FUZZY ENTROPY FROM WEAK FUZZY SUBSETHOOD MEASURES

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Abstract: In this paper, we propose a new construction method for fuzzy and weak fuzzy subsethood measures based on the aggregation of implication operators. We study the desired properties of the implication operators in order to construct these measures. We also show the relationship between fuzzy entropy and weak fuzzy subsethood measures constructed by our method.

Key words: *Implication operator, fuzzy subsethood measures, weak fuzzy subsethood measures, fuzzy entropy*

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1. Introduction

The fuzzy sets theory was introduced by Zadeh in 1965 (see [44]). Afterward he established the order relation between fuzzy sets in [45] as follows:

$$A \leq B, \text{ if and only if } \mu_A(x) \leq \mu_B(x) \text{ for all } x \in X,$$

where A and B are two fuzzy sets in the same universe X.

There have been many discussions about the non-fuzzy character of this order relation [1]. This fact led many authors (Bandler and Kohout [1], De Baets and Kerre [11], etc.) to propose different measures that provide an inclusion degree or subsethood measure of one fuzzy set in another. Mainly, three different axiomatizations of these fuzzy subsethood measures σ have been given. The first one was given by Kitainik [25], the second one by Sinha and Dougherty [36], and the last one by Young [43].

These three axiomatizations have the first axiom (Axiom 1) in common:

$$\text{(Axiom 1)} \quad \sigma(A, B) = 1 \text{ if and only if } A \leq B.$$

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While Kitainik's axiomatization was the first one, it has not been widely studied until recently. On the other hand, Sinha and Dougherty (see [36]) proposed nine axioms any subsethood measure has to fulfill. They also introduced three axioms which depend on the application. In [12, 22] there is a study of the conditions in which Kitainik's axiomatization satisfies most of Sinha and Dougherty's axioms and vice versa.

The first axiom of Sinha and Dougherty is Axiom 1 and the second axiom is:

(Axiom 2) $\sigma(A,B)=0$ if and only if $\exists x \in X : \mu_A(x)=1$ and $\mu_B(x)=0$.

This axiom was criticized because one element in the referential determines the inclusion degree [43]. However, there exist some applications, such as image processing, where this axiom leads to good results.

We are also going to consider the tenth axiom proposed by Sinha and Dougherty for approximate reasoning:

$$\text{(Axiom 10)} \quad \sigma(A,B)+\sigma(A,\bar{B}) \geq 1,$$

where $\bar{B}$ is the complementary of B associated to a strong negation n (see [6, 9]).

On the basis of Kosko's fuzzy subsethood measure [28, 29], fuzzy entropy and Willmott's work [42], Young introduced a new axiomatization for the fuzzy subsethood measures [43].

The following three considerations led Young to define the concept of *weak fuzzy subsethood measure* (see [43]):

1. There exist several widely used fuzzy subsethood measures that do not fulfill Axiom 1, for example:
   - Weak inclusion [14]:
   $$\sigma(A,B)=\frac{1}{N}\sum_{i=1}^{N}\max(1-\mu_A(x_i),\mu_B(x_i))$$
   - Inclusion from implication [41, 44]:
   $$\sigma(A,B)=\frac{1}{N}\sum_{i=1}^{N}(1-\mu_A(x_i)+\mu_A(x_i)\cdot\mu_B(x_i))$$
2. It can be proven that one can generate fuzzy entropies from measures that do not fulfill Axiom 1 (see [43]).
3. One element by itself can determine the fulfillment of the Axiom 2 of Sinha and Dougherty.

Later, Fan et al. [16] proposed an alternative definition of weak fuzzy subsethood measure. While many existing weak fuzzy subsethood measures satisfy Fan et al. definition, there is no general construction method for such measures. In this article, we study a generic construction method for weak fuzzy subsethood measures and their special classes based on aggregating implication operators. We perform an analysis of the properties of the implication operators which translate into one or another desirable property of weak fuzzy subsethood measures. We obtain many existing and new weak fuzzy subsethood measures by this process.

Additionally, taking into account considerations given by Young, we study the relation between weak fuzzy subsethood measures and fuzzy entropy, proving that we can construct fuzzy entropies using our method.

Fuzzy and weak fuzzy subsethood measures have been applied to image processing [36], neural network architecture [32], feature selection [31], defuzzification [40], fuzzy clustering validity [16], fuzzy rule tunning and inference [43]. However, there were no special studies on the effect of using different measures on the performance of algorithms in these domains. Our intention is to present a way to construct various fuzzy subsethood measures and apply them in the algorithms in order to analyze their effect in the future work.

This paper is organized as follows: in Section 2 we recall some preliminary concepts that we use in the paper. General considerations and our construction methods are proposed in Section 3. In Section 4 we present a theoretical study of the properties of the implication operators, to determine those suitable for our construction methods. These construction methods for fuzzy and weak fuzzy subsethood measures together with construction method for fuzzy entropy are presented in Section 5. Finally the conclusions and future lines are presented.

2. Preliminaries

All the results we present in this section will be used throughout the paper. We recall the concepts of fuzzy negation, t-norm and t-conorm. Then we present main properties of implication operators and relations between them. We also include some construction methods for implication operators from fuzzy negations, t-norms and t-conorms. Finally, we present aggregation functions that we use to aggregate the implication operators in order to construct subsethood measures.

2.1 Fuzzy negations, t-norms and t-conorms

A function $n : [0,1] \to [0,1]$ such that $n(0) = 1$ and $n(1) = 0$ is called a *strict negation* whenever it is strictly decreasing and continuous. If, in addition, it is involutive; that is, if $n(n(x)) = x$ for all $x \in [0,1]$, then n is said to be a *strong negation.* Any strong negation can be represented [37] in terms of automorphisms (an *automorphism* of the interval $[a,b] \subset \mathbb{R}$ is any continuous, strictly increasing function $\varphi : [a,b] \to [a,b]$ such that $\varphi(a) = a, \varphi(b) = b$). A function $n : [0,1] \to [0,1]$ is a strong negation, if and only if there exists an automorphism φ of the unit interval such that $n(x) = \varphi^{-1}(1 - \varphi(x))$. It is easy to show that any strong negation n has a unique equilibrium point $e \in (0,1)$ such that $n(e) = e$. In this paper, only strong negations are considered.

We know that a function $T : [0,1]^2 \to [0,1]$ is a t-norm (triangular norm), if it is commutative, associative and increasing, with neutral element 1. In the same way, a function $S : [0,1]^2 \to [0,1]$ is said to be a t-conorm if it is commutative, associative, increasing and with neutral element 0. From this point we denote the minimum with $\wedge$ and the maximum with $\vee$.

We say that a t-norm T and a t-conorm S are *dual* with respect to a fuzzy negation n, if and only if $n(T(x,y)) = S(n(x), n(y))$ and $n(S(x,y)) = T(n(x), n(y))$. Under these conditions we say that (T, S, n) is a De Morgan triple (see [17]).

We will denote with $\mathcal{FS}(X)$ the set of all the fuzzy sets defined on a finite non empty, referential set X ($Cardinality(X) = N$). Also, the membership degree of each element from X in the fuzzy set $A \in \mathcal{FS}(X)$ is given by $\mu_A(x) : X \to [0,1]$. We will call $\bar{A}$ the complement of a fuzzy set A given by the following expression:

$$\bar{A} = \{(x, \mu_{\bar{A}}(x)) |\, \mu_{\bar{A}}(x) = n(\mu_A(x))\ \forall x \in X\},$$

where n is a fuzzy negation.

2.2 Implication operators

Definition 1 *([1, 17]) An implication operator is a mapping* $I : [0,1]^2 \to [0,1]$*, if and only if it satisfies the boundary conditions* $I(0,0) = I(0,1) = I(1,1) = 1$ *and* $I(1,0) = 0$.

Conditions from Definition 1 are, of course, the minimal set of conditions we can demand from an implication operator. Other interesting properties of implication operators are listed in [3, 5, 15, 27, 38, 35]. All fuzzy implications are obtained by generalizing the implication operator of classical logic. Fodor and Roubens [17] provide the following definition:

Definition 2 *An implication is a function* $I : [0,1]^2 \to [0,1]$ *that satisfies the following properties:*

$I1 : x \le z$ *implies* $I(x,y) \ge I(z,y)$ *for all* $y \in [0,1]$ *(left antitonicity).*

$I2 : y \le t$ *implies* $I(x,y) \le I(x,t)$ *for all* $x \in [0,1]$ *(right isotonicity).*

$I3 : I(0,x) = 1$ *for all* $x \in [0,1]$ *(dominance of falsity or left boundary condition).*

$I4 : I(x,1) = 1$ *for all* $x \in [0,1]$ *(right boundary condition).*

$I5 : I(1,0) = 0$ *(normality condition).*

In this paper, when we write about implication operators, we always refer to the ones defined in Definition 2. In other cases, we simply talk of functions $I : [0,1]^2 \to [0,1]$ satisfying Definition 1 (see [3]).

Note that for any function $f : [0,1]^2 \to [0,1]$ satisfying $I1$ and $I2$, conditions $I3 - I5$ are equivalent to conditions in Definition 1.

Now we recall further conditions that can be required. These properties were stated in different papers (see [5, 34, 35] and look [3]) and they could be important in some applications.

$I6 : I(1,x) = x$ (neutrality of truth or left neutrality).

$I7 : I(x, I(y,z)) = I(y, I(x,z))$ (exchange principle).

$I8 : I(x,y) = 1$, if and only if $x \le y$ (ordering property).

$I9 : I(x, 0) = n(x)$ is a strong negation (natural negation).

$I10 : I(x, y) \geq y$.

$I11 : I(x, x) = 1$ (identity principle).

$I12 : I(x, y) = I(n(y), n(x))$ with a strong negation n (law of contraposition).

$I13 : I$ is a continuous function (continuity).

$I14 : I(x, n(x)) = n(x)$ for all $x \in [0, 1]$ being n a strong negation.

The last property has allowed to construct subsethood measures [5] and to prove certain theorems for the D-implications [33, 34]. In this paper, it takes an important role because we require it to construct fuzzy entropy measures.

The following three special properties will be studied in the paper in order to apply them in the construction methods:

$$\begin{array}{ll}(P1) & I(x,y) = 0, \text{ if and only if } x = 1 \text{ and } y = 0 \\ (P2) & I(x,y) + I(x,n(y)) \geq 1 \text{ with } n \text{ a strong negation} \\ (P3) & I(x,y) = 1, \text{ if and only if } x = 0 \text{ or } y = 1\end{array}$$

Remark: Clearly $(P3)$ and $I8$ are mutually exclusive.

The following results on implication operators will be used throughout the paper. The proofs of Lemma 1 and Theorem 1 can be found in [5].

Lemma 1 *Let* $f : [0, 1]^2 \to [0, 1]$.

i) If f satisfies $I1$ and $I12$, then f satisfies $I2$;

ii) If f satisfies $I2$ and $I12$, then f satisfies $I1$;

iii) If f satisfies $I3$ and $I12$, then f satisfies $I4$;

iv) If f satisfies $I4$ and $I12$, then f satisfies $I3$;

v) If f satisfies $I6$ and $I12$, then f satisfies $I9$;

vi) If f satisfies $I9$ and $I12$, then f satisfies $I6$;

vii) If f satisfies $I2$ and $I9$, then f satisfies $I3$;

viii) If f satisfies $I1$ and $I6$, then f satisfies $I10$;

ix) If f satisfies $I7$ and $I9$, then f satisfies $I12$;

x) If f satisfies $I1$, $I6$ and $I12$, then f satisfies $I2$, $I3$, $I4$, $I5$, $I9$ and $I10$;

xi) If f satisfies $I2$, $I7$ and $I8$, then f satisfies $I1$, $I3$, $I4$, $I5$, $I6$, $I10$ and $I11$.

A detailed study of other possible relations between the properties $I1 - I14$ can be found in [5, 35]. The following two theorems are proven in [5].

Theorem 1 *A function $I : [0,1]^2 \to [0,1]$ is strictly monotone and satisfies $I7, I9$ and $I13$, if and only if there exists an automorphism φ of the unit interval such that*

$$I(x,y) = n\Big(\varphi^{-1}(\varphi(x) \cdot \varphi(n(y)))\Big).$$

Theorem 2 *A function $I : [0,1]^2 \to [0,1]$ is such that $I2, I7, I9, I11$ and $I13$ are satisfied, if and only if there exists an automorphism φ of the unit interval such that*

$$I(x,y) = \varphi^{-1}(\wedge(\varphi(n(x)) + \varphi(y), 1))$$
$$n(x) \geq \varphi^{-1}(1 - \varphi(x)).$$

2.3 S-implications, R-implications and QL-implications

In this subsection, we recall three types of implications.

An *S-implication* associated with a t-conorm S and a strong negation n is a mapping I defined by

$$I(x,y) = S(n(x), y).$$

A *QL-implication* associated with a t-conorm S, a t-norm T and a strong negation n is a mapping I defined by

$$I(x,y) = S(n(x), T(x,y)).$$

An *R-implication* associated with a t-norm T is a mapping I defined by

$$I(x,y) = \sup\{z | T(x,z) \leq y\}.$$

In the special case of a left-continuous t-norm, we will refer to the R-implication as: *residual implication* [3, 24, 26].

Several characterizations of S-implications, QL-implications and R-implications can be found in [38]. It is easy to see that both S-implications and R-implications satisfy properties $I1 - I5$ for any t-norm T, t-conorm S and strong negation n, thus they are implications in Fodor and Roubens' sense. In general, QL-implications violate property $I1$. The conditions under which $I1$ is satisfied can be found in [18].

In [5] the following theorem is proven,

Theorem 3 *Let I be an S-implication associated with a t-conorm S and a strong negation n. Then*

$$I(x, n(x)) = n(x) \quad \text{for all } x \in [0,1], \quad \text{if and only if } S = \vee.$$

2.4 Aggregation functions

To end this section, we present the *aggregation functions* [6, 8, 4] that we use to aggregate the implication operators, in order to construct fuzzy subsethood measures.

Definition 3 *An aggregation function is a mapping $M : [0,1]^N \to [0,1]$ that fulfills:*

(A_1) $M(0,0,\ldots,0)=0$;

(A_2) $M(1,1,\ldots,1)=1$;

(A_3) *is non decreasing*

Definition 4 *An element a is called absorbing element of an aggregation function* $M:[0,1]^N \to [0,1]$*, if it satisfies:*

$$M(x_1,x_2,\ldots,x_N)=a \text{ if } x_i=a \text{ for some } i\in\{1,\ldots,N\}$$

3. General Considerations

In this section, we recall the existing definitions of weak fuzzy subsethood measures.

The considerations about Axiom 1 and Axiom 2 analyzed in the introduction, led Young to propose the following definition:

Definition 5 *(Young [43]) If* $\sigma:\mathcal{FS}(X)\times\mathcal{FS}(X)\to[0,1]$ *satisfies:*

(3a) σ *does not fulfill the following condition: If* $A\le B$*, then* $\sigma(A,B)=1$;

(3b) *If* $A\ge\widetilde{e}$*, then* $\sigma(A,\bar{A})=0$*, if and only if* $A=\widetilde{1}$;

(3c) *If* $A\le B\le C$*, then* $\sigma(C,A)\le\sigma(B,A)$ *and* $\sigma(C,A)\le\sigma(C,B)$ *for all* $A,B,C\in\mathcal{FS}(X)$,

where $\widetilde{e}=\{(x,\mu_{\widetilde{e}}(x))|\,\mu_{\widetilde{e}}(x)=e\ \forall x\in X\}$ *(where e is the equilibrium point of the strong negation considered) and* $\widetilde{1}=\{(x,\mu_{\widetilde{1}}(x))|\,\mu_{\widetilde{1}}(x)=1\ \forall x\in X\}$*, then we say that* σ *is a weak fuzzy subsethood measure on* X.

Later, Fan et al. [16] criticized the lack of boundary conditions and the negative first condition in Young's definition. They proposed an alternative definition for weak fuzzy subsethood measures in the following way:

Definition 6 *(Fan, Xie, Pei [16]) A function* $\sigma:\mathcal{FS}(X)\times\mathcal{FS}(X)\to[0,1]$ *is called a weak fuzzy subsethood measure, if* σ *satisfies the following properties:*

(4a) $\sigma(\widetilde{0},\widetilde{0})=\sigma(\widetilde{0},\widetilde{1})=\sigma(\widetilde{1},\widetilde{1})=1$;

(4b) $\sigma(\widetilde{1},\widetilde{0})=0$;

(4c) *If* $A\le B\le C$*, then* $\sigma(C,A)\le\sigma(B,A)$ *and* $\sigma(C,A)\le\sigma(C,B)$ *for all* $A,B,C\in\mathcal{FS}(X)$,

where $\widetilde{0}=\{(x,\mu_{\widetilde{0}}(x))|\,\mu_{\widetilde{0}}(x)=0\ \forall x\in X\}$.

In our view, this definition does not maintain the spirit of the definition given by Young. Also, there exist weak fuzzy subsethood measures in the sense of Young that are not in the sense of Fan. For example:

$$\sigma(A,B)=\frac{1}{N}\sum_{i=1}^{N}\frac{1-\mu_A(x_i)+\mu_B(x_i)}{2}.$$

Motivated by the arguments that led Young to change Axiom 2 of Sinha and Dougherty and the fact that well-known weak fuzzy subsethood measures (like the weak inclusion and the inclusion from implication) satisfy the property:

$$(4b') \quad \sigma(A,B) = 0 \text{ if and only if } A = 1 \text{ and } B = 0,$$

lead us to consider a special class of Fan et al.'s definition characterized by the conditions ($4a$), ($4b'$) and ($4c$), which we call *Class 1* of weak fuzzy subsethood measures.

Under these conditions and taking into account the following two considerations:

1. The construction methods of subsethood measures [6, 7];
2. The condition ($4a$) is similar to one of the properties usually required from the implication operators,

lead us to study construction methods of weak fuzzy subsethood measures and specifically those of Class 1, using aggregation functions that satisfy special properties. The construction is as follows:

$$(1) \quad \begin{aligned} \sigma(A,B) &= \mathop{M}_{i=1}^{N} I(\mu_A(x_i), \mu_B(x_i)) = \\ M\Big(&I(\mu_A(x_1), \mu_B(x_1)), I(\mu_A(x_2), \mu_B(x_2)), \ldots, I(\mu_A(x_N), \mu_B(x_N)) \Big) \end{aligned}$$

with $A, B \in \mathcal{FS}(X)$ and M an aggregation function in the sense of Definition 3.

This objective lead us to study under which conditions $I : [0,1]^2 \to [0,1]$ satisfy the property ($P1$), so, when we aggregate this functions with the appropriate aggregation functions (without absorbing element 0), the weak fuzzy subsethood measure satisfies ($4b'$).

Also, we are interested to know when our constructions satisfy Axiom 10 of Sinha and Dougherty, for which we need to analyze the property ($P2$).

We have obtained Class 1 from the definition given by Fan et al. by strengthening the condition ($4b$). We also want to study construction methods for fuzzy subsethood measures that satisfy Axiom 1 instead of ($4a$), maintaining ($4b'$) and ($4c$) (these fuzzy subsethood measures are non-weak). This study is needed because if the Axiom 1 is satisfied, then the condition ($4a$) is always fulfilled, but not the other way around. For this reason, we propose another special class of fuzzy subsethood measures, characterized by (Axiom 1), ($4b'$) and ($4c$), which we call *Class 2* of fuzzy subsethood measures.

Note that any fuzzy subsethood measure from Class 2 is a weak fuzzy subsethood measure from Class 1 too, but not the other way around.

At the end of the paper, we also present construction methods for these measures by aggregating functions that satisfy special properties (using equation (1)).

4. Theoretical Study

In this section, we study the properties ($P1$) and ($P2$). We first study when they are fulfilled separately, and then we consider the relations with other possible properties of implication operators. We also consider when the functions constructed from Hamacher family satisfy ($P1$) and ($P2$).

4.1 Property $(P1)$

Proposition 1 *If $f : [0,1]^2 \to [0,1]$ satisfies the property $(P1)$, then f satisfies $I5$.*

In [5] the following Lemma and Proposition are proven:

Lemma 2 *Let $f : [0,1]^2 \to [0,1]$ be any function that satisfies at least one of the following items:*

i) If f satisfies $I2$, $I6$ and $I9$, or

ii) If f satisfies $I1$, $I6$ and $I9$, or

iii) If f satisfies $I4$, $I7$, $I9$ and $I11$, or

iv) If f satisfies $I6$, $I7$, $I9$ and $f(x,x) = f(0,x)$ for all $x \in [0,1]$,

then f fulfills $(P1)$.

Proposition 2 *If I is an S-implication or a QL-implication, then I satisfies property $(P1)$.*

Proof. If I is an S-implication or a QL-implication, we know by [5] that it satisfies $I2$, $I6$ and $I9$, then by item i) of Lemma 2 it satisfies $(P1)$. ■

4.2 Property $(P2)$:

Lemma 3 *Let n be a strong negation such that $n(x) \geq 1 - x$ for all $x \in [0,1]$. Let $f : [0,1]^2 \to [0,1]$ be any function that satisfies at least one of the following items:*

i) If f satisfies $I10$, or

ii) If f satisfies $I1$, $I6$, or

iii) If f satisfies $I2$, $I7$, $I8$,

then f fulfills $(P2)$.

Proof. i) $f(x,y) + f(x,n(y)) \geq y + n(y) \geq 1$ for all $x, y \in [0,1]$. The rest of the items are proven in a similar way, taking into account Lemma 1. ■

Proposition 3 *Let n be a strong negation such that $n(x) \geq 1 - x$ for all $x \in [0,1]$. If I is an S-implication or an R-implication, then $(P2)$ holds with n and any T, S.*

Proof. If I is an S-implication, we know by [5] that it satisfies $I1$ and $I6$. By item ii) of Lemma 3 then it satisfies $(P2)$.

If I is an R-implication, we know by [5] that it satisfies $I1$. Let us see that it also satisfies $I6$ in order to apply Lemma 3. $I(1,x) = \sup\{z | T(1,z) \leq x\} = \sup\{z | z \leq x\} = x$. ■

4.3 Properties (P1), (P2) and Property $I14$

From this point, we start studying when properties $(P1)$ and $(P2)$ are fulfilled at the same time with other properties usually required from implication operators. In this subsection, we study these properties together with property $I14$. This is an important relation because the weak fuzzy subsethood measures constructed from functions satisfying these three properties allow us to construct fuzzy entropies presented in Section 5.1.1.

Theorem 4 *The only S-implication associated to a t-conorm S and a strong negation n such that $n(x) \geq 1 - x$ for all $x \in [0, 1]$ that satisfies $(P1)$, $(P2)$ and $I14$ is $I(x, y) = \vee(n(x), y)$.*

Proof. Considering Theorem 3, we only need to prove the property $(P2)$. $I(x, y) + I(x, n(y)) = \vee(n(x), y) + \vee(n(x), n(y)) \geq y + n(y) \geq y + 1 - y = 1$. ■

This is an important result, because when Theorem 4 holds, it is easy to prove the property $(P3)$. Then the functions satisfying Theorem 4 allow us to construct proper weak fuzzy subsethood measures (not fuzzy subsethood measures).

4.4 Properties (P1), (P2), (P3) and automorphisms

In this subsection, we show how to construct functions that satisfy properties $(P1)$, $(P2)$ and $(P3)$ based on automorphisms.

Theorem 5 *Let φ be an automorphism of the unit interval such that $\varphi(x)+\varphi(1-x) \leq 1$ for all $x \in [0, 1]$. Under these conditions, if $n(x) = \varphi^{-1}(1 - \varphi(x))$, then*

$$I(x, y) = \varphi^{-1} (1 - \varphi(x) + \varphi(x)\varphi(y))$$

satisfies properties $(P1)$, $(P2)$ and $(P3)$.

Proof. Since $\varphi(x)+\varphi(1-x) \leq 1$, we have $n(x) = \varphi^{-1}(1-\varphi(x)) \geq \varphi^{-1}(\varphi(1-x)) = 1 - x$ for all $x \in [0, 1]$.

By Theorem 1 we have that

$$I(x, y) = n\Big(\varphi^{-1}(\varphi(x) \cdot \varphi(n(y)))\Big) = \varphi^{-1} (1 - \varphi(x) + \varphi(x)\varphi(y))$$

is strictly monotone and fulfills $I7$, $I9$ and $I13$. By Lemma 1 we have that I also fulfills $I6$, $I12$ and $I10$. Therefore, $I(x, y) + I(x, n(y)) \geq y + n(y) \geq y + 1 - y = 1$. Also, $I(x, y) = 1 = n(\varphi^{-1}(\varphi(x) \cdot \varphi(n(y))))$, then $0 = \varphi(x) \cdot \varphi(n(y))$. Therefore, $x = 0$ or $y = 1$. Using Lemma 2, it follows that I also satisfies property $(P1)$. ■

Example 1 *Let the automorphism be $\varphi(x) = x^2$, which satisfies $\varphi(x)+\varphi(1-x) \leq 1$ and $n(x) = (1 - x^2)^{\frac{1}{2}} \geq 1 - x$. In these conditions the implication operator*

$$I(x, y) = \left(1 - x^2 + x^2y^2\right)^{\frac{1}{2}}$$

fulfills properties $(P1), (P2)$ and $(P3)$ by Theorem 1.

Evidently, if we take the identity automorphism, then $n(x) = 1 - x$ and the implication $I(x, y) = 1 - x + xy$ (Rechenbach's implication operator). It satisfies properties $(P1), (P2)$ and $(P3)$ by Theorem 5.

4.5 Properties (P1), (P2), $I8$ and automorphisms

In the following theorem, we present a characterization based on automorphisms of functions $I : [0, 1]^2 \to [0, 1]$ that satisfy axioms $I1, I7, I8, I9, I13$ and property $(P2)$.

Theorem 6 *A function $I : [0, 1]^2 \to [0, 1]$ satisfies $I1, I7, I8, I9, I13$ and $(P2)$ with a strong negation n, if and only if there exists an automorphism φ of the unit interval such that*

$$\begin{aligned} I(x, y) &= \varphi^{-1}\Big(\wedge(\varphi(n(x)) + \varphi(y), 1)\Big) \\ n(x) &= \varphi^{-1}(1 - \varphi(x)) \\ n(x) &\geq 1 - x. \end{aligned}$$

Proof. We use Theorem 2, which is a generalization of Smets and Magrez's Theorem (see [5]) with the condition $n(x) \geq 1 - x$. ■

Remark: Please note that in the conditions of Theorem 6 to say that $n(x) \geq 1-x$ for all $x \in [0, 1]$ is equivalent to the automorphism φ fulfilling the condition:

$$\varphi(x) + \varphi(1 - x) \leq 1, \quad \text{for all } \; x \in [0, 1].$$

Remark: If I satisfies conditions of Theorem 6, it also satisfies, $I2$, $I3$, $I4$, $I5$, $I6$, $I10$, $I11$ and $I12$ (by Lemma 1, see [17]-[22], [38, 39]). Also, by Lemma 2, I satisfies property $(P1)$. Moreover, since it satisfies $I8$, then it does not satisfy property $(P3)$.

Example 2 *Let the automorphism be $\varphi(x) = x^2$. Evidently:*

$$\varphi(x) + \varphi(1 - x) \leq 1.$$

Under these conditions we have:

$$\begin{aligned} n(x) &= (1 - x^2)^{\frac{1}{2}} \\ I(x, y) &= \Big(\wedge(1 - x^2 + y^2, 1)\Big)^{\frac{1}{2}} \end{aligned}$$

fulfills properties $(P1)$ and $(P2)$ by Theorem 6.

4.6 Property (P1), (P2) and the Hamacher family

The Hamacher family of t-norms, t-conorms and strong negations is defined as follows [23]:

$$T_\alpha(x,y) = \frac{xy}{\alpha + (1-\alpha)(x+y-xy)}, \quad \alpha \geq 0$$
$$S_\beta(x,y) = \frac{x+y-(\beta-1)xy}{1+\beta xy}, \quad \beta \geq -1$$
$$n_\gamma(x) = \frac{1-x}{1+\gamma x}, \quad \gamma > -1. \text{ (Sugeno's class)}$$

In this section, we study rational functions constructed from the Hamacher family that fulfill $(P1)$ and $(P2)$.

Proposition 4 *Let S_β and n_γ belong to the Hamacher family. Then*

$$I(x,y) = \frac{n(x)+y+(\beta-1)n(x)y}{1+\beta n(x)y} = n\left(\frac{xn(y)}{\alpha+(1-\alpha)(x+n(y)-xn(y))}\right)$$

with $\beta \geq -1$, $\gamma > -1$, $\alpha = \frac{1+\beta}{1+\gamma} \geq 0$, and $n = n_\gamma$, satisfies $I10$.

Proof. Evidently, I is an S-implication and thus satisfies $I1$ and $I6$, and by Lemma 1 we have that I satisfies $I10$. ■

Corollary 1 *Under the conditions of Proposition 4, if $-1 < \gamma \leq 0$, then $(P2)$ is satisfied.*

Proof. Since we know that I satisfies $I10$, then $I(x,y) + I(x,n(y)) \geq y + n(y)$. Moreover, since $-1 < \gamma \leq 0$, then

$$n(x) = \frac{1-x}{1+\gamma x} \geq 1-x.$$

Therefore, $I(x,y) + I(x,n(y)) \geq y + n(y) \geq y + 1 - y = 1.$ ■

It is important to point out that in the conditions of Corollary 1 the properties $(P1)$ and $(P3)$ are also fulfilled. So, they do not fulfill $I8$; therefore, we can use the implication operators of the Proposition 4 to construct proper weak fuzzy subsethood measures.

5. Construction Methods

In this section, we present construction methods of fuzzy and weak fuzzy subsethood measures by aggregating functions $I : [0,1]^2 \rightarrow [0,1]$ we have studied in the previous sections. We use the aggregation functions presented in Section 2.4, our construction is based on the formula (1).

We also present how to construct fuzzy entropies from weak fuzzy subsethood measures, proving that the measures that do not fulfill Axiom 1 allow us to construct fuzzy entropies, as Young pointed out.

5.1 Construction of weak fuzzy subsethood measures

The following theorem together with the subsequents corollaries allow us to construct weak fuzzy subsethood measures that also can satisfy the Axiom 10 given by Sinha and Dougherty.

Theorem 7 *Let n be a strong negation and let $\sigma : \mathcal{FS}(X)\times\mathcal{FS}(X) \to [0,1]$, given by (1), where: $M : [0,1]^N \to [0,1]$ is an aggregation function, and $I : [0,1]^2 \to [0,1]$ is a function that satisfies $I1$, $I2$ and $I(0,0) = I(0,1) = I(1,1) = 1$. Then σ is a weak fuzzy subsethood measure on X.*

Proof.

1) $\sigma(0,0) = \overset{N}{\underset{i=1}{M}}(I(0,0)) = M(1,\cdots,1) = 1;$

$\sigma(0,1) = \overset{N}{\underset{i=1}{M}}(I(0,1)) = M(1,\cdots,1) = 1;$

$\sigma(1,1) = \overset{N}{\underset{i=1}{M}}(I(1,1)) = M(1,\cdots,1) = 1;$

2) $\sigma(1,0) = \overset{N}{\underset{i=1}{M}}(I(1,0)) = M(0,\cdots,0) = 0;$

3) If $A \le B$, then $I(\mu_A(x),\mu_C(x)) \ge I(\mu_B(x),\mu_C(x))$ for all $x \in X$, considering that M is increasing, we have $\sigma(A,C) \ge \sigma(B,C)$. If $A \le B$, then $I(\mu_C(x),\mu_A(x)) \le I(\mu_C(x),\mu_A(x))$ for all $x \in X$, therefore $\sigma(C,A) \le \sigma(C,B)$. ■

Corollary 2 *Under the conditions of Theorem 7 if I satisfies $(P1)$ and M is an aggregation function without absorbing element 0, then σ is a weak fuzzy subsethood measure of Class 1.*

Corollary 3 *Under the conditions of Theorem 7 if M satisfies:*

$$M(x_1,\cdots,x_N) + M(1-x_1,\cdots,1-x_N) \ge 1$$

and I satisfies $(P2)$ with n a strong negation, then σ satisfies:

$$\sigma(A,B) + \sigma(A,\bar{B}) \ge 1.$$

Proof.

$$\sigma(A,B) + \sigma(A,\bar{B}) = \overset{N}{\underset{i=1}{M}}(I(\mu_A(x_i),\mu_B(x_i))) + \overset{N}{\underset{i=1}{M}}(I(\mu_A(x_i),n(\mu_B(x_i)))) \ge$$
$$\overset{N}{\underset{i=1}{M}}(I(\mu_A(x_i),\mu_B(x_i))) + \overset{N}{\underset{i=1}{M}}(1 - I(\mu_A(x_i),\mu_B(x_i))) \ge$$
$$\overset{N}{\underset{i=1}{M}}(I(\mu_A(x_i),\mu_B(x_i))) + 1 - \overset{N}{\underset{i=1}{M}}(I(\mu_A(x_i),\mu_B(x_i))) = 1. \quad ■$$

Corollary 4 *Under the conditions of Corollary 3 if I satisfies $(P1)$ and M is an aggregation function without absorbing element 0, then σ is a weak fuzzy subsethood measure of Class 1 that satisfies Axiom 10 of Sinha and Dougherty.*

Example 3 *It is easy to see that* $M(x_1, \cdots, x_N) = \frac{1}{N}\sum_{i=1}^{N} x_i$ *and* $I(x, y) = \vee(1 - x, y)$ *satisfy the conditions of Theorem 7 and Corollary 3. Therefore,*

$$\sigma(A, B) = \frac{1}{N}\sum_{i=1}^{N} \vee (1 - \mu_A(x_i), \mu_B(x_i))$$

is a weak fuzzy subsethood measure of Class 1 that satisfies Axiom 10 of Sinha and Dougherty for fuzzy subsethood measures.

Remark: In Example 3 we show that our construction methods allow us to construct the weak inclusion [14] presented in the introduction of this paper.

Example 4 *The minimum* $M(x_1, \cdots, x_N) = \bigwedge_{i=1}^{N} x_i$ *and the product* $M(x_1, \cdots, x_N) = \prod_{i=1}^{N} x_i$ *and* $I(x, y) = 1 - x + xy$ *satisfy the conditions of Theorem 7 but not those of Corollary 3 (both* M *have absorbing element* 0*). Therefore,*

$$\sigma(A, B) = \bigwedge_{i=1}^{N} (\vee(1 - \mu_A(x_i) + \mu_A(x_i)\mu_B(x_i)))$$

$$\sigma(A, B) = \prod_{i=1}^{N} (\vee(1 - \mu_A(x_i) + \mu_A(x_i)\mu_B(x_i)))$$

are weak fuzzy subsethood measure of Fan et al. (but not of Class 1) that satisfy Axiom 10 of Sinha and Dougherty for fuzzy subsethood measures.

Remark: In Example 3 and 4 we show that depending on the aggregation function, we can construct measures either of Class 1 or not. For the former, we have to use aggregation functions, such as arithmetic mean, quadratic mean or maximum. While for the latter, we have to use aggregation functions with absorbing element 0, such as minimum, product, harmonic mean or geometric mean. In fact, by using any continuous t-conorm, we obtain measures of Class 1, and by using any continuous t-norm, we obtain measures of Fan et al. not of Class 1.

From this point, Corollaries 5 and 6 and Theorem 8 allow us to present the construction of fuzzy entropies from weak fuzzy subsethood measure of Class 1.

Corollary 5 *Let* n *be a strong negation such that* $n(x) \geq 1 - x$ *for all* $x \in [0, 1]$ *and let* $\sigma : \mathcal{FS}(X) \times \mathcal{FS}(X) \to [0, 1]$*, given by (1), where* $M : [0, 1]^N \to [0, 1]$ *is an aggregation function without absorbing element* 0 *that satisfies*

$$M(x_1, \cdots, x_N) + M(1 - x_1, \cdots, 1 - x_N) \geq 1$$

and I *is an S-implication associated to a t-conorm* S *and a strong negation* n*. Then* σ *is a weak fuzzy subsethood measure of Class 1 on* X *that satisfies* $\sigma(A, B) + \sigma(A, \bar{B}) \geq 1$*.*

Proof. By using Propositions 2 and 3, we obtain the result. ■

Theorem 8 *Let* n *be a strong negation and let* $\sigma : \mathcal{FS}(X) \times \mathcal{FS}(X) \to [0,1]$, *given by* (1) *a weak fuzzy subsethood measure of Class 1 on* X, *where* $M : [0,1]^N \to [0,1]$ *is an aggregation function that is idempotent and without absorbing element* 0. *Under these conditions the following are equivalent*

a) $I(x,y) = \vee(n(x), y)$.

b) $\sigma(A, \bar{A}) = \underset{i=1}{\overset{N}{M}} n(\mu_A(x_i))$
if $A = \{(x, \mu_A(x)) | \ \mu_A(x) = k \ x \in X\}$, *then* $\sigma(A, \bar{A}) = n(k)$

Proof. ($a \Rightarrow b$) It is enough to take into account Theorem 4 and Corollary 5. ($b \Rightarrow a$) If $A = \{(x, \mu_A(x)) | \ \mu_A(x) = p \ \forall x \in X\}$, then we have that $I(p, n(p)) = M(I(p, n(p)), \cdots, I(p, n(p))) = \sigma(A, \bar{A}) = n(p)$. By Theorem 4 we know that the only S-implication that satisfies $I(p, n(p)) = n(p)$ for all $p \in [0,1]$ is $I(x,y) = \vee(n(x), y)$. The rest of the proof is direct. ■

Corollary 6 *Under the conditions of Theorem 8, if* n *is a strong negation such that* $n(x) \geq 1 - x$ *for all* $x \in [0,1]$ *and* $M(x_1, \cdots, x_N) + M(1-x_1, \cdots, 1-x_N) \geq 1$, *then* σ *satisfies Axiom 10.*

5.1.1 Fuzzy entropy and weak fuzzy subsethood measures

As Young pointed out, the fuzzy subsethood measures that do not fulfill the first axiom based on Zadeh's order definition allow to construct entropies from them. We present in this subsection how to construct fuzzy entropies from the weak fuzzy subsethood measures. The entropy of a fuzzy set gives us useful information about it [10], so measures that allow to construct them are very useful too.

There exist important studies about the concept of fuzzy entropy [30]. We know that a measure of fuzzy entropy assesses the amount of vagueness, or fuzziness, in a fuzzy set. In 1972 De Luca and Termini [13] formalized the properties of fuzzy entropy through the following axioms:

Definition 7 *(De Luca and Termini [13]) A real function* $E : \mathcal{FS}(X) \to [0,1]$ *is called an entropy on* $\mathcal{FS}(X)$, *if* E *has the following properties:*

($E1$) $E(A) = 0$, *if and only if* A *is non fuzzy;*

($E2$) $E(A) = 1$, *if and only if* $A = \widetilde{e}$;

($E3$) $E(A) \leq E(B)$, *if* A *refines* B; *that is,* $\mu_A(x) \leq \mu_B(x)$ *when* $\mu_B(x) \leq e$ *and* $\mu_A(x) \geq \mu_B(x)$ *when* $\mu_B(x) \geq e$;

($E4$) $E(A) = E(\bar{A})$,

where $\widetilde{e} = \{(x, \mu_{\widetilde{e}}(x)) | \ \mu_{\widetilde{e}}(x) = e \ \forall x \in X\}$ *with* e *is the equilibrium point of the strong negation considered.*

Theorem 9 *Let n be a strong negation and let $\sigma : \mathcal{FS}(X)\times\mathcal{FS}(X) \to [0,1]$, given by (1), a weak fuzzy subsethood measure of Class 1 on X where $M : [0,1]^N \to [0,1]$ is an aggregation function that is idempotent, strictly increasing and without absorbing element 0, we have*

$$E(A) = \frac{\sigma(\vee(A,\bar{A}), n(\vee(A,\bar{A})))}{\sigma(\widetilde{e},\widetilde{e})} \text{ for all } A \in \mathcal{FS}(X)$$

is a fuzzy entropy on X.

Proof.

($E1$) $E(A) = 0 \Leftrightarrow \sigma(\vee(A,\bar{A}), n(\vee(A,\bar{A}))) = 0 \Leftrightarrow \vee(\mu_A(x_i), n(\mu_A(x_i))) = 1$ for all $x_i \in X$, if and only if $\mu_A(x_i) = 1$ or $\mu_A(x_i) = 0$ for all $x_i \in X$.

($E2$) $E(A) = 1 \Leftrightarrow \sigma(\vee(A,\bar{A}), n(\vee(A,\bar{A}))) = \sigma(\widetilde{e},\widetilde{e}) \Leftrightarrow$ $\overset{N}{\underset{i=1}{M}} n(\vee(\mu_A(x_i), n(\mu_A(x_i))))) = N \cdot e$. We have two possibilities:

a) $\mu_A(x_i) = e$ for all $x_i \in X$. In this case the property is proven.

b) If $\mu_A(x_i) < e$, then $n(\mu_A(x_i)) > e > \mu_A(x_i)$. If $\mu_A(x_i) > e$, then $n(\mu_A(x_i)) < e < \mu_A(x_i)$. In any of two cases $\wedge(n(\mu_A(x_i)), \mu_A(x_i)) < e$. Taking into account that M is strict, we have a contradiction, so, only the case a) can occur.

($E3$) If $\mu_A(x_i) \leq \mu_B(x_i) \leq e$, then $n(\mu_A(x_i)) \geq n(\mu_B(x_i)) \geq e \geq \mu_B(x_i) \geq \mu_A(x_i)$, therefore $\wedge(n(\mu_A(x_i)), \mu_A(x_i)) = \mu_A(x_i) \leq \wedge(n(\mu_B(x_i)), \mu_B(x_i)) = \mu_B(x_i)$. If $\mu_A(x_i) \geq \mu_B(x_i) \geq e$, then $n(\mu_A(x_i)) \leq n(\mu_B(x_i)) \leq e \leq \mu_B(x_i) \leq \mu_A(x_i)$, therefore $\wedge(n(\mu_A(x_i)), \mu_A(x_i)) = n(\mu_A(x_i)) \leq \wedge(n(\mu_B(x_i)), \mu_B(x_i)) = n(\mu_B(x_i))$. Therefore $E(A) \leq E(B)$.

($E4$) Direct because n is a strong negation. ■

Example 5 *Taking the arithmetic mean $M(x_1,\ldots,x_N) = \frac{1}{N}\sum_{i=1}^{N} x_i$ as an aggregation function, we have:*

$$E(A) = \frac{1}{N \cdot e}\sum_{i=1}^{N} n(\vee(\mu_A(x_i), n(\mu_A(x_i))))$$

is a fuzzy entropy in the sense of De Luca and Termini.

Example 6 *Taking the dual of harmonic mean $M(x_1,\ldots,x_N) = 1 - \frac{N}{\sum_{i=1}^{N}\frac{1}{1-x_i}}$ as an aggregation function, we have:*

$$E(A) = \frac{1}{e}\cdot\left(1 - \frac{N}{\sum_{i=1}^{N}\frac{1}{1-n(\vee(\mu_A(x_i), n(\mu_A(x_i))))}}\right)$$

is a fuzzy entropy in the sense of De Luca and Termini.

5.2 Construction of fuzzy subsethood measures

Theorem 10 *Let n be a strong negation generated by the automorphism φ such that*

$$\varphi(x) + \varphi(1-x) \leq 1 \text{ for all } x \in [0,1]$$

and let $\sigma : \mathcal{FS}(X) \times \mathcal{FS}(X) \to [0,1]$, given by (1), where $M : [0,1]^N \to [0,1]$ is an aggregation function that satisfies $M(x_1, \cdots, x_N) + M(1-x_1, \cdots, 1-x_N) \geq 1$ and neither 0 nor 1 are absorbing elements of M. Under these conditions

$$\sigma(A,B) = \underset{i=1}{\overset{N}{M}}\left(\varphi^{-1}\left(\wedge(\varphi(n(\mu_A(x_i))) + \varphi(\mu_B(x_i)), 1)\right)\right)$$

is a fuzzy subsethood measures of Class 2 that satisfies Axiom 10.

Proof. The proof is similar to that of Theorem 7 and Corollary 3 taking into account Theorem 6. To show that Axiom 1 is fulfilled, by Theorem 6 I satisfies $I8$ and using the condition imposed on M, we obtain the result. ■

Example 7 *It is easy to see that taking the arithmetic mean $M(x_1, \ldots, x_N) = \frac{1}{N}\sum_{i=1}^{N} x_i$ and $\varphi(x) = x$, we have $I(x,y) = \wedge(1-x+y, 1)$ (implication of Lukasiewicz), satisfy the conditions of Theorem 10. Therefore,*

$$\sigma(A,B) = \frac{1}{N}\sum_{i=1}^{N} \wedge (1-x+y, 1)$$

is a fuzzy subsethood measure of Class 2 that fulfills Axiom 10 of Sinha and Dougherty for fuzzy subsethood measures.

Example 8 *Taking the quadratic mean $M(x_1, \ldots, x_N) = \sqrt{\frac{1}{N}\sum_{i=1}^{N} x_i^2}$ and $\varphi(x) = x^2$, we have $I(x,y) = (\wedge(1-x^2+y^2, 1))^{\frac{1}{2}}$, satisfy the conditions of Theorem 10. Therefore,*

$$\sigma(A,B) = \sqrt{\frac{1}{N}\sum_{i=1}^{N} \wedge (1-x^2+y^2, 1)}$$

is a fuzzy subsethood measure of Class 2 that fulfills Axiom 10 of Sinha and Dougherty for fuzzy subsethood measures.

6. Conclusions and Future Lines of Research

Given the similarity between the conditions required in the definitions of weak fuzzy subsethood measures and some of the conditions usually imposed on implication operators, we have proposed a method of construction of these measures by aggregating implication operators with special properties. We have carried out a systematic study of the implication operators properties which translate into useful properties of weak fuzzy subsethood measures.

We also have studied the conditions under which such measures satisfy the Axiom 10 given by Sinha and Dougherty. The results obtained will be used in approximate reasoning in the future.

Finally, we have proven that weak fuzzy subsethood measures from Class 1 allow us to construct fuzzy entropies.

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