Planar Receiver Placement for Unique Emitter Localization for Indoor Applications

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Abstract

This paper investigates two independent approaches to verify the necessary and sufficient condition to guarantee a unique solution for a passive emitter localization system based on time difference of arrival measurements from an array of sensors.

1. Introduction

Locating a signal source is the most important problem in many applications such as navigation, surveillance, and geophysics. Therefore developing an accurate and efficient approach to estimate the location of a signal sources has generated significant attention in the recent past. Among different approaches, one very useful method of localization is based on measuring the difference in signal arrival times to points whose locations are known. In such systems, an array of sensors located in known positions are used to receive the signal transmitted from a source whose position is desired to be known. The time difference of arrival (TDOA) of the received signal is measured and converted to the corresponding range difference (RD) by multiplying it by the velocity of propagation. However, the measurements are noisy and the sensor locations are often not precisely known. Numerous approaches including least square have been employed to improve system performance when subjected to such uncertainties. Traditionally, geometric solutions have been based on intersection of hyperbolic lines of position (LOP) where each measured TDOA provides one hyperbolic LOP. Many optimum processing techniques are proposed in locating a source based on intersection of hyperbolic curves defined by TDOAs such as beamformer, spherical-interpolation, divide and conquer, and iterative Taylor-series methods [6]. In [5] Smidth has proposed a formulation in which the source location is found as a focus of a conic specified by the sensor locations and RD measurements. Smidth's method is extended in [8] to a closed-form localization technique, termed the Spherical-Interpolation (SI) method. The SI method is similar in formulation to a method of Schau and Robinson [2] called Sperical-Intersection (SX) method. In [6] an alternative closed-form solution for hyperbolic fix is presented, which is noniterative and gives an explicit solution. In addition, some studies concentrate on the configuration of the sensors.

This paper investigates a derivation of closed-form solution and defines a necessary and sufficient condition to have a unique solution for the problem of localization. Although similar conditions were presented in [3], no sufficient and necessary condition was claimed using minimum number of sensors. In current research, two different approaches has been presented and the results are compared and discussed. In addition, a new method to find the optimum configuration of sensors according to unique solution area is presented. The development is considered for 2D plane to ensure simplicity. 3D space follows along similar lines.

2. Notations and Definitions

There are many ways and approaches in order to estimate the location of an emitter. One approach is to position sensors in different positions. Obviously, each sensor receives the transmitted signal at different times. In general N+1 sensors are used, which one of them is the reference sensor and all of the distances and time measurements are with respect to its location and the time it receives the signal, respectively. In 2D and 3D space at least 3 and 4 sensors are required [3]. The time differences of arrival are used to estimate the position of the transmitter. Such a system is called

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a TDOA-based positioning system.

$$\Delta t_i = t_i - t_0 \tag{1}$$

where t_0 and t_i are the absolute time of arrival to the reference and the i^{th} sensor respectively. The differences in the time of arrival (Δt_i) to the known stations can be converted to range differences (d_i) as

$$d_i = c * \Delta t_i = c(t_i - t_0) \tag{2}$$

where c is the velocity of propagation.
Let the spatial coordinate vectors be:

$$\mathbf{x}_0 \equiv \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}, \mathbf{x}_i \equiv \begin{bmatrix} x_i \\ y_i \end{bmatrix}, \mathbf{x}_s \equiv \begin{bmatrix} x_s \\ y_s \end{bmatrix} \tag{3}$$

where $\mathbf{x}_0$ is the reference sensor position and $\mathbf{x}_i$ is the i^{th} sensor position and the unknown signal source position is $\mathbf{x}_s$. The distance between the i^{th} sensor and the source is given by:

$$R_{is} = \|\mathbf{x}_i - \mathbf{x}_s\|\,. \tag{4}$$

Let $\mathbf{x} \in \Re^2$ be the position of the sensor with respect to the reference sensor and the distance between the reference sensor and the source is

$$R_s = \|\mathbf{x}\| \tag{5}$$

Hence, the distance between the reference sensor and the ith sensor is calculated using (6).

$$d_i = R_{is} - R_s \tag{6}$$

Fig. 1 shows the localization in 2D plane.

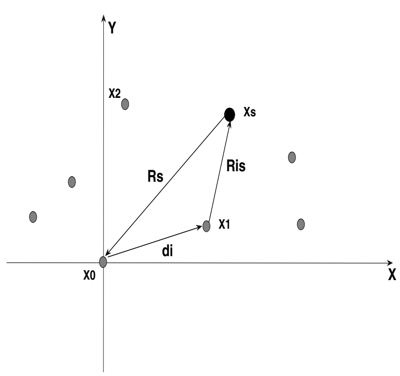

Figure 1. The localization in 2D plane

d_i can be represented in terms of x_s and y_s as shown in (7).

$$d_i = \sqrt{(x_s - x_i)^2 + (y_s - y_i)^2} - \sqrt{x_s^2 + y_s^2} \tag{7}$$

which yields

$$x_i x_s + y_i y_s + d_i \sqrt{x_s^2 + y_s^2} = \frac{1}{2}(x_i^2 + y_i^2 - d_i^2) \tag{8}$$

For a general case of N+1 sensors, following matrices are defined

$$\mathbf{S} \equiv \begin{bmatrix} x_1 & y_1 \\ . & . \\ . & . \\ . & . \\ x_N & y_N \end{bmatrix}, \mathbf{z} \equiv \tfrac{1}{2} \begin{bmatrix} x_1^2 + y_1^2 - d_1^2 \\ . \\ . \\ . \\ x_N^2 + y_N^2 - d_N^2 \end{bmatrix}$$
$$\mathbf{d} \equiv \begin{bmatrix} d_1 \\ . \\ . \\ . \\ d_N \end{bmatrix} \tag{9}$$

In other words,

$$\mathbf{Sx} = \mathbf{z} - \mathbf{d}R_s \tag{10}$$

To estimate the emitter position, we have to solve Equation (10) for source position $\mathbf{x}$

$$\begin{aligned} \mathbf{x} = (\mathbf{S}^T\mathbf{S})^{-1}\mathbf{S}^T(\mathbf{z} - \mathbf{d}R_s) = \\ (\mathbf{S}^T\mathbf{S})^{-1}\mathbf{S}^T\mathbf{z} - (\mathbf{S}^T\mathbf{S})^{-1}\mathbf{S}^T\mathbf{d}R_s \end{aligned} \tag{11}$$

Since all RDs are not measured to same accuracy, a weighting matrix $R_{N\times N}$ is in order. Therefore, (11) becomes

$$\hat{\mathbf{x}} = (\mathbf{S}^T\mathbf{R}^{-1}\mathbf{S})^{-1}\mathbf{S}^T\mathbf{R}^{-1}\mathbf{z} - (\mathbf{S}^T\mathbf{R}^{-1}\mathbf{S})^{-1}\mathbf{S}^T\mathbf{R}^{-1}\mathbf{d}R_s \tag{12}$$

To simplify (12) the following equations are defined

$$\mathbf{a} \equiv (\mathbf{S}^T\mathbf{R}^{-1}\mathbf{S})^{-1}\mathbf{S}^T\mathbf{R}^{-1}\mathbf{z} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \tag{13}$$

and

$$\mathbf{b} \equiv (\mathbf{S}^T\mathbf{R}^{-1}\mathbf{S})^{-1}\mathbf{S}^T\mathbf{R}^{-1}\mathbf{d} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{14}$$

Inserting (13) and (14) in (12) yields :

$$\hat{\mathbf{x}} = \mathbf{a} - \mathbf{b}R_s, \tag{15}$$

and the source position $\mathbf{x}$ is obtained as,

$$\mathbf{x} = \begin{bmatrix} x_s \\ y_s \end{bmatrix} = \begin{bmatrix} a_1 - b_1 R_s \\ a_2 - b_2 R_s \end{bmatrix}, \tag{16}$$

Substituting (16) in (5), the following quadratic equation results:

$$AR_s^2 + BR_s + C = 0, \tag{17}$$

where

$$\begin{aligned} A &= b_1^2 + b_2^2 - 1, \\ B &= 2(a_1 b_1 + a_1 b_1), \\ C &= a_1^2 + a_2^2. \end{aligned} \tag{18}$$

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3. The Necessary and Sufficient Condition to Have Unique Solution

In the 2D plane, if at least 3 sensors are not collinear the matrix **S**, in (9), has full rank and it is possible to solve the quadratic equation (17). However, in some cases there are 2 possible solutions for the (17). Considering the solution plane we can present the following proposition regarding a necessary and sufficient condition that guarantees the existence of a unique, real, positive solution (location of the emitter). The following will define this condition.

ASSUMPTION 1
In 2D plane, at least, 3 sensors are not collinear, respectively.

PROPOSITION 1
Assume that Assumption holds.
Then there is a unique, real, positive solution if and only if

$$A < 0 \tag{19}$$

Proof:In general, there are 2 solutions for any quadratic equation.

$$R_s = \frac{-B \pm \sqrt{B^2 - 4AC}}{A} \tag{20}$$

As $C \geq 0$, if $A < 0$ then there would be two real solutions for R_s

$$AC < 0 \Rightarrow (B^2 - 4AC) > 0 \text{ and } \sqrt{B^2 - 4AC} \text{ is real} \tag{21}$$

One solution is positive (minus sign applies) and the other one is negative (plus sign applies). Hence, the negative solution for R_s is unacceptable.

PROPOSITION 2
Assume that Assumption 1 holds with $A > 0$, then the following holds:
(I)

$$B < 0 \Leftrightarrow \text{ 2 possible solutions.} \tag{22}$$

Remark:
According to [2], the two positions are usually far enough apart that the incorrect solution can be discarded by other physical reasoning such as one solution lying outside the domain of interest.
NOTE:
If $B < 0$ and $B^2 - 4AC = 0$ then the two positive answers are equal.

(II)

$$B > 0 \Leftrightarrow \text{no solution.} \tag{23}$$

Proposition 1 claims to be necessary and sufficient condition to have a unique solution for the position of transmitter.

Although similar propositions were presented in [3], we state that $A < 0$ is the only condition for uniqueness (contrary to [3]).

For instance, let assume the there are 3 sensors at the following locations

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{24}$$

In this case A, B and C are:

$$\begin{aligned} A &= d_1^2 + d_2^2 - 1 < 0 \\ B &= d_1(1 - d_1^2) + d_2(1 - d_2^2) \\ C &= \tfrac{1}{2}(1 - d_1^2)^2 + \tfrac{1}{2}(1 - d_2^2)^2 \end{aligned} \tag{25}$$

Therefore, in 2D, the only condition to have a unique, real, positive solution is $d_1^2 + d_2^2 - 1 < 0$. The area where this condition is satisfied is illustrated by white area in Fig. 2. Hence, dark areas correspond to regions with either two possible positions or the estimation is not possible.

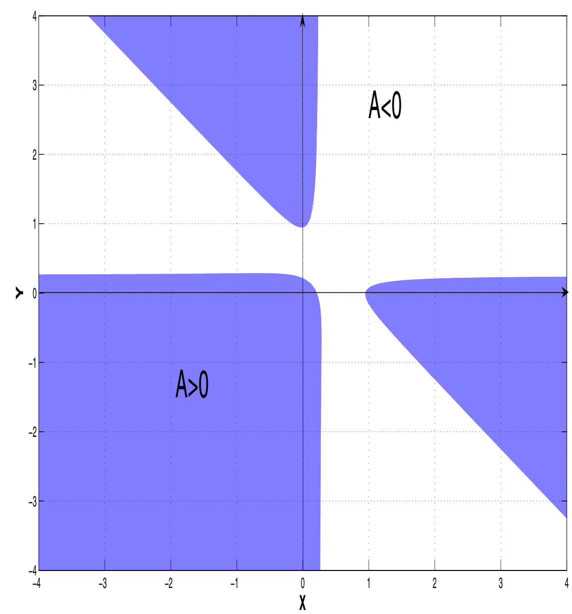

Figure 2. The unique solution area

4. Customizing The Sensor Configuration

There is a vast variety of configurations for 3 sensors in 2D. It is important to find the unique solution area for different configurations. We consider the following configuration with a variable angle for the third sensor such as:

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \tag{26}$$

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In this case the distances are

$$d_1 = \sqrt{(x_s-1)^2+y_s^2} - \sqrt{x_s^2+y_s^2}$$
$$d_2 = \sqrt{(x_s-\cos(\theta))^2+(y_s-\sin(\theta))^2} - \sqrt{x_s^2+y_s^2} \quad (27)$$

and the following relationship holds

$$\begin{bmatrix} 1 & 0 \\ \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} x_s \\ y_s \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1-d_1^2 \\ 1-d_2^2 \end{bmatrix} - R_s \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \quad (28)$$

Applying algebraic manipulations yields a new quadratic equation. Again the C term is always positive and the condition to have a unique, real, positive solution is $A < 0$.

$$d_1^2 + (-\frac{\cos(\theta)d_1}{\sin(\theta)} + \frac{d_2}{\sin(\theta)})^2 - 1 < 0 \quad (29)$$

Figs. 3 and 4 illustrate the area where this condition is satisfied for 4 configurations with different angles. The unique solution area (the white area) significantly changes as the angle alters.

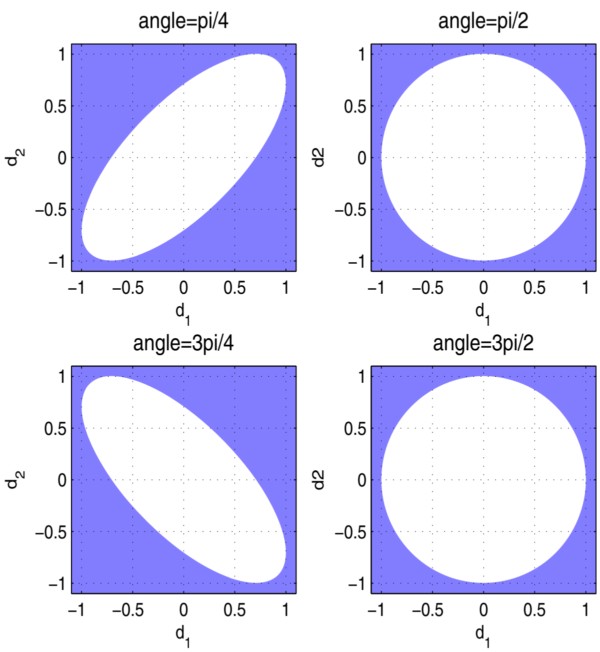

Figure 3. The unique solution area for different angles in $d_1 - d_2$ plane

One of the important applications of TDOA-based systems is emitter localization for indoor environment. Usually, the array of sensors are located at each corner of the room. Let the following points be 4 corners of the room:

$$\mathbf{c}_0 \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{c}_1 \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{c}_2 \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{c}_3 \equiv \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (30)$$

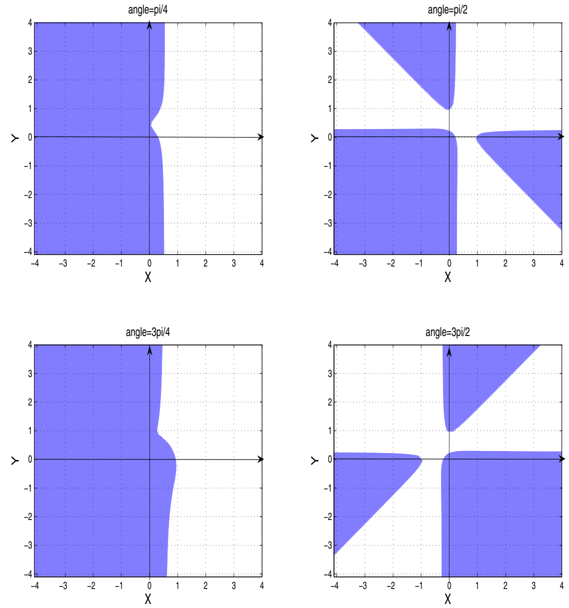

Figure 4. The unique solution area for different angles in XY plane

If the room is completely located in the white area, we guarantee that the unique solution will be met. As Fig. 3 shows, $\theta = \frac{\pi}{2}$ corresponds to the most extended white area in the room.

Hence, the configuration at $\theta = \frac{\pi}{2}$ is selected for further studies.

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (31)$$

5. A New Approach to Unique Solution

Having known the best configuration for sensors and also the necessary and sufficient condition for having a unique solution, a new approach to unique solution is presented. Finally, the results of the both approaches were compared. The distances d_1 and d_2 can be presented as a function of r_s and θ_s in polar coordinates.

$$d_1 = d_1(r_s,\theta_s) = \sqrt{r_s^2 - 2r_s\cos(\theta_s)+1} - r_s$$
$$d_2 = d_2(r_s,\theta_s) = \sqrt{r_s^2 - 2r_s\sin(\theta_s)+1} - r_s \quad (32)$$

Having more than one solution for quadratic equation means that there are two points with the same distance and time differences. Let consider two points x_{s1} and x_{s2} in the polar coordinates with the same distance and time difference with respect to the reference sensor.

$$x_{s1} = r\angle\theta$$
$$x_{s2} = R\angle\beta \quad (33)$$

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It means that the two functions d_1 and d_2 are the same for both points and the following can be stated.

$$\begin{array}{l}\sqrt{r_s^2 - 2r_s\cos(\theta_s)+1} - r_s = \sqrt{R_s^2 - 2R_s\cos(\beta_s)+1} - R_s \\ \sqrt{r_s^2 - 2r_s\sin(\theta_s)+1} - r_s = \sqrt{R_s^2 - 2R_s\sin(\beta_s)+1} - R_s\end{array} \quad (34)$$

solving these equations results in the following answers.

$$\begin{array}{c}R_s = r_s, \beta_s = \theta_s \\ R_s = F(r,\theta), \beta_s = G(r,\theta)\end{array} \quad (35)$$

where both F and G are functions of r_s and θ_s (see Appendix (40) and (41)).
The first solution is the trivial solution. This corresponds to a unique solution. On the other hand, the existence of the second pair corresponds to two possible positions. Therefore, in areas where the second pair of answers is unacceptable we can guarantee to have a unique solution. Otherwise there are two points with the same values for d_1 and d_2 functions.

Therefore, in regions where $F(x,y) < 0$ the second pair is unacceptable and only one solution exists. The area where $F(x,y) < 0$ (guaranteed unique solution) was illustrated by white areas in Fig. 5.

As Figs. 2 and 5 illustrate, the unique solution area for both conditions is the same. Although these two conditions and also the approaches to find them was completely different, figures show that they represent the same area for unique solution. In fact, it represents the accuracy and validity of both (essentially the same) conditions.

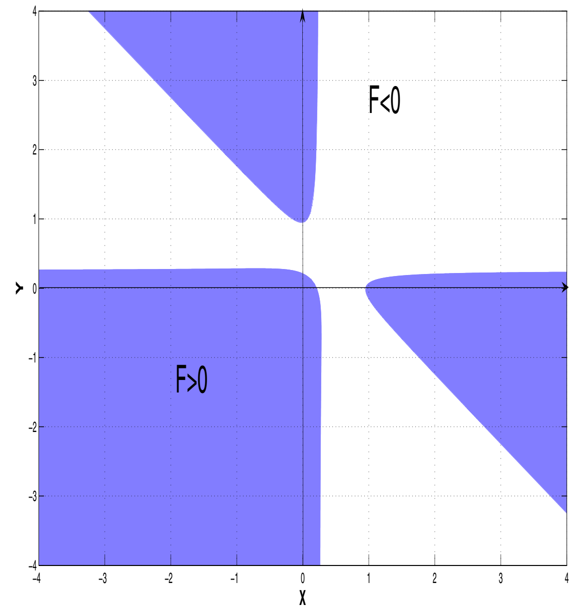

Figure 5. The unique solution area using the new approach

To Find the unique solution condition we have to solve the following equations simultaneously.

$$\begin{array}{l}\sqrt{(x_s-1)^2+y_s^2} - \sqrt{x_s^2+y_s^2} = \\ \sqrt{(X_s-1)^2+Y_s^2} - \sqrt{X_s^2+Y_s^2} \\ \\ \sqrt{x_s^2+(y_s-1)^2} - \sqrt{x_s^2+y_s^2} = \\ \sqrt{X_s^2+(Y_s-1)^2} - \sqrt{X_s^2+Y_s^2} \\ \\ \sqrt{(x_s-1)^2+(y_s-1)^2} - \sqrt{x_s^2+y_s^2} = \\ \sqrt{(X_s-1)^2+(Y_s-1)^2} - \sqrt{X_s^2+Y_s^2}\end{array} \quad (38)$$

which returns only one unique solution

$$x_s = X_s, y_s = Y_s \quad (39)$$

Similar calculations show that in the case of at least 4 sensors, in different configurations, there is no condition for having a unique solution. In other words, the location of emitter would be estimated uniquely any where in the XY plane.This configuration for sensors is illustrated in Fig. 6.

6. Extending The Unique Solution Area with an Additional Sensor

One approach to extend the unique solution area is to increase the number of sensors to 4 sensrs. Due to importance of such systems in indoor applications, assume that there are 4 sensors at each corner of the room.
Therefore, Let

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (36)$$
$$\mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Therefore the RD's are

$$d_1 = d_1(x_s,y_s) = \sqrt{(x_s-1)^2+y_s^2} - \sqrt{x_s^2+y_s^2}$$
$$d_2 = d_2(x_s,y_s) = \sqrt{x_s^2+(y_s-1)^2} - \sqrt{x_s^2+y_s^2}$$
$$d_3 = d_3(x_s,y_s) = \sqrt{(x_s-1)^2+(y_s-1)^2} - \sqrt{x_s^2+y_s^2} \quad (37)$$

7. Conclusion

We investigated the necessary and sufficient condition that guarantees the existence of a unique, real, positive closed-form solution for the location of a transmitter in a 2D plane. A comparison of regions defined by the conditions derived independently proved their accuracy and con-

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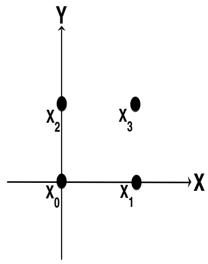

Figure 6. Configuration of 4 sensors

cordance in a graphical context. In addition, the best configuration with minimum number of sensors for emitter localization in indoor environment was presented. Our future work will be concentrated on providing a mathematical justification for the equivalence of the two conditions as well as optimal sensor placement for 3D localization in an indoor setting.

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A. Appendix

$$
\begin{aligned}
&F(r,\theta) = -r(4r^2 - 4r\cos\theta - 2r\sqrt{r^2 - 2r\sin\theta + 1} \\
&+2\sin\theta\sqrt{r^2 - 2r\sin\theta + 1} - 2r\sqrt{r^2 - 2r\cos\theta + 1} \\
&-4r\sin\theta + 3 + 2\cos\theta\sqrt{r^2 - 2rcos(t) + 1})/ \\
&(-1 + 2r\sin\theta + 2r\cos\theta + 2r\sqrt{r^2 - 2r\cos\theta + 1} \\
&-4r^2 + 2r\sqrt{r^2 - 2r\sin\theta + 1})
\end{aligned} \tag{40}
$$

$$
\begin{aligned}
&G(r,\theta) = atan(-(-2\cos\theta r\sqrt{r^2 - 2r\sin\theta + 1} \\
&-2r\sin\theta\sqrt{r^2 - 2r\sin\theta + 1} - 2r\sin^2\theta \\
&+\sin\theta + 2r^2\cos\theta + 2\sqrt{r^2 - 2r\sin\theta + 1}\sqrt{r^2 - 2r\cos\theta + 1}\cos\theta \\
&+2\sqrt{r^2 - 2r\sin\theta + 1} + 2\cos\theta r\sin\theta - 2r\cos\theta\sqrt{r^2 - 2r\cos\theta + 1} \\
&-2r + 2\sqrt{r^2 - 2r\cos\theta + 1}r\sin\theta)/ \\
&(4r^2 - 4r\cos\theta - 2r\sqrt{r^2 - 2r\sin\theta + 1} + 2\sqrt{r^2 - 2r\sin\theta + 1}\sin\theta \\
&-2r\sqrt{r^2 - 2r\cos\theta + 1} - 4r\sin\theta + 3 + 2\sqrt{r^2 - 2r\cos\theta + 1}\cos\theta) \\
&, -(2\cos\theta r\sqrt{r^2 - 2r\sin\theta + 1} + 2\sqrt{r^2 - 2r\cos\theta + 1}\sqrt{r^2 - 2r\sin\theta + 1} \\
&+2r\sin^2\theta - 4r + \cos\theta - 2r\sin\theta\sqrt{r^2 - 2r\sin\theta + 1} + 2\sqrt{r^2 - 2r\cos\theta} \\
&+2\cos\theta r\sin\theta + 2r^2\sin\theta - 2r\cos\theta\sqrt{r^2 - 2r\cos\theta + 1} \\
&-2\sqrt{r^2 - 2r\cos\theta + 1}r\sin\theta)/ \\
&(4r^2 - 4r\cos\theta - 2r\sqrt{r^2 - 2r\sin\theta + 1} + 2\sqrt{r^2 - 2r\sin\theta + 1}\sin\theta \\
&-2r\sqrt{r^2 - 2r\cos\theta + 1} - 4r\sin\theta + 3 + 2\sqrt{r^2 - 2r\cos\theta + 1}\cos\theta))
\end{aligned} \tag{41}
$$

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