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Stochastic Memristive Quaternion-Valued Neural Networks with Time Delays: An Analysis on Mean Square Exponential Input-to-State Stability

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Abstract: In this paper, we study the mean-square exponential input-to-state stability (exp-ISS) problem for a new class of neural network (NN) models, i.e., continuous-time stochastic memristive quaternion-valued neural networks (SMQVNNs) with time delays. Firstly, in order to overcome the difficulties posed by non-commutative quaternion multiplication, we decompose the original SMQVNNs into four real-valued models. Secondly, by constructing suitable Lyapunov functional and applying Itô's formula, Dynkin's formula as well as inequity techniques, we prove that the considered system model is mean-square exp-ISS. In comparison with the conventional research on stability, we derive a new mean-square exp-ISS criterion for SMQVNNs. The results obtained in this paper are the general case of previously known results in complex and real fields. Finally, a numerical example has been provided to show the effectiveness of the obtained theoretical results.

Keywords: stochastic memristive quaternion-valued neural networks; exponential input-to-state stability; Lyapunov fractional

1. Introduction

In 1971, Chua proposed the theoretical idea of memristor [1], and its prototype was produced by the HP lab successfully in 2008 [2,3]. A memristor is the fourth fundamental element of an electrical circuit that can be used to construct a new artificial neural network. It has a tremendous potential to be utilized in synopsis for simulation of the human brain by replacing a resistor with a memristor [4–6]. In view of these characteristics, a new neural network (NN) model, namely, the memristive neural network (MNN) has been widely studied, and many theoretical papers regarding various dynamics of MNNs have been published in recent years [7–13]. From the real-world application perspective, time delays inherently arise in many practical systems including NNs. Indeed, time delays appear as the main cause of instability, compromising the system performance in real environments [10,14–18].

Typically, two main types of time delays exist, i.e., either constant delays or time-varying delays. The time-varying delays can be viewed as a general case of constant time delays. Up to now, a number of investigations with respect to the dynamics of NN models along with time-varying delays are available in the literature [10,15–21].

It has been reported that real-valued neural networks (RVNNs) are useful in various scientific fields, for example, optimization, image and signal processing, as well as associative memory [10,16,17]. However, RVNN models perform poorly in tackling the XOR problem and in 2D affine transformations [18,19]. In view of this, complex properties have been incorporated into RVNNs to formulate complex-valued neural networks (CVNNs), which can effectively solve the XOR problem and 2D affine transformations [18,19]. As a result, CVNN-related models have received significant research attention in both mathematical and practical analyses [20–24]. Nevertheless, CVNN models are inefficient in handling higher dimension transformations including color night vision, color image compression, and 3D and 4D problems [25–27]. Meanwhile, several engineering problems involve quaternion-valued signals and quaternion functions, such as 3D wind forecasting, polarized signal classification, color night vision, as well as color night vision [26–28]. Undoubtedly, quaternion-based networks present good mathematical models to undertake these applications due to the quaternion features. In view of this, quaternion-valued neural networks (QVNNs) have been developed by implementing quaternion algebra into CVNNs, in order to generalize RVNN and CVNN models with quaternion-valued activation functions, connection weights, as well as signals states [11,29–32]. Therefore, the investigation of the dynamics of QVNN models is essential and important. Recently, many computational approaches for various QVNN models and their learning algorithms have been proposed. Among the studies include global μ stability, global asymptotic stability as well as global synchronization [29–31]. Other studies of QVNN models are also available, for example, exponential input-to-state stability (exp-ISS) and global Mittag-Leffler stability and synchronization [32,33]. Similarly, some other stability conditions have been defined for QVNN models [12,34,35].

In modeling problems, we have to consider the existence of noise, since it is inherent in nonlinear dynamic systems. In fact, when a system is influenced by external disturbance, the stochastic neural network (SNN) models become very useful for describing a real system, as compared with a deterministic NN. Therefore, it is crucial to examine the dynamic properties pertaining to SNN models. As a result, several research studies have extensively analyzed the dynamics of SNN models [36,37], stochastic Hopfield NN models [38], stochastic memristive NN models [13], and stochastic CVNN models [13]. On the other hand, it is clear that the external inputs can influence the dynamic behaviors of NNs in practical applications. As such, the ISS criterion is important to characterize the effects of external inputs [10,39,40]. Recently, several useful ISS results for stochastic RVNN and CVNN models have been published [13,39–41]. It should be noted that most studies on QVNN dynamic behaviors are concerned with global asymptotic stability and synchronization analysis [11,29–32]. There are only a few studies on the exponential stability of QVNN models. According to our survey, the study on mean-square exp-ISS analysis with respect to SMQVNN models with time-varying delays is new in the literature, and our paper contributes toward this research area.

Inspired by the above-mentioned discussions, our analysis mainly focuses on the mean-square exp-ISS problem for the SMQVNN models. The current SMQVNN model encompasses stochastic memristive CVNN and stochastic memristive RVNN as its special cases. We divide the original QVNN into four RVNN models, in order to avoid the issue of non-commutative of quaternion multiplication. By constructing suitable Lyapunov functional and utilizing Dynkin's and Itô's formulae, as well as other inequality methods, we establish new sufficient conditions for exp-ISS with respect to the considered SMQVNN model. We illustrate the usefulness of the obtained results by a numerical example.

This study has the following contributions: (1) this is the first analysis on the mean-square exp-ISS pertaining to SMQVNN models with time-varying delays; (2) unlike the conventional stability criteria, we establish new mean-square exp-ISS pertaining to the SMQVNN by dividing the original QVNN

into four RVNN models; (3) the main results of this paper are new and more general than those in the QVNN literature.

This paper is structured as follows. We formally define the proposed model and analyze the new exp-ISS criteria in Sections 2 and 3, respectively. We present the numerical example and the associated simulation results in Section 4. In Section 5, we present the key conclusions and some suggestions for future research.

2. Mathematical Fundamentals and Problem Formulation

2.1. Mathematical Notations

The real field, complex field, and skew field of quaternion are denoted as \mathbb{R}, \mathbb{C} , and \mathbb{Q} , respectively. Their n -dimension vector with elements of \mathbb{R}, \mathbb{C} and \mathbb{Q} are denoted as $\mathbb{R}^n, \mathbb{C}^n$ and \mathbb{Q}^n , while their $n \times n$ matrices with entries from \mathbb{R}, \mathbb{C} and \mathbb{Q} are denoted as $\mathbb{R}^{n \times n}, \mathbb{C}^{n \times n}$ and $\mathbb{Q}^{n \times n}$, respectively. In addition, the space of a continuous function mapping φ from $[-\tau, 0]$ into \mathbb{Q}^n is denoted as $\mathcal{C}([-\tau, 0]; \mathbb{Q}^n)$. The closure of the convex hull of \mathbb{Q}^n , which is formulated from quaternion numbers ∇ and Δ , is denoted as $co\{\nabla, \Delta\}$. A class of essentially bounded function u from $[0, \infty)$ to \mathbb{Q}^n with $\|u\|_\infty = \text{ess sup}_{-\tau \leq s \leq 0} |u(s)| < \infty$ is denoted as ℓ_∞ , while the family of all \mathcal{F}_0 measurable is denoted as $\mathcal{L}_{\mathcal{F}_0}^2([-\tau, 0]; \mathbb{Q}^n)$. Besides that, $\mathcal{C}([-\tau, 0]; \mathbb{Q}^n)$ -valued stochastic variables $\{\varphi(s) : -\tau \leq s \leq 0\}$ is in a way that $\int_{-\tau}^0 \mathbb{E}|\varphi(s)|^2 ds < \infty$, in which the mathematical expectation operation pertaining to a probability measure \mathcal{P} is denoted as $\mathbb{E}\{\cdot\}$. Superscripts $*$ and T represent the complex conjugate transpose and matrix transposition, while i, j, k represent the imaginary units, respectively, and $N = 1, 2, \dots, n$.

2.2. Quaternion Algebra

Firstly, we address the quaternion and its operating rules. We can express the quaternion, which consists of a real part along and three imaginary parts, as:

$$\mathbf{m} = \mathbf{m}^R + i\mathbf{m}^I + j\mathbf{m}^J + k\mathbf{m}^K \in \mathbb{Q},$$

where $\mathbf{m}^R, \mathbf{m}^I, \mathbf{m}^J, \mathbf{m}^K \in \mathbb{R}$. The multiplication rules of Hamilton are satisfied by the imaginary roots i, j, k :

$$i^2 = j^2 = k^2 = ijk = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j. \tag{1}$$

The quaternion-valued function is denoted by $\mathbf{m}(t) = \mathbf{m}^R(t) + i\mathbf{m}^I(t) + j\mathbf{m}^J(t) + k\mathbf{m}^K(t) \in \mathbb{Q}$, where $\mathbf{m}^R(t), \mathbf{m}^I(t), \mathbf{m}^J(t), \mathbf{m}^K(t) \in \mathbb{R}$. Let $\mathbf{p} = \mathbf{p}^R + i\mathbf{p}^I + j\mathbf{p}^J + k\mathbf{p}^K$ and $\mathbf{q} = \mathbf{q}^R + i\mathbf{q}^I + j\mathbf{q}^J + k\mathbf{q}^K$ are two quaternions, the addition and subtraction of $\mathbf{p} \pm \mathbf{q}$ are defined as

$$\mathbf{p} \pm \mathbf{q} = (\mathbf{p}^R \pm \mathbf{q}^R) + i(\mathbf{p}^I \pm \mathbf{q}^I) + j(\mathbf{p}^J \pm \mathbf{q}^J) + k(\mathbf{p}^K \pm \mathbf{q}^K).$$

We can express the product of $\mathbf{p}\mathbf{q}$ with respect to the multiplication rules of Hamilton as:

$$\begin{aligned} \mathbf{p}\mathbf{q} = & (\mathbf{p}^R\mathbf{q}^R - \mathbf{p}^I\mathbf{q}^I - \mathbf{p}^J\mathbf{q}^J - \mathbf{p}^K\mathbf{q}^K) + i(\mathbf{p}^R\mathbf{q}^I + \mathbf{p}^I\mathbf{q}^R + \mathbf{p}^J\mathbf{q}^K - \mathbf{p}^K\mathbf{q}^J) \\ & + j(\mathbf{p}^R\mathbf{q}^J + \mathbf{p}^J\mathbf{q}^R - \mathbf{p}^I\mathbf{q}^K + \mathbf{p}^K\mathbf{q}^I) + k(\mathbf{p}^R\mathbf{q}^K + \mathbf{p}^K\mathbf{q}^R + \mathbf{p}^I\mathbf{q}^J - \mathbf{p}^J\mathbf{q}^I). \end{aligned}$$

Note that given any $\mathbf{m}(t) \in \mathbb{Q}$, $\|\mathbf{m}\| = \sqrt{\sum_{r=1}^n (\mathbf{m}^R)^2 + \sum_{r=1}^n (\mathbf{m}^I)^2 + \sum_{r=1}^n (\mathbf{m}^J)^2 + \sum_{r=1}^n (\mathbf{m}^K)^2}$ represents the norm of \mathbf{m} , while $|\mathbf{m}| = \sqrt{\mathbf{m}\mathbf{m}^*} = \sqrt{(\mathbf{m}^R)^2 + (\mathbf{m}^I)^2 + (\mathbf{m}^J)^2 + (\mathbf{m}^K)^2}$ represents the modulus of \mathbf{m} , in which the conjugate transpose of \mathbf{m} is denoted as $\mathbf{m}^* = \mathbf{m}^R(t) - i\mathbf{m}^I(t) - j\mathbf{m}^J(t) - k\mathbf{m}^K(t)$.

2.3. Problem Formation

In this section, we define a SMQVNN model as:

$$\begin{aligned}
 d\mathbf{m}_r(t) = & [-\mathbf{c}_r \mathbf{m}_r(t) + \sum_{\mathfrak{z}=1}^n \mathbf{a}_{r\mathfrak{z}}(\mathbf{m}_r(t)) \mathbf{f}_{\mathfrak{z}}(\mathbf{m}_3(t)) + \sum_{\mathfrak{z}=1}^n \mathbf{b}_{r\mathfrak{z}}(\mathbf{m}_r(t)) \mathbf{g}_{\mathfrak{z}}(\mathbf{m}_3(t - \tau(t))) + \mathbf{u}_r(t)] dt \\
 & + \sum_{\mathfrak{z}=1}^n \sigma_{r\mathfrak{z}}(t, \mathbf{m}_3(t), \mathbf{m}_3(t - \tau(t))) dw_{\mathfrak{z}}(t), \quad t \geq 0, \quad r \in N,
 \end{aligned} \tag{2}$$

Subject to the initial condition of

$$\mathbf{m}_r(t) = \varphi_r(t), \quad t \in [-\tau, 0], \quad r \in N, \tag{3}$$

where $\mathbf{c}_r > 0$ is a constant and the state variable is $\mathbf{m}_r(t)$; the external input vector is $\mathbf{u}_r(t)$. In addition, the time delay is $\tau(t)$, which is differentiable and it is able to satisfy $0 \leq \tau(t) \leq \tau$, $\dot{\tau}(t) \leq \mu < 1$, with positive constants μ and τ ; $\varphi_r(t) \in \mathcal{C}([-\tau, 0]; \mathbb{Q}^n)$ is the initial condition. In addition, $\mathbf{f}_r(\mathbf{m}_r(t))$ and $\mathbf{g}_r(\mathbf{m}_r(t - \tau(t)))$ are quaternion-valued activation functions. The standard Brownian motion is denoted by $w_{\mathfrak{z}}$. Note that it is defined on a probability space of $(\Omega, \mathcal{F}, \mathcal{P})$ subject to a natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Besides that, $\sigma_{r\mathfrak{z}} : \mathbb{R} \times \mathbb{Q}^n \times \mathbb{Q}^n \rightarrow \mathbb{Q}^{n \times n}$ is a Borel measurable function; while the quaternion-valued connection weights are $\mathbf{a}_{r\mathfrak{z}}(\mathbf{m}_3(t))$ and $\mathbf{b}_{r\mathfrak{z}}(\mathbf{m}_3(t))$, which can be expressed as:

$$\begin{aligned}
 \mathbf{a}_{r\mathfrak{z}}(\mathbf{m}_3(t)) &= \frac{\mathbf{W}_{f_{r\mathfrak{z}}}}{\mathbf{C}_r} \times \text{sign}_{r\mathfrak{z}}, \\
 \mathbf{b}_{r\mathfrak{z}}(\mathbf{m}_3(t)) &= \frac{\mathbf{W}_{g_{r\mathfrak{z}}}}{\mathbf{C}_r} \times \text{sign}_{r\mathfrak{z}}, \\
 \text{sign}_{r\mathfrak{z}} &= \begin{cases} -1, & r = \mathfrak{z} \\ 1, & r \neq \mathfrak{z} \end{cases}
 \end{aligned}$$

with \mathbf{C}_r is the capacitor, $\mathbf{W}_{f_{r\mathfrak{z}}}$ and $\mathbf{W}_{g_{r\mathfrak{z}}}$ are the memductances of memristors $\mathbf{R}_{f_{r\mathfrak{z}}}$ and $\mathbf{R}_{g_{r\mathfrak{z}}}$, respectively, that represent a memristor from $\mathbf{f}_r(\mathbf{m}_r(t))$ and $\mathbf{m}_r(t)$ and a memristor from $\mathbf{g}_r(\mathbf{m}_r(t - \tau(t)))$ and $\mathbf{m}_r(t)$. Based on the memristor current-voltage properties, the memristive connection weights are chosen to be state-dependent switching as follows:

$$\begin{aligned}
 \mathbf{a}_{r\mathfrak{z}}(\mathbf{m}_3(t)) &= \begin{cases} \hat{\mathbf{a}}_{r\mathfrak{z}}, & |\mathbf{m}_3(t)| < \psi_{\mathfrak{z}} \\ \check{\mathbf{a}}_{r\mathfrak{z}}, & |\mathbf{m}_3(t)| > \psi_{\mathfrak{z}}, \end{cases} & \mathbf{b}_{r\mathfrak{z}}(\mathbf{m}_3(t)) &= \begin{cases} \hat{\mathbf{b}}_{r\mathfrak{z}}, & |\mathbf{m}_3(t)| < \psi_{\mathfrak{z}} \\ \check{\mathbf{b}}_{r\mathfrak{z}}, & |\mathbf{m}_3(t)| > \psi_{\mathfrak{z}}, \end{cases} \\
 \mathbf{a}_{r\mathfrak{z}}^R(\mathbf{m}_3^R(t)) &= \begin{cases} \hat{\mathbf{a}}_{r\mathfrak{z}}^R, & |\mathbf{m}_3^R(t)| < \psi_{\mathfrak{z}} \\ \check{\mathbf{a}}_{r\mathfrak{z}}^R, & |\mathbf{m}_3^R(t)| > \psi_{\mathfrak{z}}, \end{cases} & \mathbf{a}_{r\mathfrak{z}}^I(\mathbf{m}_3^I(t)) &= \begin{cases} \hat{\mathbf{a}}_{r\mathfrak{z}}^I, & |\mathbf{m}_3^I(t)| < \psi_{\mathfrak{z}} \\ \check{\mathbf{a}}_{r\mathfrak{z}}^I, & |\mathbf{m}_3^I(t)| > \psi_{\mathfrak{z}}, \end{cases} \\
 \mathbf{a}_{r\mathfrak{z}}^J(\mathbf{m}_3^J(t)) &= \begin{cases} \hat{\mathbf{a}}_{r\mathfrak{z}}^J, & |\mathbf{m}_3^J(t)| < \psi_{\mathfrak{z}} \\ \check{\mathbf{a}}_{r\mathfrak{z}}^J, & |\mathbf{m}_3^J(t)| > \psi_{\mathfrak{z}}, \end{cases} & \mathbf{a}_{r\mathfrak{z}}^K(\mathbf{m}_3^K(t)) &= \begin{cases} \hat{\mathbf{a}}_{r\mathfrak{z}}^K, & |\mathbf{m}_3^K(t)| < \psi_{\mathfrak{z}} \\ \check{\mathbf{a}}_{r\mathfrak{z}}^K, & |\mathbf{m}_3^K(t)| > \psi_{\mathfrak{z}}, \end{cases} \\
 \mathbf{b}_{r\mathfrak{z}}^R(\mathbf{m}_3^R(t)) &= \begin{cases} \hat{\mathbf{b}}_{r\mathfrak{z}}^R, & |\mathbf{m}_3^R(t)| < \psi_{\mathfrak{z}} \\ \check{\mathbf{b}}_{r\mathfrak{z}}^R, & |\mathbf{m}_3^R(t)| > \psi_{\mathfrak{z}}, \end{cases} & \mathbf{b}_{r\mathfrak{z}}^I(\mathbf{m}_3^I(t)) &= \begin{cases} \hat{\mathbf{b}}_{r\mathfrak{z}}^I, & |\mathbf{m}_3^I(t)| < \psi_{\mathfrak{z}} \\ \check{\mathbf{b}}_{r\mathfrak{z}}^I, & |\mathbf{m}_3^I(t)| > \psi_{\mathfrak{z}}, \end{cases} \\
 \mathbf{b}_{r\mathfrak{z}}^J(\mathbf{m}_3^J(t)) &= \begin{cases} \hat{\mathbf{b}}_{r\mathfrak{z}}^J, & |\mathbf{m}_3^J(t)| < \psi_{\mathfrak{z}} \\ \check{\mathbf{b}}_{r\mathfrak{z}}^J, & |\mathbf{m}_3^J(t)| > \psi_{\mathfrak{z}}, \end{cases} & \mathbf{b}_{r\mathfrak{z}}^K(\mathbf{m}_3^K(t)) &= \begin{cases} \hat{\mathbf{b}}_{r\mathfrak{z}}^K, & |\mathbf{m}_3^K(t)| < \psi_{\mathfrak{z}} \\ \check{\mathbf{b}}_{r\mathfrak{z}}^K, & |\mathbf{m}_3^K(t)| > \psi_{\mathfrak{z}}, \end{cases}
 \end{aligned}$$

for $r, \mathfrak{z} \in N$, the switching jumps $\psi_{\mathfrak{z}} > 0$, and, where $\mathbf{a}_{r\mathfrak{z}}^R(\mathbf{m}_3^R(t)) = \text{Re}(\mathbf{a}_{r\mathfrak{z}}(\mathbf{m}_3(t)))$, $\mathbf{a}_{r\mathfrak{z}}^I(\mathbf{m}_3^I(t)) = \text{Im}(\mathbf{a}_{r\mathfrak{z}}(\mathbf{m}_3(t)))$, $\mathbf{a}_{r\mathfrak{z}}^J(\mathbf{m}_3^J(t)) = \text{Im}(\mathbf{a}_{r\mathfrak{z}}(\mathbf{m}_3(t)))$, $\mathbf{a}_{r\mathfrak{z}}^K(\mathbf{m}_3^K(t)) = \text{Im}(\mathbf{a}_{r\mathfrak{z}}(\mathbf{m}_3(t)))$, $\mathbf{b}_{r\mathfrak{z}}^R(\mathbf{m}_3^R(t)) = \text{Re}(\mathbf{b}_{r\mathfrak{z}}(\mathbf{m}_3(t)))$, $\mathbf{b}_{r\mathfrak{z}}^I(\mathbf{m}_3^I(t)) = \text{Im}(\mathbf{b}_{r\mathfrak{z}}(\mathbf{m}_3(t)))$, $\mathbf{b}_{r\mathfrak{z}}^J(\mathbf{m}_3^J(t)) = \text{Im}(\mathbf{b}_{r\mathfrak{z}}(\mathbf{m}_3(t)))$, $\mathbf{b}_{r\mathfrak{z}}^K(\mathbf{m}_3^K(t)) = \text{Im}(\mathbf{b}_{r\mathfrak{z}}(\mathbf{m}_3(t)))$ and $\hat{\mathbf{a}}_{r\mathfrak{z}}, \check{\mathbf{a}}_{r\mathfrak{z}}, \hat{\mathbf{a}}_{r\mathfrak{z}}^R, \check{\mathbf{a}}_{r\mathfrak{z}}^R, \hat{\mathbf{a}}_{r\mathfrak{z}}^I, \check{\mathbf{a}}_{r\mathfrak{z}}^I, \hat{\mathbf{a}}_{r\mathfrak{z}}^J, \check{\mathbf{a}}_{r\mathfrak{z}}^J, \hat{\mathbf{a}}_{r\mathfrak{z}}^K, \check{\mathbf{a}}_{r\mathfrak{z}}^K, \hat{\mathbf{b}}_{r\mathfrak{z}}, \check{\mathbf{b}}_{r\mathfrak{z}}, \hat{\mathbf{b}}_{r\mathfrak{z}}^R, \check{\mathbf{b}}_{r\mathfrak{z}}^R, \hat{\mathbf{b}}_{r\mathfrak{z}}^I, \check{\mathbf{b}}_{r\mathfrak{z}}^I,$

$\hat{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^J, \check{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^J, \hat{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^K, \check{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^K$ are constants.

A1: [32] For any $\mathfrak{j} \in N$, $\mathbf{f}_{\mathfrak{j}}(\mathbf{m}_{\mathfrak{j}}(\cdot))$ and $\mathbf{g}_{\mathfrak{j}}(\mathbf{m}_{\mathfrak{j}}(\cdot))$ are bounded and continuous along with positive constants $\mathbf{G}_{\mathfrak{j}}, \mathbf{H}_{\mathfrak{j}}$ such that

$$\begin{cases} |\mathbf{f}_{\mathfrak{j}}(\mathbf{m}_1) - \mathbf{f}_{\mathfrak{j}}(\mathbf{m}_2)| \leq \mathbf{G}_{\mathfrak{j}}|\mathbf{m}_1 - \mathbf{m}_2|, \\ |\mathbf{g}_{\mathfrak{j}}(\mathbf{m}_1) - \mathbf{g}_{\mathfrak{j}}(\mathbf{m}_2)| \leq \mathbf{H}_{\mathfrak{j}}|\mathbf{m}_1 - \mathbf{m}_2|, \end{cases} \tag{4}$$

for all $\mathbf{m}_1, \mathbf{m}_2 \in \mathbb{Q}$ and $\mathbf{f}_{\mathfrak{j}}(0) = 0, \mathbf{g}_{\mathfrak{j}}(0) = 0$.

Remark 1. The activation functions are the fundamental factors in the structure of the NN model. The choice of appropriate quaternion-valued activation functions remains an open question in the QVNNs study. Recently, several activation functions have recently been used to study QVNNs. For example, nonmonotonic piecewise nonlinear activation functions, linear threshold activation functions, and real-imaginary separate type activation functions. The main results of this paper will be derived based on the assumption **A1** that the activation functions can be divided into real and imaginary parts.

As similar in [13,41], by using the set valued map as well as the differential inclusion theory, from Equation (2), we obtain

$$\begin{aligned} d\mathbf{m}_{\mathfrak{r}}(t) \in & [-\mathbf{c}_{\mathfrak{r}}\mathbf{m}_{\mathfrak{r}}(t) + \sum_{\mathfrak{j}=1}^n \text{co}[\bar{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}, \underline{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}]\mathbf{f}_{\mathfrak{j}}(\mathbf{m}_{\mathfrak{j}}(t)) + \sum_{\mathfrak{j}=1}^n \text{co}[\bar{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}, \underline{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}]\mathbf{g}_{\mathfrak{j}}(\mathbf{m}_{\mathfrak{j}}(t - \tau(t))) + \mathbf{u}_{\mathfrak{r}}(t)]dt \\ & + \sum_{\mathfrak{j}=1}^n \sigma_{\mathfrak{r}\mathfrak{j}}(t, \mathbf{m}_{\mathfrak{j}}(t), \mathbf{m}_{\mathfrak{j}}(t - \tau(t)))d\omega_{\mathfrak{j}}(t), \end{aligned} \tag{5}$$

or equivalently, for all $\mathfrak{r}, \mathfrak{j} \in N$, where $\bar{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}} = \max\{\hat{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}, \check{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}\}, \underline{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}} = \min\{\hat{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}, \check{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}\}, \bar{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}} = \max\{\hat{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}, \check{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}\}, \underline{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}} = \min\{\hat{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}, \check{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}\}$. There exist measurable functions $\tilde{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}} \in \text{co}[\bar{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}, \underline{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}], \tilde{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}} \in \text{co}[\bar{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}, \underline{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}]$, such that

$$\begin{aligned} d\mathbf{m}_{\mathfrak{r}}(t) = & [-\mathbf{c}_{\mathfrak{r}}\mathbf{m}_{\mathfrak{r}}(t) + \sum_{\mathfrak{j}=1}^n \tilde{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}\mathbf{f}_{\mathfrak{j}}(\mathbf{m}_{\mathfrak{j}}(t)) + \sum_{\mathfrak{j}=1}^n \tilde{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}\mathbf{g}_{\mathfrak{j}}(\mathbf{m}_{\mathfrak{j}}(t - \tau(t))) + \mathbf{u}_{\mathfrak{r}}(t)]dt \\ & + \sum_{\mathfrak{j}=1}^n \sigma_{\mathfrak{r}\mathfrak{j}}(t, \mathbf{m}_{\mathfrak{j}}(t), \mathbf{m}_{\mathfrak{j}}(t - \tau(t)))d\omega_{\mathfrak{j}}(t), \end{aligned} \tag{6}$$

Now, let $\bar{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^R = \max\{\hat{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^R, \check{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^R\}, \underline{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^R = \min\{\hat{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^R, \check{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^R\}, \bar{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^I = \max\{\hat{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^I, \check{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^I\}, \underline{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^I = \min\{\hat{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^I, \check{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^I\}, \bar{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^J = \max\{\hat{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^J, \check{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^J\}, \underline{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^J = \min\{\hat{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^J, \check{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^J\}, \bar{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^K = \max\{\hat{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^K, \check{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^K\}, \underline{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^K = \min\{\hat{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^K, \check{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^K\}, \bar{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^R = \max\{\hat{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^R, \check{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^R\}, \underline{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^R = \min\{\hat{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^R, \check{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^R\}, \bar{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^I = \max\{\hat{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^I, \check{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^I\}, \underline{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^I = \min\{\hat{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^I, \check{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^I\}, \bar{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^J = \max\{\hat{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^J, \check{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^J\}, \underline{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^J = \min\{\hat{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^J, \check{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^J\}, \bar{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^K = \max\{\hat{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^K, \check{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^K\}, \underline{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^K = \min\{\hat{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^K, \check{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^K\}$. We assume that the NN model (6) can be separated into real and imaginary parts. Therefore, there exist measurable functions $\tilde{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^R \in \text{co}[\bar{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^R, \underline{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^R], \tilde{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^I \in \text{co}[\bar{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^I, \underline{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^I], \tilde{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^J \in \text{co}[\bar{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^J, \underline{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^J], \tilde{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^K \in \text{co}[\bar{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^K, \underline{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^K], \tilde{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^R \in \text{co}[\bar{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^R, \underline{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^R], \tilde{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^I \in \text{co}[\bar{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^I, \underline{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^I], \tilde{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^J \in \text{co}[\bar{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^J, \underline{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^J], \tilde{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^K \in \text{co}[\bar{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^K, \underline{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^K]$, such that the NN model (6) can be express as follows:

$$\left\{ \begin{aligned}
 dm_{\mathfrak{r}}^R(t) &= \left[-c_{\mathfrak{r}}m_{\mathfrak{r}}^R(t) + \sum_{\mathfrak{z}=1}^n [\tilde{a}_{\mathfrak{r}\mathfrak{z}}^R f_{\mathfrak{z}}^R(\mathbf{m}_{\mathfrak{z}}^R(t)) - \tilde{a}_{\mathfrak{r}\mathfrak{z}}^I f_{\mathfrak{z}}^I(\mathbf{m}_{\mathfrak{z}}^I(t)) - \tilde{a}_{\mathfrak{r}\mathfrak{z}}^J f_{\mathfrak{z}}^J(\mathbf{m}_{\mathfrak{z}}^J(t)) - \tilde{a}_{\mathfrak{r}\mathfrak{z}}^K f_{\mathfrak{z}}^K(\mathbf{m}_{\mathfrak{z}}^K(t))] \right. \\
 &+ \sum_{\mathfrak{z}=1}^n [\tilde{b}_{\mathfrak{r}\mathfrak{z}}^R g_{\mathfrak{z}}^R(\mathbf{m}_{\mathfrak{z}}^R(t - \tau(t))) - \tilde{b}_{\mathfrak{r}\mathfrak{z}}^I g_{\mathfrak{z}}^I(\mathbf{m}_{\mathfrak{z}}^I(t - \tau(t))) - \tilde{b}_{\mathfrak{r}\mathfrak{z}}^J g_{\mathfrak{z}}^J(\mathbf{m}_{\mathfrak{z}}^J(t - \tau(t))) \\
 &\left. - \tilde{b}_{\mathfrak{r}\mathfrak{z}}^K g_{\mathfrak{z}}^K(\mathbf{m}_{\mathfrak{z}}^K(t - \tau(t)))] + \mathbf{u}_{\mathfrak{r}}^R(t) \right] dt + \sum_{\mathfrak{z}=1}^n \sigma_{\mathfrak{r}\mathfrak{z}}^R(t, \mathbf{m}_{\mathfrak{z}}^R(t), \mathbf{m}_{\mathfrak{z}}^R(t - \tau(t))) dw_{\mathfrak{z}}(t), \\
 dm_{\mathfrak{r}}^I(t) &= \left[-c_{\mathfrak{r}}m_{\mathfrak{r}}^I(t) + \sum_{\mathfrak{z}=1}^n [\tilde{a}_{\mathfrak{r}\mathfrak{z}}^R f_{\mathfrak{z}}^I(\mathbf{m}_{\mathfrak{z}}^I(t)) + \tilde{a}_{\mathfrak{r}\mathfrak{z}}^I f_{\mathfrak{z}}^R(\mathbf{m}_{\mathfrak{z}}^R(t)) + \tilde{a}_{\mathfrak{r}\mathfrak{z}}^J f_{\mathfrak{z}}^K(\mathbf{m}_{\mathfrak{z}}^K(t)) - \tilde{a}_{\mathfrak{r}\mathfrak{z}}^K f_{\mathfrak{z}}^I(\mathbf{m}_{\mathfrak{z}}^I(t))] \right. \\
 &+ \sum_{\mathfrak{z}=1}^n [\tilde{b}_{\mathfrak{r}\mathfrak{z}}^R g_{\mathfrak{z}}^I(\mathbf{m}_{\mathfrak{z}}^I(t - \tau(t))) + \tilde{b}_{\mathfrak{r}\mathfrak{z}}^I g_{\mathfrak{z}}^R(\mathbf{m}_{\mathfrak{z}}^R(t - \tau(t))) + \tilde{b}_{\mathfrak{r}\mathfrak{z}}^J g_{\mathfrak{z}}^K(\mathbf{m}_{\mathfrak{z}}^K(t - \tau(t))) \\
 &\left. - \tilde{b}_{\mathfrak{r}\mathfrak{z}}^K g_{\mathfrak{z}}^I(\mathbf{m}_{\mathfrak{z}}^I(t - \tau(t)))] + \mathbf{u}_{\mathfrak{r}}^I(t) \right] dt + \sum_{\mathfrak{z}=1}^n \sigma_{\mathfrak{r}\mathfrak{z}}^I(t, \mathbf{m}_{\mathfrak{z}}^I(t), \mathbf{m}_{\mathfrak{z}}^I(t - \tau(t))) dw_{\mathfrak{z}}(t), \\
 dm_{\mathfrak{r}}^J(t) &= \left[-c_{\mathfrak{r}}m_{\mathfrak{r}}^J(t) + \sum_{\mathfrak{z}=1}^n [\tilde{a}_{\mathfrak{r}\mathfrak{z}}^R f_{\mathfrak{z}}^J(\mathbf{m}_{\mathfrak{z}}^J(t)) + \tilde{a}_{\mathfrak{r}\mathfrak{z}}^I f_{\mathfrak{z}}^R(\mathbf{m}_{\mathfrak{z}}^R(t)) + \tilde{a}_{\mathfrak{r}\mathfrak{z}}^J f_{\mathfrak{z}}^I(\mathbf{m}_{\mathfrak{z}}^I(t)) - \tilde{a}_{\mathfrak{r}\mathfrak{z}}^I f_{\mathfrak{z}}^K(\mathbf{m}_{\mathfrak{z}}^K(t))] \right. \\
 &+ \sum_{\mathfrak{z}=1}^n [\tilde{b}_{\mathfrak{r}\mathfrak{z}}^R g_{\mathfrak{z}}^J(\mathbf{m}_{\mathfrak{z}}^J(t - \tau(t))) + \tilde{b}_{\mathfrak{r}\mathfrak{z}}^I g_{\mathfrak{z}}^R(\mathbf{m}_{\mathfrak{z}}^R(t - \tau(t))) + \tilde{b}_{\mathfrak{r}\mathfrak{z}}^K g_{\mathfrak{z}}^I(\mathbf{m}_{\mathfrak{z}}^I(t - \tau(t))) \\
 &\left. - \tilde{b}_{\mathfrak{r}\mathfrak{z}}^I g_{\mathfrak{z}}^K(\mathbf{m}_{\mathfrak{z}}^K(t - \tau(t)))] + \mathbf{u}_{\mathfrak{r}}^J(t) \right] dt + \sum_{\mathfrak{z}=1}^n \sigma_{\mathfrak{r}\mathfrak{z}}^J(t, \mathbf{m}_{\mathfrak{z}}^J(t), \mathbf{m}_{\mathfrak{z}}^J(t - \tau(t))) dw_{\mathfrak{z}}(t), \\
 dm_{\mathfrak{r}}^K(t) &= \left[-c_{\mathfrak{r}}m_{\mathfrak{r}}^K(t) + \sum_{\mathfrak{z}=1}^n [\tilde{a}_{\mathfrak{r}\mathfrak{z}}^R f_{\mathfrak{z}}^K(\mathbf{m}_{\mathfrak{z}}^K(t)) + \tilde{a}_{\mathfrak{r}\mathfrak{z}}^I f_{\mathfrak{z}}^I(\mathbf{m}_{\mathfrak{z}}^I(t)) + \tilde{a}_{\mathfrak{r}\mathfrak{z}}^J f_{\mathfrak{z}}^R(\mathbf{m}_{\mathfrak{z}}^R(t)) - \tilde{a}_{\mathfrak{r}\mathfrak{z}}^J f_{\mathfrak{z}}^I(\mathbf{m}_{\mathfrak{z}}^I(t))] \right. \\
 &+ \sum_{\mathfrak{z}=1}^n [\tilde{b}_{\mathfrak{r}\mathfrak{z}}^R g_{\mathfrak{z}}^K(\mathbf{m}_{\mathfrak{z}}^K(t - \tau(t))) + \tilde{b}_{\mathfrak{r}\mathfrak{z}}^I g_{\mathfrak{z}}^J(\mathbf{m}_{\mathfrak{z}}^J(t - \tau(t))) + \tilde{b}_{\mathfrak{r}\mathfrak{z}}^K g_{\mathfrak{z}}^R(\mathbf{m}_{\mathfrak{z}}^R(t - \tau(t))) \\
 &\left. - \tilde{b}_{\mathfrak{r}\mathfrak{z}}^J g_{\mathfrak{z}}^I(\mathbf{m}_{\mathfrak{z}}^I(t - \tau(t)))] + \mathbf{u}_{\mathfrak{r}}^K(t) \right] dt + \sum_{\mathfrak{z}=1}^n \sigma_{\mathfrak{r}\mathfrak{z}}^K(t, \mathbf{m}_{\mathfrak{z}}^K(t), \mathbf{m}_{\mathfrak{z}}^K(t - \tau(t))) dw_{\mathfrak{z}}(t),
 \end{aligned} \right. \tag{7}$$

for $t \in [0, +\infty)$, where $\mathfrak{r}, \mathfrak{z} \in N$, $\mathbf{u}_{\mathfrak{r}}^R(t) = Re(\mathbf{u}_{\mathfrak{r}}(t))$, $\mathbf{u}_{\mathfrak{r}}^I(t) = Im(\mathbf{u}_{\mathfrak{r}}(t))$, $\mathbf{u}_{\mathfrak{r}}^J(t) = Im(\mathbf{u}_{\mathfrak{r}}(t))$, $\mathbf{u}_{\mathfrak{r}}^K(t) = Im(\mathbf{u}_{\mathfrak{r}}(t))$, and

$$\begin{aligned}
 \sigma_{\mathfrak{r}\mathfrak{z}}^R(t, \mathbf{m}_{\mathfrak{z}}^R(t), \mathbf{m}_{\mathfrak{z}}^R(t - \tau(t))) &= Re(\sigma_{\mathfrak{r}\mathfrak{z}}(t, \mathbf{m}_{\mathfrak{z}}(t), \mathbf{m}_{\mathfrak{z}}(t - \tau(t))), \\
 \sigma_{\mathfrak{r}\mathfrak{z}}^I(t, \mathbf{m}_{\mathfrak{z}}^I(t), \mathbf{m}_{\mathfrak{z}}^I(t - \tau(t))) &= Im(\sigma_{\mathfrak{r}\mathfrak{z}}(t, \mathbf{m}_{\mathfrak{z}}(t), \mathbf{m}_{\mathfrak{z}}(t - \tau(t))), \\
 \sigma_{\mathfrak{r}\mathfrak{z}}^J(t, \mathbf{m}_{\mathfrak{z}}^J(t), \mathbf{m}_{\mathfrak{z}}^J(t - \tau(t))) &= Im(\sigma_{\mathfrak{r}\mathfrak{z}}(t, \mathbf{m}_{\mathfrak{z}}(t), \mathbf{m}_{\mathfrak{z}}(t - \tau(t))), \\
 \sigma_{\mathfrak{r}\mathfrak{z}}^K(t, \mathbf{m}_{\mathfrak{z}}^K(t), \mathbf{m}_{\mathfrak{z}}^K(t - \tau(t))) &= Im(\sigma_{\mathfrak{r}\mathfrak{z}}(t, \mathbf{m}_{\mathfrak{z}}(t), \mathbf{m}_{\mathfrak{z}}(t - \tau(t))).
 \end{aligned}$$

Consider Model (7), its initial condition is

$$\mathbf{m}_{\mathfrak{r}}^R(t) = \varphi_{\mathfrak{r}}^R(t), \mathbf{m}_{\mathfrak{r}}^I(t) = \varphi_{\mathfrak{r}}^I(t), \mathbf{m}_{\mathfrak{r}}^J(t) = \varphi_{\mathfrak{r}}^J(t), \mathbf{m}_{\mathfrak{r}}^K(t) = \varphi_{\mathfrak{r}}^K(t), \tag{8}$$

for $t \in [-\tau, 0]$, where $\varphi_{\mathfrak{r}}^R(t) = Re(\varphi_{\mathfrak{r}}(t))$, $\varphi_{\mathfrak{r}}^I(t) = Im(\varphi_{\mathfrak{r}}(t))$, $\varphi_{\mathfrak{r}}^J(t) = Im(\varphi_{\mathfrak{r}}(t))$, $\varphi_{\mathfrak{r}}^K(t) = Im(\varphi_{\mathfrak{r}}(t))$.

A2: [32] For any $\mathfrak{z} \in N$, $\mathbf{f}_{\mathfrak{z}}^s(\mathbf{m}_{\mathfrak{z}}(\cdot))$ and $\mathbf{g}_{\mathfrak{z}}^s(\mathbf{m}_{\mathfrak{z}}(\cdot))$, $s = R, I, J, K$ are bounded and continuous, along with $\mathbf{G}_{\mathfrak{z}}^s, \mathbf{H}_{\mathfrak{z}}^s$ as the positive constants, in which

$$\begin{cases} |\mathbf{f}_{\mathfrak{z}}^s(\mathbf{m}_1) - \mathbf{f}_{\mathfrak{z}}^s(\mathbf{m}_2)| \leq \mathbf{G}_{\mathfrak{z}}^s |\mathbf{m}_1 - \mathbf{m}_2|, \\ |\mathbf{g}_{\mathfrak{z}}^s(\mathbf{m}_1) - \mathbf{g}_{\mathfrak{z}}^s(\mathbf{m}_2)| \leq \mathbf{H}_{\mathfrak{z}}^s |\mathbf{m}_1 - \mathbf{m}_2|, \end{cases} \tag{9}$$

for all $\mathbf{m}_1, \mathbf{m}_2 \in \mathbb{R}$ and $\mathbf{f}_{\mathfrak{z}}^s(0) = 0, \mathbf{g}_{\mathfrak{z}}^s(0) = 0, s = R, I, J, K$.

Let $\mathbf{C} = diag\{c_1, \dots, c_n\} \in \mathbb{R}^n$, $\mathbf{A}^R = (\tilde{a}_{\mathfrak{r}\mathfrak{z}}^R)_{n \times n} \in \mathbb{R}^{n \times n}$, $\mathbf{A}^I = (\tilde{a}_{\mathfrak{r}\mathfrak{z}}^I)_{n \times n} \in \mathbb{R}^{n \times n}$, $\mathbf{A}^J = (\tilde{a}_{\mathfrak{r}\mathfrak{z}}^J)_{n \times n} \in \mathbb{R}^{n \times n}$, $\mathbf{A}^K = (\tilde{a}_{\mathfrak{r}\mathfrak{z}}^K)_{n \times n} \in \mathbb{R}^{n \times n}$, $\mathbf{B}^R = (\tilde{b}_{\mathfrak{r}\mathfrak{z}}^R)_{n \times n} \in \mathbb{R}^{n \times n}$, $\mathbf{B}^I = (\tilde{b}_{\mathfrak{r}\mathfrak{z}}^I)_{n \times n} \in \mathbb{R}^{n \times n}$, $\mathbf{B}^J = (\tilde{b}_{\mathfrak{r}\mathfrak{z}}^J)_{n \times n} \in \mathbb{R}^{n \times n}$, $\mathbf{B}^K = (\tilde{b}_{\mathfrak{r}\mathfrak{z}}^K)_{n \times n} \in \mathbb{R}^{n \times n}$, $\mathbf{f}^R(\mathbf{m}^R(t)) = [f_1^R(\mathbf{m}_1^R(t)), \dots, f_n^R(\mathbf{m}_n^R(t))]^T \in \mathbb{R}^n$, $\mathbf{f}^I(\mathbf{m}^I(t)) = [f_1^I(\mathbf{m}_1^I(t)), \dots, f_n^I(\mathbf{m}_n^I(t))]^T \in \mathbb{R}^n$, $\mathbf{f}^J(\mathbf{m}^J(t)) = [f_1^J(\mathbf{m}_1^J(t)), \dots, f_n^J(\mathbf{m}_n^J(t))]^T \in \mathbb{R}^n$, $\mathbf{f}^K(\mathbf{m}^K(t)) = [f_1^K(\mathbf{m}_1^K(t)), \dots, f_n^K(\mathbf{m}_n^K(t))]^T \in \mathbb{R}^n$.

$\mathbf{f}^K(\mathbf{m}^K(t)) = [\mathbf{f}_1^K(\mathbf{m}_1^K(t)), \dots, \mathbf{f}_n^K(\mathbf{m}_n^K(t))]^T \in \mathbb{R}^n$, $\mathbf{g}^R(\mathbf{m}^R(t - \tau(t))) = [\mathbf{g}_1^R(\mathbf{m}_1^R(t - \tau(t))), \dots, \mathbf{g}_n^R(\mathbf{m}_n^R(t - \tau(t)))]^T \in \mathbb{R}^n$, $\mathbf{g}^I(\mathbf{m}^I(t - \tau(t))) = [\mathbf{g}_1^I(\mathbf{m}_1^I(t - \tau(t))), \dots, \mathbf{g}_n^I(\mathbf{m}_n^I(t - \tau(t)))]^T \in \mathbb{R}^n$, $\mathbf{g}^J(\mathbf{m}^J(t - \tau(t))) = [\mathbf{g}_1^J(\mathbf{m}_1^J(t - \tau(t))), \dots, \mathbf{g}_n^J(\mathbf{m}_n^J(t - \tau(t)))]^T \in \mathbb{R}^n$, $\mathbf{g}^K(\mathbf{m}^K(t - \tau(t))) = [\mathbf{g}_1^K(\mathbf{m}_1^K(t - \tau(t))), \dots, \mathbf{g}_n^K(\mathbf{m}_n^K(t - \tau(t)))]^T \in \mathbb{R}^n$, $\mathbf{u}^R(t) = [\mathbf{u}_1^R(t), \dots, \mathbf{u}_n^R(t)]^T \in \mathbb{R}^n$, $\mathbf{u}^I(t) = [\mathbf{u}_1^I(t), \dots, \mathbf{u}_n^I(t)]^T \in \mathbb{R}^n$, $\mathbf{u}^J(t) = [\mathbf{u}_1^J(t), \dots, \mathbf{u}_n^J(t)]^T \in \mathbb{R}^n$, $\mathbf{u}^K(t) = [\mathbf{u}_1^K(t), \dots, \mathbf{u}_n^K(t)]^T \in \mathbb{R}^n$, $\sigma^R(t) = (\sigma_{\tau_3}^R(t))_{n \times n} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$, $\sigma^I(t) = (\sigma_{\tau_3}^I(t))_{n \times n} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$, $\sigma^J(t) = (\sigma_{\tau_3}^J(t))_{n \times n} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$, $\sigma^K(t) = (\sigma_{\tau_3}^K(t))_{n \times n} : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$, $\sigma_{\tau_3}^R(t) = \sigma_{\tau_3}^R(t, \mathbf{m}_3^R(t), \mathbf{m}_3^R(t - \tau(t)))$, $\sigma_{\tau_3}^I(t) = \sigma_{\tau_3}^I(t, \mathbf{m}_3^I(t), \mathbf{m}_3^I(t - \tau(t)))$, $\sigma_{\tau_3}^J(t) = \sigma_{\tau_3}^J(t, \mathbf{m}_3^J(t), \mathbf{m}_3^J(t - \tau(t)))$, $\sigma_{\tau_3}^K(t) = \sigma_{\tau_3}^K(t, \mathbf{m}_3^K(t), \mathbf{m}_3^K(t - \tau(t)))$.

Now, we can express Model (7) as

$$\left\{ \begin{aligned} d\mathbf{m}^R(t) &= [-\mathbf{C}\mathbf{m}^R(t) + \mathbf{A}^R\mathbf{f}^R(\mathbf{m}^R(t)) - \mathbf{A}^I\mathbf{f}^I(\mathbf{m}^I(t)) - \mathbf{A}^J\mathbf{f}^J(\mathbf{m}^J(t)) - \mathbf{A}^K\mathbf{f}^K(\mathbf{m}^K(t)) \\ &\quad + \mathbf{B}^R\mathbf{g}^R(\mathbf{m}^R(t - \tau(t))) - \mathbf{B}^I\mathbf{g}^I(\mathbf{m}^I(t - \tau(t))) - \mathbf{B}^J\mathbf{g}^J(\mathbf{m}^J(t - \tau(t))) \\ &\quad - \mathbf{B}^K\mathbf{g}^K(\mathbf{m}^K(t - \tau(t))) + \mathbf{u}^R(t)]dt + \sigma^R(t)d\mathbf{w}(t), \\ d\mathbf{m}^I(t) &= [-\mathbf{C}\mathbf{m}^I(t) + \mathbf{A}^R\mathbf{f}^I(\mathbf{m}^I(t)) + \mathbf{A}^I\mathbf{f}^R(\mathbf{m}^R(t)) + \mathbf{A}^J\mathbf{f}^K(\mathbf{m}^K(t)) - \mathbf{A}^K\mathbf{f}^J(\mathbf{m}^J(t)) \\ &\quad + \mathbf{B}^R\mathbf{g}^I(\mathbf{m}^I(t - \tau(t))) + \mathbf{B}^I\mathbf{g}^R(\mathbf{m}^R(t - \tau(t))) + \mathbf{B}^J\mathbf{g}^K(\mathbf{m}^K(t - \tau(t))) \\ &\quad - \mathbf{B}^K\mathbf{g}^J(\mathbf{m}^J(t - \tau(t))) + \mathbf{u}^I(t)]dt + \sigma^I(t)d\mathbf{w}(t), \\ d\mathbf{m}^J(t) &= [-\mathbf{C}\mathbf{m}^J(t) + \mathbf{A}^R\mathbf{f}^J(\mathbf{m}^J(t)) + \mathbf{A}^I\mathbf{f}^R(\mathbf{m}^R(t)) + \mathbf{A}^K\mathbf{f}^I(\mathbf{m}^I(t)) - \mathbf{A}^I\mathbf{f}^K(\mathbf{m}^K(t)) \\ &\quad + \mathbf{B}^R\mathbf{g}^J(\mathbf{m}^J(t - \tau(t))) + \mathbf{B}^I\mathbf{g}^R(\mathbf{m}^R(t - \tau(t))) + \mathbf{B}^K\mathbf{g}^I(\mathbf{m}^I(t - \tau(t))) \\ &\quad - \mathbf{B}^J\mathbf{g}^K(\mathbf{m}^K(t - \tau(t))) + \mathbf{u}^J(t)]dt + \sigma^J(t)d\mathbf{w}(t), \\ d\mathbf{m}^K(t) &= [-\mathbf{C}\mathbf{m}^K(t) + \mathbf{A}^R\mathbf{f}^K(\mathbf{m}^K(t)) + \mathbf{A}^I\mathbf{f}^J(\mathbf{m}^J(t)) + \mathbf{A}^K\mathbf{f}^R(\mathbf{m}^R(t)) - \mathbf{A}^J\mathbf{f}^I(\mathbf{m}^I(t)) \\ &\quad + \mathbf{B}^R\mathbf{g}^K(\mathbf{m}^K(t - \tau(t))) + \mathbf{B}^I\mathbf{g}^J(\mathbf{m}^J(t - \tau(t))) + \mathbf{B}^K\mathbf{g}^R(\mathbf{m}^R(t - \tau(t))) \\ &\quad - \mathbf{B}^J\mathbf{g}^I(\mathbf{m}^I(t - \tau(t))) + \mathbf{u}^K(t)]dt + \sigma^K(t)d\mathbf{w}(t), \end{aligned} \right. \tag{10}$$

Denote

$$\begin{aligned} \bar{\mathbf{m}}(t) &= [(\mathbf{m}^R(t))^T, (\mathbf{m}^I(t))^T, (\mathbf{m}^J(t))^T, (\mathbf{m}^K(t))^T]^T, \\ \bar{\mathbf{f}}(\bar{\mathbf{m}}(t)) &= [(\mathbf{f}^R(\mathbf{m}^R(t)))^T, (\mathbf{f}^I(\mathbf{m}^I(t)))^T, (\mathbf{f}^J(\mathbf{m}^J(t)))^T, (\mathbf{f}^K(\mathbf{m}^K(t)))^T]^T, \\ \bar{\mathbf{g}}(\bar{\mathbf{m}}(t - \tau(t))) &= [(\mathbf{g}^R(\mathbf{m}^R(t - \tau(t))))^T, (\mathbf{g}^I(\mathbf{m}^I(t - \tau(t))))^T, (\mathbf{g}^J(\mathbf{m}^J(t - \tau(t))))^T, (\mathbf{g}^K(\mathbf{m}^K(t - \tau(t))))^T]^T, \\ \bar{\sigma}(t) &= [(\sigma^R(t))^T, (\sigma^I(t))^T, (\sigma^J(t))^T, (\sigma^K(t))^T]^T, \\ \bar{\mathbf{U}} &= [(\mathbf{u}^R(t))^T, (\mathbf{u}^I(t))^T, (\mathbf{u}^J(t))^T, (\mathbf{u}^K(t))^T]^T, \bar{\mathbf{C}} = \text{diag}\{\mathbf{C}, \mathbf{C}, \mathbf{C}, \mathbf{C}\}, \\ \bar{\mathbf{A}} &= \begin{bmatrix} \mathbf{A}^R & -\mathbf{A}^I & -\mathbf{A}^J & -\mathbf{A}^K \\ \mathbf{A}^I & \mathbf{A}^R & -\mathbf{A}^K & \mathbf{A}^J \\ \mathbf{A}^J & \mathbf{A}^K & \mathbf{A}^R & -\mathbf{A}^I \\ \mathbf{A}^K & -\mathbf{A}^J & \mathbf{A}^I & \mathbf{A}^R \end{bmatrix}, \bar{\mathbf{B}} = \begin{bmatrix} \mathbf{B}^R & -\mathbf{B}^I & -\mathbf{B}^J & -\mathbf{B}^K \\ \mathbf{B}^I & \mathbf{B}^R & -\mathbf{B}^K & \mathbf{B}^J \\ \mathbf{B}^J & \mathbf{B}^K & \mathbf{B}^R & -\mathbf{B}^I \\ \mathbf{B}^K & -\mathbf{B}^J & \mathbf{B}^I & \mathbf{B}^R \end{bmatrix}. \end{aligned}$$

Therefore, the model in Equation (10) can be rewritten as

$$d\bar{\mathbf{m}}(t) = [-\bar{\mathbf{C}}\bar{\mathbf{m}}(t) + \bar{\mathbf{A}}\bar{\mathbf{f}}(\bar{\mathbf{m}}(t)) + \bar{\mathbf{B}}\bar{\mathbf{g}}(\bar{\mathbf{m}}(t - \tau(t))) + \bar{\mathbf{U}}]dt + \bar{\sigma}(t)d\mathbf{w}(t). \tag{11}$$

Note that the model in Equation (11) has the following initial condition

$$\bar{\mathbf{m}}(t) = \phi(t), \quad t \in [-\tau, 0], \tag{12}$$

where $\phi(t) = [\varphi^R(t), \varphi^I(t), \varphi^J(t), \varphi^K(t)]^T$, with $\|\varphi^R(t)\| = \sup_{-\tau \leq t \leq 0} |\varphi^R(t)|$, $\|\varphi^I(t)\| = \sup_{-\tau \leq t \leq 0} |\varphi^I(t)|$, $\|\varphi^J(t)\| = \sup_{-\tau \leq t \leq 0} |\varphi^J(t)|$, $\|\varphi^K(t)\| = \sup_{-\tau \leq t \leq 0} |\varphi^K(t)|$.

A3: [13] There exist nonnegative constants $\hat{\alpha}_{\mathfrak{r}\mathfrak{j}}, \check{\alpha}_{\mathfrak{r}\mathfrak{j}}, \bar{\alpha}_{\mathfrak{r}\mathfrak{j}}, \tilde{\alpha}_{\mathfrak{r}\mathfrak{j}}, \hat{\beta}_{\mathfrak{r}\mathfrak{j}}, \check{\beta}_{\mathfrak{r}\mathfrak{j}}, \bar{\beta}_{\mathfrak{r}\mathfrak{j}}, \tilde{\beta}_{\mathfrak{r}\mathfrak{j}}$ such that

$$\begin{aligned} [\sigma_{\mathfrak{r}\mathfrak{j}}^R(t, \hat{\mathbf{m}}^R, \check{\mathbf{m}}^R) - \sigma_{\mathfrak{r}\mathfrak{j}}^R(t, \hat{\mathbf{m}}^R, \check{\mathbf{m}}^R)]^2 &\leq \hat{\alpha}_{\mathfrak{r}\mathfrak{j}}(\hat{\mathbf{m}}^R - \check{\mathbf{m}}^R)^2 + \hat{\beta}_{\mathfrak{r}\mathfrak{j}}(\hat{\mathbf{m}}^R - \check{\mathbf{m}}^R)^2, \\ [\sigma_{\mathfrak{r}\mathfrak{j}}^I(t, \hat{\mathbf{m}}^I, \check{\mathbf{m}}^I) - \sigma_{\mathfrak{r}\mathfrak{j}}^I(t, \hat{\mathbf{m}}^I, \check{\mathbf{m}}^I)]^2 &\leq \check{\alpha}_{\mathfrak{r}\mathfrak{j}}(\hat{\mathbf{m}}^I - \check{\mathbf{m}}^I)^2 + \check{\beta}_{\mathfrak{r}\mathfrak{j}}(\hat{\mathbf{m}}^I - \check{\mathbf{m}}^I)^2, \\ [\sigma_{\mathfrak{r}\mathfrak{j}}^J(t, \hat{\mathbf{m}}^J, \check{\mathbf{m}}^J) - \sigma_{\mathfrak{r}\mathfrak{j}}^J(t, \hat{\mathbf{m}}^J, \check{\mathbf{m}}^J)]^2 &\leq \bar{\alpha}_{\mathfrak{r}\mathfrak{j}}(\hat{\mathbf{m}}^J - \check{\mathbf{m}}^J)^2 + \bar{\beta}_{\mathfrak{r}\mathfrak{j}}(\hat{\mathbf{m}}^J - \check{\mathbf{m}}^J)^2, \\ [\sigma_{\mathfrak{r}\mathfrak{j}}^K(t, \hat{\mathbf{m}}^K, \check{\mathbf{m}}^K) - \sigma_{\mathfrak{r}\mathfrak{j}}^K(t, \hat{\mathbf{m}}^K, \check{\mathbf{m}}^K)]^2 &\leq \tilde{\alpha}_{\mathfrak{r}\mathfrak{j}}(\hat{\mathbf{m}}^K - \check{\mathbf{m}}^K)^2 + \tilde{\beta}_{\mathfrak{r}\mathfrak{j}}(\hat{\mathbf{m}}^K - \check{\mathbf{m}}^K)^2, \end{aligned}$$

for all $\hat{\mathbf{m}}^R, \check{\mathbf{m}}^R, \hat{\mathbf{m}}^I, \check{\mathbf{m}}^I, \hat{\mathbf{m}}^J, \check{\mathbf{m}}^J, \hat{\mathbf{m}}^K, \check{\mathbf{m}}^K \in \mathbb{R}, \mathfrak{r}, \mathfrak{j} \in N$.

A4: [13] $\sigma_{\mathfrak{r}\mathfrak{j}}^R(t, 0, 0) = 0, \sigma_{\mathfrak{r}\mathfrak{j}}^I(t, 0, 0) = 0, \sigma_{\mathfrak{r}\mathfrak{j}}^J(t, 0, 0) = 0, \sigma_{\mathfrak{r}\mathfrak{j}}^K(t, 0, 0) = 0$, for all $\mathfrak{r}, \mathfrak{j} \in N$.

Definition 1. [13] There existence of positive scalars $\alpha > 0, \beta > 0, \gamma > 0$, the SMQVNN model in Equation (2) has a trivial solution that is mean-square exp-ISS, in which

$$\mathbb{E}[|\mathbf{m}(t; \varphi)|^2] \leq \alpha e^{-\beta t} \mathbb{E}[|\varphi|^2] + \gamma \|\mathbf{u}\|_\infty^2, \tag{13}$$

for any $\varphi \in \mathcal{L}_{\mathcal{F}_0}^2([-\tau, 0]; \mathbb{Q}^n), \mathbf{u}(t) \in \ell_\infty$.

Now, consider the initial data $\bar{\mathbf{m}}(t) = \phi(t)$ on $-\tau \leq t \leq 0$ in $\mathcal{L}_{\mathcal{F}_0}^2([-\tau, 0]; \mathbb{R}^{4n})$ has the solution of $\bar{\mathbf{m}}(t; \phi)$. Obviously, under assumptions **A2** and **A4**, subject to the input $\bar{\mathbf{U}} = 0$, the model in Equation (11) has a trivial solution, or a zero solution $\bar{\mathbf{m}}(t; 0) = 0$.

3. Main Results

We derive the sufficient conditions that assure the mean-square exp-ISS with respect to the trivial solution of the SMQVNN model in Equation (2).

Theorem 1. Suppose that **A1–A4** hold, subject to the existence of positive scalars $\lambda, \mathbf{p}_{\mathfrak{r}}, \mathbf{q}_{\mathfrak{r}}, \mathbf{r}_{\mathfrak{r}}, \mathbf{s}_{\mathfrak{r}}, \mathbf{d}_{\mathfrak{r}}, \mathbf{h}_{\mathfrak{r}}, \mathbf{u}_{\mathfrak{r}}, \mathbf{v}_{\mathfrak{r}}, \mathfrak{r} \in N$, the SMQVNN model in Equation (2) has a trivial solution, which is mean-square exp-ISS, in which

$$\begin{aligned} 2\mathbf{p}_{\mathfrak{r}}\mathbf{c}_{\mathfrak{r}} &\geq \lambda\mathbf{p}_{\mathfrak{r}} + \sum_{\mathfrak{j}=1}^n \mathbf{p}_{\mathfrak{j}}\hat{\alpha}_{\mathfrak{j}\mathfrak{r}} + \mathbf{d}_{\mathfrak{r}} + \mathbf{p}_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\hat{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^R| \mathbf{G}_{\mathfrak{j}}^R + \mathbf{p}_{\mathfrak{j}} \sum_{\mathfrak{j}=1}^n |\hat{\mathbf{a}}_{\mathfrak{j}\mathfrak{r}}^R| \mathbf{G}_{\mathfrak{r}}^R + \mathbf{p}_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\hat{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^I| \mathbf{G}_{\mathfrak{j}}^I \\ &+ \mathbf{p}_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\hat{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^J| \mathbf{G}_{\mathfrak{j}}^J + \mathbf{p}_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\hat{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^K| \mathbf{G}_{\mathfrak{j}}^K + \mathbf{p}_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\hat{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^R| \mathbf{H}_{\mathfrak{j}}^R + \mathbf{p}_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\hat{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^I| \mathbf{H}_{\mathfrak{j}}^I \\ &+ \mathbf{p}_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\hat{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^J| \mathbf{H}_{\mathfrak{j}}^J + \mathbf{p}_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\hat{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^K| \mathbf{H}_{\mathfrak{j}}^K + \mathbf{p}_{\mathfrak{r}} + \mathbf{q}_{\mathfrak{j}} \sum_{\mathfrak{j}=1}^n |\hat{\mathbf{a}}_{\mathfrak{j}\mathfrak{r}}^I| \mathbf{G}_{\mathfrak{r}}^I + \mathbf{r}_{\mathfrak{j}} \sum_{\mathfrak{j}=1}^n |\hat{\mathbf{a}}_{\mathfrak{j}\mathfrak{r}}^J| \mathbf{G}_{\mathfrak{r}}^J + \mathbf{s}_{\mathfrak{j}} \sum_{\mathfrak{j}=1}^n |\hat{\mathbf{a}}_{\mathfrak{j}\mathfrak{r}}^K| \mathbf{G}_{\mathfrak{r}}^K, \end{aligned} \tag{14}$$

$$(1 - \mu)\mathbf{d}_{\mathfrak{r}} \geq e^{\lambda\tau} \sum_{\mathfrak{j}=1}^n \mathbf{p}_{\mathfrak{j}}\hat{\beta}_{\mathfrak{j}\mathfrak{r}} + e^{\lambda\tau} \sum_{\mathfrak{j}=1}^n \mathbf{p}_{\mathfrak{j}}|\hat{\mathbf{b}}_{\mathfrak{j}\mathfrak{r}}^R| \mathbf{H}_{\mathfrak{r}}^R + e^{\lambda\tau} \sum_{\mathfrak{j}=1}^n \mathbf{q}_{\mathfrak{j}}|\hat{\mathbf{b}}_{\mathfrak{j}\mathfrak{r}}^I| \mathbf{H}_{\mathfrak{r}}^I + e^{\lambda\tau} \sum_{\mathfrak{j}=1}^n \mathbf{r}_{\mathfrak{j}}|\hat{\mathbf{b}}_{\mathfrak{j}\mathfrak{r}}^J| \mathbf{H}_{\mathfrak{r}}^J + e^{\lambda\tau} \sum_{\mathfrak{j}=1}^n \mathbf{s}_{\mathfrak{j}}|\hat{\mathbf{b}}_{\mathfrak{j}\mathfrak{r}}^K| \mathbf{H}_{\mathfrak{r}}^K, \tag{15}$$

$$\begin{aligned} 2\mathbf{q}_{\mathfrak{r}}\mathbf{c}_{\mathfrak{r}} &\geq \lambda\mathbf{q}_{\mathfrak{r}} + \sum_{\mathfrak{j}=1}^n \mathbf{q}_{\mathfrak{j}}\check{\alpha}_{\mathfrak{j}\mathfrak{r}} + \mathbf{h}_{\mathfrak{r}} + \mathbf{q}_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\check{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^R| \mathbf{G}_{\mathfrak{j}}^I + \mathbf{q}_{\mathfrak{j}} \sum_{\mathfrak{j}=1}^n |\check{\mathbf{a}}_{\mathfrak{j}\mathfrak{r}}^R| \mathbf{G}_{\mathfrak{r}}^I + \mathbf{p}_{\mathfrak{j}} \sum_{\mathfrak{j}=1}^n |\check{\mathbf{a}}_{\mathfrak{j}\mathfrak{r}}^I| \mathbf{G}_{\mathfrak{r}}^I \\ &+ \mathbf{q}_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\check{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^I| \mathbf{G}_{\mathfrak{j}}^R + \mathbf{q}_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\check{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^J| \mathbf{G}_{\mathfrak{j}}^K + \mathbf{q}_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\check{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^K| \mathbf{G}_{\mathfrak{j}}^J + \mathbf{q}_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\check{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^R| \mathbf{H}_{\mathfrak{j}}^I \\ &+ \mathbf{q}_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\check{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^I| \mathbf{H}_{\mathfrak{j}}^R + \mathbf{q}_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\check{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^J| \mathbf{H}_{\mathfrak{j}}^K + \mathbf{q}_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\check{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^K| \mathbf{H}_{\mathfrak{j}}^J + \mathbf{q}_{\mathfrak{r}} + \mathbf{r}_{\mathfrak{j}} \sum_{\mathfrak{j}=1}^n |\check{\mathbf{a}}_{\mathfrak{j}\mathfrak{r}}^K| \mathbf{G}_{\mathfrak{r}}^I + \mathbf{s}_{\mathfrak{j}} \sum_{\mathfrak{j}=1}^n |\check{\mathbf{a}}_{\mathfrak{j}\mathfrak{r}}^J| \mathbf{G}_{\mathfrak{r}}^I, \end{aligned} \tag{16}$$

$$(1 - \mu)\mathbf{h}_r \geq e^{\lambda\tau} \sum_{j=1}^n \mathbf{q}_j \check{\beta}_{3r} + e^{\lambda\tau} \sum_{j=1}^n \mathbf{p}_j |\tilde{\mathbf{b}}_{3r}^I| \mathbf{H}_r^I + e^{\lambda\tau} \sum_{j=1}^n \mathbf{q}_j |\tilde{\mathbf{b}}_{3r}^R| \mathbf{H}_r^I + e^{\lambda\tau} \sum_{j=1}^n \mathbf{r}_j |\tilde{\mathbf{b}}_{3r}^K| \mathbf{H}_r^I + e^{\lambda\tau} \sum_{j=1}^n \mathbf{s}_j |\tilde{\mathbf{b}}_{3r}^J| \mathbf{H}_r^I, \tag{17}$$

$$\begin{aligned} 2\mathbf{r}_r \mathbf{c}_r &\geq \lambda \mathbf{r}_r + \sum_{j=1}^n \mathbf{r}_j \check{\alpha}_{3r} + \mathbf{u}_r + \mathbf{r}_r \sum_{j=1}^n |\check{\mathbf{a}}_{r3}^R| \mathbf{G}_j^I + \mathbf{r}_j \sum_{j=1}^n |\check{\mathbf{a}}_{3r}^R| \mathbf{G}_r^I + \mathbf{p}_j \sum_{j=1}^n |\check{\mathbf{a}}_{3r}^I| \mathbf{G}_r^I \\ &+ \mathbf{q}_j \sum_{j=1}^n |\check{\mathbf{a}}_{3r}^K| \mathbf{G}_r^I + \mathbf{r}_r \sum_{j=1}^n |\check{\mathbf{a}}_{r3}^I| \mathbf{G}_j^R + \mathbf{r}_r \sum_{j=1}^n |\check{\mathbf{a}}_{r3}^K| \mathbf{G}_j^I + \mathbf{r}_r \sum_{j=1}^n |\check{\mathbf{a}}_{r3}^I| \mathbf{G}_j^K \\ &+ \mathbf{r}_r \sum_{j=1}^n |\tilde{\mathbf{b}}_{r3}^I| \mathbf{H}_j^I + \mathbf{r}_r \sum_{j=1}^n |\tilde{\mathbf{b}}_{r3}^I| \mathbf{H}_j^R + \mathbf{r}_r \sum_{j=1}^n |\tilde{\mathbf{b}}_{r3}^K| \mathbf{H}_j^I + \mathbf{r}_r \sum_{j=1}^n |\tilde{\mathbf{b}}_{r3}^I| \mathbf{H}_j^K + \mathbf{r}_r + \mathbf{s}_j \sum_{j=1}^n |\check{\mathbf{a}}_{3r}^I| \mathbf{G}_r^I, \end{aligned} \tag{18}$$

$$(1 - \mu)\mathbf{u}_r \geq e^{\lambda\tau} \sum_{j=1}^n \mathbf{r}_j \check{\beta}_{3r} + e^{\lambda\tau} \sum_{j=1}^n \mathbf{p}_j |\tilde{\mathbf{b}}_{3r}^J| \mathbf{H}_r^I + e^{\lambda\tau} \sum_{j=1}^n \mathbf{q}_j |\tilde{\mathbf{b}}_{3r}^K| \mathbf{H}_r^I + e^{\lambda\tau} \sum_{j=1}^n \mathbf{r}_j |\tilde{\mathbf{b}}_{3r}^R| \mathbf{H}_r^I + e^{\lambda\tau} \sum_{j=1}^n \mathbf{s}_j |\tilde{\mathbf{b}}_{3r}^I| \mathbf{H}_r^I, \tag{19}$$

$$\begin{aligned} 2\mathbf{s}_r \mathbf{c}_r &\geq \lambda \mathbf{s}_r + \sum_{j=1}^n \mathbf{s}_j \check{\alpha}_{3r} + \mathbf{v}_r + \mathbf{s}_r \sum_{j=1}^n |\check{\mathbf{a}}_{r3}^R| \mathbf{G}_j^K + \mathbf{s}_j \sum_{j=1}^n |\check{\mathbf{a}}_{3r}^R| \mathbf{G}_r^K + \mathbf{p}_j \sum_{j=1}^n |\check{\mathbf{a}}_{3r}^K| \mathbf{G}_r^K \\ &+ \mathbf{q}_j \sum_{j=1}^n |\check{\mathbf{a}}_{3r}^I| \mathbf{G}_r^K + \mathbf{r}_j \sum_{j=1}^n |\check{\mathbf{a}}_{r3}^I| \mathbf{G}_j^K + \mathbf{s}_r \sum_{j=1}^n |\check{\mathbf{a}}_{r3}^I| \mathbf{G}_j^I + \mathbf{s}_r \sum_{j=1}^n |\check{\mathbf{a}}_{r3}^K| \mathbf{G}_j^R \\ &+ \mathbf{s}_r \sum_{j=1}^n |\check{\mathbf{a}}_{r3}^I| \mathbf{G}_j^I + \mathbf{s}_r \sum_{j=1}^n |\tilde{\mathbf{b}}_{r3}^I| \mathbf{H}_j^K + \mathbf{s}_r \sum_{j=1}^n |\tilde{\mathbf{b}}_{r3}^I| \mathbf{H}_j^I + \mathbf{s}_r \sum_{j=1}^n |\tilde{\mathbf{b}}_{r3}^K| \mathbf{H}_j^R + \mathbf{s}_r \sum_{j=1}^n |\tilde{\mathbf{b}}_{r3}^I| \mathbf{H}_j^I + \mathbf{s}_r, \end{aligned} \tag{20}$$

$$(1 - \mu)\mathbf{v}_r \geq e^{\lambda\tau} \sum_{j=1}^n \mathbf{s}_j \check{\beta}_{3r} + e^{\lambda\tau} \sum_{j=1}^n \mathbf{p}_j |\tilde{\mathbf{b}}_{3r}^K| \mathbf{H}_r^K + e^{\lambda\tau} \sum_{j=1}^n \mathbf{q}_j |\tilde{\mathbf{b}}_{3r}^I| \mathbf{H}_r^K + e^{\lambda\tau} \sum_{j=1}^n \mathbf{r}_j |\tilde{\mathbf{b}}_{3r}^I| \mathbf{H}_r^K + e^{\lambda\tau} \sum_{j=1}^n \mathbf{s}_j |\tilde{\mathbf{b}}_{3r}^R| \mathbf{H}_r^K. \tag{21}$$

Proof. A Lyapunov functional is formulated as

$$\mathcal{V}(t, \mathbf{m}(t)) = e^{\lambda t} \sum_{r=1}^{4n} \zeta_r \mathbf{m}_r^2(t) + \int_{t-\tau(t)}^t e^{\lambda s} \sum_{r=1}^{4n} \zeta_r \mathbf{m}_r^2(s) ds, \tag{22}$$

where

$$\zeta_r = \begin{cases} \mathbf{p}_r, & r = 1, 2, \dots, n, \\ \mathbf{q}_r, & r = n + 1, n + 2, \dots, 2n, \\ \mathbf{r}_r, & r = 2n + 1, 2n + 2, \dots, 3n, \\ \mathbf{s}_r, & r = 3n + 1, 3n + 2, \dots, 4n, \end{cases},$$

$$\zeta_r = \begin{cases} \mathbf{d}_r, & r = 1, 2, \dots, n, \\ \mathbf{h}_r, & r = n + 1, n + 2, \dots, 2n, \\ \mathbf{u}_r, & r = 2n + 1, 2n + 2, \dots, 3n, \\ \mathbf{v}_r, & r = 3n + 1, 3n + 2, \dots, 4n. \end{cases}$$

Then,

$$\mathcal{V}(t, \mathbf{m}(t)) = \mathcal{V}_1(t, \mathbf{m}^R(t)) + \mathcal{V}_1(t, \mathbf{m}^I(t)) + \mathcal{V}_1(t, \mathbf{m}^J(t)) + \mathcal{V}_1(t, \mathbf{m}^K(t)), \tag{23}$$

where

$$\begin{aligned} \mathcal{V}_1(t, \mathbf{m}^R(t)) &= e^{\lambda t} \sum_{\mathfrak{r}=1}^n \mathbf{p}_{\mathfrak{r}} \mathbf{m}_{\mathfrak{r}}^{R^2}(t) + \int_{t-\tau(t)}^t e^{\lambda s} \sum_{\mathfrak{r}=1}^n \mathbf{d}_{\mathfrak{r}} \mathbf{m}_{\mathfrak{r}}^{R^2}(s) ds, \\ \mathcal{V}_2(t, \mathbf{m}^I(t)) &= e^{\lambda t} \sum_{\mathfrak{r}=1}^n \mathbf{q}_{\mathfrak{r}} \mathbf{m}_{\mathfrak{r}}^{I^2}(t) + \int_{t-\tau(t)}^t e^{\lambda s} \sum_{\mathfrak{r}=1}^n \mathbf{h}_{\mathfrak{r}} \mathbf{m}_{\mathfrak{r}}^{I^2}(s) ds, \\ \mathcal{V}_3(t, \mathbf{m}^J(t)) &= e^{\lambda t} \sum_{\mathfrak{r}=1}^n \mathbf{r}_{\mathfrak{r}} \mathbf{m}_{\mathfrak{r}}^{J^2}(t) + \int_{t-\tau(t)}^t e^{\lambda s} \sum_{\mathfrak{r}=1}^n \mathbf{u}_{\mathfrak{r}} \mathbf{m}_{\mathfrak{r}}^{J^2}(s) ds, \\ \mathcal{V}_4(t, \mathbf{m}^K(t)) &= e^{\lambda t} \sum_{\mathfrak{r}=1}^n \mathbf{s}_{\mathfrak{r}} \mathbf{m}_{\mathfrak{r}}^{K^2}(t) + \int_{t-\tau(t)}^t e^{\lambda s} \sum_{\mathfrak{r}=1}^n \mathbf{v}_{\mathfrak{r}} \mathbf{m}_{\mathfrak{r}}^{K^2}(s) ds. \end{aligned}$$

Based on the Itô’s formula, we have

$$d\mathcal{V}(t, \bar{\mathbf{m}}(t)) = \mathcal{L}\mathcal{V}(t, \bar{\mathbf{m}}(t))dt + \mathcal{V}_{\bar{\mathbf{m}}}(t, \bar{\mathbf{m}}(t))\bar{\sigma}(t)dw(t), \tag{24}$$

where

$$\mathcal{V}_{\bar{\mathbf{m}}}(t, \bar{\mathbf{m}}(t)) = \left(\frac{\partial \mathcal{V}(t, \bar{\mathbf{m}}(t))}{\partial \bar{\mathbf{m}}_1}, \dots, \frac{\partial \mathcal{V}(t, \bar{\mathbf{m}}(t))}{\partial \bar{\mathbf{m}}_{4n}} \right).$$

where the Itô’s operator is \mathcal{L} , such that

$$\begin{aligned} \mathcal{L}\mathcal{V}_1(t, \mathbf{m}^R(t)) &= \lambda e^{\lambda t} \sum_{\mathfrak{r}=1}^n \mathbf{p}_{\mathfrak{r}} \mathbf{m}_{\mathfrak{r}}^{R^2}(t) + 2e^{\lambda t} \sum_{\mathfrak{r}=1}^n \mathbf{p}_{\mathfrak{r}} \mathbf{m}_{\mathfrak{r}}^R(t) \left[-\mathbf{c}_{\mathfrak{r}} \mathbf{m}_{\mathfrak{r}}^R(t) + \sum_{\mathfrak{j}=1}^n \tilde{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^R \mathbf{f}_{\mathfrak{j}}^R(\mathbf{m}_{\mathfrak{j}}^R(t)) \right. \\ &\quad - \sum_{\mathfrak{j}=1}^n \tilde{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^I \mathbf{f}_{\mathfrak{j}}^I(\mathbf{m}_{\mathfrak{j}}^I(t)) - \sum_{\mathfrak{j}=1}^n \tilde{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^J \mathbf{f}_{\mathfrak{j}}^J(\mathbf{m}_{\mathfrak{j}}^J(t)) \\ &\quad \left. + \sum_{\mathfrak{j}=1}^n \tilde{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^R \mathbf{g}_{\mathfrak{j}}^R(\mathbf{m}_{\mathfrak{j}}^R(t-\tau(t))) - \sum_{\mathfrak{j}=1}^n \tilde{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^I \mathbf{g}_{\mathfrak{j}}^I(\mathbf{m}_{\mathfrak{j}}^I(t-\tau(t))) - \sum_{\mathfrak{j}=1}^n \tilde{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^J \mathbf{g}_{\mathfrak{j}}^J(\mathbf{m}_{\mathfrak{j}}^J(t-\tau(t))) \right. \\ &\quad \left. - \sum_{\mathfrak{j}=1}^n \tilde{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^K \mathbf{g}_{\mathfrak{j}}^K(\mathbf{m}_{\mathfrak{j}}^K(t-\tau(t))) + \mathbf{u}_{\mathfrak{r}}^R(t) \right] + e^{\lambda t} \sum_{\mathfrak{r}=1}^n \mathbf{p}_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n [\sigma_{\mathfrak{r}\mathfrak{j}}^{R^2}(t, \mathbf{m}_{\mathfrak{j}}^R(t), \mathbf{m}_{\mathfrak{j}}^R(t-\tau(t)))]^2 \\ &\quad + e^{\lambda t} \sum_{\mathfrak{r}=1}^n \mathbf{d}_{\mathfrak{r}} \mathbf{m}_{\mathfrak{r}}^{R^2}(t) - (1 - \tilde{\tau}(t))e^{\lambda(t-\tau(t))} \sum_{\mathfrak{r}=1}^n \mathbf{d}_{\mathfrak{r}} \mathbf{m}_{\mathfrak{r}}^{R^2}(t-\tau(t)). \end{aligned} \tag{25}$$

According to **A2** and **A3**, we have

$$\begin{aligned} \mathcal{L}\mathcal{V}_1(t, \mathbf{m}^R(t)) &= e^{\lambda t} \sum_{\mathfrak{r}=1}^n (\lambda \mathbf{p}_{\mathfrak{r}} - 2\mathbf{p}_{\mathfrak{r}} \mathbf{c}_{\mathfrak{r}} + \sum_{\mathfrak{j}=1}^n \mathbf{p}_{\mathfrak{j}} \hat{\alpha}_{\mathfrak{j}\mathfrak{r}} + \mathbf{d}_{\mathfrak{r}}) \mathbf{m}_{\mathfrak{r}}^{R^2}(t) + 2e^{\lambda t} \sum_{\mathfrak{r}=1}^n \mathbf{p}_{\mathfrak{r}} |\mathbf{m}_{\mathfrak{r}}^R(t)| \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^R| \mathbf{G}_{\mathfrak{j}}^R |\mathbf{m}_{\mathfrak{j}}^R(t)| \\ &\quad + 2e^{\lambda t} \sum_{\mathfrak{r}=1}^n \mathbf{p}_{\mathfrak{r}} |\mathbf{m}_{\mathfrak{r}}^R(t)| \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^I| \mathbf{G}_{\mathfrak{j}}^I |\mathbf{m}_{\mathfrak{j}}^I(t)| + 2e^{\lambda t} \sum_{\mathfrak{r}=1}^n \mathbf{p}_{\mathfrak{r}} |\mathbf{m}_{\mathfrak{r}}^R(t)| \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^J| \mathbf{G}_{\mathfrak{j}}^J |\mathbf{m}_{\mathfrak{j}}^J(t)| \\ &\quad + 2e^{\lambda t} \sum_{\mathfrak{r}=1}^n \mathbf{p}_{\mathfrak{r}} |\mathbf{m}_{\mathfrak{r}}^R(t)| \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{r}\mathfrak{j}}^K| \mathbf{G}_{\mathfrak{j}}^K |\mathbf{m}_{\mathfrak{j}}^K(t)| + 2e^{\lambda t} \sum_{\mathfrak{r}=1}^n \mathbf{p}_{\mathfrak{r}} |\mathbf{m}_{\mathfrak{r}}^R(t)| \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^R| \mathbf{H}_{\mathfrak{j}}^R |\mathbf{m}_{\mathfrak{j}}^R(t-\tau(t))| \\ &\quad + 2e^{\lambda t} \sum_{\mathfrak{r}=1}^n \mathbf{p}_{\mathfrak{r}} |\mathbf{m}_{\mathfrak{r}}^R(t)| \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^I| \mathbf{H}_{\mathfrak{j}}^I |\mathbf{m}_{\mathfrak{j}}^I(t-\tau(t))| + 2e^{\lambda t} \sum_{\mathfrak{r}=1}^n \mathbf{p}_{\mathfrak{r}} |\mathbf{m}_{\mathfrak{r}}^R(t)| \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^J| \mathbf{H}_{\mathfrak{j}}^J |\mathbf{m}_{\mathfrak{j}}^J(t-\tau(t))| \\ &\quad + 2e^{\lambda t} \sum_{\mathfrak{r}=1}^n \mathbf{p}_{\mathfrak{r}} |\mathbf{m}_{\mathfrak{r}}^R(t)| \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{b}}_{\mathfrak{r}\mathfrak{j}}^K| \mathbf{H}_{\mathfrak{j}}^K |\mathbf{m}_{\mathfrak{j}}^K(t-\tau(t))| + 2e^{\lambda t} \sum_{\mathfrak{r}=1}^n \mathbf{p}_{\mathfrak{r}} |\mathbf{m}_{\mathfrak{r}}^R(t)| |\mathbf{u}_{\mathfrak{r}}^R(t)| \\ &\quad + e^{\lambda t} \sum_{\mathfrak{r}=1}^n \left[-(1 - \mu) e^{-\lambda \tau} \mathbf{d}_{\mathfrak{r}} + \sum_{\mathfrak{j}=1}^n \mathbf{p}_{\mathfrak{j}} \hat{\beta}_{\mathfrak{j}\mathfrak{r}} \right] \mathbf{m}_{\mathfrak{r}}^{R^2}(t-\tau(t)). \end{aligned} \tag{26}$$

$$\begin{aligned}
 \mathcal{L}\mathcal{V}_4(t, \mathbf{m}^K(t)) = & \lambda e^{\lambda t} \sum_{\tau=1}^n (\lambda \mathbf{s}_\tau - 2\mathbf{s}_\tau \mathbf{c}_\tau + \sum_{\beta=1}^n \mathbf{s}_\beta \tilde{\alpha}_{\beta\tau} + \mathbf{v}_\tau) \mathbf{m}_\tau^{K^2}(t) + e^{\lambda t} \sum_{\tau=1}^n \sum_{\beta=1}^n \mathbf{s}_\tau |\tilde{\mathbf{a}}_{\tau\beta}^R| \mathbf{G}_\beta^K (|\mathbf{m}_\tau^K(t)|^2 + \mathbf{m}_\beta^K(t)|^2) \\
 & + e^{\lambda t} \sum_{\tau=1}^n \sum_{\beta=1}^n \mathbf{s}_\tau |\tilde{\mathbf{a}}_{\tau\beta}^I| \mathbf{G}_\beta^I (|\mathbf{m}_\tau^K(t)|^2 + \mathbf{m}_\beta^I(t)|^2) + e^{\lambda t} \sum_{\tau=1}^n \sum_{\beta=1}^n \mathbf{s}_\tau |\tilde{\mathbf{a}}_{\tau\beta}^K| \mathbf{G}_\beta^K (|\mathbf{m}_\tau^K(t)|^2 + \mathbf{m}_\beta^K(t)|^2) \\
 & + e^{\lambda t} \sum_{\tau=1}^n \sum_{\beta=1}^n \mathbf{s}_\tau |\tilde{\mathbf{a}}_{\tau\beta}^I| \mathbf{G}_\beta^I (|\mathbf{m}_\tau^K(t)|^2 + \mathbf{m}_\beta^I(t)|^2) + e^{\lambda t} \sum_{\tau=1}^n \sum_{\beta=1}^n \mathbf{s}_\tau |\tilde{\mathbf{b}}_{\tau\beta}^R| \mathbf{H}_\beta^K (|\mathbf{m}_\tau^K(t)|^2 + \mathbf{m}_\beta^K(t - \tau(t))|^2) \\
 & + e^{\lambda t} \sum_{\tau=1}^n \sum_{\beta=1}^n \mathbf{s}_\tau |\tilde{\mathbf{b}}_{\tau\beta}^I| \mathbf{H}_\beta^I (|\mathbf{m}_\tau^K(t)|^2 + \mathbf{m}_\beta^I(t - \tau(t))|^2) + e^{\lambda t} \sum_{\tau=1}^n \sum_{\beta=1}^n \mathbf{s}_\tau |\tilde{\mathbf{b}}_{\tau\beta}^K| \mathbf{H}_\beta^K (|\mathbf{m}_\tau^K(t)|^2 + \mathbf{m}_\beta^K(t - \tau(t))|^2) \\
 & + e^{\lambda t} \sum_{\tau=1}^n \sum_{\beta=1}^n \mathbf{s}_\tau |\tilde{\mathbf{b}}_{\tau\beta}^I| \mathbf{H}_\beta^I (|\mathbf{m}_\tau^K(t)|^2 + \mathbf{m}_\beta^I(t - \tau(t))|^2) + e^{\lambda t} \sum_{\tau=1}^n \mathbf{s}_\tau (|\mathbf{m}_\tau^K(t)|^2 + |\mathbf{u}_\tau^K(t)|^2) \\
 & + e^{\lambda t} \sum_{\tau=1}^n [-(1 - \mu)e^{-\lambda\tau} \mathbf{v}_\tau + \sum_{\beta=1}^n \mathbf{s}_\beta \tilde{\beta}_{\beta\tau}] \mathbf{m}_\tau^{K^2}(t - \tau(t)). \tag{33}
 \end{aligned}$$

With further simplification, we have

$$\begin{aligned}
 \mathcal{L}\mathcal{V}(t, \hat{\mathbf{m}}(t)) = & -e^{\lambda t} \min_{1 \leq \tau \leq n} [2\mathbf{p}_\tau \mathbf{c}_\tau - \lambda \mathbf{p}_\tau - \sum_{\beta=1}^n \mathbf{p}_\beta \hat{\alpha}_{\beta\tau} - \mathbf{d}_\tau - \mathbf{p}_\tau \sum_{\beta=1}^n |\tilde{\mathbf{a}}_{\tau\beta}^R| \mathbf{G}_\beta^K - \mathbf{p}_\beta \sum_{\beta=1}^n |\tilde{\mathbf{a}}_{\beta\tau}^R| \mathbf{G}_\tau^K \\
 & - \mathbf{p}_\tau \sum_{\beta=1}^n |\tilde{\mathbf{a}}_{\tau\beta}^I| \mathbf{G}_\beta^I - \mathbf{p}_\beta \sum_{\beta=1}^n |\tilde{\mathbf{a}}_{\beta\tau}^I| \mathbf{G}_\tau^I - \mathbf{p}_\tau \sum_{\beta=1}^n |\tilde{\mathbf{a}}_{\tau\beta}^K| \mathbf{G}_\beta^K - \mathbf{p}_\beta \sum_{\beta=1}^n |\tilde{\mathbf{b}}_{\tau\beta}^R| \mathbf{H}_\beta^K - \mathbf{p}_\tau \sum_{\beta=1}^n |\tilde{\mathbf{b}}_{\tau\beta}^I| \mathbf{H}_\beta^I \\
 & - \mathbf{p}_\tau \sum_{\beta=1}^n |\tilde{\mathbf{b}}_{\tau\beta}^K| \mathbf{H}_\beta^K - \mathbf{p}_\beta \sum_{\beta=1}^n |\tilde{\mathbf{b}}_{\beta\tau}^K| \mathbf{H}_\tau^K - \mathbf{p}_\tau - \mathbf{q}_\beta \sum_{\beta=1}^n |\tilde{\mathbf{a}}_{\beta\tau}^I| \mathbf{G}_\tau^K - \mathbf{r}_\beta \sum_{\beta=1}^n |\tilde{\mathbf{a}}_{\beta\tau}^I| \mathbf{G}_\tau^I \\
 & - \mathbf{s}_\beta \sum_{\beta=1}^n |\tilde{\mathbf{a}}_{\beta\tau}^K| \mathbf{G}_\tau^K] \sum_{\tau=1}^n \mathbf{m}_\tau^{R^2}(t) - e^{\lambda t} \min_{1 \leq \tau \leq n} [(1 - \mu)\mathbf{d}_\tau - e^{\lambda\tau} \sum_{\beta=1}^n \mathbf{p}_\beta \hat{\beta}_{\beta\tau} - e^{\lambda\tau} \sum_{\beta=1}^n \mathbf{p}_\beta |\tilde{\mathbf{b}}_{\tau\beta}^R| \mathbf{H}_\tau^K \\
 & - e^{\lambda\tau} \sum_{\beta=1}^n \mathbf{q}_\beta |\tilde{\mathbf{b}}_{\tau\beta}^I| \mathbf{H}_\tau^K - e^{\lambda\tau} \sum_{\beta=1}^n \mathbf{r}_\beta |\tilde{\mathbf{b}}_{\tau\beta}^I| \mathbf{H}_\tau^K - e^{\lambda\tau} \sum_{\beta=1}^n \mathbf{s}_\beta |\tilde{\mathbf{b}}_{\tau\beta}^K| \mathbf{H}_\tau^K] \sum_{\tau=1}^n \mathbf{m}_\tau^{R^2}(t - \tau(t)) \\
 & - e^{\lambda t} \min_{1 \leq \tau \leq n} [2\mathbf{q}_\tau \mathbf{c}_\tau - \lambda \mathbf{q}_\tau - \sum_{\beta=1}^n \mathbf{q}_\beta \tilde{\alpha}_{\beta\tau} - \mathbf{h}_\tau - \mathbf{q}_\tau \sum_{\beta=1}^n |\tilde{\mathbf{a}}_{\tau\beta}^R| \mathbf{G}_\beta^I - \mathbf{q}_\beta \sum_{\beta=1}^n |\tilde{\mathbf{a}}_{\beta\tau}^R| \mathbf{G}_\tau^I \\
 & - \mathbf{p}_\beta \sum_{\beta=1}^n |\tilde{\mathbf{a}}_{\beta\tau}^I| \mathbf{G}_\tau^I - \mathbf{q}_\tau \sum_{\beta=1}^n |\tilde{\mathbf{a}}_{\tau\beta}^I| \mathbf{G}_\beta^K - \mathbf{q}_\tau \sum_{\beta=1}^n |\tilde{\mathbf{a}}_{\tau\beta}^I| \mathbf{G}_\beta^I - \mathbf{q}_\tau \sum_{\beta=1}^n |\tilde{\mathbf{b}}_{\tau\beta}^R| \mathbf{H}_\beta^K \\
 & - \mathbf{q}_\tau \sum_{\beta=1}^n |\tilde{\mathbf{b}}_{\tau\beta}^I| \mathbf{H}_\beta^K - \mathbf{q}_\tau \sum_{\beta=1}^n |\tilde{\mathbf{b}}_{\tau\beta}^I| \mathbf{H}_\beta^K - \mathbf{q}_\tau \sum_{\beta=1}^n |\tilde{\mathbf{b}}_{\tau\beta}^K| \mathbf{H}_\beta^K - \mathbf{q}_\tau - \mathbf{r}_\beta \sum_{\beta=1}^n |\tilde{\mathbf{a}}_{\beta\tau}^K| \mathbf{G}_\tau^K \\
 & - \mathbf{s}_\beta \sum_{\beta=1}^n |\tilde{\mathbf{a}}_{\beta\tau}^I| \mathbf{G}_\tau^I] \sum_{\tau=1}^n \mathbf{m}_\tau^{I^2}(t) - e^{\lambda t} \min_{1 \leq \tau \leq n} [(1 - \mu)\mathbf{h}_\tau - e^{\lambda\tau} \sum_{\beta=1}^n \mathbf{q}_\beta \tilde{\beta}_{\beta\tau} - e^{\lambda\tau} \sum_{\beta=1}^n \mathbf{p}_\beta |\tilde{\mathbf{b}}_{\tau\beta}^I| \mathbf{H}_\tau^K \\
 & - e^{\lambda\tau} \sum_{\beta=1}^n \mathbf{q}_\beta |\tilde{\mathbf{b}}_{\tau\beta}^R| \mathbf{H}_\tau^K - e^{\lambda\tau} \sum_{\beta=1}^n \mathbf{r}_\beta |\tilde{\mathbf{b}}_{\tau\beta}^K| \mathbf{H}_\tau^K - e^{\lambda\tau} \sum_{\beta=1}^n \mathbf{s}_\beta |\tilde{\mathbf{b}}_{\tau\beta}^I| \mathbf{H}_\tau^K] \sum_{\tau=1}^n \mathbf{m}_\tau^{I^2}(t - \tau(t)) \\
 & - e^{\lambda t} \min_{1 \leq \tau \leq n} [2\mathbf{r}_\tau \mathbf{c}_\tau - \lambda \mathbf{r}_\tau - \sum_{\beta=1}^n \mathbf{r}_\beta \tilde{\alpha}_{\beta\tau} - \mathbf{u}_\tau - \mathbf{r}_\tau \sum_{\beta=1}^n |\tilde{\mathbf{a}}_{\tau\beta}^R| \mathbf{G}_\beta^I - \mathbf{r}_\beta \sum_{\beta=1}^n |\tilde{\mathbf{a}}_{\beta\tau}^R| \mathbf{G}_\tau^I \\
 & - \mathbf{p}_\beta \sum_{\beta=1}^n |\tilde{\mathbf{a}}_{\beta\tau}^I| \mathbf{G}_\tau^I - \mathbf{q}_\beta \sum_{\beta=1}^n |\tilde{\mathbf{a}}_{\beta\tau}^K| \mathbf{G}_\tau^K - \mathbf{r}_\tau \sum_{\beta=1}^n |\tilde{\mathbf{a}}_{\tau\beta}^I| \mathbf{G}_\beta^K - \mathbf{r}_\tau \sum_{\beta=1}^n |\tilde{\mathbf{a}}_{\tau\beta}^I| \mathbf{G}_\beta^I - \mathbf{r}_\tau \sum_{\beta=1}^n |\tilde{\mathbf{a}}_{\tau\beta}^I| \mathbf{G}_\beta^K \\
 & - \mathbf{r}_\tau \sum_{\beta=1}^n |\tilde{\mathbf{b}}_{\tau\beta}^R| \mathbf{H}_\beta^K - \mathbf{r}_\tau \sum_{\beta=1}^n |\tilde{\mathbf{b}}_{\tau\beta}^I| \mathbf{H}_\beta^K - \mathbf{r}_\tau \sum_{\beta=1}^n |\tilde{\mathbf{b}}_{\tau\beta}^K| \mathbf{H}_\beta^K - \mathbf{r}_\tau \sum_{\beta=1}^n |\tilde{\mathbf{b}}_{\tau\beta}^I| \mathbf{H}_\beta^K - \mathbf{r}_\tau
 \end{aligned}$$

$$\begin{aligned}
 & -s_3 \sum_{j=1}^n |\tilde{a}_{3j}^I| G_j^I \sum_{r=1}^n m_r^I(t) - e^{\lambda t} \min_{1 \leq r \leq n} [(1-\mu)u_r - e^{\lambda \tau} \sum_{j=1}^n r_j \tilde{\beta}_{3j} - e^{\lambda \tau} \sum_{j=1}^n p_j |\tilde{b}_{rj}^I| H_r^I] \\
 & - e^{\lambda \tau} \sum_{j=1}^n q_j |\tilde{b}_{rj}^K| H_r^I - e^{\lambda \tau} \sum_{j=1}^n r_j |\tilde{b}_{rj}^R| H_r^I - e^{\lambda \tau} \sum_{j=1}^n s_j |\tilde{b}_{rj}^I| H_r^I \sum_{r=1}^n m_r^I(t - \tau(t)) \\
 & - e^{\lambda t} \min_{1 \leq r \leq n} [2s_r c_r - \lambda s_r - \sum_{j=1}^n s_j \tilde{\alpha}_{3j} - v_r - s_r \sum_{j=1}^n |\tilde{a}_{rj}^R| G_j^K - s_j \sum_{j=1}^n |\tilde{a}_{rj}^R| G_j^K \\
 & - p_j \sum_{j=1}^n |\tilde{a}_{rj}^I| G_j^K - q_j \sum_{j=1}^n |\tilde{a}_{rj}^I| G_j^K - r_j \sum_{j=1}^n |\tilde{a}_{rj}^I| G_j^K - s_r \sum_{j=1}^n |\tilde{a}_{rj}^I| G_j^I - s_r \sum_{j=1}^n |\tilde{a}_{rj}^I| G_j^R \\
 & - s_r \sum_{j=1}^n |\tilde{a}_{rj}^I| G_j^I - s_r \sum_{j=1}^n |\tilde{b}_{rj}^R| H_j^K - s_r \sum_{j=1}^n |\tilde{b}_{rj}^I| H_j^I - s_r \sum_{j=1}^n |\tilde{b}_{rj}^K| H_j^R \\
 & - s_r \sum_{j=1}^n |\tilde{b}_{rj}^I| H_j^I - s_r] \sum_{r=1}^n m_r^K(t) - e^{\lambda t} \min_{1 \leq r \leq n} [(1-\mu)v_r - e^{\lambda \tau} \sum_{j=1}^n s_j \tilde{\beta}_{3j} - e^{\lambda \tau} \sum_{j=1}^n p_j |\tilde{b}_{rj}^K| H_r^K \\
 & - e^{\lambda \tau} \sum_{j=1}^n q_j |\tilde{b}_{rj}^I| H_r^K - e^{\lambda \tau} \sum_{j=1}^n r_j |\tilde{b}_{rj}^I| H_r^K - e^{\lambda \tau} \sum_{j=1}^n s_j |\tilde{b}_{rj}^R| H_r^K] \sum_{r=1}^n m_r^K(t - \tau(t)) + e^{\lambda t} \sum_{r=1}^{4n} \zeta_r |\bar{U}(t)|^2, \tag{34}
 \end{aligned}$$

where

$$\bar{U}(t) = \begin{cases} \mathbf{u}_r^R(t), & r = 1, 2, \dots, n, \\ \mathbf{u}_r^I(t), & r = n + 1, n + 2, \dots, 2n, \\ \mathbf{u}_r^I(t), & r = 2n + 1, 2n + 2, \dots, 3n, \\ \mathbf{u}_r^K(t), & r = 3n + 1, 3n + 2, \dots, 4n. \end{cases}$$

According to Equations (14)–(21), we can easily obtain

$$\mathcal{L}\mathcal{V}(t, \bar{\mathbf{m}}(t)) \leq e^{\lambda t} \sum_{r=1}^{4n} \zeta_r |\bar{U}(t)|^2 \leq e^{\lambda t} \max_{1 \leq r \leq 4n} \zeta_r \|\bar{U}\|_\infty^2. \tag{35}$$

Now, similar to the study in [13], the Markov time or stopping time is defined as $q_h := \inf\{s \geq 0 : |\bar{\mathbf{m}}(s)| \geq h\}$. The Dynkin formula is then used to yield

$$\mathbb{E}[\mathcal{V}(t \wedge q_h, \bar{\mathbf{m}}(t \wedge q_h))] = \mathbb{E}[\mathcal{V}(0, \bar{\mathbf{m}}(0))] + \mathbb{E}\left[\int_0^{t \wedge q_h} \mathcal{L}\mathcal{V}(s, \bar{\mathbf{m}}(s)) ds\right]. \tag{36}$$

With respect to the monotone convergence theorem, we have the following when h goes to ∞

$$\mathbb{E}[\mathcal{V}(t, \bar{\mathbf{m}}(t))] \leq \sum_{r=1}^{4n} \zeta_r \mathbb{E}[\bar{\mathbf{m}}_r^2(0)] + \int_{-\tau(0)}^0 e^{\lambda s} \mathbb{E}\left[\sum_{r=1}^{4n} \zeta_r \bar{\mathbf{m}}_r^2(s)\right] ds + \frac{1}{\lambda} \|\bar{U}\|_\infty^2 \min_{1 \leq r \leq 4n} \zeta_r (e^{\lambda t} - 1). \tag{37}$$

Based on Equation (22), we have

$$\begin{aligned}
 \mathbb{E}[\mathcal{V}(t, \bar{\mathbf{m}}(t))] & \geq \mathbb{E}\left[e^{\lambda t} \sum_{r=1}^{4n} \zeta_r \mathbb{E}[\bar{\mathbf{m}}_r^2(t)]\right] \\
 & \geq e^{\lambda t} \min_{1 \leq r \leq 4n} \zeta_r \mathbb{E}[|\bar{\mathbf{m}}(t)|^2]. \tag{38}
 \end{aligned}$$

Combining Equations (37) and (38), we have

$$e^{\lambda t} \min_{1 \leq r \leq 4n} \zeta_r \mathbb{E}[|\bar{\mathbf{m}}(t)|^2] \leq \max_{1 \leq r \leq 4n} \zeta_r \sum_{r=1}^{4n} \mathbb{E}[\bar{\mathbf{m}}_r^2(0)] + \max_{1 \leq r \leq 4n} \zeta_r \int_{-\tau}^0 e^{\lambda s} \sum_{r=1}^{4n} \mathbb{E}[\bar{\mathbf{m}}_r^2(s)] ds + \frac{1}{\lambda} e^{\lambda t} \|\bar{U}\|_\infty^2 \max_{1 \leq r \leq 4n} \zeta_r. \tag{39}$$

Consequently, we can derive

$$\mathbb{E}[|\bar{\mathbf{m}}(t)|^2] \leq \alpha e^{-\lambda t} \mathbb{E}[|\phi|^2] + \gamma \|\bar{U}\|_\infty^2, \tag{40}$$

where

$$\alpha = \frac{\max_{1 \leq r \leq 4n} \zeta_r + \tau \max_{1 \leq r \leq 4n} \zeta_r}{\min_{1 \leq r \leq 4n} \zeta_r}, \quad \gamma = \frac{\max_{1 \leq r \leq 4n} \zeta_r}{\lambda \min_{1 \leq r \leq 4n} \zeta_r},$$

which implies that the SMQVNN model in Equation (2) has a trivial solution, which is mean-square exp ISS. The proof is completed. \square

Remark 2. Suppose that the time-varying delay $\tau(t)$ is assumed to satisfy $0 \leq \tau(t) \leq \tau$, where τ is a positive real constant. As we mentioned in the introduction, constant delays can be seen as a special case of time-varying delays when the delays vary in a fixed point. Therefore, the mean-square exp-ISS criterion of the NN Model (2) can be derived based on the same approaches as in Theorem (1) with constant delays in the following Corollary (1).

Corollary 1. Suppose that A1-A4 hold, and subject to the existence of positive scalars $\lambda, \mathbf{p}_r, \mathbf{q}_r, \mathbf{r}_r, \mathbf{s}_r, \mathbf{d}_r, \mathbf{h}_r, \mathbf{u}_r, \mathbf{v}_r, r \in N$, the SMQVNN model in Equation (2) with a constant delay has a trivial solution, which is mean-square exp-ISS, in which

$$\begin{aligned} 2\mathbf{p}_r \mathbf{c}_r &\geq \lambda \mathbf{p}_r + \sum_{j=1}^n \mathbf{p}_j \hat{\alpha}_{jr} + \mathbf{d}_r + \mathbf{p}_r \sum_{j=1}^n |\tilde{\mathbf{a}}_{rj}^R| \mathbf{G}_j^R + \mathbf{p}_j \sum_{j=1}^n |\tilde{\mathbf{a}}_{rj}^R| \mathbf{G}_r^R + \mathbf{p}_r \sum_{j=1}^n |\tilde{\mathbf{a}}_{rj}^I| \mathbf{G}_j^I \\ &+ \mathbf{p}_r \sum_{j=1}^n |\tilde{\mathbf{a}}_{rj}^J| \mathbf{G}_j^J + \mathbf{p}_r \sum_{j=1}^n |\tilde{\mathbf{a}}_{rj}^K| \mathbf{G}_j^K + \mathbf{p}_r \sum_{j=1}^n |\tilde{\mathbf{b}}_{rj}^R| \mathbf{H}_j^R + \mathbf{p}_r \sum_{j=1}^n |\tilde{\mathbf{b}}_{rj}^I| \mathbf{H}_j^I \\ &+ \mathbf{p}_r \sum_{j=1}^n |\tilde{\mathbf{b}}_{rj}^J| \mathbf{H}_j^J + \mathbf{p}_r \sum_{j=1}^n |\tilde{\mathbf{b}}_{rj}^K| \mathbf{H}_j^K + \mathbf{p}_r + \mathbf{q}_j \sum_{j=1}^n |\tilde{\mathbf{a}}_{jr}^I| \mathbf{G}_r^I + \mathbf{r}_j \sum_{j=1}^n |\tilde{\mathbf{a}}_{jr}^J| \mathbf{G}_r^J + \mathbf{s}_j \sum_{j=1}^n |\tilde{\mathbf{a}}_{jr}^K| \mathbf{G}_r^K, \end{aligned} \quad (41)$$

$$\mathbf{d}_r \geq e^{\lambda\tau} \sum_{j=1}^n \mathbf{p}_j \hat{\beta}_{jr} + e^{\lambda\tau} \sum_{j=1}^n \mathbf{p}_j |\tilde{\mathbf{b}}_{jr}^R| \mathbf{H}_r^R + e^{\lambda\tau} \sum_{j=1}^n \mathbf{q}_j |\tilde{\mathbf{b}}_{jr}^I| \mathbf{H}_r^I + e^{\lambda\tau} \sum_{j=1}^n \mathbf{r}_j |\tilde{\mathbf{b}}_{jr}^J| \mathbf{H}_r^J + e^{\lambda\tau} \sum_{j=1}^n \mathbf{s}_j |\tilde{\mathbf{b}}_{jr}^K| \mathbf{H}_r^K, \quad (42)$$

$$\begin{aligned} 2\mathbf{q}_r \mathbf{c}_r &\geq \lambda \mathbf{q}_r + \sum_{j=1}^n \mathbf{q}_j \check{\alpha}_{jr} + \mathbf{h}_r + \mathbf{q}_r \sum_{j=1}^n |\tilde{\mathbf{a}}_{rj}^R| \mathbf{G}_j^I + \mathbf{q}_j \sum_{j=1}^n |\tilde{\mathbf{a}}_{rj}^R| \mathbf{G}_r^I + \mathbf{p}_j \sum_{j=1}^n |\tilde{\mathbf{a}}_{rj}^I| \mathbf{G}_r^I \\ &+ \mathbf{q}_r \sum_{j=1}^n |\tilde{\mathbf{a}}_{rj}^J| \mathbf{G}_j^R + \mathbf{q}_r \sum_{j=1}^n |\tilde{\mathbf{a}}_{rj}^J| \mathbf{G}_j^K + \mathbf{q}_r \sum_{j=1}^n |\tilde{\mathbf{a}}_{rj}^K| \mathbf{G}_j^I + \mathbf{q}_r \sum_{j=1}^n |\tilde{\mathbf{b}}_{rj}^R| \mathbf{H}_j^I \\ &+ \mathbf{q}_r \sum_{j=1}^n |\tilde{\mathbf{b}}_{rj}^I| \mathbf{H}_j^R + \mathbf{q}_r \sum_{j=1}^n |\tilde{\mathbf{b}}_{rj}^J| \mathbf{H}_j^K + \mathbf{q}_r \sum_{j=1}^n |\tilde{\mathbf{b}}_{rj}^K| \mathbf{H}_j^I + \mathbf{q}_r + \mathbf{r}_j \sum_{j=1}^n |\tilde{\mathbf{a}}_{jr}^K| \mathbf{G}_r^I + \mathbf{s}_j \sum_{j=1}^n |\tilde{\mathbf{a}}_{jr}^I| \mathbf{G}_r^I, \end{aligned} \quad (43)$$

$$\mathbf{h}_r \geq e^{\lambda\tau} \sum_{j=1}^n \mathbf{q}_j \check{\beta}_{jr} + e^{\lambda\tau} \sum_{j=1}^n \mathbf{p}_j |\tilde{\mathbf{b}}_{jr}^I| \mathbf{H}_r^I + e^{\lambda\tau} \sum_{j=1}^n \mathbf{q}_j |\tilde{\mathbf{b}}_{jr}^R| \mathbf{H}_r^I + e^{\lambda\tau} \sum_{j=1}^n \mathbf{r}_j |\tilde{\mathbf{b}}_{jr}^K| \mathbf{H}_r^I + e^{\lambda\tau} \sum_{j=1}^n \mathbf{s}_j |\tilde{\mathbf{b}}_{jr}^J| \mathbf{H}_r^I, \quad (44)$$

$$\begin{aligned} 2\mathbf{r}_r \mathbf{c}_r &\geq \lambda \mathbf{r}_r + \sum_{j=1}^n \mathbf{r}_j \bar{\alpha}_{jr} + \mathbf{u}_r + \mathbf{r}_r \sum_{j=1}^n |\tilde{\mathbf{a}}_{rj}^R| \mathbf{G}_j^J + \mathbf{r}_j \sum_{j=1}^n |\tilde{\mathbf{a}}_{rj}^R| \mathbf{G}_r^J + \mathbf{p}_j \sum_{j=1}^n |\tilde{\mathbf{a}}_{rj}^I| \mathbf{G}_r^I \\ &+ \mathbf{q}_j \sum_{j=1}^n |\tilde{\mathbf{a}}_{rj}^K| \mathbf{G}_r^I + \mathbf{r}_r \sum_{j=1}^n |\tilde{\mathbf{a}}_{rj}^J| \mathbf{G}_j^R + \mathbf{r}_r \sum_{j=1}^n |\tilde{\mathbf{a}}_{rj}^K| \mathbf{G}_j^I + \mathbf{r}_r \sum_{j=1}^n |\tilde{\mathbf{a}}_{rj}^I| \mathbf{G}_j^K \\ &+ \mathbf{r}_r \sum_{j=1}^n |\tilde{\mathbf{b}}_{rj}^R| \mathbf{H}_j^I + \mathbf{r}_r \sum_{j=1}^n |\tilde{\mathbf{b}}_{rj}^J| \mathbf{H}_j^R + \mathbf{r}_r \sum_{j=1}^n |\tilde{\mathbf{b}}_{rj}^K| \mathbf{H}_j^I + \mathbf{r}_r \sum_{j=1}^n |\tilde{\mathbf{b}}_{rj}^I| \mathbf{H}_j^K + \mathbf{r}_r + \mathbf{s}_j \sum_{j=1}^n |\tilde{\mathbf{a}}_{jr}^I| \mathbf{G}_r^I, \end{aligned} \quad (45)$$

$$\mathbf{u}_r \geq e^{\lambda\tau} \sum_{j=1}^n \mathbf{r}_j \bar{\beta}_{jr} + e^{\lambda\tau} \sum_{j=1}^n \mathbf{p}_j |\tilde{\mathbf{b}}_{jr}^J| \mathbf{H}_r^I + e^{\lambda\tau} \sum_{j=1}^n \mathbf{q}_j |\tilde{\mathbf{b}}_{jr}^K| \mathbf{H}_r^I + e^{\lambda\tau} \sum_{j=1}^n \mathbf{r}_j |\tilde{\mathbf{b}}_{jr}^R| \mathbf{H}_r^I + e^{\lambda\tau} \sum_{j=1}^n \mathbf{s}_j |\tilde{\mathbf{b}}_{jr}^I| \mathbf{H}_r^I, \quad (46)$$

$$\begin{aligned}
 2s_{\mathfrak{r}}c_{\mathfrak{r}} \geq & \lambda s_{\mathfrak{r}} + \sum_{\mathfrak{j}=1}^n s_{\mathfrak{j}}\tilde{\alpha}_{\mathfrak{j}\mathfrak{r}} + \mathbf{v}_{\mathfrak{r}} + s_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{j}\mathfrak{r}}^R|G_{\mathfrak{j}}^K + s_{\mathfrak{j}} \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{j}\mathfrak{r}}^R|G_{\mathfrak{r}}^K + \mathbf{p}_{\mathfrak{j}} \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{j}\mathfrak{r}}^K|G_{\mathfrak{r}}^K \\
 & + \mathbf{q}_{\mathfrak{j}} \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{j}\mathfrak{r}}^I|G_{\mathfrak{r}}^K + \mathbf{r}_{\mathfrak{j}} \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{j}\mathfrak{r}}^I|G_{\mathfrak{r}}^K + s_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{j}\mathfrak{r}}^I|G_{\mathfrak{j}}^I + s_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{j}\mathfrak{r}}^K|G_{\mathfrak{j}}^R \\
 & + s_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{j}\mathfrak{r}}^I|G_{\mathfrak{j}}^I + s_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{b}}_{\mathfrak{j}\mathfrak{r}}^R|H_{\mathfrak{j}}^K + s_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{b}}_{\mathfrak{j}\mathfrak{r}}^I|H_{\mathfrak{j}}^I + s_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{b}}_{\mathfrak{j}\mathfrak{r}}^K|H_{\mathfrak{j}}^R + s_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{b}}_{\mathfrak{j}\mathfrak{r}}^I|H_{\mathfrak{j}}^I + s_{\mathfrak{r}}, \quad (47)
 \end{aligned}$$

$$\mathbf{v}_{\mathfrak{r}} \geq e^{\lambda\tau} \sum_{\mathfrak{j}=1}^n s_{\mathfrak{j}}\tilde{\beta}_{\mathfrak{j}\mathfrak{r}} + e^{\lambda\tau} \sum_{\mathfrak{j}=1}^n \mathbf{p}_{\mathfrak{j}}|\tilde{\mathbf{b}}_{\mathfrak{j}\mathfrak{r}}^K|H_{\mathfrak{r}}^K + e^{\lambda\tau} \sum_{\mathfrak{j}=1}^n \mathbf{q}_{\mathfrak{j}}|\tilde{\mathbf{b}}_{\mathfrak{j}\mathfrak{r}}^I|H_{\mathfrak{r}}^K + e^{\lambda\tau} \sum_{\mathfrak{j}=1}^n \mathbf{r}_{\mathfrak{j}}|\tilde{\mathbf{b}}_{\mathfrak{j}\mathfrak{r}}^I|H_{\mathfrak{r}}^K + e^{\lambda\tau} \sum_{\mathfrak{j}=1}^n s_{\mathfrak{j}}|\tilde{\mathbf{b}}_{\mathfrak{j}\mathfrak{r}}^R|H_{\mathfrak{r}}^K, \quad (48)$$

Remark 3. The exp-ISS criteria pertaining to the SMQVNN model is yet to be reported in the study of QVNNs dynamics. In order to fill this gap, Theorem (1) and Corollary (1) reveal several new sufficient conditions for the SMQVNN model.

Based on the complex number definition, the model in Equation (10) becomes:

$$\begin{cases}
 d\mathbf{m}_{\mathfrak{r}}^R(t) = [-c_{\mathfrak{r}}\mathbf{m}_{\mathfrak{r}}^R(t) + \sum_{\mathfrak{j}=1}^n [\tilde{\mathbf{a}}_{\mathfrak{j}\mathfrak{r}}^R f_{\mathfrak{j}}^R(\mathbf{m}_{\mathfrak{j}}^R(t)) - \tilde{\mathbf{a}}_{\mathfrak{j}\mathfrak{r}}^I f_{\mathfrak{j}}^I(\mathbf{m}_{\mathfrak{j}}^I(t))] + \sum_{\mathfrak{j}=1}^n [\tilde{\mathbf{b}}_{\mathfrak{j}\mathfrak{r}}^R \mathbf{g}_{\mathfrak{j}}^R(\mathbf{m}_{\mathfrak{j}}^R(t - \tau(t))) \\
 - \tilde{\mathbf{b}}_{\mathfrak{j}\mathfrak{r}}^I \mathbf{g}_{\mathfrak{j}}^I(\mathbf{m}_{\mathfrak{j}}^I(t - \tau(t)))] + \mathbf{u}_{\mathfrak{r}}^R(t)]dt + \sum_{\mathfrak{j}=1}^n \sigma_{\mathfrak{r}\mathfrak{j}}^R(t, \mathbf{m}_{\mathfrak{j}}^R(t), \mathbf{m}_{\mathfrak{j}}^R(t - \tau(t)))d\omega_{\mathfrak{j}}(t), \\
 d\mathbf{m}_{\mathfrak{r}}^I(t) = [-c_{\mathfrak{r}}\mathbf{m}_{\mathfrak{r}}^I(t) + \sum_{\mathfrak{j}=1}^n [\tilde{\mathbf{a}}_{\mathfrak{j}\mathfrak{r}}^R f_{\mathfrak{j}}^I(\mathbf{m}_{\mathfrak{j}}^I(t)) + \tilde{\mathbf{a}}_{\mathfrak{j}\mathfrak{r}}^I f_{\mathfrak{j}}^R(\mathbf{m}_{\mathfrak{j}}^R(t))] + \sum_{\mathfrak{j}=1}^n [\tilde{\mathbf{b}}_{\mathfrak{j}\mathfrak{r}}^R \mathbf{g}_{\mathfrak{j}}^I(\mathbf{m}_{\mathfrak{j}}^I(t - \tau(t))) \\
 + \tilde{\mathbf{b}}_{\mathfrak{j}\mathfrak{r}}^I \mathbf{g}_{\mathfrak{j}}^R(\mathbf{m}_{\mathfrak{j}}^R(t - \tau(t)))] + \mathbf{u}_{\mathfrak{r}}^I(t)]dt + \sum_{\mathfrak{j}=1}^n \sigma_{\mathfrak{r}\mathfrak{j}}^I(t, \mathbf{m}_{\mathfrak{j}}^I(t), \mathbf{m}_{\mathfrak{j}}^I(t - \tau(t)))d\omega_{\mathfrak{j}}(t).
 \end{cases} \quad (49)$$

Corollary 2. Suppose that A1-A4 hold, and subject to the existence of positive scalars $\lambda, \mathbf{p}_{\mathfrak{r}}, \mathbf{q}_{\mathfrak{r}}, \mathbf{d}_{\mathfrak{r}}, \mathbf{h}_{\mathfrak{r}}, \mathfrak{r} \in N$, the SMCVNN model in Equation (49) has a trivial solution that is the mean-square exp ISS, such that

$$\begin{aligned}
 2\mathbf{p}_{\mathfrak{r}}c_{\mathfrak{r}} \geq & \lambda \mathbf{p}_{\mathfrak{r}} + \sum_{\mathfrak{j}=1}^n \mathbf{p}_{\mathfrak{j}}\hat{\alpha}_{\mathfrak{j}\mathfrak{r}} + \mathbf{d}_{\mathfrak{r}} + \mathbf{p}_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{j}\mathfrak{r}}^R|G_{\mathfrak{j}}^R + \mathbf{p}_{\mathfrak{j}} \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{j}\mathfrak{r}}^R|G_{\mathfrak{r}}^R + \mathbf{p}_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{j}\mathfrak{r}}^I|G_{\mathfrak{j}}^I \\
 & + \mathbf{p}_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{b}}_{\mathfrak{j}\mathfrak{r}}^R|H_{\mathfrak{j}}^R + \mathbf{p}_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{b}}_{\mathfrak{j}\mathfrak{r}}^I|H_{\mathfrak{j}}^I + \mathbf{p}_{\mathfrak{r}} + \mathbf{q}_{\mathfrak{j}} \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{j}\mathfrak{r}}^I|G_{\mathfrak{r}}^R, \quad (50)
 \end{aligned}$$

$$(1 - \mu)\mathbf{d}_{\mathfrak{r}} \geq e^{\lambda\tau} \sum_{\mathfrak{j}=1}^n \mathbf{p}_{\mathfrak{j}}\hat{\beta}_{\mathfrak{j}\mathfrak{r}} + e^{\lambda\tau} \sum_{\mathfrak{j}=1}^n \mathbf{p}_{\mathfrak{j}}|\tilde{\mathbf{b}}_{\mathfrak{j}\mathfrak{r}}^R|H_{\mathfrak{r}}^R + e^{\lambda\tau} \sum_{\mathfrak{j}=1}^n \mathbf{q}_{\mathfrak{j}}|\tilde{\mathbf{b}}_{\mathfrak{j}\mathfrak{r}}^I|H_{\mathfrak{r}}^R, \quad (51)$$

$$\begin{aligned}
 2\mathbf{q}_{\mathfrak{r}}c_{\mathfrak{r}} \geq & \lambda \mathbf{q}_{\mathfrak{r}} + \sum_{\mathfrak{j}=1}^n \mathbf{q}_{\mathfrak{j}}\check{\alpha}_{\mathfrak{j}\mathfrak{r}} + \mathbf{h}_{\mathfrak{r}} + \mathbf{q}_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{j}\mathfrak{r}}^R|G_{\mathfrak{j}}^I + \mathbf{q}_{\mathfrak{j}} \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{j}\mathfrak{r}}^R|G_{\mathfrak{r}}^I + \mathbf{p}_{\mathfrak{j}} \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{j}\mathfrak{r}}^I|G_{\mathfrak{r}}^I \\
 & + \mathbf{q}_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{j}\mathfrak{r}}^I|G_{\mathfrak{j}}^R + \mathbf{q}_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{b}}_{\mathfrak{j}\mathfrak{r}}^R|H_{\mathfrak{j}}^I + \mathbf{q}_{\mathfrak{r}} \sum_{\mathfrak{j}=1}^n |\tilde{\mathbf{b}}_{\mathfrak{j}\mathfrak{r}}^I|H_{\mathfrak{j}}^R + \mathbf{q}_{\mathfrak{r}}, \quad (52)
 \end{aligned}$$

$$(1 - \mu)\mathbf{h}_{\mathfrak{r}} \geq e^{\lambda\tau} \sum_{\mathfrak{j}=1}^n \mathbf{q}_{\mathfrak{j}}\check{\beta}_{\mathfrak{j}\mathfrak{r}} + e^{\lambda\tau} \sum_{\mathfrak{j}=1}^n \mathbf{p}_{\mathfrak{j}}|\tilde{\mathbf{b}}_{\mathfrak{j}\mathfrak{r}}^I|H_{\mathfrak{r}}^I + e^{\lambda\tau} \sum_{\mathfrak{j}=1}^n \mathbf{q}_{\mathfrak{j}}|\tilde{\mathbf{b}}_{\mathfrak{j}\mathfrak{r}}^R|H_{\mathfrak{r}}^I. \quad (53)$$

Remark 4. When the external inputs $\mathbf{u}^R(t), \mathbf{u}^I(t), \mathbf{u}^I(t)$ and $\mathbf{u}^K(t)$ in Equation (10) are bounded, then $\bar{\mathbf{U}} = [(\mathbf{u}^R(t))^T, (\mathbf{u}^I(t))^T, (\mathbf{u}^I(t))^T, (\mathbf{u}^K(t))^T]^T$ is bounded in Equation (12). From Equation (40), we can acquire that $\mathbb{E}|\bar{\mathbf{m}}(t)|^2$ is bounded. Therefore, the model in Equation (2) is stable with respect to a bounded input bounded output (BIBO) condition subject to validity of the conditions of Equations (14)–(21).

Remark 5. In [39,40], the authors have studied the exp-ISS stability criteria for a class of stochastic RVNNs. Recently, in [13,41], the authors have extended the results proposed in [39,40], to the complex domain. The exp-ISS stability criteria for stochastic CVNNs has also been analyzed. Compared to the previous results, in this paper, we have generalized the results proposed in [13,39–41] to the quaternion domain. Furthermore, the results obtained in previous studies [13,39–41] can be seen as special cases of this paper.

4. Illustrative Examples

A simulation example is presented to ascertain the efficiency of the results.

Example 1. A 2D SMQVNN model with a time-varying delay is considered as follows:

$$d\mathbf{m}(t) = [-\mathbf{C}\mathbf{m}(t) + \mathbf{A}(\mathbf{m}(t))\mathbf{f}(\mathbf{m}(t)) + \mathbf{B}(\mathbf{m}(t))\mathbf{g}(\mathbf{m}(t - \tau(t))) + \mathbf{U}]dt + \sigma(t, \mathbf{m}(t), \mathbf{m}(t - \tau(t)))dw(t), \quad (54)$$

where

$$\begin{aligned} \mathbf{a}_{11}(\mathbf{m}_1(t)) &= \begin{cases} 0.5 - 0.5i + 0.3j - 0.4k, & |\mathbf{m}_1(t)| < 0.5 \\ 0.4 - 0.5i + 0.3j - 0.4k, & |\mathbf{m}_1(t)| > 0.5, \end{cases} \\ \mathbf{a}_{12}(\mathbf{m}_2(t)) &= \begin{cases} 0.5 + 0.4i - 0.4j + 0.3k, & |\mathbf{m}_2(t)| < 0.5 \\ 0.4 + 0.3i - 0.4j + 0.3k, & |\mathbf{m}_2(t)| > 0.5, \end{cases} \\ \mathbf{a}_{21}(\mathbf{m}_1(t)) &= \begin{cases} 0.4 + 0.5i - 0.4j - 0.5k, & |\mathbf{m}_1(t)| < 0.5 \\ 0.2 + 0.4i - 0.4j - 0.4k, & |\mathbf{m}_1(t)| > 0.5, \end{cases} \\ \mathbf{a}_{22}(\mathbf{m}_2(t)) &= \begin{cases} 0.5 - 0.5i + 0.4j + 0.4k, & |\mathbf{m}_2(t)| < 0.5 \\ 0.3 - 0.3i + 0.2j + 0.2k, & |\mathbf{m}_2(t)| > 0.5, \end{cases} \\ \mathbf{b}_{11}(\mathbf{m}_1(t)) &= \begin{cases} 0.5 - 0.4i + 0.4j + 0.3k, & |\mathbf{m}_1(t)| < 0.5 \\ 0.4 - 0.3i + 0.3j + 0.3k, & |\mathbf{m}_1(t)| > 0.5, \end{cases} \\ \mathbf{b}_{12}(\mathbf{m}_2(t)) &= \begin{cases} 0.4 + 0.4i - 0.4j - 0.5k, & |\mathbf{m}_2(t)| < 0.5 \\ 0.3 + 0.4i - 0.3j - 0.5k, & |\mathbf{m}_2(t)| > 0.5, \end{cases} \\ \mathbf{b}_{21}(\mathbf{m}_1(t)) &= \begin{cases} 0.4 + 0.3i - 0.4j - 0.5k, & |\mathbf{m}_1(t)| < 0.5 \\ -0.3 + 0.3i - 0.3j - 0.4k, & |\mathbf{m}_1(t)| > 0.5, \end{cases} \\ \mathbf{b}_{22}(\mathbf{m}_2(t)) &= \begin{cases} 0.5 - 0.5i + 0.4j + 0.5k, & |\mathbf{m}_2(t)| < 0.5 \\ 0.4 - 0.4i + 0.3j + 0.4k, & |\mathbf{m}_2(t)| > 0.5, \end{cases} \end{aligned}$$

and $\mathbf{c}_1 = 6, \mathbf{c}_2 = 6$. The the activation functions and external input are as follows $\mathbf{f}(\mathbf{m}(\cdot)) = \mathbf{g}(\mathbf{m}(\cdot)) = \tanh \mathbf{m}^R(\cdot) + i \tanh \mathbf{m}^I(\cdot) + j \tanh \mathbf{m}^J(\cdot) + k \tanh \mathbf{m}^K(\cdot)$ and $\mathbf{U} = [0.1 \sin(t) + i \sin(0.1t) + j \sin(0.1t) + k \sin(0.1t), 0.1 \cos(t) + i \cos(0.1t) + j \cos(0.1t) + k \cos(0.1t)]^T$.

According to the above analysis, we have $\sigma(t, \mathbf{m}(t), \mathbf{m}(t - \tau(t))) = \sigma^R(t, \mathbf{m}^R(t), \mathbf{m}^R(t - \tau(t))) + i\sigma^I(t, \mathbf{m}^I(t), \mathbf{m}^I(t - \tau(t))) + j\sigma^J(t, \mathbf{m}^J(t), \mathbf{m}^J(t - \tau(t))) + k\sigma^K(t, \mathbf{m}^K(t), \mathbf{m}^K(t - \tau(t)))$, where

$$\begin{aligned} \sigma^R(t, \mathbf{m}^R(t), \mathbf{m}^R(t - \tau(t))) &= \begin{bmatrix} 0.5\mathbf{m}_1^R(t) & 0.7\mathbf{m}_1^R(t - \tau(t)) \\ 0.4\mathbf{m}_2^R(t) + 0.5\mathbf{m}_2^R(t - \tau(t)) & 0.3\mathbf{m}_2^R(t - \tau(t)) \end{bmatrix}, \\ \sigma^I(t, \mathbf{m}^I(t), \mathbf{m}^I(t - \tau(t))) &= \begin{bmatrix} 0.4\mathbf{m}_1^I(t) & 0.3\mathbf{m}_1^I(t - \tau(t)) \\ 0.2\mathbf{m}_2^I(t) + 0.3\mathbf{m}_2^I(t - \tau(t)) & 0.1\mathbf{m}_2^I(t - \tau(t)) \end{bmatrix}, \\ \sigma^J(t, \mathbf{m}^J(t), \mathbf{m}^J(t - \tau(t))) &= \begin{bmatrix} 0.5\mathbf{m}_1^J(t) & 0.3\mathbf{m}_1^J(t - \tau(t)) \\ 0.4\mathbf{m}_2^J(t) + 0.1\mathbf{m}_2^J(t - \tau(t)) & 0.5\mathbf{m}_2^J(t - \tau(t)) \end{bmatrix}, \\ \sigma^K(t, \mathbf{m}^K(t), \mathbf{m}^K(t - \tau(t))) &= \begin{bmatrix} 0.2\mathbf{m}_1^K(t) & 0.4\mathbf{m}_1^K(t - \tau(t)) \\ 0.4\mathbf{m}_2^K(t) + 0.4\mathbf{m}_2^K(t - \tau(t)) & 0.2\mathbf{m}_2^K(t - \tau(t)) \end{bmatrix}. \end{aligned}$$

Take, $\tau = 0.3$, $\tau(t) = 0.2 \cos(t) + 0.1$, $\lambda = 0.1$, $\mathbf{p}_1 = 0.4$, $\mathbf{p}_2 = 0.5$, $\mathbf{q}_1 = 0.3$, $\mathbf{q}_2 = 0.4$, $\mathbf{r}_1 = 0.6$, $\mathbf{r}_2 = 0.5$, $\mathbf{s}_1 = 0.4$, $\mathbf{s}_2 = 0.5$, $\mathbf{d}_1 = 1.5$, $\mathbf{d}_2 = 1.6$, $\mathbf{h}_1 = 1.5$, $\mathbf{h}_2 = 1.3$, $\mathbf{u}_1 = 1.4$, $\mathbf{u}_2 = 1.6$, $\mathbf{v}_1 = 1.2$, $\mathbf{v}_2 = 1.4$, $\mu = 0.1$, $\mathbf{G}_1^s = \mathbf{G}_2^s = 0.2$, $\mathbf{H}_1^s = \mathbf{H}_2^s = 0.2$, ($s = R, I, J, K$). By simple computation, the following conditions as given in Equations (55)–(62) are true for all $\mathfrak{r}, \mathfrak{s} = 1, 2$.

$$\begin{aligned} 2\mathbf{p}_\mathfrak{r}\mathbf{c}_\mathfrak{r} &> \lambda\mathbf{p}_\mathfrak{r} + \sum_{\mathfrak{s}=1}^n \mathbf{p}_\mathfrak{s}\hat{\alpha}_{\mathfrak{s}\mathfrak{r}} + \mathbf{d}_\mathfrak{r} + \mathbf{p}_\mathfrak{r} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{r}\mathfrak{s}}^R| \mathbf{G}_\mathfrak{s}^R + \mathbf{p}_\mathfrak{s} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{s}\mathfrak{r}}^R| \mathbf{G}_\mathfrak{r}^R + \mathbf{p}_\mathfrak{r} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{r}\mathfrak{s}}^I| \mathbf{G}_\mathfrak{s}^I \\ &+ \mathbf{p}_\mathfrak{r} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{r}\mathfrak{s}}^J| \mathbf{G}_\mathfrak{s}^J + \mathbf{p}_\mathfrak{r} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{r}\mathfrak{s}}^K| \mathbf{G}_\mathfrak{s}^K + \mathbf{p}_\mathfrak{r} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{b}}_{\mathfrak{r}\mathfrak{s}}^R| \mathbf{H}_\mathfrak{s}^R + \mathbf{p}_\mathfrak{r} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{b}}_{\mathfrak{r}\mathfrak{s}}^I| \mathbf{H}_\mathfrak{s}^I \\ &+ \mathbf{p}_\mathfrak{r} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{b}}_{\mathfrak{r}\mathfrak{s}}^J| \mathbf{H}_\mathfrak{s}^J + \mathbf{p}_\mathfrak{r} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{b}}_{\mathfrak{r}\mathfrak{s}}^K| \mathbf{H}_\mathfrak{s}^K + \mathbf{p}_\mathfrak{r} + \mathbf{q}_\mathfrak{s} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{s}\mathfrak{r}}^I| \mathbf{G}_\mathfrak{r}^I + \mathbf{r}_\mathfrak{s} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{s}\mathfrak{r}}^J| \mathbf{G}_\mathfrak{r}^J + \mathbf{s}_\mathfrak{s} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{s}\mathfrak{r}}^K| \mathbf{G}_\mathfrak{r}^K, \end{aligned} \tag{55}$$

$$(1 - \mu)\mathbf{d}_\mathfrak{r} > e^{\lambda\tau} \sum_{\mathfrak{s}=1}^n \mathbf{p}_\mathfrak{s}\hat{\beta}_{\mathfrak{s}\mathfrak{r}} + e^{\lambda\tau} \sum_{\mathfrak{s}=1}^n \mathbf{p}_\mathfrak{s}|\tilde{\mathbf{b}}_{\mathfrak{s}\mathfrak{r}}^R| \mathbf{H}_\mathfrak{r}^R + e^{\lambda\tau} \sum_{\mathfrak{s}=1}^n \mathbf{q}_\mathfrak{s}|\tilde{\mathbf{b}}_{\mathfrak{s}\mathfrak{r}}^I| \mathbf{H}_\mathfrak{r}^I + e^{\lambda\tau} \sum_{\mathfrak{s}=1}^n \mathbf{r}_\mathfrak{s}|\tilde{\mathbf{b}}_{\mathfrak{s}\mathfrak{r}}^J| \mathbf{H}_\mathfrak{r}^J + e^{\lambda\tau} \sum_{\mathfrak{s}=1}^n \mathbf{s}_\mathfrak{s}|\tilde{\mathbf{b}}_{\mathfrak{s}\mathfrak{r}}^K| \mathbf{H}_\mathfrak{r}^K, \tag{56}$$

$$\begin{aligned} 2\mathbf{q}_\mathfrak{r}\mathbf{c}_\mathfrak{r} &> \lambda\mathbf{q}_\mathfrak{r} + \sum_{\mathfrak{s}=1}^n \mathbf{q}_\mathfrak{s}\check{\alpha}_{\mathfrak{s}\mathfrak{r}} + \mathbf{h}_\mathfrak{r} + \mathbf{q}_\mathfrak{r} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{r}\mathfrak{s}}^R| \mathbf{G}_\mathfrak{s}^I + \mathbf{q}_\mathfrak{s} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{s}\mathfrak{r}}^R| \mathbf{G}_\mathfrak{r}^I + \mathbf{p}_\mathfrak{s} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{s}\mathfrak{r}}^I| \mathbf{G}_\mathfrak{r}^I \\ &+ \mathbf{q}_\mathfrak{r} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{r}\mathfrak{s}}^I| \mathbf{G}_\mathfrak{s}^R + \mathbf{q}_\mathfrak{r} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{r}\mathfrak{s}}^J| \mathbf{G}_\mathfrak{s}^K + \mathbf{q}_\mathfrak{r} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{r}\mathfrak{s}}^K| \mathbf{G}_\mathfrak{s}^I + \mathbf{q}_\mathfrak{r} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{b}}_{\mathfrak{r}\mathfrak{s}}^R| \mathbf{H}_\mathfrak{s}^I \\ &+ \mathbf{q}_\mathfrak{r} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{b}}_{\mathfrak{r}\mathfrak{s}}^I| \mathbf{H}_\mathfrak{s}^R + \mathbf{q}_\mathfrak{r} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{b}}_{\mathfrak{r}\mathfrak{s}}^J| \mathbf{H}_\mathfrak{s}^K + \mathbf{q}_\mathfrak{r} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{b}}_{\mathfrak{r}\mathfrak{s}}^K| \mathbf{H}_\mathfrak{s}^I + \mathbf{q}_\mathfrak{r} + \mathbf{r}_\mathfrak{s} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{s}\mathfrak{r}}^K| \mathbf{G}_\mathfrak{r}^I + \mathbf{s}_\mathfrak{s} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{s}\mathfrak{r}}^I| \mathbf{G}_\mathfrak{r}^I, \end{aligned} \tag{57}$$

$$(1 - \mu)\mathbf{h}_\mathfrak{r} > e^{\lambda\tau} \sum_{\mathfrak{s}=1}^n \mathbf{q}_\mathfrak{s}\check{\beta}_{\mathfrak{s}\mathfrak{r}} + e^{\lambda\tau} \sum_{\mathfrak{s}=1}^n \mathbf{p}_\mathfrak{s}|\tilde{\mathbf{b}}_{\mathfrak{s}\mathfrak{r}}^I| \mathbf{H}_\mathfrak{r}^I + e^{\lambda\tau} \sum_{\mathfrak{s}=1}^n \mathbf{q}_\mathfrak{s}|\tilde{\mathbf{b}}_{\mathfrak{s}\mathfrak{r}}^R| \mathbf{H}_\mathfrak{r}^I + e^{\lambda\tau} \sum_{\mathfrak{s}=1}^n \mathbf{r}_\mathfrak{s}|\tilde{\mathbf{b}}_{\mathfrak{s}\mathfrak{r}}^K| \mathbf{H}_\mathfrak{r}^I + e^{\lambda\tau} \sum_{\mathfrak{s}=1}^n \mathbf{s}_\mathfrak{s}|\tilde{\mathbf{b}}_{\mathfrak{s}\mathfrak{r}}^J| \mathbf{H}_\mathfrak{r}^I, \tag{58}$$

$$\begin{aligned} 2\mathbf{r}_\mathfrak{r}\mathbf{c}_\mathfrak{r} &> \lambda\mathbf{r}_\mathfrak{r} + \sum_{\mathfrak{s}=1}^n \mathbf{r}_\mathfrak{s}\bar{\alpha}_{\mathfrak{s}\mathfrak{r}} + \mathbf{u}_\mathfrak{r} + \mathbf{r}_\mathfrak{r} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{r}\mathfrak{s}}^R| \mathbf{G}_\mathfrak{s}^J + \mathbf{r}_\mathfrak{s} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{s}\mathfrak{r}}^R| \mathbf{G}_\mathfrak{r}^J + \mathbf{p}_\mathfrak{s} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{s}\mathfrak{r}}^I| \mathbf{G}_\mathfrak{r}^J \\ &+ \mathbf{q}_\mathfrak{s} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{s}\mathfrak{r}}^K| \mathbf{G}_\mathfrak{r}^I + \mathbf{r}_\mathfrak{r} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{r}\mathfrak{s}}^I| \mathbf{G}_\mathfrak{s}^R + \mathbf{r}_\mathfrak{r} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{r}\mathfrak{s}}^K| \mathbf{G}_\mathfrak{s}^I + \mathbf{r}_\mathfrak{r} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{r}\mathfrak{s}}^I| \mathbf{G}_\mathfrak{s}^K \\ &+ \mathbf{r}_\mathfrak{r} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{b}}_{\mathfrak{r}\mathfrak{s}}^R| \mathbf{H}_\mathfrak{s}^J + \mathbf{r}_\mathfrak{r} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{b}}_{\mathfrak{r}\mathfrak{s}}^I| \mathbf{H}_\mathfrak{s}^R + \mathbf{r}_\mathfrak{r} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{b}}_{\mathfrak{r}\mathfrak{s}}^K| \mathbf{H}_\mathfrak{s}^I + \mathbf{r}_\mathfrak{r} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{b}}_{\mathfrak{r}\mathfrak{s}}^I| \mathbf{H}_\mathfrak{s}^K + \mathbf{r}_\mathfrak{r} + \mathbf{s}_\mathfrak{s} \sum_{\mathfrak{s}=1}^n |\tilde{\mathbf{a}}_{\mathfrak{s}\mathfrak{r}}^I| \mathbf{G}_\mathfrak{r}^J, \end{aligned} \tag{59}$$

$$(1 - \mu)\mathbf{u}_\mathfrak{r} > e^{\lambda\tau} \sum_{\mathfrak{s}=1}^n \mathbf{r}_\mathfrak{s}\bar{\beta}_{\mathfrak{s}\mathfrak{r}} + e^{\lambda\tau} \sum_{\mathfrak{s}=1}^n \mathbf{p}_\mathfrak{s}|\tilde{\mathbf{b}}_{\mathfrak{s}\mathfrak{r}}^J| \mathbf{H}_\mathfrak{r}^J + e^{\lambda\tau} \sum_{\mathfrak{s}=1}^n \mathbf{q}_\mathfrak{s}|\tilde{\mathbf{b}}_{\mathfrak{s}\mathfrak{r}}^K| \mathbf{H}_\mathfrak{r}^J + e^{\lambda\tau} \sum_{\mathfrak{s}=1}^n \mathbf{r}_\mathfrak{s}|\tilde{\mathbf{b}}_{\mathfrak{s}\mathfrak{r}}^R| \mathbf{H}_\mathfrak{r}^J + e^{\lambda\tau} \sum_{\mathfrak{s}=1}^n \mathbf{s}_\mathfrak{s}|\tilde{\mathbf{b}}_{\mathfrak{s}\mathfrak{r}}^I| \mathbf{H}_\mathfrak{r}^J, \tag{60}$$

$$\begin{aligned}
 2\mathbf{s}_r \mathbf{c}_r &> \lambda \mathbf{s}_r + \sum_{j=1}^n \mathbf{s}_j \tilde{\mathbf{a}}_{jr} + \mathbf{v}_r + \mathbf{s}_r \sum_{j=1}^n |\tilde{\mathbf{a}}_{jr}^R| \mathbf{G}_j^K + \mathbf{s}_j \sum_{j=1}^n |\tilde{\mathbf{a}}_{jr}^R| \mathbf{G}_r^K + \mathbf{p}_j \sum_{j=1}^n |\tilde{\mathbf{a}}_{jr}^K| \mathbf{G}_r^K \\
 &+ \mathbf{q}_j \sum_{j=1}^n |\tilde{\mathbf{a}}_{jr}^J| \mathbf{G}_r^K + \mathbf{r}_j \sum_{j=1}^n |\tilde{\mathbf{a}}_{jr}^I| \mathbf{G}_r^K + \mathbf{s}_r \sum_{j=1}^n |\tilde{\mathbf{a}}_{jr}^I| \mathbf{G}_j^I + \mathbf{s}_r \sum_{j=1}^n |\tilde{\mathbf{a}}_{jr}^K| \mathbf{G}_j^R \\
 &+ \mathbf{s}_r \sum_{j=1}^n |\tilde{\mathbf{a}}_{jr}^J| \mathbf{G}_j^I + \mathbf{s}_r \sum_{j=1}^n |\tilde{\mathbf{b}}_{jr}^R| \mathbf{H}_j^K + \mathbf{s}_r \sum_{j=1}^n |\tilde{\mathbf{b}}_{jr}^I| \mathbf{H}_j^J + \mathbf{s}_r \sum_{j=1}^n |\tilde{\mathbf{b}}_{jr}^K| \mathbf{H}_j^R + \mathbf{s}_r \sum_{j=1}^n |\tilde{\mathbf{b}}_{jr}^J| \mathbf{H}_j^I + \mathbf{s}_r, \quad (61)
 \end{aligned}$$

$$(1 - \mu) \mathbf{v}_r > e^{\lambda \tau} \sum_{j=1}^n \mathbf{s}_j \tilde{\mathbf{b}}_{jr} + e^{\lambda \tau} \sum_{j=1}^n \mathbf{p}_j |\tilde{\mathbf{b}}_{jr}^K| \mathbf{H}_r^K + e^{\lambda \tau} \sum_{j=1}^n \mathbf{q}_j |\tilde{\mathbf{b}}_{jr}^J| \mathbf{H}_r^K + e^{\lambda \tau} \sum_{j=1}^n \mathbf{r}_j |\tilde{\mathbf{b}}_{jr}^I| \mathbf{H}_r^K + e^{\lambda \tau} \sum_{j=1}^n \mathbf{s}_j |\tilde{\mathbf{b}}_{jr}^R| \mathbf{H}_r^K. \quad (62)$$

From Theorem (1), it is clear that the model in Equation (54) is mean-square exp-ISS. Referring to the model in Equation (54) in the presence of inputs, its state responses with respect to the real and imaginary parts are plotted in Figures 1–4. On the other hand, the model in Equation (54) in the absence of the inputs has the state responses with respect to the real and imaginary parts shown in Figures 5–8. Therefore, it is clear that the model in Equation (54) is mean-square exponentially stable.

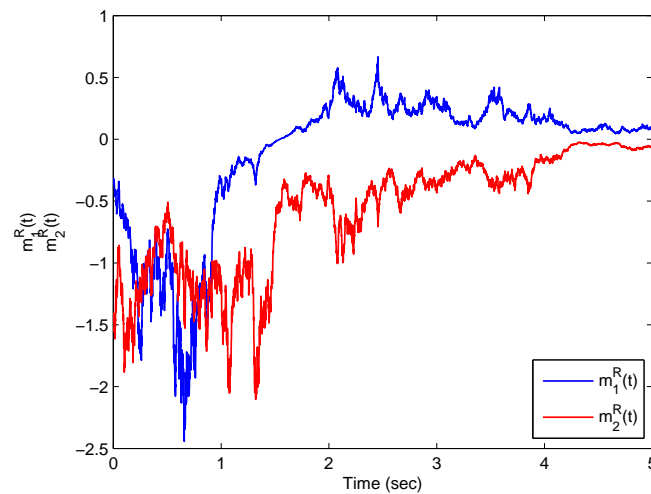


Figure 1. A time response plot with respect to the states $\mathbf{m}_1^R(t)$, $\mathbf{m}_2^R(t)$ pertaining to the SMQVNN model in Equation (54) in which $\mathbf{U} \neq 0$.

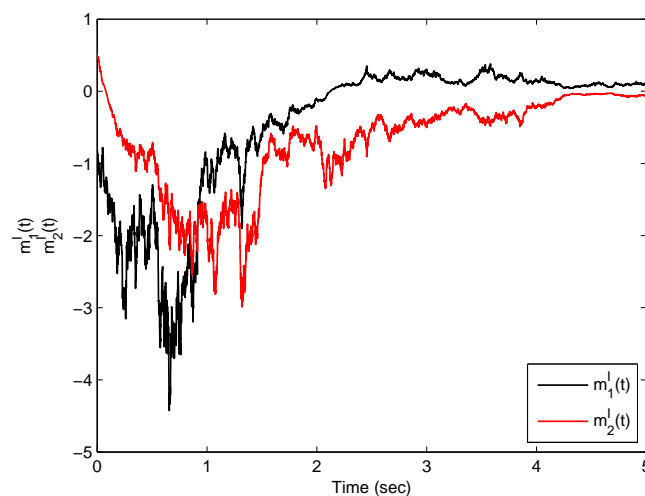


Figure 2. A time response plot with respect to the states $\mathbf{m}_1^I(t)$, $\mathbf{m}_2^I(t)$ pertaining to the SMQVNN model in Equation (54) in which $\mathbf{U} \neq 0$.

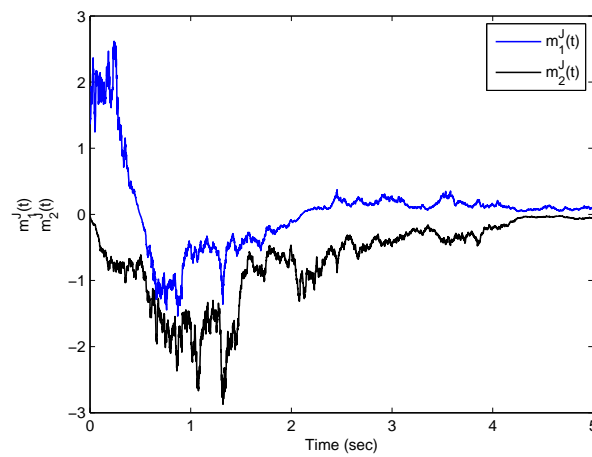


Figure 3. A time response plot with respect to the states $m_1^J(t)$, $m_2^J(t)$ pertaining to the SMQVNN model in Equation (54) in which $U \neq 0$.

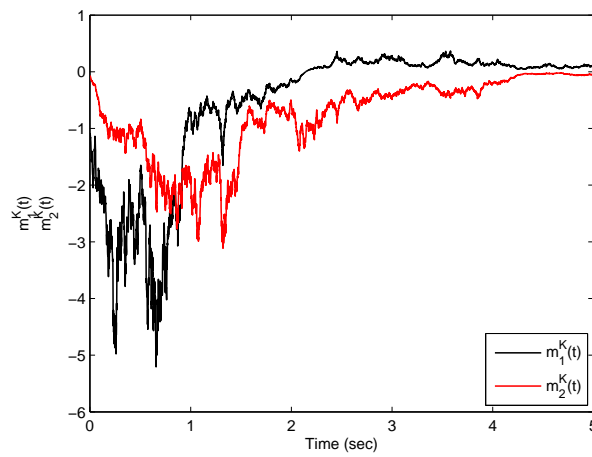


Figure 4. A time response plot with respect to the states $m_1^K(t)$, $m_2^K(t)$ pertaining to the SMQVNN model in Equation (54) in which $U \neq 0$.

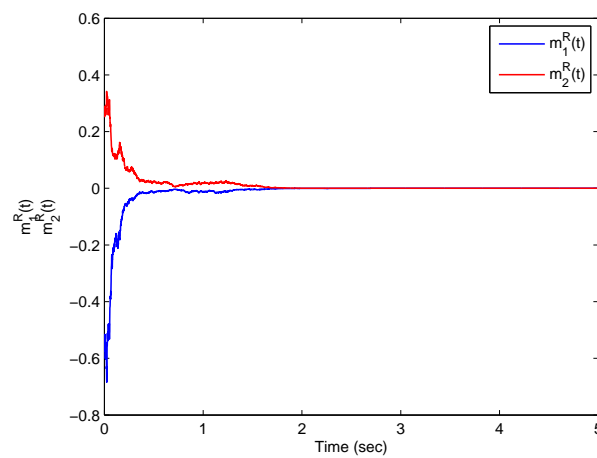


Figure 5. A time response plot with respect to the states $m_1^R(t)$, $m_2^R(t)$ pertaining to the SMQVNN model in Equation (54) in which $U = 0$.

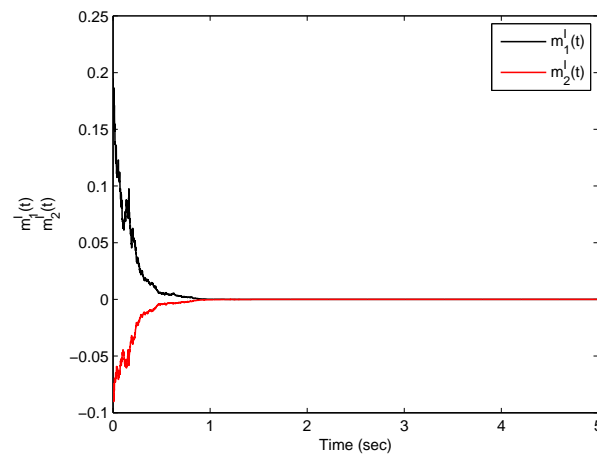


Figure 6. A time response plot with respect to the states $m_1^I(t)$, $m_2^I(t)$ pertaining to the SMQVNN model in Equation (54) in which $U = 0$.

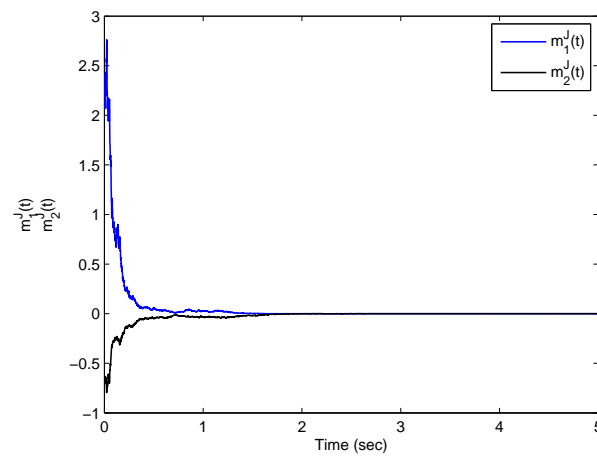


Figure 7. A time response plot with respect to the states $m_1^J(t)$, $m_2^J(t)$ pertaining to the SMQVNN model in Equation (54) in which $U = 0$.

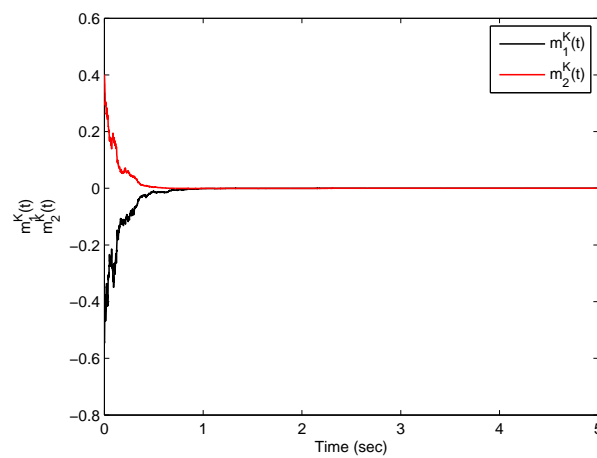


Figure 8. A time response plot with respect to the states $m_1^K(t)$, $m_2^K(t)$ pertaining to the SMQVNN model in Equation (54) in which $U = 0$.

5. Summary

In this paper, we have studied the mean-square exponential input-to-state stability problem for a new class of SMQVNNs with time-varying delays. Firstly, we decomposed the original SMQVNNs into four real-valued models, in order to avoid the difficulties posed by non-commutative quaternion multiplication. Secondly, by constructing suitable Lyapunov functional and applying Itô's formula, Dynkin's formula as well as inequality techniques, several new sufficient conditions have been obtained to guarantee that the considered system model is mean-square exp-ISS. The results obtained in this paper are the general case of previously known results in complex and real fields. Finally, a numerical example shows the effectiveness of the obtained theoretical results.

Based on the proposed method in this paper, it is possible to analyze different QVNN models. The proposed approach can be applied to the analysis of stochastic QVNN models for the discrete-time case. In future studies, the stability and synchronization analysis for discrete-time stochastic QVNN models will also be investigated. The results will be useful for the dynamical analysis of discrete-time stochastic QVNN models.

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