Impact of Context and Representation on Year 10 Students' Expression of Conceptions of Rate

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Rate is an important, but difficult mathematical concept. More than twenty years of research, especially with calculus students, report difficulties with this concept. This paper reports on an alternative analysis, from the perspective of multiple representations and context, of interviews probing twenty Victorian Year 10 students' conceptions of rate. This analysis shows that multiple representations of functions provide different rate-related information for different students. Understandings of rate in one representation or context are not necessarily transferred to another representation or context.

Rate is an important mathematical concept needed for everyday numeracy and important for more advanced areas of study, such as calculus. Despite considerable research in the area it remains a troublesome concept to teach and learn. Even many calculus students have found rate particularly troublesome (Ubuz, 2007). Rate is a complex concept comprising many interwoven ideas. It expresses the change in the dependent variable resulting from a unit change in the independent variable and involves the ideas of change in a quantity; co-ordination of two quantities; and the simultaneous covariation of the quantities (Thompson, 1994). Improvement of conceptions of rate held by students who have not yet studied calculus may, in turn, also address some of the difficulties experienced by students studying calculus identified by many researchers. Functions, and hence, rate may be represented numerically, graphically and symbolically. Kaput (1999) emphasises the importance of connections between everyday experiences and these multiple representations. This study investigates the nature of these connections in two particular real world contexts.

This paper builds on Herbert and Pierce (2009) which describes the educationally critical aspects of rate (see Figure 1) resulting from the phenomenographic analysis of the interviews. These aspects provide a framework for further content analysis of the same interview data from the different perspective of context and associated multiple representations of functions resulting from the simulations.

1. rate as a relationship between two changing quantities.
2. rate as a relationship between two changing quantities which may vary.
3. rate as a numerical relationship between two changing quantities which may vary.
4. rate as a numerical relationship between any two changing quantities which may vary.

Figure 1. Critical aspects of rate.

This re-examination of the data is considered appropriate since Boulton-Lewis and Wilss (2007) suggest that data collected for a phenomenographic study can also be analysed in other ways to investigate other research questions, which, in conjunction with phenomenographic results, allow a fuller understanding of the data and, so, determine the impact of context and representation on Year 10 students' expression of their conceptions of rate.

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Method

The twenty Year 10 participants, from five different Victorian secondary schools, were selected by their teachers, to represent a range of mathematical ability and a mix of gender. It was expected that these participants had previously experienced constant rate in the form of linear functions (VCAA, 2007). Two interactive computer simulations, one in *Geometer's Sketchpad* (GSP) (Key Curriculum Press, 2006) of a blind partially covering a window (see Figure 2), and the other in *JavaMathWorld* (JMW) (Mathematics Education Researchers Group, 2004) of a frog and clown walking (see Figure 3) were prepared.

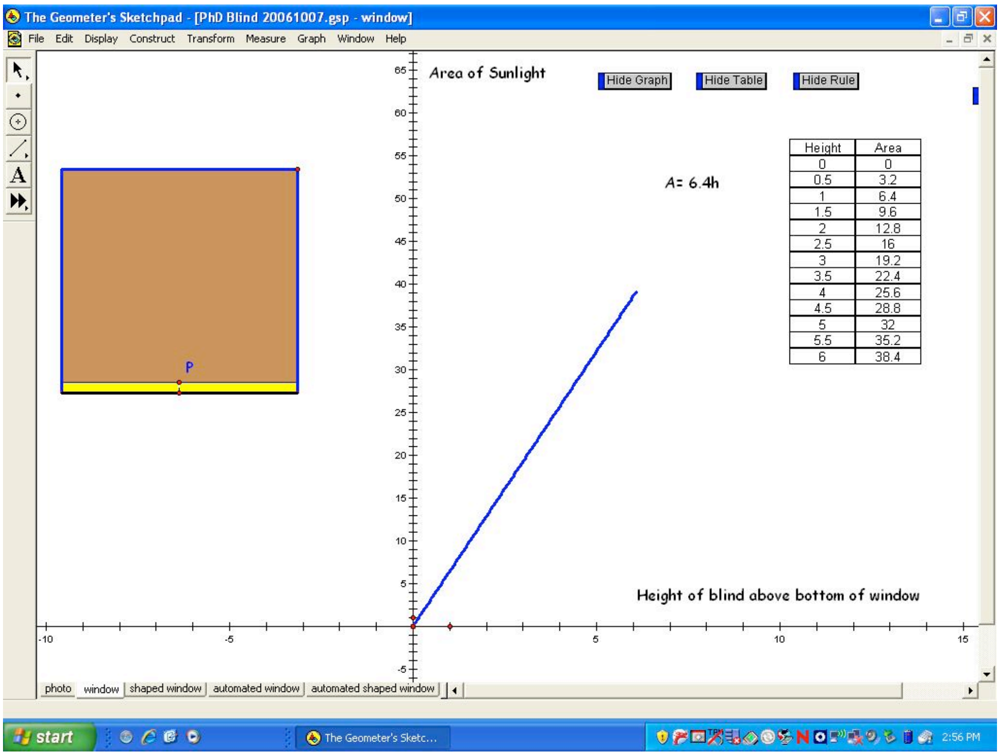

Figure 2. Screen dump of the GSP simulation showing representations.

These two simulations provided participants with concrete examples of rate to facilitate discussion in video-recorded interviews of approximately forty-five minutes in duration. GSP and JMW were particularly well suited for this purpose as they provide easy access to numeric, graphic and symbolic representations, so participants' conceptions of rate could be explored more broadly. In particular, they were chosen to explore participants' expression of their understanding of a rate involving motion compared with a rate where time is not a variable. In the GSP simulation, the context of the blind partially covering a window (see Figure 2) facilitated discussion of the participants' understanding of constant rate, where area and height are the rate-related variables. This facilitated discussion of the participants' understanding of constant rate in a different context where distance and time are the rate-related variables. The JMW simulation involved frog and clown walking at constant speed (see Figure 3) enabling comparison of their speeds. Like GSP, the dynamic links between the numeric, graphic and symbolic representations in JMW enabled the relationship between distance and time to be explored and provided opportunities for discussion of participants' understanding of constant speed in these representations. Participants were encouraged to explain their reasoning and think aloud as they were presented with different representational forms of rate: the simulation, table of values, graph and rule.

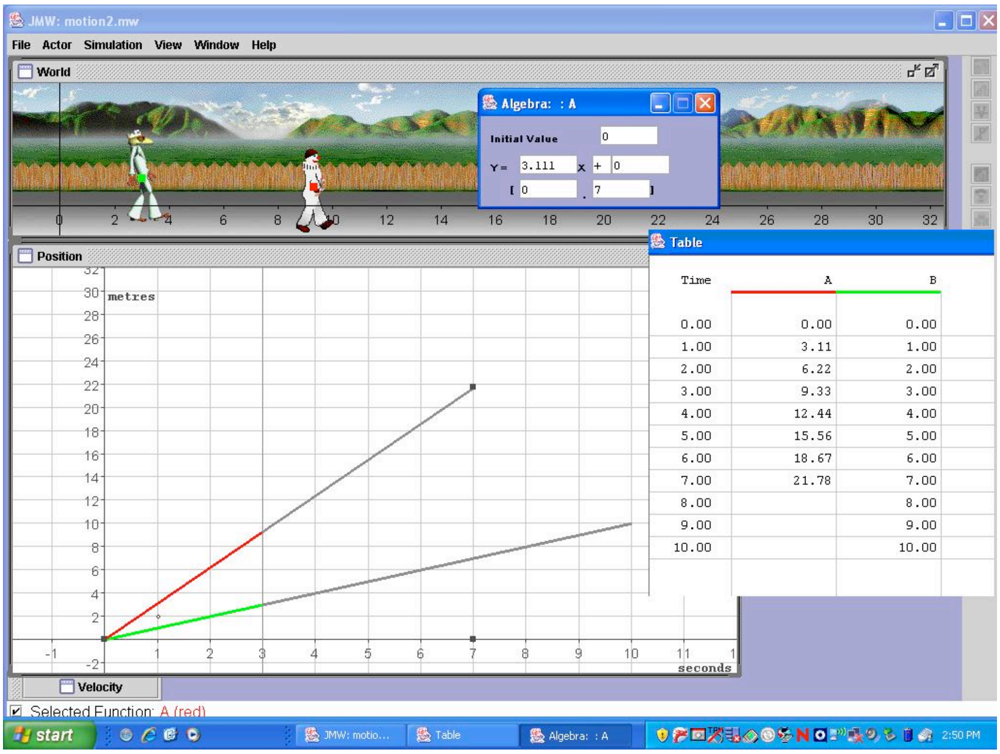

Figure 3. Screen dump of the JMW simulation showing representations.

The content analysis began with a re-examination of the data. Each participant's responses relating to rate were coded by representation (numeric, graphic or symbolic) and context (blind or walking). All responses were examined to discern any impact of the multiple representations and contexts on individual participant's expression of their understanding of rate.

Results

The results are illustrated by appropriate quotations from the interview data. Excerpts were chosen for inclusion within the discussion to: exemplify influences identified in the participants' responses; clarify and justify inferences made by the researcher; and demonstrate the detail of the data. The following sections present a comparison of responses between different representations and contexts demonstrating the relative influence of representations and contexts in participants' expression of their thinking about rate.

Comparison of Participants' Responses to Numeric Representations

The numeric representation in the GSP context (see Figure 2) provided rate-related information to most participants (13). This is illustrated by the responses given to the question "What does the table tell you about the rate that the area of the sunlight is changing?". Eleven participants could see the pattern in the columns of the table and use it to express the numeric relationship between the variables, for example: "Well, yeah, three point two, there's a difference between um, [pause] so for every half a metre you get three point two extra" (I15). Whilst two participants provided a qualitative description, for example: "The more height, the more area of sunlight. It's just going up at a steady rate" (I9). Six participants were unable to give a rate-related response, focusing instead on only one variable, for example: "It's always like 3.2, it's always adding 3.2 the area [pause] like 6.4 plus 3. [pause] as a guess. I don't know. What I'm looking at, it's from 6.4 to 9.6 its grown 3.2 [pause] in that sort of pattern, mainly" (I2). Three other participants made other responses indicating their lack of awareness of any relationship between the variables of height and area. "It [table] tells you the different heights and the different area of sunlight"(I1).

The numeric representation in the JMW context (see Figure 3) provided rate-related information to all participants. Every participant could express some understanding of the relationship between distance and time. Most participants (14) could even use the information in the table to calculate a numeric rate. This is seen in their responses to the question "What does the table tell you about the rate that the frog and clown are walking?", for example: "The red one [clown] is walking faster. He's [clown] going 3.1 metres in 1 minute. He's [frog] only going 1 metre in 1 minute" (I1). Other participants expressed other interesting aspects of their quantitative understanding of rate in the JMW context. These responses indicate the perceptual nature of participants' understanding of speed influences their discussion of the relationship between the variables of distance and time, for example: "I can see the clown is walking faster. If you look at the time column at 7 time and look at A. He has covered more distance than B. [I chose 7] because that's when [clown stopped] I could choose any number before seven because they would still be walking at the same time, I am comparing that with the same time" (I2). Only two participants were unable to give a quantitative description of rate, for example: "Both of them are increasing. I would need distance and that's the time" (I4).

Comparison of Participants' Responses to Graphic Representations

In the GSP context the graphic representation provided some information about rate for about half (9) of the participants. This is illustrated by the following quotes, in response to the question, "What does the graph tell you about the rate?". Four participants could read approximate values from the graph and give a quantitative relationship between the variables, for example: "Well after it every 5 it's going up roughly, I'd roughly say 30, yeah sorry about 32 and the area of sunlights going up is about 32 [every] 5 metres" (I21). Two participants were aware of the two variables, area and height, involved in the rate, and responded with a description of the relationship between the variables in qualitative terms, for example: "That when the height of the blind, [from] the bottom of the window goes up, the area of sunlight increases" (I15). Six participants expressed their understanding of rate in terms of the shape of the graph, for example: "Just that like it's always, I dunno, that it's in a straight line, it means that [rate] is always going to be, like the same" (I5). The graph provided little or no information about rate for more than half (11 out of 20) of the participants. Seven participants demonstrated a lack of awareness of the relationship between the variables involved in the rate and the actual rate, for example: "Um, that the height is 7" (I1). Four participants were aware of the two variables involved in the rate and could read values off the graph, but not see what these values had to with the rate, for example:

I12: Ah, the area of sunlight is 40 and the height of the blind above the bottom of the window is 6.

R: What does that tell you about the rate?
I12: I'm not sure. Would it be the rate increasing?

In the JMW context, the graphic representation context provided some rate-related information to every participant. Five participants could read values off the graph and express them as the numeric rate, for example: "Well he [clown] goes 3 metres per second. Divide 22 by 7. [For the frog] just divide the 10, this number here which is just 10, wait that'd be 1 yeah, yep it'd [speed of frog] be 1" (I13). Five participants demonstrated awareness that the relationship in constant rate is the same regardless of which points a chosen from the graph, for example: "Well a total of 10 metres, that was the maximum, you can really use any point" (I17). Points on the graph were used to give values needed to quantify rate and there was less emphasis on the shape of the graph. Only three participants commented on the connection between the shape of the graph and rate, for example: "The rate stays pretty much the same because the lines don't have any curves in them, they are nice and straight" (I11). Almost all (15) participants attempted to quantify rate as a numerical relationship between the variables of distance and time, for example: "It takes him [frog] 2 metres, 2 seconds to do 2 metres where it takes the clown roughly to do 2 it takes him 6" (I21). Only one participant's description of rate was limited to merely qualitative terms, for example: "It looks like the clown is walking faster Um, because he's covered more metres" (I18).

Some confusion between rate, distance and time is illustrated by the following exchanges. It seems that these two participants have some understanding of the concept of rate, but do not connect it to the word 'rate' in this context. The following two exchanges illustrate this confusion.

R: Can you tell me anything more about the rate the frog and the clown are walking?
I10: The frog is walking about 2 metres per 2 seconds and the clown's walking 6 metres for 2 seconds.
R: Anything else you can tell me about the rate that they're walking?
I10: Frog walked for longer. Clown walked more seconds.

I1: Well, he[clown] walks 22 metres in 7 seconds and then he [frog] only makes 7 metres in 7 seconds
R: So who's walking faster?
I1: That one, the red one.
R: What rate is he walking do you think?
I1: I don't know.

Two participants referred to rate as the result of a formula calculation possibly remembered from science education, for example: "There's a formula for that [pause] yes we did that in science. We have done velocity and that. We have got time and distance, so we have got velocity, distance and time, so 22 metres divided by 7 is [pause] he's (clown) going at 3.14 metres per second. For the frog, I don't know if that would make a difference, I will do it up to 7 [pause] it's doing 1 metre per second. You would get those measurements, of course, and divide the distance by the time" (I4).

Comparison of Participants' Responses to Symbolic Representations

In the GSP context, the participants expressed almost no rate-related comments in response to the question, "what does the rule tell you about the rate?", for example: "That might even be [pause] to be quite honest, I have no idea" (I22). Other participants responded to this same question by translating the symbolic representation into words. Two participants went further by describing the process of substitution of values for height into the rule to calculate the corresponding values of area. These participants did not isolate the rate from the symbolic representation. Possibly they were trying to tell the researcher as much as they could about the symbolic representation and did not restrict their discussion to rate, for example: “Um, the, for every area of sunlight It’s six point four of the H, so for example one point five the height from the bottom times it six point four and gives how much area of sunlight area there is. I don't know it’s [the rule] just kind of representing that, every time you lift it up you find the area of the thing, of sunlight increases by six point four times the height, so it’s just a matter of the mathematical formula at the end of how to work out how to do that” (I6). Other responses demonstrated confusion between rate and the changes in the variables, for example: “That the rate of them the height rate is [pause] gets higher as the area of sunlight does” (I5).

In the JMW context, the symbolic representation did provide some rate-related information to most participants. Seven participants could correctly identify the walker from the symbolic representation suggesting that they could connect the rate of the walker with the symbolic representation, for example when responding to the question “This is the rule for one of my walkers, who is it for, the frog or the clown?”, for example: “It would be the clown, the clown’s rule because the frog is nice and easy to work out. He's just 1. He's [clown] got roughly 3 here, so 3.1 fits in nicely” (I11). It is difficult to discern what rate-related information participants gained from the rule as the rate had not been varied from the questions relating to the table and the graph, for example:

I18: The clown.
R: How can you tell?
I18: Um, I remembered that.

When a different question was asked “What do you think the rule might be for the clown?” two participants were able to describe the symbolic representation from the previously known rate, for example: “Well if it was around three units per metre, if it was the clown it would be M for metres equals three F, I’m not really sure” (I19).

Seven participant’s choice of response to the question “This is the rule for one of my walkers, who is it for, the frog or the clown?” demonstrated little rate-related reasoning, for example: “I guess the clown” (I2). The following quote demonstrates that this participant was seeking a formula to use to determine their choice of matching a walker to the given symbolic representation. “Clown, because when I worked it out before it is going up by 3.1 and the frog is, it’s quite [pause], I dunno, straight forward. Well for the time, well we can always put distance over time. I can't remember it. I don’t know if this is going to work. I can’t really remember it but, I’m trying to remember a rule with a particular formula and you need the gradient. Maybe I just, I don't know” (I17).

Discussion

Analysis of data showed that many participants could obtain information about constant rate from the numeric representation. The graphic representation was often used to determine co-ordinate points along the line to calculate constant rate. Only one participant referred to the gradient of the line and its relationship to rate. Whilst many of the participants were able to communicate ideas about rate in the form of tables and graphs, no participant linked the symbolic representation to rate in the GSP context. It is surprising that no participant expressed the connection of the symbolic representation to gradient, hence rate, as it was expected that their prior experience with the symbolic representation of linear functions (for example see Bull et al., 2004) would have emphasised the meaning of the co-efficient of the variable as gradient. However, in the

JMW context, four participants expressed some connection of the symbolic representation with rate, possibly because of the understanding of speed participants brought to their discussions of rate. These findings indicate that the participants did not move seamlessly between representations and that understandings demonstrated in one representation do not necessarily transfer to other representations. This is consistent with Amit and Fried's (2005) report that questions whether the potential of multiple representations, to enhance students' understanding of functions, is realised in the classroom. It appears that the participants of this study do not transfer their understanding of rate demonstrated in tables and graphs to the corresponding symbolic representation.

Stronger understandings of rate demonstrated in the JMW context were not evident in the GSP context. The GSP context preceded the JMW context for all participants as required by phenomenographic interviewing requirements, so it is possible that this influenced some participants' responses to questions relating to the JMW context. However, this is considered unlikely because the participants were not considered to be gifted mathematically, so it is doubtful that they could have learnt something new so quickly. Even the weakest participants demonstrated better understanding of rate in the JMW context than in the GSP context. These findings indicate that understanding of rate in a JMW context is much stronger than in the GSP context. This is more than would have been expected if understanding developed the GSP context had any effect on the understanding in the JMW context. This suggests that speed is better understood as a rate expressing a numeric relationship between distance and time, than the relationship between area and height in the GSP context. Findings of this study indicate that whilst rate appears to be quite well understood in the context of walking, this understanding does not automatically transfer to the blind context. So, specific instruction connecting rate in a motion context to other rates would be necessary to capitalise on the understanding of rate in a motion context, which many participants seemed to bring from their prior experiences in science classrooms, or experiences outside of the classroom. Alternatively, initial treatment of rate emphasising the covariational approach suggested by Carlson et al. (2002), and utilising speed, density and other rates as examples, rather than substitutions in formulae, may support the development of rate as a numeric relationship. Although the researcher's questions used the word 'rate', some participants spontaneously used the word 'speed' when discussing rate in the JMW context, so for these participants, rate, in this context, seems to be synonymous with speed. So this may be an example the difficulties associated with rate having a label in natural language suggested by Lamon (1999). It appears that speed may be seen as a single entity with little emphasis on the covariance of the variables of distance and time.

Whilst the selection of the participants was intended to result in capturing a wide diversity of conceptions, limiting factors may have been the teachers' selection of the participants from their school; the inarticulate responses given by many participants; or their lack of appropriate vocabulary with which to discuss rate.

Conclusions

This study shows that multiple representations of functions provide different rate-related information for different students and understandings of rate in one representation or context are not necessarily transferred to other representations. Results indicate that speed as a phenomenon is quite well understood by these participants, but this understanding was not necessarily helpful in understanding rate in a context not involving speed. In addition, results indicate that numeric and graphic representations support the expression of rate-related reasoning, but the symbolic representation was of little value in facilitating participants' expression of their conceptions of rate. This study confirms the notion that rate is a complex concept and informs teachers of the different ways in which students read meaning into the concepts they are learning.

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