Functional Observability

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Abstract—A simple theorem for Functional Observability is presented considering the observable and unobservable states of a system based on Kalman decomposition. The proposed theorem is also consistent with two other theorems on Functional Observability which was based on eigen decomposition [6]. The paper also reports a new definition for Functional Observability which is consistent with previously reported definitions and theorems [4], [5], [6].

I. INTRODUCTION

While state Observability is a precondition to design state observers, state Observability is not a necessary precondition to design functional observers, it should be Functional Observability instead. When the system is not entirely observable it is still possible design functional observers. Consider a system in statespace form with with A being the system matrix, C being the output matrix and $L_0$ being the functions to be estimated. The concept of Functional Observability, and also a necessary and sufficient condition for a triple $(A, C, L_0)$ to be Functional Observable was introduced in [4]. The concept of Functional Observability is a generalization of the concept of Observability i.e., when $L_0$ is chosen as the identity matrix then the concept of Functional Observability reduces to state Observability. In [6], two theorems for a triple $(A, C, L_0)$ to be Functional Observable was presented based on eigen decomposition. The contribution of this paper is in reporting a new equivalent definition for Functional Observability and also a simple theorem for Functional Observability considering observable and unobservable states of a system based on Kalman decomposition. Both, the definition and the theorem for Functional Observability reported in this paper are intuitive and draws a connection to our previously reported results.

II. MAIN RESULTS

Consider an $n$-th order linear time-invariant dynamical system, without loss of generality we can assume that the dynamical system is in the following Kalman decomposition form:

$$
\begin{bmatrix}
\dot{x}_o(t) \\
\bar{x}_o(t)
\end{bmatrix} =
\begin{bmatrix}
A_{11} & 0 \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
x_o(t) \\
x_\bar{o}(t)
\end{bmatrix} +
\begin{bmatrix}
B_1 \\
B_2
\end{bmatrix} u(t) \tag{1}
$$

$$
y(t) =
\begin{bmatrix}
C_1 & 0
\end{bmatrix}
\begin{bmatrix}
x_o(t) \\
x_\bar{o}(t)
\end{bmatrix} \tag{2}
$$

where $x_o(t) \in \mathbb{R}^m$ and $x_\bar{o}(t) \in \mathbb{R}^{(n-m)}$ represent the observable and unobservable states of the system respectively. Let the function to be estimated given by:

$$
z(t) = L_0 \begin{bmatrix} x_o(t) \\ x_\bar{o}(t) \end{bmatrix} = \begin{bmatrix} L_0' & L''_0 \end{bmatrix} \begin{bmatrix} x_o(t) \\ x_\bar{o}(t) \end{bmatrix} \tag{3}
$$

where $L_0'$ represent the linear combinations of the observable states and $L''_0$ represent the linear combinations of the unobservable states in the function to be estimated $z(t)$. Let us consider the following definitions:

**Definition 1:** The linear function $L_0 x(t)$ is Functional Observable if and only if there exists a finite time $T$ such that the initial value of the function $L_0 x(t)$ can be determined from the observation history $y(t)$ given the control $u(t), 0 \leq t \leq T$.

**Definition 2:** The entire state vector $x(t)$ is Observable if and only if there exists a finite time $T$ such that the initial value of the state vector $x(0)$ can be determined from the observation history $y(t)$ given the control $u(t), 0 \leq t \leq T$.

**Definition 3:** The entire state vector $x(t)$ is Unobservable if and only if the initial value of the state vector $x(0)$ (and hence any functional relationship of $x(0)$) cannot be determined from the system output $y(t)$ that has been observed through any finite time interval $T$ given the control $u(t), 0 \leq t \leq T$.

**Theorem 1:** The triple $(A, C, L_0)$ is Functional Observable if and only if the linear function to be estimated is a function of observable states only.

**Proof:** According to Definition 2 all the initial observable states $x_o(0)$ can be determined from the output $y(t)$ that has been observed through a finite time interval $T$, hence any linear combination of $x_o(0)$ can also be determined, and furthermore according to Definition 1 it follows that any linear combination of $x_o(t)$ is Functional Observable. The necessity of the proof follows from Definition 3.

Now consider the following definition for Functional Observability.

**Definition 4:** The triple $(A, C, L_0)$ is Functional Observable if and only if there exists an $L \in \mathbb{R}^{m \times n}$, rows$(L_0) \leq m \leq n$, such that $\mathcal{R}(L) \supseteq \mathcal{R}(L_0)$ and $L$ satisfies conditions 1 and 2 below (the row space of a matrix is written using the symbol $\mathcal{R}$ and rows$(\cdot)$ represents the number of rows of $(\cdot)$).
Based on Definition 4 and the existence of a minimum dimension matrix \( L \) that satisfies Condition 1 and 2, a necessary and sufficient condition for the triple \((A, C, L_0)\) to be Functional Observable was reported in [4]. Based on Definition 4 a necessary and sufficient condition for the triple \((A, C, L_0)\) to be Functional Observable was reported in [6].

In the following we draw a connection to previously reported results by using Definition 1 and also Theorem 1 to provide simple proofs to the two theorems reported in [6] which in turn also shows that all the three theorems on Functional Observability and also the definitions for Functional Observability are equivalent.

**Theorem 2**: [5] The triple \((A, C, L_0)\) is Functional Observable if and only if

\[
\text{rank} \begin{bmatrix} L_0 & C \\ sI - A & sI - A \end{bmatrix} = \text{rank} \begin{bmatrix} C \\ C \end{bmatrix} \quad \forall s \in \mathbb{C}. 
\]

**Proof**: Clearly, if \( L_0'' = 0 \) (i.e., \( z(t) \) is a linear combination of observable states only) then LHS and RHS of (6) is \( q \). If \( L_0'' \neq 0 \) (i.e., \( z(t) \) includes a linear combination of unobservable states) then LHS of (6) is greater than \( q \) but the RHS of (6) is \( q \).

**Theorem 3**: [6] The triple \((A, C, L_0)\) is Functional Observable if and only if

\[
\text{rank} \begin{bmatrix} C \\ CA \\
\vdots \\
CA^{n-1} \\
L_0 \\
L_0A \\
\vdots \\
L_0A^{n-1} \end{bmatrix} = \text{rank} \begin{bmatrix} C \\ CA \\
\vdots \\
CA^{n-1} \end{bmatrix}. 
\]

**Proof**: Clearly, if \( L_0'' = 0 \) then LHS and RHS of (7) is \( q \). If \( L_0'' \neq 0 \) then LHS of (7) is greater than \( q \) but the RHS of (7) is \( q \).

**Remark 1**: To test Functional Observability of a triple \((A, C, L_0)\) based on Theorem 1 or 2 it does not require any special structure for \( A, C \) or \( L_0 \), however requires the computation of eigen values and matrix ranks for Theorem 2 and matrix ranks only for Theorem 3. On the other hand ascertaining Functional Observability based on Theorem 1 requires transforming the system based on Kalman decomposition and checking if unobservable states are present in \( z(t) \) (i.e., requires checking if \( L_0'' \neq 0 \)), no rank computation is required.

**Remark 2**: When \( L_0 \) is chosen as the identity matrix then Definition 1 reduces to the state observability as in Definition 2. Furthermore, when \( L_0 \) is chosen as the identity matrix Condition 1 is satisfied for \( L = L_0 \) and Condition 2 reduces to the well known state Observability condition

\[
\begin{bmatrix} C \\ sI - A \end{bmatrix} = n, \forall s \in \mathbb{C},
\]

so Definition 4 also reduces to state Observability just as Definition 1.

**Remark 3**: When \( L_0 \) is chosen as the identity matrix then all three theorems reduces to the well known theorems in state Observability. Theorem 1 reduces to the system having no uncontrollable states, and also Theorem 2 and 3 reduces to (8) because the LHS of (6) and (7) is \( n \).

Ascertaining Functional Observability of a triple \((A, C, L_0)\) is a first step in the design of a Functional Observer just like ascertaining observability is a first step in the design of a state observer. However, it provides no information about the minimum order possible for the functional observer. Designing the minimum order functional observer requires finding the minimum dimension matrix \( L \) which satisfies Condition 1 and 2, and it requires the computation of some auxiliary matrices for a given \( A, C \) and \( L_0 \) on which the Functional Observability criteria is also based in [4]. Ascertaining Functional Observability based on Theorems 1 or 2 or 3 servers as a quick check of Functional Observability of a triple \((A, C, L_0)\).

### III. NUMERICAL EXAMPLES

**Example 1**: Consider \( A = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 1 & -1 & 2 \end{bmatrix} \), \( C = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \) and \( L_0 = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \). The pair \((A, C)\) is not Observable. The system has 2 observable states and one unobservable state, so \( q = 2 \). According to Theorem 1 the triple \((A, C, L_0)\) is Functional Observable because the linear function to be estimated is only a linear combination of Observable states. We also note that Theorem 2 and 3 also provide the same conclusion regrading the Functional Observability of the triple \((A, C, L_0)\) because the LHS and RHS of both (6) and (7) is \( q = 2 \).

**Example 2**: Now consider the same \( A \) and \( C \) as in Example 1 with \( L_0 = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \). Still the system has 2 observable states and one unobservable state, so \( q = 2 \). According to Theorem 1 the triple \((A, C, L_0)\) is not Functional Observable because the linear function to be estimated is a function of unobservable states. We also note that Theorem 2 and 3 also provide the same conclusion regrading the Functional Observability of the triple \((A, C, L_0)\) because the LHS of both (6) and (7) is 3 while the RHS of both (6) and (7) is only 2.

### IV. CONCLUSION

The paper presents a new definition for Functional Observability and also presents a simple theorem for Functional
Observability in terms of observable and unobservable states of a system. We also show that the proposed theorem is consistent with previously reported two theorems, Theorem 2 and 3, on Functional Observability and provides simple proofs based on the results of this paper.

REFERENCES