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Design of Reduced-Order State/Unknown Input Observers: A Descriptor System Approach

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Abstract—**This paper addresses the problem of estimating simultaneously a linear function of both the state and unknown input of linear system with unknown inputs. By adopting the descriptor system approach, the problem can be conveniently solved. Observers proposed in this paper are of low-order and do not include the derivatives of the outputs. New conditions for the existence of reduced-order observers are derived. A design procedure for the determination of the observer parameters can also be easily derived based on the derived existence conditions.**

I. Introduction and Problem Statement

In this paper, we present some new results on designing reduced-order observers that can estimate simultaneously a linear function of both the state and unknown input of linear system with unknown inputs. Let us consider a system described by

$$\dot{x}(t) = Ax(t) + Bu(t) + Dd(t) \tag{1a}$$

$$y(t) = Cx(t) + Wd(t) \tag{1b}$$

where $x(t) \in \Re^n$, $y(t) \in \Re^r$, $u(t) \in \Re^m$ and $d(t) \in \Re^q$ are the state, measured output, input and unknown input, respectively. Matrices A, B, D, C and W are known real constant and of appropriate dimensions. It is assumed that $rank(C) = r$, $r \ge q$ and the system (A, D, C, W) satisfies the well-known minimum phase condition.

For the above system (1), the problem of estimating the state and unknown input has been a subject of widespread interest in the literature (see, for example [1]-[8] and the references therein). Unlike the existing work in the literature [1]-[8], the aim of this paper is to design a reduced-order asymptotic observer to estimate *a linear function* of the state, $x(t)$, and the unknown input, $d(t)$. The approach adopted in this paper is based on the Descriptor System Approach [9].

Let us define an augmented state vector, $\omega(t)$, where $\omega(t) = \begin{bmatrix} x(t) \\ d(t) \end{bmatrix} \in \Re^{(n+q)}$. Accordingly, the system (1) can be expressed as

$$E\dot{\omega}(t) = \bar{A}\,\omega(t) + Bu(t) \tag{2a}$$

$$y(t) = \bar{C}\,\omega(t) \tag{2b}$$

where $E = [I_n \quad 0_{n\times q}]$, $\bar{A} = [A \quad D]$ and $\bar{C} = [C \quad W]$.

Remark 1: System (2) belongs to a class of descriptor systems. The problem of designing a reduced-order observer to estimate *a linear function* of the state, $x(t)$, and the unknown input, $d(t)$, now becomes that of designing a reduced-order observer to estimate *a linear function* of the state, $\omega(t)$, of system (2). Thus this work is essentially the design of linear functional observers for the descriptor systems (2).

Let us define the following functional state vector, $z(t) \in \Re^p$, where

$$z(t) = L\omega(t), \tag{3}$$

and $L \in \Re^{p\times(n+q)}$ is a given constant matrix. Without loss of generality, it is assumed that $rank(L) = p$ and $rank\begin{bmatrix} \bar{C} \\ L \end{bmatrix} = (r+p) \le (n+q)$.

Here, we are interested in designing a reduced-order observer to estimate $z(t) \in \Re^p$. Let us consider the following observer structure of order p for the system (2)

$$\hat{z}(t) = \xi(t) + My(t) \tag{4a}$$

$$\dot{\xi}(t) = N\xi(t) + Jy(t) + Hu(t) \tag{4b}$$

where $\xi(t) \in \Re^p$ and $\hat{z}(t)$ denotes the estimate of $z(t)$. Matrices N, J, H and M are to be determined such that $\hat{z}(t)$ converges asymptotically to $z(t)$ (i.e. $\hat{z}(t) \to z(t)$ as $t \to \infty$).

Remark 2: If we are only interested in the estimation of the unknown input (i.e. $d(t)$), then matrix L is chosen as: $L = [0_{q\times n} \quad I_q]$. Likewise, if we are only interested in estimating a linear combination of the states (say, $L_1 x(t)$), then L takes the form: $L = [L_1 \quad 0_{p\times q}]$. Thus, (3) provides a

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great degree of flexibility and allows any linear combination of the state and unknown input be defined and estimated. Note that when $rank\begin{bmatrix}\bar{C}\\ L\end{bmatrix} = (n+q)$ (i.e. $\begin{bmatrix}\bar{C}\\ L\end{bmatrix}$ is a square and nonsingular matrix), then an $(n+q-r)$-order observer for $\omega(t)$ can be obtained, where $\hat{\omega}(t) = \begin{bmatrix}\hat{x}(t)\\ \hat{d}(t)\end{bmatrix} = \begin{bmatrix}\bar{C}\\ L\end{bmatrix}^{-1}\begin{bmatrix}\hat{z}(t)\\ y(t)\end{bmatrix}$.

II. MAIN RESULTS

Let $X \in \Re^{p\times n}$ be a full-row rank matrix and define error vectors $\varepsilon(t) \in \Re^{p}$ and $e(t) \in \Re^{p}$ as

$$\varepsilon(t) = \xi(t) - XE\omega(t)\,, \tag{5a}$$

and

$$e(t) = \hat{z}(t) - z(t)\,. \tag{5b}$$

The following theorem provides a sufficient condition for ensuring that $\hat{z}(t)$ converges asymptotically to $z(t)$.

Theorem 1: *There exists an observer of the form (4) for the system (2) so that $\hat{z}(t) \to z(t)$ as $t \to \infty$ provided that the following matrix equations hold.*

$$NXE + J\bar{C} - X\bar{A} = 0,\ \ N \text{ is Hurwitz,} \tag{6}$$

$$XE + M\bar{C} - L = 0\,, \tag{7}$$

$$H = XB\,. \tag{8}$$

Proof: From (5a), the following error dynamics equation is obtained

$$\dot{\varepsilon}(t) = \dot{\xi}(t) - XE\dot{\omega}(t) = N\varepsilon(t) + (NXE + J\bar{C} - X\bar{A})\omega(t)$$
$$+(H - XB)u(t)\,. \tag{9}$$

From (5b), the error vector $e(t)$ can be expressed as

$$e(t) = \varepsilon(t) + (XE + M\bar{C} - L)\omega(t)\,. \tag{10}$$

From (9) and (10), it is clear that $e(t) \to 0$ as $t \to \infty$ if the matrix equations (6)-(8) of Theorem 1 are satisfied. This completes the proof of Theorem 1.

Remark 3: In order to derive the parameters of the linear functional observer (4), we will need to solve equations (6)-(8) of Theorem 1 for the unknown matrices N, X, J, M and H. The following theorem will provide the necessary and sufficient conditions for the solvability of matrix equations (6)-(7) of Theorem 1 (note that matrix H is obtained from (8) once matrix X is solved).

Theorem 2: *The matrix equations (6)-(7) of Theorem 1 are completely solvable if and only if the following two conditions hold*

Condition 1: $rank\begin{bmatrix} LA_u & L\\ \bar{C}A_u & \bar{C}\\ \bar{C} & 0\\ 0 & E\\ L & 0\end{bmatrix} = rank\begin{bmatrix}\bar{C}A_u & \bar{C}\\ \bar{C} & 0\\ 0 & E\\ L & 0\end{bmatrix}$,

$$A_u = \begin{bmatrix}\bar{A}\\ 0_{q\times(n+q)}\end{bmatrix}; \tag{11}$$

Condition 2: $rank\begin{bmatrix}(sL - LA_u) & -L\\ \bar{C}A_u & \bar{C}\\ \bar{C} & 0\\ 0 & E\end{bmatrix} = rank\begin{bmatrix}\bar{C}A_u & \bar{C}\\ \bar{C} & 0\\ 0 & E\\ L & 0\end{bmatrix}$,

$$\forall s \in \boldsymbol{C}, \mathrm{Re}(s) \geq 0\,. \tag{12}$$

Proof: By letting $A_u = \begin{bmatrix}\bar{A}\\ 0_{q\times(n+q)}\end{bmatrix}$, then matrix $\bar{A}$ can be expressed as

$$\bar{A} = EA_u\,. \tag{13}$$

Substituting (7) and (13) into (6), the following equation is obtained

$$NL = LA_u - [M\quad T]\begin{bmatrix}\bar{C}A_u\\ \bar{C}\end{bmatrix}, \tag{14}$$

where

$$T = J - NM\,. \tag{15}$$

Post-multiply both sides of (14) by the following full-row rank matrix

$$S = [L^{+}\quad (I_{(n+q)} - L^{+}L)] = [S_1\quad S_2]\,, \tag{16}$$

(note: here $S_1 = L^{+}$ (the pseudo inverse of L) and $S_2 = I_{(n+q)} - L^{+}L$ (the projection) with the property $S_2S_1 = 0$, hence $rank(S) = rank(S_1) + rank(S_2) = n+q$) yields the following two equations

$$N = LA_uS_1 - [M\quad T]\begin{bmatrix}\bar{C}A_u\\ \bar{C}\end{bmatrix}S_1 \tag{17}$$

and

$$[M\quad T]\begin{bmatrix}\bar{C}A_u\\ \bar{C}\end{bmatrix}S_2 = LA_uS_2\,. \tag{18}$$

Equations (18) and (7) can now be written in an augmented matrix equation as follows

$$[M \quad T \quad X]\Omega = \Psi\,, \tag{19}$$

where $\Omega \in \Re^{(n+2r)\times 2(n+q)}$ and $\Psi \in \Re^{p\times 2(n+q)}$ are known matrices and are defined as

$$\Omega = \begin{bmatrix} \overline{C}A_uS_2 & \overline{C} \\ \overline{C}S_2 & 0 \\ 0 & E \end{bmatrix}, \tag{20}$$

and

$$\Psi = [LA_uS_2 \quad L]. \tag{21}$$

From (19), we can derive the necessary and sufficient condition for the existence of a solution of the unknown matrix, i.e. $[M \quad T \quad X]$. Then, by substituting the solution (i.e. $[M \quad T]$) into (17), the necessary and sufficient condition for ensuring that matrix N be Hurwitz can be derived. Finally, by using (15), matrix J can be derived. As a result, all the unknown matrices N, X, J, M and H that satisfy the matrix equations (6)-(8) of the Theorem 1 are obtained.

Now, in (19), there exists a solution to the unknown matrix $[M \quad T \quad X]$ if and only if the following condition holds

$$rank\begin{bmatrix} \Psi \\ \Omega \end{bmatrix} = rank(\Omega)\,,$$

i.e.

$$rank\begin{bmatrix} LA_uS_2 & L \\ \overline{C}A_uS_2 & \overline{C} \\ \overline{C}S_2 & 0 \\ 0 & E \end{bmatrix} = rank\begin{bmatrix} \overline{C}A_uS_2 & \overline{C} \\ \overline{C}S_2 & 0 \\ 0 & E \end{bmatrix}. \tag{22}$$

It is easy to show that the Condition 1 of Theorem 2 is equivalent to the condition (22) (note: To show that (11) is equivalent to (22), post-multiply both sides of (11) by a full row-rank matrix $\begin{bmatrix} S_1 & S_2 & 0 \\ 0 & 0 & I_{(n+q)} \end{bmatrix}$). Therefore upon the satisfaction of (11), a general solution to (19) is

$$[M \quad T \quad X] = \Psi\Omega^+ + Z(I_{(n+2r)} - \Omega\Omega^+)\,, \tag{23}$$

where $Z \in \Re^{p\times(n+2r)}$ is an arbitrary matrix.

Let us now substituting (23) into (17) to give

$$N = N_1 - ZN_2\,, \tag{24}$$

where

$$N_1 = LA_uS_1 - \Psi\Omega^+\Gamma\,,\ N_2 = (I_{(n+2r)} - \Omega\Omega^+)\Gamma \text{ and}$$

$$\Gamma = \begin{bmatrix} \overline{C}A_uS_1 \\ \overline{C}S_1 \\ 0 \end{bmatrix}. \tag{25}$$

In (24), N_1 and N_2 are known matrices, if and only if the pair (N_2, N_1) is detectable, i.e.

$$rank\begin{bmatrix} sI_p - N_1 \\ N_2 \end{bmatrix} = p\,,\ \forall s \in \boldsymbol{C}, \mathrm{Re}(s) \geq 0\,, \tag{26}$$

then the matrix N can be made Hurwitz by the eigenvalue assignment via an injection matrix Z.

In the following, we will show that the Condition 2 of Theorem 2 is equivalent to the condition (26) and therefore ensuring that matrix N is Hurwitz.

First, post-multiply the RHS of (12) by a full row-rank matrix $\begin{bmatrix} S_1 & S_2 & 0 \\ 0 & 0 & I_{(n+q)} \end{bmatrix}$ to give

$$rank\begin{bmatrix} \overline{C}A_u & \overline{C} \\ \overline{C} & 0 \\ 0 & E \\ L & 0 \end{bmatrix} = rank\left(\begin{bmatrix} \overline{C}A_u & \overline{C} \\ \overline{C} & 0 \\ 0 & E \\ L & 0 \end{bmatrix}\begin{bmatrix} S_1 & S_2 & 0 \\ 0 & 0 & I_{(n+q)} \end{bmatrix}\right) = p + rank(\Omega)\,. \tag{27}$$

Now, the LHS of (12) can be expressed as follows

$$\begin{aligned}
&rank\begin{bmatrix} (sL - LA_u) & -L \\ \overline{C}A_u & \overline{C} \\ \overline{C} & 0 \\ 0 & E \end{bmatrix} \\
&= rank\left(\begin{bmatrix} (sL - LA_u) & -L \\ \overline{C}A_u & \overline{C} \\ \overline{C} & 0 \\ 0 & E \end{bmatrix}\begin{bmatrix} S_1 & S_2 & 0 \\ 0 & 0 & I_{(n+q)} \end{bmatrix}\right) \\
&= rank\begin{bmatrix} (sI_p - LA_uS_1) & -\Psi \\ \Gamma & \Omega \end{bmatrix} \\
&= rank\left(\begin{bmatrix} I_p & \Psi\Omega^+ \\ 0 & (I_{(n+2r)} - \Omega\Omega^+) \\ 0 & \Omega\Omega^+ \end{bmatrix}\begin{bmatrix} (sI_p - LA_uS_1) & -\Psi \\ \Gamma & \Omega \end{bmatrix}\right) \\
&= rank\begin{bmatrix} sI_p - N_1 & 0 \\ N_2 & 0 \\ \Omega\Omega^+\Gamma & \Omega \end{bmatrix}
\end{aligned}$$

$$= rank\left(\begin{bmatrix} sI_p - N_1 & 0 \\ N_2 & 0 \\ \Omega\Omega^+\Gamma & \Omega \end{bmatrix}\begin{bmatrix} I_p & 0 \\ -\Omega^+\Gamma & I_{2(n+q)} \end{bmatrix}\right)$$

$$= rank\begin{bmatrix} sI_p - N_1 \\ N_2 \end{bmatrix} + rank[\Omega]\,,\ \forall s \in \boldsymbol{C}, \mathrm{Re}(s) \ge 0\,. \quad (28)$$

It is clear from (27) and (28) that Condition (12) is equivalent to (26) and therefore provides the necessary and sufficient condition for matrix N to be Hurwitz. Upon the satisfaction of the Conditions 1&2 of Theorem 2, then from (24), we can easily find a matrix gain Z and a stable matrix N. Once matrix Z is obtained, the unknown matrix $[M \quad T \quad X]$ is derived from (23). From (8) and (15), matrices H and J can be derived. This completes the proof of Theorem 2.

Remark 4: The design of a reduced-order linear functional observer (4) depends on the satisfaction of the Conditions 1&2 of Theorem 2. Accordingly, for any given matrix L, the Conditions 1&2 of Theorem 2 can be easily tested. If both Conditions hold, then all the observer's parameters can be systematically derived, as explained in the proof of Theorem 2.

In the following, let us further exam the Conditions 1 and 2 of Theorem 2. In doing so, we will look at one special case where the Conditions 1 and 2 of Theorem 2 can be simplified further and they will be reduced to the well-known existing conditions.

Special Case: $rank\begin{bmatrix} \bar{C} \\ L \end{bmatrix} = (n+q)$. For this case, it is clear that an $(n+q-r)$-order observer for $\omega(t)$ can be designed, where $\hat{\omega}(t) = \begin{bmatrix} \hat{x}(t) \\ \hat{d}(t) \end{bmatrix} = \begin{bmatrix} \bar{C} \\ L \end{bmatrix}^{-1} \begin{bmatrix} \hat{z}(t) \\ y(t) \end{bmatrix}$. The following Corollary provides the necessary and sufficient conditions for the solvability of the matrix equations (6)-(7) of Theorem 1.

Corollary 1: *When* $rank\begin{bmatrix} \bar{C} \\ L \end{bmatrix} = (n+q)$, *then the matrix equations (6)-(7) of Theorem 1 are completely solvable if and only if the following two conditions hold*

Condition 1: Matrix W is a full-column rank matrix, i.e.

$$rank(W) = q\,; \quad (29)$$

Condition 2:

$$rank\begin{bmatrix} sI_n - A & -D \\ C & W \end{bmatrix} = (n+q)\,,\ \forall s \in \boldsymbol{C}, \mathrm{Re}(s) \ge 0\,. \quad (30)$$

Proof: First, let us consider the Condition 1 of Theorem 2. When $rank\begin{bmatrix} \bar{C} \\ L \end{bmatrix} = (n+q)$, the RHS of (11) can be written as

$$rank\begin{bmatrix} \bar{C}A_u & \bar{C} \\ \bar{C} & 0 \\ 0 & E \\ L & 0 \end{bmatrix} = rank\begin{bmatrix} \bar{C}A_u & \bar{C} \\ 0 & E \\ \bar{C} & 0 \\ L & 0 \end{bmatrix} = (n+q) + rank\begin{bmatrix} \bar{C} \\ E \end{bmatrix}$$

$$= (2n+q) + rank(W)\,. \quad (31)$$

Similarly, the LHS of (11) is

$$rank\begin{bmatrix} LA_u & L \\ \bar{C}A_u & \bar{C} \\ \bar{C} & 0 \\ 0 & E \\ L & 0 \end{bmatrix} = rank\begin{bmatrix} LA_u & L \\ \bar{C}A_u & \bar{C} \\ 0 & E \\ \bar{C} & 0 \\ L & 0 \end{bmatrix}$$

$$= (n+q) + rank\begin{bmatrix} L \\ \bar{C} \\ E \end{bmatrix} = (2n+2q) \quad (32)$$

From (31) and (32), it is clear that (11) holds if and only if $rank(W) = q$.

Now, the LHS of (12) can be expressed as follows

$$rank\begin{bmatrix} (sL - LA_u) & -L \\ \bar{C}A_u & \bar{C} \\ \bar{C} & 0 \\ 0 & E \end{bmatrix} =$$

$$rank\left(\begin{bmatrix} G_1 & -G_2 & sG_2 & 0 \\ 0 & 0 & I_r & 0 \\ 0 & 0 & 0 & I_n \end{bmatrix}\begin{bmatrix} (sL - LA_u) & -L \\ \bar{C}A_u & \bar{C} \\ \bar{C} & 0 \\ 0 & E \end{bmatrix}\right)$$

$$= rank\begin{bmatrix} (sI_{(n+q)} - A_u) & -I_{(n+q)} \\ \bar{C} & 0 \\ 0 & E \end{bmatrix}$$

$$= rank\begin{bmatrix} (sI_n - A) & -D \\ C & W \end{bmatrix} + (n+q)\,,\ \forall s \in \boldsymbol{C}, \mathrm{Re}(s) \ge 0\,, \quad (33)$$

where $[G_1 \quad G_2] = \begin{bmatrix} L \\ \bar{C} \end{bmatrix}^{-1}$. From (31)-(33), Condition 2 of Theorem 2 is thus reduced to (30). This completes the proof of Corollary 1.

Remark 5: Condition 1 of Corollary 1 implies that

$W \neq 0$ (i.e. the measured output vector must contain components of direct connection to the unknown inputs) and the number of outputs must be equal or greater than that of the unknown inputs (i.e. $r \geq q$). Notably, the later requirement is a well-known necessary condition in the design of observers for linear systems with unknown inputs. Condition 2 of Corollary 1 is also a well-known minimum phase condition of the system (A, D, C, W) (see, [6]).

Upon the satisfaction of the Conditions 1&2 of Corollary 1, the design of a $(n+q-r)$-order observer for $\omega(t)$ can now be performed according to the following step-by-step design procedure.

Step 1: Choose any $(n+q-r)\times(n+q)$ matrix L such that $\begin{bmatrix} \overline{C} \\ L \end{bmatrix} \in \Re^{(n+q)\times(n+q)}$ is nonsingular. This can always be easily done since matrix $\overline{C}$ is a full row-rank matrix.

Step 2: Use equations (25) to obtain matrices N_1 and N_2. From (24), obtain a matrix gain Z and a stable matrix N.

Step 3: From (23), obtain the unknown matrix $[M \quad T \quad X]$.

Step 4: Obtain matrices H and J from (8) and (15), respectively.

Step 5: Finally, a $(n+q-r)$-order observer for $\omega(t)$ is obtained, where

$$\hat{\omega}(t) = \begin{bmatrix} \hat{x}(t) \\ \hat{d}(t) \end{bmatrix} = \begin{bmatrix} \overline{C} \\ L \end{bmatrix}^{-1} \begin{bmatrix} \hat{z}(t) \\ y(t) \end{bmatrix}.$$

III. Conclusion

This paper has addressed the problem of estimating simultaneously a linear function of both the state and unknown input of linear system with unknown inputs. Here, the problem has been solved by adopting the descriptor system approach. The observers proposed in this paper have the advantages of having the order the same as the dimension of the vector to be estimated. It has been shown that when the dimension of the vector to be estimated is full, then the conditions derived in this paper are the same as those derived in the literature. The results of this paper can be regarded as more general and thus extend the existing work [1]-[8] to include estimation of a linear function of both state/unknown input.

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