_{13}) = 0
\]

(A-27)

Independent of the coordinate transformation you apply to the stress tensor, its principal stress will be the same three values. Therefore, the above equation is always be the same, regardless the transformation. This means that the combinations of stress components, which serve as coefficients of the \( \lambda \)'s, must be invariant under coordinate transformations. Thus, the equation can be written as
\( \lambda^3 - l_1 \lambda^2 + l_2 \lambda - l_3 = 0 \) \hspace{1cm} (A-28)

where,

\[ l_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} \] \hspace{1cm} (A-29)

\[ l_2 = \sigma_{11} \sigma_{22} + \sigma_{22} \sigma_{33} + \sigma_{33} \sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2 \] \hspace{1cm} (A-30)

\[ l_3 = \sigma_{11} \sigma_{22} \sigma_{33} - \sigma_{12} \sigma_{23} \sigma_{31} - \sigma_{23} \sigma_{31} \sigma_{12} - \sigma_{31} \sigma_{12} \sigma_{23} + 2 \sigma_{12} \sigma_{23} \sigma_{31} \] \hspace{1cm} (A-31)

The invariants can also be written as

\[ l_1 = \text{tr}(\sigma) = \sigma_{kk} \] \hspace{1cm} (A-32)

\[ l_2 = \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{23} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{13} & \sigma_{33} \end{vmatrix} = \frac{1}{2} (\sigma_{ij} \sigma_{jk} - \sigma_{ij} \sigma_{kj}) \] \hspace{1cm} (A-33)

\[ l_3 = \det(\sigma) = \varepsilon_{ijk} \sigma_{i1} \sigma_{j2} \sigma_{k3} \] \hspace{1cm} (A-34)

### A.10 CAYLEY–HAMILTON THEOREM

The Cayley–Hamilton theorem states that every square matrix satisfies its own characteristic equation. If \( A \) is a given \( n \times n \) matrix and \( I_n \) is the \( n \times n \) identity matrix, then the characteristic polynomial of \( A \) is defined as

\[ p(\lambda) = \det(A - \lambda I) \] \hspace{1cm} (A-35)

where \( \det \) is the determinant operation and \( \lambda \) is a scalar element. The Cayley–Hamilton theorem also states that substituting the matrix \( A \) for \( \lambda \) in this polynomial results in the zero matrix,

\[ p(A) = 0 \] \hspace{1cm} (A-36)

Given the 3x3 matrix below,

\[
A = \begin{bmatrix}
2 & 0 & 1 \\
-2 & 3 & 4 \\
-5 & 5 & 6
\end{bmatrix}
\] \hspace{1cm} (A-37)

the characteristic equation is

\[
\lambda^3 - (\text{tr}(A)) \lambda^2 + (C_{11} + C_{22} + C_{33}) \lambda - \det(A) = 0
\] \hspace{1cm} (A-38)
\[ \lambda^3 - (\text{tr}(A))\lambda^2 + \frac{1}{2}\left(\text{tr}(A)^2 - \text{tr}(A^2)\right)\lambda - \text{det}(A) = 0 \]  
(A-39)

where \((C_{11} + C_{22} + C_{33})\) is the sum of the diagonal cofactors and \(\text{det}(A)\) is

\[ \text{det}(A)_{3x3} = \frac{1}{6}\left(\text{tr}(A)^3 - 3\text{tr}(A)^2 \text{tr}(A) + 2\text{tr}(A^3)\right) \]  
(A-40)

\[ A^3 = \begin{bmatrix} -52 & 55 & 67 \\ -288 & 257 & 290 \\ -455 & 390 & 436 \end{bmatrix}, \quad A^2 = \begin{bmatrix} -1 & 5 & 8 \\ -30 & 29 & 34 \\ -50 & 45 & 51 \end{bmatrix}, \quad A = \begin{bmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{bmatrix} \]  
(A-41)

\[ \text{tr}(A) = 11 \]
\[ \text{tr}(A^2) = 79 \implies \text{det}(A) = 1 \]
\[ \text{tr}(A^3) = 631 \]  
(A-42)

\[ \lambda^3 - 11\lambda^2 + 21\lambda - 1 = 0 \]  
(A-43)

\[ p(A) = A^3 - 11A^2 + 21A - I = 0 \]  
(A-44)

\[ p(A) = \begin{bmatrix} -52 & 55 & 67 \\ -288 & 257 & 290 \\ -455 & 390 & 436 \end{bmatrix} - 11\begin{bmatrix} -1 & 5 & 8 \\ -30 & 29 & 34 \\ -50 & 45 & 51 \end{bmatrix} + 21\begin{bmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{bmatrix} - 1\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0 \]  
(A-45)

\[ A^3 - 11A^2 + 21A = I \]  
(A-46)

Multiplying by \(A^{-1}\) gives

\[ A^2 - 11A + 21 = IA^{-1} \implies A^{-1} = A^2 - 11A + 21 \]  
(A-47)

\[ A^{-1} = \begin{bmatrix} -1 & 5 & 8 \\ -30 & 29 & 34 \\ -50 & 45 & 51 \end{bmatrix} - 11\begin{bmatrix} 2 & 0 & 1 \\ -2 & 3 & 4 \\ -5 & 5 & 6 \end{bmatrix} + 21\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 5 & -3 \\ -8 & 17 & -10 \\ 5 & -10 & 6 \end{bmatrix} \]  
(A-48)

For a 2x2 matrix, the characteristic equation is

\[ \lambda^2 - (\text{tr}(A))\lambda + \frac{1}{2}\left(\text{tr}(A)^2 - \text{tr}(A^2)\right) = 0 \]  
(A-49)
A.11 DEVIATORIC STRESS

Any state of stress can be decomposed into a hydrostatic (or mean) stress $\sigma_m$ and a deviatoric stress $S$, according to

$$
\begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{12} & \sigma_{22} & \sigma_{23} \\
\sigma_{13} & \sigma_{23} & \sigma_{33}
\end{bmatrix} =
\begin{bmatrix}
\sigma_m & 0 & 0 \\
0 & \sigma_m & 0 \\
0 & 0 & \sigma_m
\end{bmatrix} +
\begin{bmatrix}
s_{11} & s_{12} & s_{13} \\
s_{12} & s_{22} & s_{23} \\
s_{13} & s_{23} & s_{33}
\end{bmatrix}
$$

(A-50)

where

$$\sigma_m = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3}$$

(A-51)

and

$$
\begin{bmatrix}
s_{11} & s_{12} & s_{13} \\
s_{12} & s_{22} & s_{23} \\
s_{13} & s_{23} & s_{33}
\end{bmatrix} =
\begin{bmatrix}
\frac{(2\sigma_{11} - \sigma_{22} - \sigma_{33})/3}{\sigma_{11}} & \sigma_{12} & \sigma_{13} \\
\sigma_{12} & \frac{(2\sigma_{22} - \sigma_{11} - \sigma_{33})/3}{\sigma_{22}} & \sigma_{23} \\
\sigma_{13} & \sigma_{23} & \frac{(2\sigma_{33} - \sigma_{11} - \sigma_{22})/3}{\sigma_{33}}
\end{bmatrix}
$$

(A-52)

Figure A-6 shows a stress state decomposed into hydrostatic and deviatoric components. In index notation,

$$\sigma_{ij} = \sigma_m \delta_{ij} + s_{ij}$$

(A-53)

The first part or isotropic component is the mean hydrostatic stress, and is responsible for the type of deformation mechanism, as well as dilation. The second component is the deviatoric stress and is what actually causes distortion of the body.

As it is a second order tensor, the stress deviator tensor also has a set of invariants, which can be obtained using the same procedure used to calculate the invariants of the stress tensor. It can be shown that the principal directions of the stress deviator tensor, $s_{ij}$, are the same as the principal directions of the stress tensor, $\sigma_{ij}$. Thus, the characteristic equation is

$$s^3 - J_1 s^2 - J_2 s - J_3 = 0$$

(A-54)

where the first, second, and third deviatoric stress invariants, $J_1, J_2, J_3$ respectively, are
\[ J_1 = s_{11} + s_{22} + s_{33} = s_1 + s_2 + s_3 = 0 \]  
\[ J_2 = -\left( s_{11}s_{22} + s_{22}s_{33} + s_{33}s_{11} - s_{12}^2 - s_{23}^2 - s_{31}^2 \right) = -(s_1s_2 + s_2s_3 + s_3s_1) \]  
\[ J_3 = s_{11}s_{22}s_{33} - s_{11}s_{23}^2 - s_{12}s_{23}^2 - s_{23}s_{31}^2 + 2s_{12}s_{23}s_{31} = s_1s_2s_3 \]

which can be re-written as

\[ J_2 = \frac{1}{6} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] \]  
\[ J_2 = \frac{1}{3} (I_1^2 - 3I_2) \quad J_3 = \frac{1}{27} (2I_1^3 - 9I_1I_2 + 27I_3) \]

\[ \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \]

and the characteristic equation is

\[ \begin{vmatrix} \sigma_{11} - \lambda & \sigma_{12} \\ \sigma_{12} & \sigma_{22} - \lambda \end{vmatrix} = 0 \]

\[ \lambda^2 - (\sigma_{11} + \sigma_{22}) \lambda + (\sigma_{11}\sigma_{22} - \sigma_{12}^2) = 0 \]

The resulting eigenvalues are

**Figure A-6. Stress decomposition into hydrostatic and deviatoric components.**

**A.12 MOHRS CIRCLE**

For a plane stress condition, the stress tensor becomes

\[ \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \]

and the characteristic equation is

\[ \begin{vmatrix} \sigma_{11} - \lambda & \sigma_{12} \\ \sigma_{12} & \sigma_{22} - \lambda \end{vmatrix} = 0 \]

\[ \lambda^2 - (\sigma_{11} + \sigma_{22}) \lambda + (\sigma_{11}\sigma_{22} - \sigma_{12}^2) = 0 \]

The resulting eigenvalues are
\[ \lambda = \left( \frac{\sigma_{11} + \sigma_{22}}{2} \right) \pm \sqrt{\left( \frac{\sigma_{11} + \sigma_{22}}{2} \right)^2 + \left( \frac{\sigma_{12} - \sigma_{21}}{2} \right)^2} = \left( \frac{\sigma_{11} + \sigma_{22}}{2} \right) \pm \sqrt{\left( \frac{\sigma_{11} - \sigma_{22}}{2} \right)^2 + \sigma_{12}^2} \]  

\[ \lambda = c + R \]

where

\[ c = \sigma_{avg} = \left( \frac{\sigma_{11} + \sigma_{22}}{2} \right), \quad R = \sqrt{\left( \frac{\sigma_{11} - \sigma_{22}}{2} \right)^2 + \sigma_{12}^2} \]  

These are equivalent to the equations for Mohr’s circle (Figure A-7).

Figure A-7. Schematic of Mohr’s circle.

The plane stress transformation equations to transform a state of plane stress into a new coordinate system are

\[
\begin{align*}
\sigma'_x &= \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta \\
\sigma'_y &= \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta \\
\tau'_{xy} &= \frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta
\end{align*}
\]  

(A-64)
APPENDIX B

CONTINUUM PLASTICITY

B.1 INTRODUCTION

This appendix reviews some basic concepts of continuum plasticity required for developing material models and forming limit curves (FLC) and fracture forming limit curves (FFLC) in the main chapters. Formulations for the description of the kinematics in solid continuum mechanics, the deformation gradient, the definition of Cauchy stress, stress transformations, various strain rates used in commercial finite element codes, and a co-rotational stress rate are reviewed. Comprehensive reference books written by Bathe (1996), Belytschko et al. (2000), Hughes (2000), Cook et al. (2001), Fish and Belytschko (2007), Bonet and Wood (2008), Wriggers (2008), Zienkiewicz et al. (2013) are recommended for an in-depth study.

B.2 KINEMATICS AND MOTION

In general, there are two different formulations for the description of the kinematics in solid continuum mechanics. These are the Lagrangian and the Eulerian formulation. For the Lagrangian (material) formulation, a fixed point in the material, which is usually not fixed in space is observed. The properties of this particular point are described over time. Thus, after the discretization of a material continuum into a finite element mesh, the material and the mesh are directly coupled. This gives the possibility to describe complex geometries as well as the history of material points with a high accuracy. The position of a particle in an arbitrarily chosen coordinate system, \( X \), and the time, \( t \), are the independent variables. A mapping function \( \chi \) describes the relation between the actual position \( x \) and the reference position \( X \) of a material point \( P \).

\[
    x = \chi(X, t) \quad \text{(B-1)}
\]
where $x$ is the position vector of a particle $P$, and $X$ is the position vector of the same particle in the reference or initial configuration. The reference configuration is assumed to be the configuration at $t = 0$.

In the case of the Eulerian (spatial) formulation, a fixed point in space is observed. The properties of the material which are at or pass this particular point are described. This way of describing the properties is particularly suited for fluid dynamic problems if a fixed control volume is considered. For solid mechanical problems, the considered volume is mostly bounded by the surface of the continuum body which may vary its shape with ongoing deformation, hence the Lagrangian formulation is more suitable. Throughout this thesis, the Lagrangian system is utilized. Additionally, rectangular cartesian coordinate systems will be used, for the sake of simplicity.

The displacement vector (Figure B-1) can thus be rewritten in both material and spatial descriptions as

$$u(X,t) = \chi(X,t) - X \quad \text{(B-2)}$$

$$u(x,t) = x - X(x,t) \quad \text{(B-3)}$$

Figure B-1. Reference and current configurations of a continuum solid and displacement vectors.

Rigid body rotation should not give rise to additional strains or stresses. The formulations should be objective or frame invariant, as well as the strain and stress measures in these formulations. Various different measures of stresses and strains are defined in the literature, some of the most relevant and frequently used in this dissertation will be introduced in the following sections.
## B.3 DEFORMATION GRADIENT

The deformation gradient, $F$, is the fundamental measure of deformation in continuum mechanics. It is the second order tensor which maps line elements in the reference configuration into line elements (consisting of the same material particles) in the current configuration and plays a pivotal role in the field of continuum mechanics. This tensor carries all information about the deformation of a material element from its known reference state to its current “spatial” state. The polar decomposition theorem uniquely quantifies how any general deformation can be viewed as a combination of material re-orientation (i.e. rotation), material distortion (i.e. a change in shape), and material dilation (i.e. a change in size).

$$F = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \frac{\partial x_i}{\partial X_j} \Rightarrow F_{ij} = \frac{\partial x_i}{\partial X_j} \Rightarrow [F] = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \quad (B-4)$$

where the indices $(i, j)$ represent the two sets of coordinate systems, material and spatial, respectively. The polar decomposition theorem: for any non-singular second-order tensor $F$ there exist unique positive definite, symmetric second-order tensors $U$ and $V$, and an orthogonal second-order tensor, $R$, such that

$$F = R \ast U = V \ast R \quad (B-5)$$

Where $R$ is an orthogonal rotation tensor, and therefore,

$$RR^T = I \quad (B-6)$$

And $U$ and $V$ are symmetric tensors where

$$U = U^T \quad \text{and} \quad V = V^T \quad (B-7)$$

The essential idea behind the polar decomposition theorem is illustrated in Figure B-2 where the deformation can be decomposed into a sequence of two separate steps: a stretch $U$ followed by a rotation, $R$, such that

$$F = R \ast U \quad (B-8)$$
The tensor is called the “right stretch” because it appears on the right in the expression. Alternatively, the deformation may be decomposed in similar steps applied in the opposite order: a rotation, \( R \), followed by a “left stretch” \( V \), such that

\[
F = V \circ R
\]  
(B-9)

The right and left stretch tensors are related to each other by (Simo and Hughes)

\[
U = R^T V R
\]  
(B-10)

\[
V = R U R^T
\]  
(B-11)

A rotation is a special kind of deformation in which material vectors permissibly change orientation, but they don’t change length. In this case, it can be shown that the associated deformation gradient will be orthogonal (its inverse will equal its transpose). Furthermore, since element inversions are prohibited, rotations will be proper orthogonal (determinants will equal). A stretch is a completely different special kind of deformation in which there exist three material vector orientations in the 3D initial configuration that will change in length but not in orientation. In this case, the deformation gradient tensor will be both symmetric and positive definite. Being symmetric, a stretch is diagonal in its principal basis. The principal values, called principal stretches, equal the ratio of deformed to undeformed lengths of the three non-rotating material fibers.

![Figure B-2. Representation of the polar decomposition of the deformation gradient.](image-url)
B.4 CAUCHY STRESS

Cauchy’s Law states that there exists a Cauchy stress tensor, $\sigma$, which maps the normal to a surface to the traction vector acting on that surface, according to

$$t = \sigma n$$  \hspace{0.5cm} (B-12)

$$
\begin{align*}
t_1 &= \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3 \\
t_2 &= \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3 \\
t_3 &= \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3
\end{align*}
$$

\Rightarrow t_j = \sigma_j n_j  \hspace{0.5cm} (B-13)

The Cauchy stress refers to the current configuration, that is, it is a measure of force per unit area acting on a surface in the current configuration (Figure B-3).

![Figure B-3. Cauchy stress tetrahedral element. Stress vector acting on a plane with normal unit vector $n$.](image)

B.5 3D STRESS TRANSFORMATION

It is known that any symmetric matrix can be decomposed as Equation B-14 where $D$ is a diagonal matrix containing the eigenvalues of $A$, and $Q$ is an orthogonal matrix whose columns consist of the corresponding eigenvectors of $A$. This theorem from matrix analysis has an interpretation in tensor analysis as a change-of-basis operation.

$$[A] = [Q^T][D][Q]  \hspace{0.5cm} (B-14)$$

It can be shown that the stress tensor is a contravariant second order tensor, which is a statement of how it transforms under a change of the coordinate system. That is, the matrix that transforms the vector of
components must be the inverse of the matrix that transforms the basis vectors. From an \( x_i \)-system to an \( x'_i \)-system, the components \( \sigma_{ij} \) in the initial system are transformed into the components \( \sigma'_{ij} \) in the new system according to the tensor transformation rule (Figure B-4)

\[
\sigma'_{ij} = a_{im} a_{jn} \sigma_{mn} \text{ or } \sigma' = [A][\sigma][A]^T
\]

where \( A \) is a rotation matrix with components \( a_{ij} \). In matrix form this is

\[
\begin{bmatrix}
\sigma'_{11} & \sigma'_{12} & \sigma'_{13} \\
\sigma'_{21} & \sigma'_{22} & \sigma'_{23} \\
\sigma'_{31} & \sigma'_{32} & \sigma'_{33}
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{21} & \sigma_{22} & \sigma_{23} \\
\sigma_{31} & \sigma_{32} & \sigma_{33}
\end{bmatrix}
\begin{bmatrix}
a_{11} & a_{21} & a_{31} \\
a_{12} & a_{22} & a_{32} \\
a_{13} & a_{23} & a_{33}
\end{bmatrix}
\]

(Figure B-4. Schematic of 3D Cauchy stress transformation.

**B.6 STRAIN MEASURES**

The right and left Cauchy-Green tensors are respectively defined as

\[
B = FF^T = B_{ij} e_i \otimes e_j = V^2
\]

\[
C = F^TF = C_{ij} E_i \otimes E_j = U^2
\]

As opposed to the tensor \( F \), which is not necessarily symmetric, the \( FF^T \) tensor is always symmetric. The right and left Cauchy-Green tensors are related to each other by

\[
C = R^T BR
\]
\( B = R C R^T \) \hspace{1cm} (B-20)

The following expressions can be derived to relate the reverse of the left and right Cauchy-Green tensors to the deformation gradient

\[
B^{-1} = (FF^T) = (F^T)^{-1} = V^{-2}
\]

\[
C^{-1} = (F^T F) = F^{-1} F^{-T}
\]

The tensor \( B^{-1} \) is also called the Finger deformation tensor.

For rigid body motions (translation and rotation), the right and left stretch tensors become identity tensors, thus the Cauchy-Green tensors have non-zero strain values. Therefore, proper definitions of strain that depend solely on stretch, such that zero strains are obtained for rigid body motions, are necessary. Accordingly, alternative strain measures such as Euler-Almansi, Green-Lagrange, and true strains have been defined.

The Euler-Almansi strain is defined as

\[
E = \frac{1}{2} (I - B^{-1})
\]

The Green-Lagrange strain is defined as

\[
E_1 = \frac{1}{2} (C - I) = \frac{1}{2} (F^T F - I)
\]

And the true strain is defined as

\[
\varepsilon = \frac{1}{2} \ln(B^{-1})
\]

These strain measures result in zero strain for rigid body motions.

The principal of scalar invariants suggests that the invariants of \( C \) and \( U \) are equal. In order to evaluate \( U \) and \( U^{-1} \), it is necessary to evaluate \( \sqrt{C} \). To evaluate the square-root, \( C \) must first be obtained in relation to its principal axes (from the characteristic equation), so that it is diagonal. Then, the square
root can be taken of the diagonal elements, since its eigenvalues will be positive. Finally, the tensor needs to be transformed back to the original coordinate system.

\[
C = \begin{bmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{bmatrix}
\]  

(B-26)

The characteristic equation is

\[
v^2 - (C_{11} + C_{22})v + \left(C_{11}C_{22} - C_{12}^2\right)I = 0
\]  

(B-27)

The resulting eigenvalues are

\[
v_{1,2} = \frac{C_{11} + C_{22}}{2} \pm \sqrt{\left(\frac{C_{11} - C_{22}}{2}\right)^2 + \left(C_{12}\right)^2}
\]  

(B-28)

Using the Cayley Hamilton theorem and \( U^2 = C \)

\[
U^2 - I_u U + I_u I = 0 \quad \therefore U = I_u^{-1}(I_u + C) \quad \text{and} \quad U^{-1} = I_u^{-1}(I_u - U)
\]  

(B-29)

where

\[
I_u = \sqrt{V_1} + \sqrt{V_2}
\]

\[
I_u = \sqrt{V_1} \sqrt{V_2}
\]  

(B-30)

Using Equation B-29 and rearranging terms gives

\[
U^{-1} = (I_u I_u)^{-1} \left(I_u^2 - I_u \right) \right) I - C)
\]  

(B-31)

**B.7 JAUMANN STRESS RATE**

A distinction is made between the total and the updated Lagrangian formulation. For the total Lagrangian formulation, the initial configuration is always the reference, while for the updated Lagrange formulation, the configuration from the last deformation increment is used as reference. In sheet metal forming processes, as discussed in this thesis, the deformations are typically in a range which is well described by the Lagrangian formulation. In order to capture strain histories and be able to deal with nonlinearities due to large deformations, an updated Lagrange formulation is usually applied. In order to compute strains (in LS-DYNA), the so-called rate of deformation tensor is used.
Given the dependency on plastic deformation history of the stress, constitutive equations of elastic-plastic materials are usually written in an incremental form. Hence, instead of stress tensors, their material derivatives are used in the formulations. However, it can be demonstrated that even though the Cauchy stress tensor is objective, its material derivative is not and consequently cannot be used in constitutive formulations. This is also true for Kirchhoff stress rate. To overcome this, a Cauchy stress rate (or co-rotational stress rate) was developed by Jaumann, standing out among other objective stress rates for use in numerical computations. By decomposing $L$ into a symmetric and skew part

$$ L = \frac{\partial V}{\partial X_j} e_j \otimes e_j = \dot{F} F^{-1} = \frac{1}{2} (L + L') + \frac{1}{2} (L - L') = D + W $$ \hspace{1cm} (B-32) $$

The rate of deformation, $D$, and the spin tensor, $W$, are obtained.

$$ D = \frac{1}{2} (L + L') \hspace{1cm} (B-33) $$

$$ W = \frac{1}{2} (L - L') \hspace{1cm} (B-34) $$

The rate of deformation tensor is a measure of the rate of change of the square of the length of infinitesimal material line segments while the spin tensor can be shown to provide a measure of the average angular velocity of all material fibers passing through a material point. The Jaumann (co-rotational) rate of the Cauchy stress tensor is determined using

$$ \sigma' = \dot{\sigma} - W \sigma + \sigma W \hspace{1cm} (B-35) $$
B.8 REFERENCES

C.1 INTRODUCTION

Today, finite element simulations are used extensively to analyze the formability and structural performance of sheet metal products. The accuracy of the numerical simulations depends on many factors including the selected constitutive model and the determination of the parameters used in the material models. Various phenomenological yield functions have been proposed to simulate the isotropic and anisotropic behaviors of metals, most based on the associated flow rule (AFR) hypothesis obeying the normality rule. These plasticity models can be divided into multiple groups, each having its own range of application. Isotropic yield functions include Tresca, (1864), Mises (1913), Hershey (1954), and Hosford (1972). Several anisotropic yield functions include Hill (1948), Hill (1979), Bassani (1977), Gotoh (1977), Logan and Hosford (1980), Budianski (1984), Barlat et al., (1991), Barlat et. al. (1997), Karafillis and Boyce (1993), Cazacu and Barlat (2004), Cazacu et al. (2006), and Vegter and van den Boogaard (2006). The common approach has been to increase the flexibility of the constitutive model by using a larger number of parameters requiring additional experimental testing and complex routines to calculate the material constants. Most efforts focus on formulating the plastic deformation based on the normality rule, in which the plastic strain increment is normal to the yield surface, called the associated flow rule. The plastic potential to describe the plastic strain increment by the normality rule is identical to the plastic stress function to define the yield surface.

Anisotropy has an important effect on the strain distribution in aluminum alloy sheet forming, and it is closely related to thinning and formability of sheet metals. Thus, the anisotropy of the material should be properly considered for the realistic analyses of aluminum sheet forming processes. The anisotropy of a sheet metal during sheet forming is a combination of the initial anisotropy due to its previous history of thermo-mechanical processing and to the plastic deformation during the stamping operation. The former
leads to asymmetry with the orthotropic character while the latter, called deformation-induced anisotropy, can destroy this symmetry when principal material symmetry and deformation axes are not superimposed. Therefore, the modeling of plastic anisotropy itself and its implementation in finite element (FE) code can be complex. For practical purpose, the assumption that the change of anisotropic properties during sheet forming is small and negligible when compared to the anisotropy induced by rolling and heat treatment has been widely adopted in the analysis of sheet metal forming. This is particularly important for industrial applications, where user-friendliness and computation time are important factors to consider. In this case, it is convenient to use the concepts of anisotropic yield functions and isotropic hardening. Although this approach has its limitation (e.g., Tuğcu and Neale, 1999; Tuğcu et al., 2002), it has produced results in good agreements with experimental data (Worswick and Finn, 2000). Therefore, when plastic deformation is moderate, like in sheet forming, a simple description of anisotropy based on experimental data as input might be as accurate for sheet forming simulation than the more sophisticated anisotropy model, and at a much lower cost. For these reasons, a constitutive behavior based on an anisotropic yield function and isotropic hardening appears to be a reasonable choice for performing sheet forming simulations.

This appendix starts with a brief introduction on yield functions and an overview of some well-known isotropic yield functions. The main part of this chapter presents a review of various anisotropic yield functions. Classical yield functions developed by Hill, and more modern yield functions developed by Barlat, and some polynomial-based models are described. Hill’s and Barlat’s families of yield functions are described in detail because of their acknowledged contributions to the development of new anisotropic yield functions and also because of the large popularity of these models for finite element simulations. Section C.2 reviews the concepts of Lankford coefficient, normality hypothesis, and associated flow rule. In section C.3, various isotropic and anisotropic yield functions are described. Finally, section C.4 discusses various work hardening models commonly used in the sheet metal forming industry today.

**C.2 PLASTIC ANISOTROPY AND AFR**

The influence of plastic anisotropy on sheet metal forming has been studied with the help of FEM codes combined with appropriate anisotropic yield functions. Many such functions have been proposed. The quadratic yield function by Hill (1948) has long been one of the popular choices to represent planar anisotropy and has been widely used in FEM forming simulation. Several non-quadratic criteria were developed by Hill (1979, 1990), Hershey (1954), Hosford (1972), Bassani (1977), Gotoh (1977), Logan and Hosford (1980), Barlat and Lian (1989), Karafillis and Boyce (1993), Bron and Besson (2004), Banabic et al.
(2005) and Barlat et al. (1991, 1997, 2003, 2005). In general, Hill (1948) has been useful for explaining phenomena associated to anisotropic plasticity, particularly for steels, while the others can be used to improve the yielding description of aluminum alloys. In many circumstances Yld89 (Barlat and Lian 1989), or Yld91 (Barlat et al. 1991) can be used for steels or aluminum alloys. In particular, the yield criteria, Yld89, for planar anisotropy, and Hill (1990) have three stress components and are applicable to plane stress condition. The analytical forms of these criteria are relatively simple. The criteria Yld91 account for six stress components and can be applied to general 3D elasto-plastic continuum codes. Barlat et al. (2007) have made detailed discussions about the relationship between these yield criteria. In fact, Yld91 is a particular case of Yld2004-18P (Barlat et al. 2005), Yld89 is a particular case of Yld2000-2D (Barlat et al. 2003), and the yield function proposed by Banabic et al. (2005) is identical to Yld2000-2D. In some special conditions, Yld91 can reduce to the Hill (1948), Mises, or Tresca yield function.

The comprehensive constitutive description of material plastic behavior in a general stress-state must include three elements:

1. A yield criterion which determines the limit at which the materials begin to plastically deform,
2. A flow rule defining the relation between the plastic strain rate tensor and the stress tensor. A plasticity model is called associative if the yield function is considered as a plastic potential and its derivative provides the strain rate direction.
3. A hardening rule describing the evolution of the shape, the size, and the position of the yield surface during the deformation. It is mainly divided in two categories: isotropic and kinematic hardening. Isotropic hardening models the expansion of the yield surface with no shape distortion while kinematic hardening also called anisotropic hardening computes the yield surface displacement in the stress space.

C.2.1 LANKFORD COEFFICIENT

Due to their crystallographic structure and the characteristics of the rolling process (Figure C-1), sheet metals generally exhibit a significant anisotropy of mechanical properties. In fact, the rolling process induces a particular anisotropy characterized by the symmetry of the mechanical properties with respect to three orthogonal planes. Such a mechanical behavior is called orthotropy. The intersection lines of the symmetry planes are the orthotropy axes. In the case of the rolled sheet metals, their orientation is as follows (Figure C-2): rolling direction (RD); transverse direction (TD); normal direction (ND).
Anisotropy is generally described on the basis of the Lankford coefficients (also called $r$-values) and/or the yield stresses along the orthotropic (rolling and transverse) and diagonal directions of the metallic sheets. The Lankford coefficient at any orientation with respect to the rolling direction is determined as the ratio of width to through thickness plastic strain (increments). Due to practical difficulties in measuring the through thickness plastic strain in sheet metals, this value is conventionally determined using the incompressibility hypothesis of metals. Lankford coefficient ($r_\theta$) at degrees from the rolling direction (RD) and through thickness plastic strain increment ($d\varepsilon_{zz}^p$) are given by

$$ r_\theta = \frac{d\varepsilon_{\theta,90}^p}{d\varepsilon_{zz}^p} \quad (C-1) $$

$$ d\varepsilon_{zz}^p = -d\varepsilon_{\theta,90}^p - d\varepsilon_{\phi,90}^p \quad (C-2) $$

The normal r-value is taken to be the average

$$ r = \frac{r_{90} + 2r_{45} + r_{90}}{4} \quad (C-3) $$
The planar anisotropy coefficient, or planar r-value, is a measure of the variation of $r$ with the angle from the rolling direction. This quantity is defined as

$$\Delta r = \frac{r_{00} - 2r_{45} + r_{90}}{4}$$  \hspace{1cm} (C-4)$$

A higher normal anisotropy, results in more resistance against thinning and is thus preferable for deep drawing applications. A higher planar anisotropy can be observed by more pronounced earing in a deep drawn cup.

**C.2.2 DRUCKER’S POSTULATE**

The postulate defines a work-hardening plastic material as one where the work during incremental loading is positive and work done in a loading / unloading cycle is non-negative as shown in Figure C-3. In equation form this can be shown as an inequality

$$d\sigma d\varepsilon^p \geq 0$$  \hspace{1cm} (C-5)$$

The inequality also holds for any incremental stress states which can be denoted as $\sigma^\prime$ and $\sigma^{\prime\prime}$ and can be written as

$$\left(\sigma^\prime - \sigma^{\prime\prime}\right) d\varepsilon^p \geq 0$$  \hspace{1cm} (C-6)$$

The yield surface is considered smooth, so every point has a tangential plane thus making the yield surface convex. This is known as the normality rule and indicates that the flow rule is associated with the yield criterion.

In summary, Drucker’s Postulate defines a work-hardening or “stable” plastic material as one in which the work done during incremental loading is positive, and the work done in the loading unloading cycle is nonnegative and implies normality (associative flow rule) and convexity (Figure C-3).
C.2.3 NORMALITY RULE

A flow rule defines the magnitude and direction of the plastic strain increment for a given (infinitesimal) increment of all the stress components when yielding is taking place. The direction of plastic flow is determined by the Normality Rule (normal to the yield surface – Figure C-4) \( \frac{\partial \sigma}{\partial \sigma_{\alpha\beta}} \) and the magnitude is controlled by the plastic strain increment \( \Delta \bar{\varepsilon}_p \).

\[
\Delta \varepsilon_{\alpha\beta}^{(p)} = \Delta \bar{\varepsilon}_p \frac{\partial \sigma}{\partial \sigma_{\alpha\beta}} \tag{C-7}
\]

Thus, for example, for a Mises material which hardens isotropically, \( \frac{\partial \sigma}{\partial \sigma_{\alpha\beta}} \) is radial at the current point on the Mises circle. For a kinematically hardened material, the normal to the displaced Mises circle also gives the direction of \( \frac{\partial \sigma}{\partial \sigma_{\alpha\beta}} \), but this will no longer be radial from the origin. For a Tresca material, \( \frac{\partial \sigma}{\partial \sigma_{\alpha\beta}} \) is normal to whichever side of the hexagon the stress ‘vector’ currently lies upon.
C.2.4 ASSOCIATED FLOW RULE (AFR)

In associated flow rule, the yield function and the plastic potential functions are equal. In addition, the plastic strain increment is always normal to the yield surface (normality rule). Drucker’s postulate is applicable to elastic-plastic materials and states that in a cycle of plastic deformation the plastic work is always positive. This postulate can be expressed in incremental form as

\[ d\sigma : d\varepsilon_p \geq 0 \]  \hspace{1cm} (C-8)

where \( d\varepsilon_p \) is the incremental plastic strain tensor.

According to Bishop and Hill (1951), the normality hypothesis was theoretically valid for polycrystalline materials. In addition, Hecker (1976) described that the normality hypothesis is reasonable for most single phase-like materials based on an extensive review of experimental yield surface results. Moreover, the AFR hypothesis was strengthened by experimental observations of Bridgman (1947, 1952). He performed a series of tensile tests on metals in the presence of very high hydrostatic pressure and noticed that this pressure had no influence on the yielding of the material. In addition, a negligible permanent volume change was shown to exist (Khan and Huang, 1995). Due to the absence of pressure sensitivity in the plastic deformation, only the deviatoric stress is involved in the formulation of the yield function. On the other hand, the zero plastic dilatancy (zero permanent volume change) will not be violated by using the same formulation for yield function and plastic potential function (equivalence of yield and plastic potential functions).

In order to accurately describe both yielding and plastic flow of sheet metals, the coefficients of anisotropic yield functions commonly need to be optimized explicitly or iteratively from experimentally...
determined tensile, shear and/or bi-axial yield stresses and Lankford coefficients. It is worth noting that in the last two decades the isotropic plasticity equivalent theory generalized by Karafillis and Boyce (1993) has been a popular approach in the development of new yield functions. Recently, Soare and Barlat (2010) have proven that these orthotropic yield functions obtained through a linear transformation method are homogeneous polynomials, which brings potential benefits for numerical implementation and development of new yield functions. In the following section, various commonly used yield functions are briefly described.

C.3 ISOTROPIC AND ANISOTROPIC YIELD FUNCTIONS

Aggregates of single crystals constitute the crystallographic structure of most metals. Considering a single crystal, considerable anisotropy of mechanical properties such as different yield stresses at different orientations is observed. The mechanical anisotropy at crystal level turns into isotropy at macro-scale level in a polycrystalline aggregate with a sufficiently random distribution of crystal orientations (Neto et al., 2008). In other words, the average behavior of all single crystals represents the total material behavior. Thus an isotropic yield function seems to be a sufficient assumption for the description of macroscopic behavior for finite element simulations. However, sheet metals undergo severe plastic deformations during manufacturing processes such as cold rolling. This introduces a preferential orientation of the grains. Therefore, isotropy is no longer the appropriate assumption to represent the mechanical behavior of a rolled sheet metal. Moreover, the anisotropic behavior has been known to have a great influence on the shape of the specimen after the deformation. Earing at the rim of a deep drawn part is an example of distinct anisotropic behavior.

Focusing on the material constitutive models in general, and more specifically on the yield function, there are two major approaches to describe this behavior for polycrystalline materials. The first approach is crystal plasticity and the second one is the phenomenological approach. In the first approach, the behavior of one grain or a distribution of grains is used to describe the polycrystalline behavior (Arminjon, 1991; Van Houtte, 1994; Gambin and Barlat, 1997). In the phenomenological approach, on the other hand, the average behavior of all grains determines the global material behavior. According to Barlat (1991), using a phenomenological yield function has advantages over its microstructure based equivalent. For instance, (Barlat et al., 1991):

a) They are easy to implement in FEM and lead to efficient computations;
b) They can describe global anisotropy whereas microstructure based models account for crystallographic texture;

c) They are easy to adapt for different materials.

Many phenomenological yield functions have been successfully proposed for use in finite element codes to simulate the isotropic or anisotropic mechanical behavior of a material. The different yield functions generally make use of different combinations of yield stresses and Lankford coefficients to represent a multi-dimensional surface determining the transition between elastic and plastic deformation. First some basic concepts needed for the formulation of yield functions are introduced.

**C.3.1 TRESCA 1864**

The Tresca isotropic yield function proposed in 1864 is known to be the oldest yield criterion and is also known as the *maximum shear stress theory* (MSST). The criterion indicates that plastic deformation occurs when the maximum shear stress reaches a critical value. The yield surface is a hexagon having infinite length and suggests that the material remains elastic when all three principal stresses are roughly equivalent (a hydrostatic pressure), no matter how much it is compressed or stretched. However, when one of the principal stresses becomes smaller (or larger) than the others the material is subject to shearing. In such situations, if the shear stress reaches the yield limit then the material enters the plastic domain. Mathematically, it can be written as

\[
\begin{align*}
\tau_{\text{max}} &= \kappa \quad \text{elastic–plastic deformation} \\
\tau_{\text{max}} &< \kappa \quad \text{pure elastic deformation}
\end{align*}
\]  

(C-9)

which, in terms of principal stresses, becomes

\[
\max \left( \frac{\sigma_1 - \sigma_2}{2}, \frac{\sigma_2 - \sigma_3}{2}, \frac{\sigma_3 - \sigma_1}{2} \right) = \kappa
\]  

(C-10)

The material constant, \( \kappa \), can be determined by performing a uniaxial tension test, with

\[
\sigma_2 = \sigma_3 = 0 \quad \text{and} \quad \sigma_1 = \sigma_y
\]  

(C-11)

where \( \sigma_y \) is the yield stress in simple tension. By substitution in Equation C-10, one obtains
\[ \kappa = \frac{\sigma_y}{2} \]  
\[ \text{(C-12)} \]

### C.3.2 HUBER-MISES

The von Mises Criterion (1913), also known as the maximum distortion energy criterion, octahedral shear stress theory, or Maxwell-Huber-Hencky-von Mises theory, is often used to estimate the yield stress of ductile materials. The von Mises yield criterion suggests that the yielding of materials begins when the second deviatoric stress invariant \( J_2 \) reaches a critical value. For this reason, it is sometimes called the \( J_2 \)-plasticity or \( J_2 \) flow theory. It is part of a plasticity theory that applies best to ductile materials, such as metals. Prior to yield, material response is assumed to be elastic. A material is said to start yielding when its von Mises stress reaches a critical value known as the yield strength, \( \sigma_y \). The von Mises stress is used to predict yielding of materials under any loading condition from results of simple uniaxial tensile tests. The von Mises stress satisfies the property that two stress states with equal distortion energy have equal von Mises stress.

\[
\phi(\sigma) = (\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2 + 6(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2) = 2Y^2 = 6\kappa^2
\]  
\[ \text{(C-13)} \]

In principal stress form,

\[
\phi(\sigma) = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2Y^2 = 6\kappa^2
\]  
\[ \text{(C-14)} \]

For plane stress conditions, the equation reduces to

\[
\phi(\sigma) = \sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = Y^2 = 3\kappa^2
\]  
\[ \text{(C-15)} \]

Similarly to the procedure for Tresca yield criterion, the material constant, \( \kappa \), can be obtained by performing a uniaxial tension test. Substituting Equation C-11 into Equation C-14 results in

\[
\kappa = \frac{\sigma_y}{\sqrt{3}}
\]  
\[ \text{(C-16)} \]

where \( \sigma_y \) is the yield stress in uniaxial tension.
C.3.3 HERSHEY

An identical non-quadratic isotropic yield function was independently proposed by Hosford (1972) and Hershey (1954)

\[ \sigma_y^n = \frac{1}{2} |\sigma_2 - \sigma_3|^n + \frac{1}{2} |\sigma_3 - \sigma_1|^n + \frac{1}{2} |\sigma_1 - \sigma_2|^n \] (C-17)

where \( n \) is a constant that depends on crystallographic structure. For FCC and BCC materials, \( n \) is 8 and 6, respectively. Hershey’s formulation can be reduced to the Mises yield criterion for \( a = 2 \), whereas for \( a = 1 \) or \( a = \infty \), it resembles Tresca. It can be seen that the presence of shear stress is not accommodated in this model. Hosford (1985) unsuccessfully attempted to add shear stress to his in-plane isotropic model. Since the model was not based on stress tensor invariants he only obtained a proper in-plane isotropic function when (Barlat and Lian, 1989). Additionally, for \( 1 < a < 2 \) and \( a > 4 \) the yield surface will fall between Tresca and Mises.

C.3.4 HILL 1948

One of the first phenomenological anisotropic yield functions was proposed by Hill (1948). Von Mises (1928) had already proposed an anisotropic yield function, but for single crystals. Hill’s first anisotropic yield model is a generalization of the von Mises criterion and, due to its quadratic nature, can predict two or four ears for a deep drawn cup. The parameters of the Hill 1948 quadratic function can either be calibrated using directional plastic strain ratios (referred to as r-based Hill 1948) or using directional yield stresses (referred to as stress-based Hill 1948). Adopting the principal axes of anisotropy as the axes of reference, the criterion takes the quadratic form

\[ \phi(\sigma) = F(\sigma_{yy} - \sigma_{xx})^2 + G(\sigma_{zz} - \sigma_{xx})^2 + H(\sigma_{xx} - \sigma_{yy})^2 + 2L\sigma_{xx}^2 + 2M\sigma_{yy}^2 + 2N\sigma_{xy}^2 = 2\bar{\sigma}^2 \] (C-18)

where \( F, G, H, L, M \) and \( N \) are material constants that describe the current state of anisotropy. In fact, these can be derived if three tensile yield stresses in the three principal anisotropy axes and three yield stresses for pure shear on each of the orthogonal planes of anisotropy are measured. Additionally, it can easily be noticed that for \( L = M = N = 3F = 3G = 3H \), Equation C-18 becomes equal to the von Mises criterion for isotropic materials.

For a plane stress state, Hill’s formulation can be expressed as

\[ \phi(\sigma) = (G + H)\sigma_{xx}^2 + (F + H)\sigma_{yy}^2 - 2H\sigma_{xx}\sigma_{yy} + 2N\sigma_{xy}^2 = 2\bar{\sigma}^2 \] (C-19)
The stress-based Hill 1948 model requires uniaxial yield stresses corresponding to rolling direction, diagonal direction, and transverse direction as well as the balanced biaxial yield stress. The stress-based Hill 1948 material constants are defined as

\[
F = \left( \frac{\sigma}{\sigma_0} \right)^2 + \left( \frac{\sigma}{\sigma_0} \right)^2 - \left( \frac{\sigma_0}{\sigma_0} \right)^2
\]

\[
G = \left( \frac{\sigma}{\sigma_0} \right)^2 + \left( \frac{\sigma}{\sigma_0} \right)^2 - \left( \frac{\sigma_0}{\sigma_0} \right)^2
\]

\[
H = \left( \frac{\sigma}{\sigma_0} \right)^2 + \left( \frac{\sigma}{\sigma_0} \right)^2 - \left( \frac{\sigma_0}{\sigma_0} \right)^2
\]

\[
N = 4 \left( \frac{\sigma}{\sigma_{45}} \right)^2 - \left( \frac{\sigma}{\sigma_b} \right)^2
\]

\[
\phi(\sigma) = \left[ \sigma_{xx}^2 + \sigma_{yy}^2 \left( \frac{\sigma}{\sigma_{90}} \right)^2 + \left( \frac{\sigma_{xx}^2 \sigma_{yy}^2}{\sigma_{90}^2 \sigma_{90}^2} - 1 \right) \sigma_{xx} \sigma_{yy} + \left( \frac{4 \sigma_{45}^2}{\sigma_{45}^2} - \sigma_{b}^2 \right) \sigma_{xy}^2 \right] = \bar{\sigma}_0^2
\]

The r-based Hill 1948 model requires r-values corresponding to rolling direction, diagonal direction, and transverse direction as well as the balanced biaxial yield stress. The r-based Hill 1948 material constants are defined as

\[
F = \frac{2r_o}{r_{90}(1 + r_o)} \left( \frac{\sigma}{\sigma_0} \right)^2
\]

\[
G = \frac{2}{1 + r_o} \left( \frac{\sigma}{\sigma_0} \right)^2
\]

\[
H = \frac{2r_o}{1 + r_o} \left( \frac{\sigma}{\sigma_0} \right)^2
\]

\[
N = \frac{(r_o + r_{90})(2r_{45} + 1)}{r_{90}(1 + r_o)} \left( \frac{\sigma}{\sigma_0} \right)^2
\]

where \( r_o \), \( r_{45} \), and \( r_{90} \) are the r-values for 0°, 45°, and 90° from the rolling direction.

\[
\phi(\sigma) = \left[ \sigma_{xx}^2 \left( \frac{r_o + r_{90}}{r_{90}(1 + r_o)} \right) \sigma_{yy}^2 - 2 \frac{r_o}{r_{90}(1 + r_o)} \sigma_{xx} \sigma_{yy} + \frac{(r_o + r_{90})(2r_{45} + 1)}{r_{90}(1 + r_o)} \sigma_{xy}^2 \right] = \bar{\sigma}_0^2
\]

In some applications, the r ratio can be disregarded by considering a state of planar isotropy \( (r = r = r) \) with a uniform mean \( \bar{r} \) value. The expression then becomes (in principal stress format)

\[
\phi(\sigma) = \left[ \sigma_1^2 + \sigma_2^2 - \frac{2\bar{r}}{(1 + \bar{r})} \sigma_1 \sigma_2 \right] = \bar{\sigma}_0^2
\]
C.3.5 BARLAT YLD89

Barlat et al. (1989) successfully extended and generalized the Hosford 1972 criterion to capture the influence of the shear stress. The new anisotropic yield function (so-called Yld89) is a non-quadratic yield criterion with four coefficients under plane stress condition and is based on a linear transformation of stress tensors:

\[
\phi(\sigma) = a[K_1 + K_2] \sigma^m + a[K_1 - K_2] \sigma^m + c[K_2] \sigma^m = 2\bar{\sigma}^m
\]  

(C-25)

where

\[
K_1 = \frac{\sigma_{xx} + h\sigma_{xy}}{2} \quad \text{and} \quad K_2 = \sqrt{\frac{(\sigma_{xx} - h\sigma_{xy})^2}{2} + p^2\sigma_{xy}^2}
\]  

(C-26)

Here, \(a\), \(h\), and \(p\) are material constants and \(m\) is a parameter comparable to \(n\) in Equation C-17. This function is shown to be unconditionally convex for \(m > 1\). For \(m = 2\), the criterion reduces to the Hill-criterion. The parameter \(m\) also determines the curvature near points of deviatoric uniaxial tensile and compressive stress. \(r_0\), \(r_{45}\), and \(r_{90}\) can be used to fit the parameters \(a\), \(h\), and \(p\).

From the advantages of the Yld89 function one can point out the easy parameter identification, except for \(p\), and suitable results for moderately anisotropic metals (Geng and Wagoner, 2002). As disadvantages, the poor prediction of balanced biaxial yield stress for highly anisotropic metals and the requirement of numerical treatment for finding parameter \(p\) can be mentioned. More importantly, the variation of in-plane yield stresses and Lankford coefficients cannot be simultaneously predicted.

C.3.6 BARLAT YLD2000-2D

To overcome the drawback of the Yld96 function (relative convexity,) and also to obtain a better prediction of in-plane variation of yield stresses and Lankford coefficients, Barlat and coworkers (2003) proposed a new plane stress Yld2000-2D yield function. This model gained considerable popularity mainly because of its accurate prediction of yield stresses and Lankford coefficients at rolling, diagonal and transverse directions, as well as balanced biaxial stress state. The non-quadratic Yld2000-2D yield function is based on a linear transformation of two unconditionally convex functions and of the deviatoric stress tensor.
The non-quadratic yield function by Barlat et al. (2003) (called Yld2000-2D) is widely used in order to describe the anisotropic material behavior of aluminum alloys due to the balance in accuracy and computational time.

Yld2000-2D yield function was proposed to overcome the limitations associated with the previously suggested formulation Yld96, which could not ensure convexity and required quite complex finite element implementation procedures. It consists of a non-quadratic plane stress yield function with eight material parameters (four stress-ratios and four r-values), based on an expansion of the yield criteria introduced by Hershey (1954) and Hosford (1972) for isotropic materials.

\[ \varphi = \varphi' + \varphi'' = 2\sigma^a \]  \hspace{1cm} (C-27)

where exponent “a” is a material coefficient and

\[ \varphi' = \left| X'_1 - X'_2 \right|^p \]  \hspace{1cm} (C-28)

\[ \varphi'' = \left| 2X'_2 + X'_1 \right|^p + \left| 2X'_1 + X'_2 \right|^p \]  \hspace{1cm} (C-29)

The yield surface is convex when \( a \geq 1 \). The exponent “a” in Equation C-27 is mainly associated with the crystal structure. A higher “a” value has the effect of increasing the curvature of the rounded vertices of the yield surface. In Equation C-28 and Equation C-29, \( X'_j \) and \( X''_k \) are the principal values of \( \mathbf{X}' \) and \( \mathbf{X}'' \) (two linear transformations on the stress deviator) defined as

\[
\begin{bmatrix}
X_{xx} \\
X_{xy} \\
X_{yx} \\
X_{yy}
\end{bmatrix} =
\begin{bmatrix}
C_{11} & 0 & 0 & s_{xx} \\
0 & C_{22} & 0 & s_{yy} \\
0 & 0 & C_{33} & s_{xy}
\end{bmatrix}
\]

\[
\begin{bmatrix}
X'_{xx} \\
X'_{xy} \\
X'_{yx} \\
X'_{yy}
\end{bmatrix} =
\begin{bmatrix}
C'_{11} & C'_{12} & 0 & s'_{xx} \\
C'_{21} & C'_{22} & 0 & s'_{yy} \\
0 & 0 & C'_{33} & s'_{xy}
\end{bmatrix}
\]

where subscripts x and y represent the rolling and transverse directions of the sheet, respectively. The transformation can also apply on the stress tensor; i.e.,

\[ \mathbf{X}' = \mathbf{C}' \mathbf{s} = \mathbf{C}' \mathbf{T} \mathbf{\sigma} = \mathbf{L}' \mathbf{\sigma} \]  \hspace{1cm} (C-31)

\[ \mathbf{X}'' = \mathbf{C}'' \mathbf{s} = \mathbf{C}'' \mathbf{T} \mathbf{\sigma} = \mathbf{L}'' \mathbf{\sigma} \]  \hspace{1cm} (C-32)

with
The yield function reduces to an isotropic expression when the matrices $C'$ and $C''$ representing these linear transformations are both taken as the identity matrix. For convenience, the tensors $L'$ and $L''$ representing linear transformations of the stress tensor are also considered. The expressions of the anisotropy coefficients of $L'$ and $L''$ are given as functions of independent coefficients $\alpha_k$ (for $k$ from 1 to 8) which all reduce to 1 in the isotropic case, i.e.,

\[
\begin{align*}
L'_{11} &= 2\alpha_1 / 3 \\
L'_{12} &= -\alpha_1 / 3 \\
L'_{21} &= -\alpha_2 / 3 \\
L'_{22} &= 2\alpha_2 / 3 \\
L'_{33} &= \alpha_7 \\
L''_{11} &= (8\alpha_5 - 2\alpha_3 - 2\alpha_6 + 2\alpha_4) / 9 \\
L''_{12} &= (4\alpha_6 - 4\alpha_4 - 4\alpha_5 + \alpha_3) / 9 \\
L''_{21} &= (4\alpha_3 - 4\alpha_5 - 4\alpha_4 + \alpha_6) / 9 \\
L''_{22} &= (8\alpha_4 - 2\alpha_6 - 2\alpha_3 + 2\alpha_5) / 9 \\
L''_{33} &= \alpha_8 
\end{align*}
\]

Equation C-34 includes eight coefficients. For the plane stress case, eight experimental input data $(\sigma_a, \sigma_{45}, \sigma_{90}, \sigma_p, r_{00}, r_{45}, r_{90}, r_p)$ are required for the calculation of eight independent output coefficients $\alpha'_{1-8}$.

### C.3.7 YLD2004-18P Model

For cubic metals, there are usually enough potentially active slip systems to accommodate any shape change. Compressive and tensile yield strengths are virtually identical and yielding is not influenced by the hydrostatic pressure. The yield surface of such materials is usually represented adequately by an even function of the principal values $S_k$ of the stress deviator $\mathbf{s}$ suggested by Hosford (1972), i.e.,

\[
\phi = |S_1 - S_2|^\sigma + |S_2 - S_3|^\sigma + |S_3 - S_1|^\sigma = 2\bar{\sigma}^\sigma
\]

The exponent $\sigma$ is connected to the crystal structure of the material, i.e., 6 for BCC and 8 for FCC. Extensions of Equation C-35 for the case of planar anisotropy are briefly summarized for a general stress state. The formulation is based on two linear transformations of the stress deviator. The two linear transformations can be expressed as

\[
\tilde{\mathbf{s}}' = C'\mathbf{s} = C''T\mathbf{s} = L'\mathbf{s}
\]

\[
\begin{bmatrix}
2/3 & -1/3 & 0 \\
-1/3 & 2/3 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]
\[ \bar{s}'' = C'' s = C'' T \sigma = L'' \sigma \]  

(C-37)

where \( T \) is a matrix that transforms the Cauchy stress tensor \( \sigma \) to its deviator \( s \). \( \bar{s}' \) and \( \bar{s}'' \) are the linearly transformed stress deviators and \( C' \) and \( C'' \) (or \( L' \) and \( L'' \)) are the matrices containing the anisotropy coefficients.

For a full stress state, the linear transformations can be expressed in the most general form using the following matrices

\[
C' = \begin{bmatrix}
0 & -c_{12}' & -c_{13}' & 0 & 0 & 0 \\
-c_{21}' & 0 & -c_{23}' & 0 & 0 & 0 \\
-c_{31}' & -c_{32}' & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_{44}' & 0 & 0 \\
0 & 0 & 0 & 0 & c_{55}' & 0 \\
0 & 0 & 0 & 0 & 0 & c_{66}'
\end{bmatrix}, \quad C'' = \begin{bmatrix}
0 & -c_{12}'' & -c_{13}'' & 0 & 0 & 0 \\
-c_{21}'' & 0 & -c_{23}'' & 0 & 0 & 0 \\
-c_{31}'' & -c_{32}'' & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_{44}'' & 0 & 0 \\
0 & 0 & 0 & 0 & c_{55}'' & 0 \\
0 & 0 & 0 & 0 & 0 & c_{66}''
\end{bmatrix}
\]  

(C-38)

For use in 3D cases, there are 18 material parameters that have to be optimized by numerical methods such as Newton-Raphson iterations. Tensile yield stress and Lankford coefficient at each 15° from rolling to transverse direction as well as at balanced biaxial stress state provide sixteen experimental inputs for parameter identification. To obtain the Lankford coefficient at balanced biaxial state, the disk compression test proposed by Barlat et al is preferred due to less error compared to the viscous pressure bulge test (Barlat et al., 2003). The other inputs for parameter identification could be out-of-plane yield stresses at 45° tension in TD-ND and ND-RD planes or simple shear tests at TD-ND and ND-RD planes (Barlat et al., 2005). Since this is very challenging from a practical point of view, it is recommended to perform this parameter identification based on polycrystalline simulations. In case such simulations are not available, Barlat et al suggested that all out of plane yield stresses can be assumed equal to that corresponding to rolling direction and for pure shear tests. (Barlat et al., 2005).

### C.4 WORK HARDENING

Work hardening, also known as strain hardening or cold working, is the strengthening of a metal by plastic deformation. This strengthening occurs because of dislocation glide or slip on crystallographic planes and directions. During deformation of the material, a gradual lattice rotation is caused by the dislocation slip, leading to accumulation of dislocations at microstructural obstacles, resulting in an increase in the slip resistance and is characterized as hardening of the material (Cardoso and Yoon, 2009). This hardening of the stress-stain curve is depicted in Figure C-5.
Work hardening is characterized by a combination of isotropic hardening (in which the hardening effect is the same in all material directions) and kinematic hardening (which refers to a constant yield stress magnitude where the entire yield diagram is shifted in the direction of plastic strain) as shown in Figure C-6. Illustration of the evolution of the yield stress is represented in Figures C-6b and Figure C-6c for isotropic and kinematic hardening, respectively. These models can be coupled in most sheet metals and thus mixed isotropic-kinematic hardening models are often implemented allowing both translation and expansion of the yield surface, as illustrated in Figure C-6d.
Figure C-6. Work hardening characterized by a combination of isotropic and kinematic hardening.

For simplicity in developing the analytical forms that define strain hardening, this discussion is limited to the assumption of isotropic strain hardening, otherwise known as flow stress. A summary of available strain hardening functions that represent true flow stress (σ) are listed below (the interested reader can find a thorough treatment by Sung, et. al (2010)):

<table>
<thead>
<tr>
<th>Hardening Law</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ludwig (1909)</td>
<td>σ = σ₀ + Kεⁿ</td>
</tr>
<tr>
<td>Hollomon (1945)</td>
<td>σ = Kεⁿ</td>
</tr>
<tr>
<td>Swift (1952)</td>
<td>σ = K(ε₀ + ε)ⁿ</td>
</tr>
<tr>
<td>Voce (1948)</td>
<td>σ = A - B exp(−Cεₚ)</td>
</tr>
<tr>
<td>Hartley and Srinivasan (1983)</td>
<td>σ = σ₀ + K(ε₀ + ε)ⁿ</td>
</tr>
<tr>
<td>Ludwigson (1971)</td>
<td>σₗ = K₁εⁿ + exp(K₂ + n₁ε)</td>
</tr>
<tr>
<td>Ghosh (1977)</td>
<td>σ = A(ε₀ + εₚ)ⁿ - C</td>
</tr>
<tr>
<td>Hockett-Sherby (1975)</td>
<td>σ = A - B exp(−Cεₚ)ⁿ</td>
</tr>
</tbody>
</table>
where $\sigma_0$, $e_0$, $K$, $n$, $A$, $B$, $C$, $K_1$, $K_2$, $n_1$, and $n_2$ are material parameters obtained from standard material testing. The constant “$n$” is called the strain hardening exponent, whereas the constant “$e$” is referred to as true plastic strain.
C.5 REFERENCES


Hill's 1948 Yield Function - Derivatives

D.1 - BACKGROUND

Hill's 1948 quadratic yield function for plane stress becomes:

\[
\phi(\sigma) = \left[ \sigma_{xx}^2 + \frac{r_0(1 + r_90)}{r_{90}(1 + r_0)} \sigma_{yy}^2 - 2 \frac{r_0}{(1 + r_0)} \sigma_{xx} \sigma_{yy} + \frac{(r_0 + r_90)(2r_{45} + 1)}{r_{90}(1 + r_0)} \right] \left( \frac{\sigma}{\sigma_o} \right)^2 = \bar{\sigma}^2 \tag{D-1}
\]

In principal stress form

\[
\phi(\sigma) = \left[ \sigma_1^2 + \frac{r_0(1 + r_{90})}{r_{90}(1 + r_0)} \sigma_2^2 - \frac{2r_0}{(1 + r_0)} \sigma_1 \sigma_2 \right] \left( \frac{\sigma}{\sigma_o} \right)^2 = \bar{\sigma}^2 \tag{D-2}
\]

For the special case of transverse isotropy \((r = r_90 = r_{90})\) we get

\[
\phi(\sigma) = \left[ \sigma_1^2 + \sigma_2^2 - \frac{2r}{(1 + r)} \sigma_1 \sigma_2 \right] \left( \frac{\sigma}{\sigma_o} \right)^2 = \bar{\sigma}^2 \tag{D-3}
\]

\[
\bar{\sigma} = \left( \frac{\sigma}{\sigma_o} \right) \sqrt{r_{90}(1 + r_0) \sigma_{xx}^2 + \frac{r_0(1 + r_{90})}{r_{90}(1 + r_0)} \sigma_{yy}^2 - \frac{2r_0 r_{90}}{r_{90}(1 + r_0)} \sigma_{xx} \sigma_{yy} + \frac{(r_0 + r_{90})(2r_{45} + 1)}{r_{90}(1 + r_0)} \sigma_{xy}^2} \tag{D-4}
\]

Defining the following:
The yield function becomes

\[
\bar{\sigma} = \sqrt{a_1 \sigma_{ss}^2 + a_2 \sigma_{yy}^2 - a_3 \sigma_{ss} \sigma_{yy} + a_4 \sigma_{yy}^2}
\] (D-6)

Using the following relations:

\[
\frac{d}{dx} \left( u^n \right) = nu^{n-1} \frac{du}{dx}
\]
\[
\frac{d}{dx} \left( \sqrt{u} \right) = \frac{1}{2\sqrt{u}} \frac{du}{dx}
\]
\[
\frac{d}{dx} \left( uv \right) = v \frac{du}{dx} + u \frac{dv}{dx}
\]
\[
\frac{d}{dx} \left( \frac{1}{\sqrt{u}} \right) = -\frac{1}{2\sqrt{u}} \frac{du}{dx}
\] (D-7)

Defining \( u \)

\[
u = a_1 \sigma_{ss}^2 + a_2 \sigma_{yy}^2 - a_3 \sigma_{ss} \sigma_{yy} + a_4 \sigma_{yy}^2
\] (D-8)

Then

\[
\bar{\sigma} = \sqrt{u}
\] (D-9)

And

\[
\frac{\partial \bar{\sigma}}{\partial u} = \frac{1}{2\sqrt{u}}
\] (D-10)
\begin{align*}
\frac{\partial u}{\partial \sigma_{xx}} &= (2a_4 \sigma_{xx} - a_3 \sigma_{yy}) \\
\frac{\partial u}{\partial \sigma_{yy}} &= (2a_4 \sigma_{yy} - a_3 \sigma_{xx}) \\
\frac{\partial u}{\partial \sigma_{xy}} &= (2a_4 \sigma_{xy})
\end{align*}
(D-11)

The 1\textsuperscript{st} derivatives are
\begin{align*}
\frac{\partial \bar{\sigma}}{\partial \sigma_{xx}} &= \frac{\partial \bar{\sigma}}{\partial u} \frac{\partial u}{\partial \sigma_{xx}} = \frac{1}{2\sqrt{u}} (2a_1 \sigma_{xx} - a_3 \sigma_{yy}) = \frac{1}{2\bar{\sigma}} (2a_1 \sigma_{xx} - a_3 \sigma_{yy}) \\
\frac{\partial \bar{\sigma}}{\partial \sigma_{yy}} &= \frac{\partial \bar{\sigma}}{\partial u} \frac{\partial u}{\partial \sigma_{yy}} = \frac{1}{2\sqrt{u}} (2a_2 \sigma_{yy} - a_3 \sigma_{xx}) = \frac{1}{2\bar{\sigma}} (2a_2 \sigma_{yy} - a_3 \sigma_{xx}) \\
\frac{\partial \bar{\sigma}}{\partial \sigma_{xy}} &= \frac{\partial \bar{\sigma}}{\partial u} \frac{\partial u}{\partial \sigma_{xy}} = \frac{1}{2\sqrt{u}} (2a_4 \sigma_{xy}) = \frac{1}{2\bar{\sigma}} (2a_4 \sigma_{xy})
\end{align*}
(D-12)

The 2\textsuperscript{nd} derivatives are
\begin{align*}
\frac{\partial^2 \bar{\sigma}}{\partial \sigma_{xx}\partial \sigma_{xx}} &= \frac{1}{2\sqrt{u}} (2a_2) - \frac{1}{4\sqrt{u}} (2a_1 \sigma_{xx} - a_3 \sigma_{yy})^2 = \frac{1}{2\bar{\sigma}} (2a_2) - \frac{1}{2\sigma} \frac{\partial \bar{\sigma}}{\partial \sigma_{xx}} \frac{\partial \bar{\sigma}}{\partial \sigma_{xx}} \\
\frac{\partial^2 \bar{\sigma}}{\partial \sigma_{xx}\partial \sigma_{yy}} &= \frac{1}{2\sqrt{u}} (2a_2) - \frac{1}{4\sqrt{u}} (2a_1 \sigma_{xx} - a_3 \sigma_{yy})(2a_2 \sigma_{yy} - a_3 \sigma_{xx}) = \frac{1}{2\bar{\sigma}} (2a_2) - \frac{1}{2\sigma} \frac{\partial \bar{\sigma}}{\partial \sigma_{xx}} \frac{\partial \bar{\sigma}}{\partial \sigma_{yy}} \\
\frac{\partial^2 \bar{\sigma}}{\partial \sigma_{xx}\partial \sigma_{xy}} &= \frac{1}{4\sqrt{u}} (2a_2) = \frac{1}{\bar{\sigma}} \frac{\partial \bar{\sigma}}{\partial \sigma_{xx}} \frac{\partial \bar{\sigma}}{\partial \sigma_{xy}} \\
\frac{\partial^2 \bar{\sigma}}{\partial \sigma_{yy}\partial \sigma_{yy}} &= \frac{1}{2\sqrt{u}} (2a_2) - \frac{1}{4\sqrt{u}} (2a_2 \sigma_{yy} - a_3 \sigma_{xx})^2 = \frac{1}{2\bar{\sigma}} (2a_2) - \frac{1}{2\sigma} \frac{\partial \bar{\sigma}}{\partial \sigma_{yy}} \frac{\partial \bar{\sigma}}{\partial \sigma_{yy}} \\
\frac{\partial^2 \bar{\sigma}}{\partial \sigma_{yy}\partial \sigma_{xy}} &= \frac{1}{4\sqrt{u}} (2a_2) = \frac{1}{\bar{\sigma}} \frac{\partial \bar{\sigma}}{\partial \sigma_{yy}} \frac{\partial \bar{\sigma}}{\partial \sigma_{xy}} \\
\frac{\partial^2 \bar{\sigma}}{\partial \sigma_{xy}\partial \sigma_{xy}} &= \frac{1}{2\sqrt{u}} (2a_2) - \frac{1}{4\sqrt{u}} (2a_4 \sigma_{xy})^2 = \frac{1}{2\bar{\sigma}} (2a_2) - \frac{1}{2\sigma} \frac{\partial \bar{\sigma}}{\partial \sigma_{xy}} \frac{\partial \bar{\sigma}}{\partial \sigma_{xy}}
\end{align*}
(D-13)
E.1 BACKGROUND

A nonlinear least squares is an unconstrained minimization problem of the form

\[ \min_x f(x) = \frac{1}{2} \sum_{i=1}^{m} (f_i(x))^2 = \frac{1}{2} F(x)^T F(x) \]  

(E-1)

where \( F(x) \) is a vector-valued function defined as:

\[ F(x) = [f_1(x), f_2(x), \ldots, f_m(x)]^T \]

(E-2)

It is called least squares because of the minimization the sum of squares of these functions.

The gradient \( \nabla \) of a multi-variable function \( f(x) \) is a vector consisting of the function’s partial derivatives

\[ \nabla f(x_1, x_2, \ldots, x_n) = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right) \]

(E-3)

The Hessian matrix \( H(f) \) of a function \( f(x) \) is the square matrix of second-order partial derivatives of \( f(x) \)

\[
H(f(x_1, x_2, \ldots, x_n)) = \\
\begin{bmatrix}
\frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\
\frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2}
\end{bmatrix}
\]

(E-4)
The gradient and Hessian can be expressed in terms of the Jacobian. The gradient is defined by

\[ \nabla f(x) = \sum_{i=1}^{m} f_i(x) \nabla f_i(x) = \nabla F(x) F(x) = J(x)^T F(x) \]  \hspace{1cm} (E-5)

where the Jacobian matrix \( J(x) \) is defined as:

\[
J(x_1, x_2, \ldots, x_n) =
\begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n}
\end{bmatrix}
\hspace{1cm} (E-6)
\]

The Hessian is determined using

\[ \nabla^2 f(x) = \sum_{i=1}^{m} \nabla^2 f_i(x) \nabla f_i(x)^T + \sum_{i=1}^{m} F(x) \nabla^2 f_i(x) = J(x)^T J(x) + \sum_{i=1}^{m} F(x) \nabla^2 F(x) = J(x)^T J(x) \]  \hspace{1cm} (E-7)

Note that the second-order term in the Hessian \( H(x) \) is multiplied by the residual \( F(x) \). In most problems, the residuals will typically be small. Also, at the minimum, the residuals will typically be distributed with mean = 0. For these reasons, the second-order term is often ignored, giving the Gauss-Newton approximation to the Hessian:

\[ H(x) \approx J(x)^T J(x) \]  \hspace{1cm} (E-8)

Hence, explicit computation of the full Hessian can again be avoided.
E.2 EXAMPLE - VOCE EQUATION CURVE FITTING

The data in Table E-1 will be used for curve fitting the Voce equation defined by:

\[ \sigma = A - B \exp(-C \varepsilon) \]  \hspace{1cm} \text{(E-9)}

where \( A \) is the saturation stress, \( B \) is the work hardening, and \( C \) is the hardening rate parameter. The derivatives of the function with respect to \( A, B, C \) are

\[
\frac{\partial \sigma}{\partial A} = 1 \\
\frac{\partial \sigma}{\partial B} = -\exp(-C \varepsilon) \\
\frac{\partial \sigma}{\partial C} = (B \varepsilon) \exp(-C \varepsilon)
\]

\[
\begin{align*}
\frac{\partial^2 \sigma}{\partial A \partial A} & = 0 \\
\frac{\partial^2 \sigma}{\partial A \partial B} & = -\exp(-C \varepsilon) \\
\frac{\partial^2 \sigma}{\partial A \partial C} & = 0 \\
\frac{\partial^2 \sigma}{\partial B \partial B} & = -C \exp(-C \varepsilon) \\
\frac{\partial^2 \sigma}{\partial B \partial C} & = B \varepsilon \exp(-C \varepsilon) \\
\frac{\partial^2 \sigma}{\partial C \partial C} & = -2C \exp(-C \varepsilon)
\end{align*}
\hspace{1cm} \text{(E-10)}
\]

\[
\begin{array}{c|cccccc}
 m & m=1 & m=2 & m=3 & m=4 & m=5 & m=6 \\
 \varepsilon_i & 0.05 & 0.10 & 0.15 & 0.20 & 0.25 & 0.30 \\
 \sigma_i & 36967.3 & 38160.6 & 38884.3 & 39323.3 & 39589.6 & 39751.1 \\
\end{array}
\]

The residuals \( F(x) \) are calculated using

\[
F(x) = \begin{bmatrix}
\sigma(\varepsilon_1) \\
\sigma(\varepsilon_2) \\
\sigma(\varepsilon_3) \\
\sigma(\varepsilon_4) \\
\sigma(\varepsilon_5) \\
\sigma(\varepsilon_6)
\end{bmatrix} - \begin{bmatrix}
A - B \exp(-C \varepsilon_1) - \sigma_1 \\
A - B \exp(-C \varepsilon_2) - \sigma_2 \\
A - B \exp(-C \varepsilon_3) - \sigma_3 \\
A - B \exp(-C \varepsilon_4) - \sigma_4 \\
A - B \exp(-C \varepsilon_5) - \sigma_5 \\
A - B \exp(-C \varepsilon_6) - \sigma_6
\end{bmatrix}
\hspace{1cm} \text{(E-12)}
\]

The gradient \( \nabla F(x) \) is determined
The formula for the least squares objective function is

\[
f(x) = \frac{1}{2} \sum_{i=1}^{6} (A - B \exp(-C \epsilon_i) - \sigma_i)^2 = \frac{1}{2} F(x)^T F(x) \tag{E-14}
\]

The gradient of the least squares objective function is

\[
\nabla f(x) = \frac{1}{2} \sum_{i=1}^{6} (A - B \exp(-C \epsilon_i) - \sigma_i)(1) = \nabla F(x) \tag{E-15}
\]

Using an initial guess for the \(A,B,C\) parameters

\[
\{ x \} = \{ A = 30000.0, B = 5000.0, C = 5.0 \} \tag{E-16}
\]

The residuals are calculated as:

\[
F(x) = \begin{bmatrix}
30000 - 6000 \exp(-5 \epsilon_1) - \sigma_1 \\
30000 - 6000 \exp(-5 \epsilon_2) - \sigma_2 \\
30000 - 6000 \exp(-5 \epsilon_3) - \sigma_3 \\
30000 - 6000 \exp(-5 \epsilon_4) - \sigma_4 \\
30000 - 6000 \exp(-5 \epsilon_5) - \sigma_5 \\
30000 - 6000 \exp(-5 \epsilon_6) - \sigma_6 \\
\end{bmatrix} = \begin{bmatrix}
-11640.2 \\
-11799.8 \\
-11718.5 \\
-11530.6 \\
-11308.6 \\
-11089.8 \\
\end{bmatrix} \tag{E-17}
\]

The gradient \( \nabla F(x) \) is

\[
\nabla F(x) = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-0.7788 & -0.60653 & -0.47237 & -0.36788 & -0.2865 & -0.22313 \\
233.6402 & 363.9184 & 425.1299 & 441.4553 & 429.7572 & 401.6343 \\
\end{bmatrix} \tag{E-18}
\]
\[
\n\nabla F(x) \nabla F(x)^T = \begin{bmatrix}
6.0000 & -2.7352 & 2295.5353 \\
-2.7352 & 1.4647 & -978.6505 \\
2295.5353 & -978.6505 & 908643.9442
\end{bmatrix}
\]

The gradient of the of the least squares function is

\[
\nabla F(x)F(x) = \begin{bmatrix}
-69087.536 \\
31714.061 \\
-2639933.525
\end{bmatrix}
\]

The Gauss-Newton search direction \( p \) is obtained by solving the linear equation

\[
\nabla F(x) \nabla F(x)^T p = -\nabla F(x)F(x)
\]

which gives

\[
p = \left[ \nabla F(x) \nabla F(x)^T \right]^{-1} \left[-\nabla F(x)F(x)\right] = \begin{bmatrix}
9021.7 \\
-2213.5 \\
3.878
\end{bmatrix}
\]

The new estimate of the solution is

\[
x_k^{i+1} = x_k^{i} + p_i^k = \begin{bmatrix}
30000 \\
6000 \\
5
\end{bmatrix} + \begin{bmatrix}
9021.7 \\
-2213.5 \\
3.878
\end{bmatrix} = \begin{bmatrix}
39021.7 \\
3786.5 \\
8.878
\end{bmatrix}
\]

A summary of the iteration process is provided in Table E-2. After iteration 5, the theoretical solution has been reached.

<table>
<thead>
<tr>
<th>Iteration</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>Residual</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30000.0</td>
<td>6000.0</td>
<td>5.000</td>
<td>3.979E+08</td>
</tr>
<tr>
<td>2</td>
<td>39021.7</td>
<td>3786.5</td>
<td>8.878</td>
<td>2.102E+06</td>
</tr>
<tr>
<td>3</td>
<td>39984.6</td>
<td>4970.6</td>
<td>10.293</td>
<td>3.511E+03</td>
</tr>
<tr>
<td>4</td>
<td>39999.2</td>
<td>4998.2</td>
<td>9.994</td>
<td>3.972E+00</td>
</tr>
<tr>
<td>5</td>
<td>40000.0</td>
<td>5000.0</td>
<td>10.000</td>
<td>8.192E-08</td>
</tr>
</tbody>
</table>
E.3 EXAMPLE - SWIFT EQUATION PARTIAL DERIVATIVES

The Swift equation is defined as

\[ \sigma = K(\varepsilon_0 + \varepsilon)^n \]  

(E-24)

The partial derivatives are

\[
\begin{align*}
\frac{\partial \sigma}{\partial K} &= (\varepsilon_0 + \varepsilon)^n \\
\frac{\partial \sigma}{\partial \varepsilon_0} &= nK(\varepsilon_0 + \varepsilon)^{n-1} \\
\frac{\partial \sigma}{\partial n} &= K(\varepsilon_0 + \varepsilon)^n \ln(\varepsilon_0 + \varepsilon)
\end{align*}
\]  

(E-25)

The 2nd order partial derivatives are determined using

\[
\begin{align*}
\frac{\partial^2 \sigma}{\partial K \partial K} &= 0 \\
\frac{\partial^2 \sigma}{\partial K \partial \varepsilon_0} &= (\varepsilon_0 + \varepsilon)^n \\
\frac{\partial^2 \sigma}{\partial K \partial n} &= (\varepsilon_0 + \varepsilon)^n \\
\frac{\partial^2 \sigma}{\partial \varepsilon_0 \partial \varepsilon_0} &= (n(n-1)K)(\varepsilon_0 + \varepsilon)^{n-2} \\
\frac{\partial^2 \sigma}{\partial \varepsilon_0 \partial n} &= \left( K(\varepsilon_0 + \varepsilon)^{n-1} \right) \left( n\ln(\varepsilon_0 + \varepsilon) + 1 \right) \\
\frac{\partial^2 \sigma}{\partial n \partial n} &= \left( K(\varepsilon_0 + \varepsilon)^n \right) \left( n\ln^2(\varepsilon_0 + \varepsilon) \right)
\end{align*}
\]  

(E-26)

Following the same procedure defined above, the Swift equation can be used for curve fitting stress-strain data.
APPENDIX F

LIST OF PUBLICATIONS

F.1 PEER REVIEWED JOURNALS


F.2 INTERNATIONAL CONFERENCES


F.3 ADDITIONAL PUBLICATIONS


