Non-Uniform Sparsity
in
Magnetic Resonance Imaging (MRI)

by

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Submitted in fulfilment of the requirements for the degree of
Doctor of Philosophy

Deakin University
March, 2014
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## Acronyms

<table>
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<th>Acronym</th>
<th>Description</th>
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<tbody>
<tr>
<td>AWGN</td>
<td>Additive White Gaussian Noise</td>
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<tr>
<td>CDF</td>
<td>Cumulative Distribution Function</td>
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<tr>
<td>CS</td>
<td>Compressive Sensing</td>
</tr>
<tr>
<td>DCT</td>
<td>Discrete Cosine Transform</td>
</tr>
<tr>
<td>DFT</td>
<td>Discrete Fourier Transform</td>
</tr>
<tr>
<td>FD</td>
<td>Finite Difference</td>
</tr>
<tr>
<td>fMRI</td>
<td>Functional Magnetic Resonance Imaging</td>
</tr>
<tr>
<td>GRAPPA</td>
<td>Generalized auto-calibrating partially parallel acquisition</td>
</tr>
<tr>
<td>MRI</td>
<td>Magnetic Resonance Imaging</td>
</tr>
<tr>
<td>MSE</td>
<td>Mean Square Error</td>
</tr>
<tr>
<td>MSSIM</td>
<td>Mean Structural SIMilarity index</td>
</tr>
<tr>
<td>NMR</td>
<td>Nuclear Magnetic Resonance</td>
</tr>
<tr>
<td>PDF</td>
<td>Probability Density Function</td>
</tr>
<tr>
<td>PILS</td>
<td>Partially parallel imaging with localized sensitivities</td>
</tr>
<tr>
<td>pMRI</td>
<td>Parallel Magnetic Resonance Imaging</td>
</tr>
<tr>
<td>PSF</td>
<td>Point Spread Function</td>
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<tr>
<td>PSNR</td>
<td>Peak Signal-to-Noise Ratio</td>
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RF  Radio Frequency
SENSE  Sensitivity encoding
SMASH  Simultaneous acquisition of spatial harmonics
SNR  Signal-to-Noise Ratio
SPIR-iT  iterative Self-consistent Parallel Imaging Reconstruction
Abstract

Magnetic Resonance Imaging (MRI) is one of the mostly used imaging techniques in hospitals for capturing images of human body for disease diagnosis and analysis. It differentiates very well between different kinds of tissues which makes it very useful for brain and cancer imaging. Ideally, MRI can be used for capturing live video stream which can be used during surgery, for diagnosis and educational purposes. However, there are some limitations in MR imaging. The imaging process is slow and bound to hardware constraints. It is costly in terms of time as well as motion sensitive which makes it hard for patients. This thesis contributes towards improving MR imaging process in terms of imaging speed and quality. Enhancing software capabilities can overcome hardware limitations to some extent. Thus, this work is based on the software, signal and image processing module of MRI.

This research explores sparsity distribution MR images. Sparsity of any image can be defined as the information content in that image. MR machines capture Fourier signals which are later converted into images. The first part of thesis analyses and identifies sparsity distribution of MR images. Different kinds of Images are used for analysis to understand sparsity distribution in more generic ways rather than making it application specific. Moreover, sparsity is also analysed in different domains other than image and Fourier. The experiments were further extended to localising the sparsity with sub-region of images thus, getting a better understanding of non-uniform nature of MR image sparsity.

The second part of thesis presents a novel method to use localise sparsity for MR image de-noising. MR images are corrupted by random Gaussian or Rician Noise. The proposed technique uses a simple method to remove this noise based
on rules and understanding of localised sparsity which was developed earlier. This method analyses and preserves energy contents of image after dividing it into a multiple local sections. The simple idea behind this technique is to maximise energy while minimising the number of non-zero coefficients. Thus, discarding as much noise data as possible and keeping only few carefully chosen coefficients.

The third part of thesis uses local sparsity and combines it with Compressive Sensing to achieve Rapid Imaging. The modified proposed approach to Compressive Sensing is named as Locally Sparsified Compressive Sensing. It uses multiple local sparsity constraints and L1 minimisation to reconstruct image from under-sampled data. Measuring fewer samples and reconstructing image from under-sampled data means reducing the image acquisition time and delays caused by MRI hardware. Moreover, a structured framework is presented to define shape, size n number of regions to use Compressive Sensing with local sparsity constraints. Different kinds of MR images were used for experiments and results were compared to simple Compressive Sensing. In comparison to simple Compressive Sensing, this method resulted in reducing sample set up to 30%.

In last part of thesis, Locally Sparsified Compressive Sensing was extended for two further applications. Firstly, to improve image quality and decreasing noise occurred due to under-sampled data measurements in simple Compressive Sensing. The basic idea was to use Locally Sparsified Compressive Sensing and exploiting the freedom of using multiple sparsity constraints and sampling levels within an image to improve image quality and reduce noise. Secondly, this developed framework is extended for Dynamic MRI which deals with multiple images captured closely over time to capture some change and motion like cardiac sequence. Detailed theory, analysis and experimental results are presented in this thesis.
Research Publications


Chapter 1

Introduction

MRI is one of the prominent medical imaging techniques. The aim of this research is to exploit non-uniform sparsity of MR images to reduce the imaging process time of MRI as well de-noising and getting better image quality.

1.1 Magnetic Resonance Imaging (MRI)

Magnetic Resonance Imaging (MRI) has become a vital tool for diagnosis of complex diseases. The internal composition of human organs and tissues can be effectively studied and explored with MR imaging. MRI is built on the principles of nuclear magnetic resonance (NMR). MRI uses electromagnetic properties of the hydrogen atoms (spins) for imaging. Human body is abundant with Hydrogen atoms. Spins inside human body act like magnets when contained in an outer magnetic field. Furthermore, gradient magnets are used along spatial axis to map real life objects into images. They change magnetic field of spins linearly. This allows us to target a specific area out of whole body. Radio Frequency (RF) waves are applied to excite spins. After excitation, spin release energy and go to their relaxation state. This released energy is captured and mapped as a signal to generate images. These signals vary due
to type of tissues. In other words, the relaxation time and amount of energy release vary based on proton density in tissues. Thus, captured signal can be mapped against different contrast levels. Received RF signals are composed of sine waves with frequency and phase information of spins. Digitisation of these signals returns a 2-D discrete Fourier which can be converted into image by applying inverse Fourier transform.

1.2 Sparsity

The idea of sparsity deals with the amount of useful information within a signal. To construct MR image from sensors, frequency domain is used. There are only a small number of coefficients which are actually significant and used to represent the image. Whereas, others coefficients are of no use at all or they have small significance that the effect of discarding them is negligible. This idea leads to another domain which says that if the total useful information lies within few significant coefficients than an image can be compressed to a very high level. Two types of sparsity are observed and hence used in image reconstruction techniques. Strongly sparse data set is the first category in which most of the coefficients are exactly zero whereas, if the coefficients are almost zero this is called weakly sparse data set.

1.3 Compressive Sensing

Compressive Sensing is the study of acquiring a signal in its compressed form. It works as if it is possible to acquire the required information directly. The literature survey in [3,4] investigates all the previous work done on the theory
of Compressive Sensing. The commonly used Nyquist sampling theorem states that, to regenerate an image without errors, the number of Fourier samples must be equal to the number of image pixels. However, signals and images have an important property of sparsity or compression which is responsible for the size reduction and efficient regeneration. Compressive Sensing is based on the idea that the information content of images is far lesser than their original size. Since, most of the data is constant or zero, it can be ignored. Traditional process of image generation acquires complete signal without ignoring anything. These large signals are needed to be compressed for size reduction and to make them communication and memory efficient. Compression procedures only retain the important information and discard the rest. Compressive sensing suggests if it is already known that most of the image values/coefficients will be discarded after acquiring than there is no need to acquire them in first place. Measuring all the image coefficients can be costly in terms of time, memory or money. Measurements should be done on the basis of compressed size of image/signal rather than its original or uncompressed size thus sensing and transmitting only a small number of adaptively chosen coefficients and discarding nothing. However, sparsity is a prerequisite in compressive sensing; it means that underlying signal should be sparse enough to be written as a superposition of small number of vectors in some transform domain. Transform sparsity generalize the compressive sensing even more. There is no need for signal to be sparse in its sensing domain. It can be sparse in any domain which is orthonormal to its sensing domain. This dose not only make the process general and easily adaptable but also orthonormal domain helps in better
recovery of under-sampled data. If the signal is S-sparse with total N elements, one can almost always reconstruct the exact signal by collecting K randomly chosen samples where

$$K > S \log N$$  \hspace{1cm} (1.1)

These samples are far fewer than N. The literature shows that using equi-spaced samples result in coherent noise and recovering the original signal is not possible. Therefore, K samples must be randomly chosen. The chance of recovery error in this case is not much worse than L2 norm of the signal. Study shows substituting this L2 with L1 give better result. Thus minimizing the L1 norm recovers S sparse signal x of size N using only K measurements or samples and the approximation error can be as good as one can achieve by knowing all entries of the signal. The literature shows that getting 5 random samples for each nonzero term of the signal and using nonlinear reconstruction method (L1 minimization) can recover the almost exact signal. The useful applications for Compressive Sensing are

- Data compression
- Channel coding
- Inverse problems
- Data acquisition

1.4 Motivation

Medical imagery is crucial for diagnosis and treatment. Many imaging techniques are currently being implemented in hospitals. Magnetic Resonance
Imaging is used to capture images of internal systems of the human body. It has an edge over other techniques; it differentiates clearly between all kinds of tissues, which make it extremely useful for brain and cancer. It works for all kinds of hard and soft tissues. Unlike X-Rays and CT scans, it does not expose patients to any harmful radiation. The real potential of MRI is far beyond this; theoretically it can even be used to capture videos of internal human structures rather than just static images. However, MRI is a time taking procedure. The amount of time it takes to generate images, makes it difficult to fully utilise the potential of this technique.

Moreover, resultant images from MRI are motion-sensitive. Patients must remain motionless for a diagnostic quality image, which is very challenging to achieve, particularly when it can take up to 30-40 minutes - the longer it takes the harder it is for patients to stay still. A particular challenge is dealing with pediatric patients. It is more feasible for doctors to administer other types of scans than to give anesthesia, especially to children and critical patients. Another problem related to traditional MRI, is the slow image acquisition process cause long queues. Patients have to wait, which delays diagnosis and proper treatment. Increasing the number of MRI machines can reduce the patient’s wait intervals, but this is not feasible, as machines are costly and hard to install. To reduce the waiting intervals, need for anesthesia and to utilize its full potential researchers are working on Rapid MRI. It is the study to speed up MR imaging process. This research also deals with Rapid MRI with better quality image acquisition.
1.5 Research Objectives

This research is based on non-uniform Sparsity of MR Image and how it can be used for image de-noising and rapid MRI. There are three main topics which are addressed in this thesis. First is to de-noise MR images by exploiting its local sparsity. The idea is to analyze small regions within an image and exploit its energy contents, then using combination of these local areas to generate improved quality image. Second is to develop a novel framework for making MR imaging time efficient using the theory of compressing sensing in collaboration with local sparsity. Compressive sensing states that the actual amount of information in a sparse signal is far lesser than what we traditionally measure. So, most of it can be discarded with negligible compromise on quality. Thus, signal measurements should be based on compressed size or information content rather than actual size. Lastly, the framework developed earlier was expanded and modified for dynamic MRI. Dynamic MRI handles an array of images and works in 3-D where motion of a targeted object is captured by acquiring multiple image frames.

1.6 Outline of Thesis

Chapter 2 of thesis presents basic principal, architectural, technical details and tear down view of MRI technique. Block level description and mathematical modeling of an MRI machine and how is MR image acquired is covered in this chapter. Last part of Chapter 2 is literature survey. It discusses brief introduction of time saving MRI techniques, quantification of MR image’s quality and details about noise in MR images.
Chapter 3 of the thesis discusses non-uniform sparsity of MR images, how sparsity varies within an image and in different domains. Experimental results for localised transform sparsity in MR images and its application using different sparsity levels. The non-uniform nature of sparsity is analysed in generic terms so that it can later be used for all different kinds of MR images.

Chapter 4 presents a novel and easy noise removal method for random thermal Rician and Gaussian noise. It uses the key features of non-uniform energy distribution of MR images as identified in chapter 3. The noise removal technique is applied with different noise levels and for different kinds of tissue images.

Applying idea of compressive sensing in collaboration with local sparsity into MR images for Rapid imaging is described in Chapter 5. A framework is developed for Locally Sparsified Compressive Sensing. Moreover, how to determine the size and number of sub regions before image acquisition. Experimentation for testing this algorithm for different kinds of images in different sparse domains.

In Chapter 6, the work of previous chapters is applied and extended for improving image quality and reducing noise cause by under-sampling. Another application of the algorithm developed in chapter 5 is Rapid Dynamic MRI. The framework of Locally Sparsified Compressive Sensing was modified to use with dynamic MRI using three different methodologies which are as follows: acquiring an array of images and applying Locally Sparsified Compressive Sensing to each image frame individually, in second approach it was applied collectively on block of all frames. In the end, Locally sparsified CS was
applied on difference images. Experimental results and procedure for applying these methods are defined in this chapter.

Chapter 7 concludes all the work and research finding and present ideas for future work.
Chapter 2

THEORY AND LITERATURE REVIEW

2.1 Introduction

This chapter presents the basic principles [5–7] and physics [8] of Magnetic Resonance Imaging (MRI) [9–11]. MRI is a medical imaging technique which works very well for soft tissues specifically for detection of malignant tissues. However, it is a time consuming method. Whole MR image cannot be captured in one acquisition due to rapid signal loss and other constraints. The image acquisition process is limited by hardware constraints [12] and physiological factors [13]. Magnets are bound to their slew rates and amplitudes. Excessive strength of magnetic field can cause damage to human nervous system [14]. It is crucial to speed up imaging process for better and quicker medical imagery. MRI has tremendous potential it can capture 2D and 3D images as well as videos if image acquisition is rapid enough. For speeding up MRI, many researchers are working on rapid MRI by exploiting the redundancy in data or image domain [15, 16] or using parallel imaging [17–19]. Moreover, MRI is effected by signal dependent Rician distribution noise [20]. De-noising MRI
is difficult due to signal dependent properties of noise as well as the noise in MRI varies based on spatial locations [21]. This chapter presents a brief review of previous literature, technologies and their advantages and drawbacks to acknowledge previous work as well as to point out gaps in literature.

2.2 Basic Principles of Magnetic Resonance Imaging (MRI)

MRI is an imaging technique employed in advanced medical facilities to study and generate images of internal structures of the human body. The history of MR imaging goes back in 1946. It was discovered that atomic nucleus exhibits a magnet like behavior when matter is placed within a magnetic field [22]. This behavior of atomic nucleus is known as Nuclear Magnetic Resonance (NMR). Later in early 1970s, Raymond Damadian discovered that tumor tissues have different NMR properties than normal tissues. Thus, NMR can be used to characterize malignant tissues. He and his team has built first image scanner based on NMR principles [23]. The name was later change to MRI because of word NUCLEAR in NMR.

2.2.1 Spins

The idea behind MRI is to use electromagnetic properties of the spins (hydrogen atoms) inside the human body. Human body is made up approximately 75% of water which makes Hydrogen atoms, the most abundant element. This large proportion of $H^+$ spin makes human imaging possible [1].
2.2.2 Spin Magnetism

In normal circumstances these spins are aligned in such a way that they cancel out each others magnetisation as shown in Fig.2.1. Spins with magnetisation vectors in opposite directions equalise the total force. Thus, net magnetisation of our body remains zero [24]. This net magnetisation is affected by $B_0$. When an outer homogenous magnetic field $B_0$ is applied to the whole body, all the spins align with $B_0$ [25]. This alignment happens in two directions, parallel and anti-parallel to $B_0$ as shown in Fig.2.2. More, Spins align parallel to $B_0$. Thus, generating a net magnetisation which produces NMR signals and can be used to generate images [26].

2.2.3 Boltzmann distribution

As, MRI machines capture signals from human body. This, signals should be strong enough that sensors can capture it. The signal strength is dependent on
outer magnetic field $B_0$, which can be explained using Boltzmann distribution.

$$N_{AP}/N_P = exp(\Delta E/kT)$$ (2.1)

Here $N_{AP}$ are spins in anti-parallel state and $N_P$ in parallel state and $N_{AP}/N_P$ is the ratio or difference between two. $\Delta E$ is the energy difference between two states which is directly proportional to outer magnetic field strength, $k$ is Boltzmann constant and $T$ is absolute temperature. In other words, greater field strength means higher energy as well as stronger and better NMR signals.

![Figure 2.3: Resonance of $H^+$ spin](image)

### 2.2.4 Resonance

In addition to their alignment along $B_0$, spins also resonate at a frequency based on $B_0$ as shown in Fig.2.3. This frequency is called Larmor Frequency and it can be calculated as

$$f_0 = (\gamma/2\pi)B_0$$ (2.2)

where $f_0$ is Larmor Frequency or Spin resonance frequency, $\gamma$ is a constant called Gyro Magnetic Ratio which is 42.58 MHz/T for Hydrogen [27].
2.2.5 Equilibrium

Due to outer homogenous magnetic field $B_0$, body shows net magnetisation in direction of $B_0$ which is represented by a vector along z-axis and the magnetisation is called longitudinal magnetisation or $M_z$. While, transverse magnetisation $M_{xy} = 0$ which means there is no magnetisation in X-Y Plane. This state is called as equilibrium state of spins and equilibrium magnetisation $M_0$ is equal to $M_z$ as shown in Fig.2.4.

![Equilibrium state](image)

Figure 2.4: Equilibrium state

2.2.6 Excitation

When Radio Frequency (RF) waves are applied, spins absorb this energy and jump to their excitation state. Two things happen in excitation state. Firstly, longitudinal magnetisation vector flips its direction to a certain angle depending on applied pulse.
$M_z$ becomes zero and spins show magnetisation in XY-Plane and $M_{xy}$ becomes active as shown in Fig.2.5. Spins in their equilibrium state precess in different angles or phases. While during excitation, they all get phase coherent with the applied frequency.

### 2.2.7 Relaxation

Once the RF waves are switched off, spins return to their equilibrium state thus releasing the gained energy. In this process $M_z$ grows back to its original state while $M_{xy}$ again becomes zero. The time in which magnetisation vector recovers its original state in called $T1$ or Spin-Lattice Relaxation as shown in Fig.2.6. The time constant $T1$ and recovery of $M_z$ can be stated as a function of time $t$

$$M_z(t) = M_0(1 - e^{-t/T1})$$  \hspace{1cm} (2.3)
Also, spins get de-phased or phase incoherent again as in their equilibrium state. This relaxation is called $T_2$ or Spin-Spin Relaxation as shown in Fig.2.7. It can be described using function of $M_{xy}$ at time $t$

$$M_{xy}(t) = M_{xy0}e^{-t/T_2} \tag{2.4}$$

Different contrast levels in any image are result of varying proton density,
the relaxation time $T_1$ and $T_2$ and many other physical properties of tissues [28]. RF sensors capture these emitted signals which are later digitised and used to generate images [29].

2.2.8 Bloch Equation

Bloch equation is a differential equation which describes X, Y and Z component of magnetisation under outer magnetic field $B_0$ as a function of time $t$

$$\frac{dM}{dt} = M(\gamma B_0 + (M_0 - M_z)/T_1 + M_{xy}/T_2)$$  \hspace{1cm} (2.5)

2.3 MRI Hardware

MRI hardware consists of a number of components. Major components are: outer magnet, gradient coils, RF coils, gradient and RF controllers. These controllers handle pulse sequences and amplification. Along with that, a computer component for processing and image generation, refer Fig.2.8.
2.3.1 Main Magnet

Most of the modern MRI machines use a super-conducting Magnet to generate outer magnetic field $B_0$ [30]. Super-conducting magnets are not permanent magnets. Instead, these are electromagnets which means they work as magnets when electric current is passing through them. These are very high power magnets ranging from 0.3T to 7T. A coolant like liquid helium is also used to handle all the heat generated due to this process [31]. This magnet generates it field along longitudinal axis.

2.3.2 RF coils

RF coils [32] are used to transmit and receive RF signals. When RF pulse is applied, transverse Radio Frequency field $B_1$ is created. $B_1$ rotates the net magnetisation of $B_0$ from longitudinal direction to X-Y Plane. The receiver RF coils capture RF signals during relaxation. Refer Fig.2.9.

![Figure 2.9: Magnetic field $B_0$ and Transverse RF field $B_1$](image)

2.3.3 Gradients

As MR machines, put whole human body under magnetic field, it is crucial to determine which signals are emitting from desired area i.e brain, knee etc.
Inside the main magnet, three gradients ($G_x$, $G_y$, $G_z$) are used to add linear variations in $B_0$ along each spatial direction x, y, z axis [33] respectively as shown in Fig.2.10. Gradients change the frequency and phase along spatial axis, which allows capturing signals from a precise spatial coordinate of targeted area [34]. These gradients are used to determine the image resolution, slice thickness and spatial location of the desired area [27].

![Figure 2.10: Gradient coils $G_x$, $G_y$ and $G_z$ [1]](image)

### 2.4 MR Imaging

#### 2.4.1 K-Space Mapping

To Map spatial coordinates into image coordinates gradients are applied. When $G_x$ is applied it changes magnetic field strength of $B_0$ along x-axis.

\[
M(x) = B_0 + G_x.x \tag{2.6}
\]

and the resonance/precession will become

\[
f(x) = \gamma/2\pi(B_0 + G_x.x) \tag{2.7}
\]

The variations in the amplitude of gradients generate changes in frequency and phase of the spins which in return can be mapped to a unique spatial
The net effect induced by all gradients on spin frequency, can be described as

\[ f(i) = \gamma / 2\pi (G(r).i) \]  

(2.8)

where \( G \) is a vector of amplitudes for all gradients. The change in phase can be written as integral of frequency from time zero to \( t \)

\[ \theta(i, t) = 2\pi \int_0^t (\gamma / 2\pi) Gr.idr \]  

(2.9a)

\[ \theta(i, t) = 2\pi i. k(t) \]  

(2.9b)

where

\[ k(t) = (\gamma / 2\pi) \int_0^t Gr.dr \]  

(2.9c)

Thus the overall signal equation for entire volume will be

\[ y(t) = \int_S x(s)e^{-i2\pi k(t)s} ds \]  

(2.9d)

This received signal is Fourier transform which can be converted into image. Here \( y \) is received signal at time \( t \) from input image or object \( x(s) \) sampled at frequency \( k \). Later these signals are digitized and each spatial location represents a cell in K-space. The received signals are composed of sine waves with varying amplitude, phase and frequency. The signal is then mapped such that each frequency and phase corresponds to a unique position in K-space with specific amplitude or value in it. MR signal mappings based on spatial coordinates/location is called K-space mapping as shown in Fig.2.11. K-space is a 2D array which stores values of signals after digitisation so that, it can later be converted into image. The amplitude of the returning signals varies with tissue composition inside the body. After mapping is completed the resultant 2D array is 2D-fourier transform which is then converted into image [36].
2.4.2 Sampling K-Space

Traditional K-space sampling methods are dependent on required image resolution [37] and Field of view (FOV). Such that, the acquired signal follows Nyquist sampling rate [38]. Different kinds of sampling [39] trajectories are currently being used in clinics i.e. Cartesian and non-Cartesian [40] like radial, spiral. Each of these techniques has their own advantages and drawback. Cartesian sampling makes Image generation really easy, simple inverse Fourier Transform [41] is used to reconstruct image. Whereas, computations are extensive for non-Cartesian methods [42–44]. Spiral is popular for faster image acquisition [45–48]. Radial works better with high contrast objects [49,50] even with under-sampled data [51–54] and it is less prone to motion noise [55]. Thus, each technique can be used for specific clinical settings and applications [56].
2.4.3 Acquiring MR Image

After K-space sampling, computer softwares are used to reconstruct images from the sampled K-space [57]. Different kinds of reconstruction Algorithms can be used for image reconstruction [58–61] which depends on hardware, sampling trajectory and pulse sequence [62].

2.5 Rapid MRI

MRI is important technique used for medical imagery. MR images provide detailed information of tissues in human body leading to better diagnosis of diseases and its further treatment. However, it is time costly. This limitation of MRI has brought scientists and researchers to work on techniques and methods for Rapid MRI [63]. Past research shows that Rapid MRI techniques can be categorized in two classes: exploiting redundancy in data [64] or exploiting redundancy in original signal either using time domain or spatial locations [65].

Parallel MRI (pMRI) [66] exploits spatial-temporal redundancy. The key idea behind parallel imaging is to increase the number of receiver coils/sensors [67] to get multiple readings at a time rather than taking just one [68]. Parallel MR imaging techniques collect the signals from multiple coils [69,70] to reconstruct the image, the major objective is to accelerate the imaging process and reduce scan time [71–73]. Typically gradients are used to determine the spatial location of data [74]. During the acquisition process only limited k-space values can be measured at a time [75], as each k-space value corresponds to a unique spatial location [76]. The idea of using multiple receiver coils emerged from the fact that the received signal varies on the basis of distance between receiver
and signal source [77]. That is, if the sensitivities (position and distance) of receiver coils are exploited for filling k-space in combination with gradients [78]. It can overcome (to some extend) the MRI hardware limitation as all coils work in parallel. Clinically most widely used methods are Partially parallel imaging with localized sensitivities (PILS) [79], Sensitivity encoding (SENSE) [80], Generalized auto-calibrating partially parallel acquisition (GRAPPA) [81] and Simultaneous acquisition of spatial harmonics (SMASH) [82]. However, there are different other methods as well [83–86]. All these methods involve extra information about coil sensitivity to overcome the effect of under-sampled k-space [2].

Figure 2.12: Image acquisition in PILS [2]
PILS use different local independent coils. Each coil focuses on a distinct portion of overall spatial domain thus generating distinct sub-images. All the sub-images are unique and disjoint there is no overlapping part. A combined imaged is generated using all the sub-images and knowledge of their corresponding coil sensitivities as shown in Fig.2.12 [79]. PILS is sensitive to coils location, geometry and direction which is difficult to estimate exactly. Thus, it is difficult to achieve accuracy completely [2].

SENSE is a method for image reconstruction form multiple receiver coils. SENSE is a generic and flexible method. Unlike PILS, it does not use different local and non-relating coils, all the coils work together on overlapping portions of image which makes it independent of individual coil and slice geometry and can considerably reduce the scan time [80]. However, it explicitly requires the coil sensitivities to be known which is an often reason for visible artefacts and reduced Signal-to-Noise Ratio (SNR) [87]. Measuring a coil sensitivity with high accuracy is a very difficult task thus amplification of errors in results [88]. Research is still on-going to improve this method [89–93].

![Image acquisition in SENSE](image.png)

Figure 2.13: Image acquisition in SENSE [2]
In SMASH, detector arrays are used during imaging process. Instead of sampling whole k space, it skips some phase encoding steps which speed up the process. Numerical approximations of coil sensitivities are later used to generate spatial harmonics which replaces the remaining phase encoding steps [94]. Improvements has been made to extend SMASH for better use [95–99]. However, this requires prior information about coil sensitivities and it is highly dependent on coil configurations. Measuring coil sensitivities accurately is an extremely difficult task which makes this process error prone [2].

GRAPPA works with individual coils in frequency domain. Data measured from each coil is used to reconstruct image separately. It is an auto-calibrating method hence it does not required any prior knowledge of coil sensitivities [81]. However, it makes some additional signal acquisition due to self-calibration, which increases the scan time [2]. Extension to basic Algorithm of GRAPPA for better imaging can be found in [100–104].

2.5.1 Data Redundancy in MRI

Medical images are redundant [105–107] in terms of information. This property is usually exploited in images for compression [108–110]. However, this same property can be used for Rapid MRI. It was revealed that the sample set can be reduced up to 30% and a satisfactory image quality can be achieved if the information content or sparsity is low. The samples with equidistant gaps perform very poor. On the other hand small unevenly gaped samples worked well with variety of different objects providing a good quality and reduced scan time [111, 112]. A lot of work has been done in this category [113–118]. Compressive Sensing in MRI also falls under this category.
2.5.2 Compressive Sensing (CS) MRI

M. Lustig, D. L. Donoho and J. M. Pauly investigated the compatibility of Compressed Sensing in MRI that how it naturally fits for MRI domain. For implementation of Compressed Sensing, 3 basic properties mentioned below are needed:

- Transform Sparsity
- Incoherent artefacts in sparsifying domain
- A possible non-linear reconstruction method to exploit the sparsity and incoherence.

The core idea behind incorporating Compressive Sensing with MRI lie in the fact that MRI machines gives flexibility in choosing different sampling techniques and echo sequences. These parameters depend on the end user while image acquisition. This implies that the acquired data/samples can be analysed without any hardware change and additional cost. This makes Compressive Sensing algorithm ideal for MRI [13, 119].

The idea of compressive sensing was introduced in last decade [3, 4, 120], so compressive sensing MRI is in initial research phase thus there are many areas that can be explored. In Compressive Sensing MRI, a sampling technique must be used that can acquire a diagnostic quality image with almost no visible noise. Different sampling techniques and reconstruction methods are used in [121–126]. The work related to an optimal sampling technique which can maximise the time efficiency without degrading the image quality is yet to be done.
Compressive Sensing was also being used in combination with parallel imaging [127–131]. The strengths of both methods are utilized in the data acquisition and image reconstruction process to achieve increased acceleration and good quality images [132]. For image reconstruction, L1-SPIRiT was used in [133]. It enforced data consistency and joint sparsity of all receiver coils to ensure high acceleration and good quality image recovery. This method was also combined with other reconstruction methods [134–138]. Vasanawala summarized his two year experimental experience to combine Compressive Sensing with auto-calibrating parallel imaging for paediatric patients. The experiments were carried out in clinical environment and images were analysed by clinical staff [139]. The results show that this imaging technique is indeed feasible in a clinical setup. Poison disc random under-sampling [140] of phase encodes is used to ensure incoherent artefacts. In this sampling technique, the gap between samples is a user provided parameter. This algorithm takes linear time $O(N)$ to generate desired samples with non-uniform spacing. The main idea is to find the next sample point within the radius equal to the gap size. Instead of exactly computing the allowed region, rejection sampling is used to discover next sample. In parallel imaging it can be merged with SENSE, simply by replacing Fourier k-space data with SENSE encoding matrix that includes samples and coil sensitivities. Modified Compressive Sensing requirements for parallel imaging are

- Sparsity
- Incoherence
Sparsity

For array of correlated coils, a joint sparsity function is used. Wavelet is applied as the sparsifying transform. Since, the L1 norm for one coil is the sum of absolute values of all transform coefficients for that coil, thus the modified joint sparsity function is a vector and sum of magnitudes of all coils. This ensures the correlated sparsity for all coils.

Incoherence

Random sampling ensures high level of incoherence. But in case of correlated coils, large gaps in sampled k space increase noise and reduce the reconstruction conditioning. So, sampling is done using Poisson-Disc distribution. It provides uniform distance between samples and a high degree of incoherence as well. As a nonlinear reconstruction method, SPIR-iT (iterative Self-consistent Parallel Imaging Reconstruction) is used. It enforces auto-calibration and correlation of coils. To ensure and enforce sparsity modified L1 norm was minimized.

Merging CS with pMRI is simple and effective but computationally intensive. It enables faster and higher resolution MRI at the cost of some added computational hardware. However, it is only initial study having vast research and improvement opportunities which are yet to be explored and researchers are working on it i.e. different MR sampling trajectories, multiple dimensions and MR imaging of a specific kind [141–147].

2.5.3 Mean Structural Similarity Index (MSSIM)

Due to random under-sampling in Compressive Sensing, apparent SNR may decrease but generated artefacts are less visually apparent which does not
change quality much. So, MSSIM is used to measure the visible errors only, by comparing the structural similarities. To quantify the structural differences between a distorted image and original image, different properties of human visual system are used. It is known that human visual system is highly adaptable to structural information. Mean Structural Similarity Index (MSSIM) is used to quantify the quality of reconstructed MR images. MSSIM measures structural similarities between original image and reconstructed image.

\[
\text{SSIM}(x, y) = \frac{(2\mu_x\mu_y + C_1)(2\sigma_{xy} + C_2)}{(\mu_x^2 + \mu_y^2 + C_1)(\sigma_x^2 + \sigma_y^2 + C_2)}
\]  \hspace{1cm} (2.10a)

where \(C_1\) and \(C_2\) are the constants and \(\mu\) and \(\sigma\) are mean and standard deviation respectively. MSSIM is mean of SSIM for all local windows within image.

\[
\text{MSSIM}(x, y) = \frac{1}{M} \sum_{j=1}^{M} \text{SSIM}(x_j, y_j)
\]  \hspace{1cm} (2.10b)

where \(x_j\) and \(y_j\) are the image contents at \(j^{th}\) local window.

MSSIM gives values between \([-1,1]\) where -1 represents the lowest possible quality and 1 shows the exact match (when the reconstructed image is same as the fully sampled/original image) [148].

2.6 Noise in MRI

MRI can only achieve limited Signal-to-Noise ratio (SNR) due to its physical and hardware limitations [21]. The SNR in MRI is dependent on image acquisition time and resolution or volume of object in spatial domain [87]. The magnetic signals are acquired using Radio Frequency (RF) sensors and the
spatial domain is mapped into frequency data i.e. K-space. The data is collected in two channels real and imaginary. Due to hardware issues as well as thermal noise from patient [149–151], these channels get affected by additive White Gaussian Noise (AWGN). Later, this frequency data is converted using Inverse Discrete Fourier transform and magnitude images are calculated using absolute values from real and imaginary data components. During this process, the noise distribution also gets affected and the Gaussian noise transforms into signal dependent Rician noise [152,153]. Managing and removing noise in MRI is a difficult because the noise is dependent on signal itself. Moreover, the noise in MRI varies spatially. The simple additive Gaussian noise in original signal tends to vary spatially in resultant magnitude image. The noise in high intensity regions remains Gaussian while in low intensity image regions it acts as Rayleigh distribution [21].

2.6.1 De-noising MRI

The Signal to Noise Ratio of MR images is restricted by hardware and application limitations. Thus, noise removal methods are used to enhance imaging. It was suggested to use complex MRI data for noise removal rather than magnitude images. This makes noise removal easy as complex data only has additive Gaussian noise. However, in most real time cases complex MRI data is not readily available [154]. One major category of such methods is based on Gaussian filter and spatial pattern redundancy which is most often used in functional MRI (fMRI) [155]. However, it causes blur edges. Later on to avoid these issues, edge preserving filters were introduced into this method [156–158]. The edge preserving filters caused missing features for the low magnitude image
areas.

2.6.2 Wavelets based noise removal methods

Another category of de-noising methods used wavelets to exploit its multi scale representation for de-noising. The basic procedure is to convert image into wavelet domain, using the transformed wavelet coefficients for noise removal and converting the de-noised wavelet data back into image. Wavelets were used in different range of methods from thresholds to complex filtering [21,159,160]. A wavelets based thresholding was applied in [161]. In another approach coefficients were squared which made noise independent of signal and thus easily removable [21]. In another method, the multi-scale representation of wavelets was used as correlation information for noise removal [159] Wiener filtering was also applied in wavelet domain for de-noising [162]. However, wavelets based processing generates artefacts which are dependent on the type of wavelets being used [163].

2.7 Conclusion

The literature study shows that a lot of work has been done to improve the acquisition speed and image quality of MRI but there are still many gaps. Rapid MRI methods which use Parallel Imaging are dependent on the sensitivity of the coils which is hard to measure and can affect accuracy of the image. Compressive sensing is one of the finest ways in rapid MRI methods based on data redundancy. It can also be used in combination with parallel imaging for further enhancements. However, CS based research has still many gaps and room for improvements. Moreover, the MRI data is prone to thermal noise which
result in signal dependent Rician Noise. The literature shows many simple and complex techniques each with its own advantages and drawback.

The literature study and review was used to better understand the gaps in methods which are currently being used for Rapid MR imaging and de-noising. This work focuses on non-uniform sparsity in MR images and will exploit it for improved quality rapid MR imaging. Next chapter will present detailed analysis and key features of Sparsity of MR images.
Chapter 3

NON-UNIFORM SPARSITY IN MR IMAGES

3.1 Introduction

MR images are sparse in their sensing as well as other transform domains. This chapter will analyse Sparsity of MR images to answer the following question. How it differs in different kinds of images or in different domains? Moreover, how it varies within an image? What are the key features of MR image sparsity distribution? All the experiments are done on different kinds of hard and soft tissue images, to deal with the MRI sparsity in more generic terms rather than making it application specific. This chapter concludes the key points related to the sparsity of MR images and will develop basis for work presented in next chapters.

3.2 Sparsity

A sparse signal is mainly consisted of zeros and has few non-zero elements. In other words, a less dense signal can be stated as sparse signals. Sparsity is beneficial in networking, data storage and computations where the cost of high
data rates is very crucial. When sparsity is taken into consideration in signal processing and utilised properly, it can make signals easy to process, store and transmit. In general, sparsity can be categorised in two types i.e. strong or weak. In strongly sparse data, most of the coefficients are exactly zero with few non-zero coefficients. While in weak sparsity, coefficients are nearly zero with very small magnitude but not exactly zero.

Taking weak sparsity into consideration, sparsity can be defined as the amount of information in the signal. A signal may not be composed of a lot of zero elements, still it can be sparse if the signal is structured or it takes some specific form and only few coefficients are needed to represent it while rests of the coefficients are not important in reconstruction of the signal. The study shows that all the natural images as well as MRI images are highly compressible which means the information content is very low in these images [13]. For real life imaging techniques, like natural images [164] or biomedical images [116] it is not possible to have strong sparsity. However, they exhibit weak sparsity.

3.3 Transform Sparsity

There can be images which are not sparse in their sensing domain i.e. Image domain for natural images or Fourier domain in case of MR imaging. However, they might exhibit sparsity in some other domains. This kind of sparsity is known as transform sparsity. Transform sparsity means signals are not sparse in their original domain but have a sparse representation in some other domain. A sparsifying transform operator is used to convert an image into a vector or sparse coefficients in some fixed orthonormal basis. Fig. 3.1 shows a brain MRI
and its respective representations in Fourier and Wavelets. Heat maps are used to show the sparsity. A cool blue colour shows low temperature or zero values while yellow and red colours show elements with high magnitudes. The original image does not demonstrate much of sparsity and shows a large image area is covered with significant high value data. Whereas, when converted into Fourier and Wavelet a large portion shows blue colour depicting that most of the data is zero or nearly zero while only small amount of coefficients contain all the significant values. A lot of research has been done for making images memory and transmission efficient by compressing them using different kinds of transform domains e.g. Wavelets \[166–168\], DCT \[169, 170\].

Figure 3.1: Illustration of transform sparsity: Varying sparsity levels in MR Brain Image in different domains. (a) Image domain (b) Frequency domain (c) Wavelets. Blue colour shows zero elements or no data. Image domain is the least sparse or most dense domain among all three. The Frequency and Wavelets show highly sparse representation of the image. Thus, the image used is not sparse in its original domain but it shows transforms sparsity.

3.4 Sparsity of MR images

MR images are sampled from spatial domain to frequency domain. The amount of sampling depends on overall image size or resolution. Whereas, assuming if an image is sparse than most of the measured samples are not significant
and can be discarded and it will not affect image quality. Sparsity of MR images can be stated as percent coefficients which are required to generate a diagnostic quality image [13].

\[ Y = \alpha . I \]  

(3.1)

where \( I \) is the image, \( \alpha \) is transform operator which will be 1 if \( I \) is sparse in its sensing domain and \( Y \) is the sparse representation of \( I \) in some transform domain. \( Y \) will be strongly sparse if \( \alpha \) measure \( I \) just for significant coefficient and replace rest of the elements with zero which is

\[ Y = \alpha_\varrho . I \]  

(3.2)

here \( \varrho = \{1, 2, ..., K\} \) is set of locations of significant coefficients in image \( I \) and size of \( \varrho \) is \(|\varrho| = K\) and \( Y \) will become

\[ Y(i) = \begin{cases} 0 & i \notin \varrho \\ \text{val} & i \in \varrho \end{cases} \]  

(3.3)

if \( Y \) is of size \( N \) than sparsity of \( Y \) will be the ratio or percentage \( S_g \) where

\[ S_g = \frac{K}{N}. \]

This means \( Y \) will be \( S_g \) sparse if all of coefficients in \( Y \) are discarded except \( S_g \) highest coefficients. MR images can be of different kind some of them show sparsity in image domain while others are sparse in their sensing domains. They also exhibits transform sparsity but this varies based on image type and selection of sparsifying operator.

### 3.5 Experimental Results: Transform Sparsity in MR images

The sensing matrix/operator used in MRI is Fourier. As MRI works with Fourier coefficients so an orthonormal basis for transform sparsity needs to be
determined. The literature shows different transforms were used to sparsify MR images [171]. Finite Differences were used for MR angiography whereas Wavelet was used for brain imaging [13]. However, this research does not focus on any specific type of imaging. To analyse sparsity of MR images following points are considered:

1. Experiments were done on different kinds of images so that MR can be analysed on more generic terms rather than any specific kind e.g. brain imaging or cardiac imaging. Six kinds of images were used brain, angiography, heart, spine, knee and wrist. The size of images used was 512x512.

2. Different sparsity levels were used to find a level that is best suited for all types of images. From fully sampled images only 1%, 5%, 10%, 20%, 30%, 40% and 50% largest coefficients were taken and rest of them were discarded. Images were reconstructed again to estimate the sparsity. Results were compared with the original images to determine the quality of the recovered images.

3. All the experiments were done in Image, Fourier, Discrete Cosine Transform (DCT) and Wavelet domain to identify the most suitable sparsifying transform.

4. In this section, MSSIM, Peak SNR (PSNR) and Mean Square Error (MSE) are used to quantify image quality. Results are presented using all three measures to analyse and decide which index best represents quality of under-sampled MR images and differentiate and quantify the results
better. So that, it can be used later in further chapters for quantification of proposed methods.

3.5.1 Brain

The experiments on brain MRI were performed in four domains with image size of 512x512. The results revealed that a good quality image can be regenerated with only 10% of DCT and wavelet coefficients. The Fourier requires 20% of the coefficients for good recovery. While image domain could not recover the image even with 50% coefficients. Figure 3.2 shows the reconstructed brain images. Table 3.1 shows quality indices. MSE was unable to quantify image errors in brain MRI as most of data showed zero. Whereas, PSNR gave results but due to under-sampling SNR was effected even in a good reconstruction SNR is low, so these values are not correctly representing image quality.

![Image](image.png)

Figure 3.2: Image recovery of Brain using 1%, 5%, 10%, 20% coefficients in different transform domains
Table 3.1: Sparsity in Brain MRI

<table>
<thead>
<tr>
<th>Sparsity</th>
<th>MSSIM</th>
<th>Image</th>
<th>Fourier</th>
<th>DCT</th>
<th>Wavelet</th>
</tr>
</thead>
<tbody>
<tr>
<td>1%</td>
<td>0.0262</td>
<td>0.5279</td>
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<td>0.7055</td>
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<td>0.7148</td>
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<td>0.1339</td>
<td>0.8152</td>
<td>0.9209</td>
<td>0.9223</td>
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<td>0.2380</td>
<td>0.9249</td>
<td>0.9695</td>
<td>0.9647</td>
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<tr>
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<td>0.3401</td>
<td>0.9719</td>
<td>0.9878</td>
<td>0.9825</td>
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<td>0.4415</td>
<td>0.9896</td>
<td>0.9951</td>
<td>0.9911</td>
<td></td>
</tr>
<tr>
<td>50%</td>
<td>0.5646</td>
<td>0.9957</td>
<td>0.9981</td>
<td>0.9956</td>
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<th>Wavelet</th>
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<td>0.0090</td>
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</tr>
<tr>
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<td>0.0684</td>
<td>0.0021</td>
<td>0.0008</td>
<td>0.0006</td>
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<tr>
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<td>0.0009</td>
<td>0.0002</td>
<td>0.0002</td>
<td></td>
</tr>
<tr>
<td>20%</td>
<td>0.0255</td>
<td>0.0003</td>
<td>0.0001</td>
<td>0.0001</td>
<td></td>
</tr>
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<td>0.0000</td>
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<th>DCT</th>
<th>Wavelet</th>
</tr>
</thead>
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<td>20.4383</td>
<td>25.0004</td>
<td></td>
</tr>
<tr>
<td>5%</td>
<td>11.6483</td>
<td>26.7646</td>
<td>31.2485</td>
<td>32.3502</td>
<td></td>
</tr>
<tr>
<td>10%</td>
<td>13.3487</td>
<td>30.3086</td>
<td>36.3195</td>
<td>36.4067</td>
<td></td>
</tr>
<tr>
<td>20%</td>
<td>15.9261</td>
<td>35.6308</td>
<td>41.8975</td>
<td>41.1690</td>
<td></td>
</tr>
<tr>
<td>30%</td>
<td>18.3645</td>
<td>40.2792</td>
<td>46.2161</td>
<td>44.7166</td>
<td></td>
</tr>
<tr>
<td>40%</td>
<td>21.4698</td>
<td>44.5599</td>
<td>50.2928</td>
<td>47.9289</td>
<td></td>
</tr>
<tr>
<td>50%</td>
<td>25.3978</td>
<td>48.4016</td>
<td>54.4035</td>
<td>51.1715</td>
<td></td>
</tr>
</tbody>
</table>

3.5.2 Angiography

Similar experiments were carried out on Angiography. The results again depicts that good image recovery in wavelet and DCT is achieved with 10% coefficients only. The Fourier require 20% and in image domain even 50% coefficients were not enough. Table 3.2 and Fig. 3.3 shows the experimental results.
Table 3.2: Sparsity in Angiography

<table>
<thead>
<tr>
<th>Sparsity</th>
<th>MSSIM</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Image</td>
<td>Fourier</td>
<td>DCT</td>
<td>Wavelet</td>
</tr>
<tr>
<td>1%</td>
<td>0.0175</td>
<td>0.6499</td>
<td>0.3264</td>
<td>0.7548</td>
</tr>
<tr>
<td>5%</td>
<td>0.0796</td>
<td>0.7900</td>
<td>0.8630</td>
<td>0.8767</td>
</tr>
<tr>
<td>10%</td>
<td>0.1503</td>
<td>0.8825</td>
<td>0.9320</td>
<td>0.9294</td>
</tr>
<tr>
<td>20%</td>
<td>0.2462</td>
<td>0.9604</td>
<td>0.9788</td>
<td>0.9708</td>
</tr>
<tr>
<td>30%</td>
<td>0.3369</td>
<td>0.9837</td>
<td>0.9926</td>
<td>0.9864</td>
</tr>
<tr>
<td>40%</td>
<td>0.4363</td>
<td>0.9921</td>
<td>0.9972</td>
<td>0.9936</td>
</tr>
<tr>
<td>50%</td>
<td>0.5323</td>
<td>0.9961</td>
<td>0.9989</td>
<td>0.9971</td>
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</table>

<table>
<thead>
<tr>
<th>Sparsity</th>
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<th></th>
<th></th>
</tr>
</thead>
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<td></td>
<td>Image</td>
<td>Fourier</td>
<td>DCT</td>
<td>Wavelet</td>
</tr>
<tr>
<td>1%</td>
<td>0.0403</td>
<td>0.0030</td>
<td>0.0070</td>
<td>0.0013</td>
</tr>
<tr>
<td>5%</td>
<td>0.0206</td>
<td>0.0008</td>
<td>0.0003</td>
<td>0.0003</td>
</tr>
<tr>
<td>10%</td>
<td>0.0144</td>
<td>0.0003</td>
<td>0.0001</td>
<td>0.0001</td>
</tr>
<tr>
<td>20%</td>
<td>0.0101</td>
<td>0.0001</td>
<td>0.0000</td>
<td>0.0001</td>
</tr>
<tr>
<td>30%</td>
<td>0.0075</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>40%</td>
<td>0.0053</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>50%</td>
<td>0.0036</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Sparsity</th>
<th>PSNR</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Image</td>
<td>Fourier</td>
<td>DCT</td>
<td>Wavelet</td>
</tr>
<tr>
<td>1%</td>
<td>13.9485</td>
<td>25.1694</td>
<td>21.5782</td>
<td>28.9729</td>
</tr>
<tr>
<td>5%</td>
<td>16.8581</td>
<td>30.9343</td>
<td>34.8307</td>
<td>35.2727</td>
</tr>
<tr>
<td>10%</td>
<td>18.4307</td>
<td>34.8600</td>
<td>38.7986</td>
<td>38.4454</td>
</tr>
<tr>
<td>20%</td>
<td>19.9499</td>
<td>40.3940</td>
<td>44.2342</td>
<td>42.8285</td>
</tr>
<tr>
<td>30%</td>
<td>21.2522</td>
<td>44.4092</td>
<td>48.9218</td>
<td>46.4232</td>
</tr>
<tr>
<td>40%</td>
<td>22.7956</td>
<td>47.8018</td>
<td>53.2728</td>
<td>49.8702</td>
</tr>
<tr>
<td>50%</td>
<td>24.4350</td>
<td>51.1124</td>
<td>57.3401</td>
<td>53.4340</td>
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</table>
3.5.3 Spine

The experiments performed on Spine MRI revealed that image can be recovered even with 1% of wavelet coefficients. It means this image is highly compressible and 99% of the coefficients can be discarded. DCT and Fourier showed a moderate compressibility with 5% and 10% coefficients respectively. However, in Image domain it was recovered with 50% coefficients as in Table 3.3. It further depicts that choosing the right transform for sparsifying the image is very crucial because results varies largely from one transform to another.
The image recovery with different sparsity levels and transforms are combined in Fig.3.4.

Table 3.3: Sparsity in Spine MRI

<table>
<thead>
<tr>
<th>Sparsity</th>
<th>MSSIM</th>
<th>MSE</th>
<th>PSNR</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Image</td>
<td>Fourier</td>
<td>DCT</td>
</tr>
<tr>
<td>1%</td>
<td>0.3595</td>
<td>0.7569</td>
<td>0.6265</td>
</tr>
<tr>
<td>5%</td>
<td>0.4111</td>
<td>0.9180</td>
<td>0.9781</td>
</tr>
<tr>
<td>10%</td>
<td>0.4697</td>
<td>0.9600</td>
<td>0.9907</td>
</tr>
<tr>
<td>20%</td>
<td>0.5798</td>
<td>0.9866</td>
<td>0.9970</td>
</tr>
<tr>
<td>30%</td>
<td>0.7184</td>
<td>0.9947</td>
<td>0.9988</td>
</tr>
<tr>
<td>40%</td>
<td>0.8692</td>
<td>0.9976</td>
<td>0.9995</td>
</tr>
<tr>
<td>50%</td>
<td>0.9625</td>
<td>0.9987</td>
<td>0.9998</td>
</tr>
</tbody>
</table>
3.5.4 Heart

In Heart MRI, Wavelets, Fourier and DCT showed good recovery with 5% coefficients. However, Image domain was unable to recover a good quality image. The results are summarized below.
Table 3.4: Sparsity in Heart MRI

<table>
<thead>
<tr>
<th>Sparsity</th>
<th>Image</th>
<th>Fourier</th>
<th>DCT</th>
<th>Wavelet</th>
</tr>
</thead>
<tbody>
<tr>
<td>1%</td>
<td>0.0151</td>
<td>0.7900</td>
<td>0.4035</td>
<td>0.8667</td>
</tr>
<tr>
<td>5%</td>
<td>0.0499</td>
<td>0.9308</td>
<td>0.9509</td>
<td>0.9618</td>
</tr>
<tr>
<td>10%</td>
<td>0.0939</td>
<td>0.9759</td>
<td>0.9827</td>
<td>0.9845</td>
</tr>
<tr>
<td>20%</td>
<td>0.1943</td>
<td>0.9950</td>
<td>0.9960</td>
<td>0.9952</td>
</tr>
<tr>
<td>30%</td>
<td>0.3074</td>
<td>0.9982</td>
<td>0.9985</td>
<td>0.9978</td>
</tr>
<tr>
<td>40%</td>
<td>0.4139</td>
<td>0.9991</td>
<td>0.9993</td>
<td>0.9990</td>
</tr>
<tr>
<td>50%</td>
<td>0.5140</td>
<td>0.9995</td>
<td>0.9996</td>
<td>0.9995</td>
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</table>

<table>
<thead>
<tr>
<th>Sparsity</th>
<th>Image</th>
<th>Fourier</th>
<th>DCT</th>
<th>Wavelet</th>
</tr>
</thead>
<tbody>
<tr>
<td>1%</td>
<td>0.0730</td>
<td>0.0014</td>
<td>0.0046</td>
<td>0.0006</td>
</tr>
<tr>
<td>5%</td>
<td>0.0492</td>
<td>0.0002</td>
<td>0.0001</td>
<td>0.0001</td>
</tr>
<tr>
<td>10%</td>
<td>0.0335</td>
<td>0.0001</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
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<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>30%</td>
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<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
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<td>0.0000</td>
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<td>0.0000</td>
</tr>
<tr>
<td>50%</td>
<td>0.0038</td>
<td>0.0000</td>
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<table>
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<th>Fourier</th>
<th>DCT</th>
<th>Wavelet</th>
</tr>
</thead>
<tbody>
<tr>
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<td>11.3683</td>
<td>28.5438</td>
<td>23.3530</td>
<td>32.3254</td>
</tr>
<tr>
<td>5%</td>
<td>13.0780</td>
<td>36.6996</td>
<td>39.9315</td>
<td>41.2375</td>
</tr>
<tr>
<td>10%</td>
<td>14.7510</td>
<td>42.4510</td>
<td>45.4197</td>
<td>45.8332</td>
</tr>
<tr>
<td>20%</td>
<td>17.4975</td>
<td>50.1030</td>
<td>52.1849</td>
<td>51.4551</td>
</tr>
<tr>
<td>30%</td>
<td>20.0518</td>
<td>55.1046</td>
<td>56.6005</td>
<td>55.2142</td>
</tr>
<tr>
<td>40%</td>
<td>22.2328</td>
<td>58.3082</td>
<td>59.9548</td>
<td>58.4681</td>
</tr>
<tr>
<td>50%</td>
<td>24.2413</td>
<td>60.8827</td>
<td>63.1069</td>
<td>61.6915</td>
</tr>
</tbody>
</table>
Figure 3.5: Image recovery of Heart using 1%, 5%, 10%, 20% coefficients in different transform domains

3.5.5 Wrist

The results for image recovery with wavelets were extraordinarily good. The wavelet was able to sparsify image up to 1%. Fourier and DCT showed a good recovery at sparsity level 5% while image domain failed to sparsify it. Results are summarized below in Table 3.5 and Fig.3.6.
Table 3.5: Sparsity in Wrist MRI

<table>
<thead>
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<th>MSSIM</th>
<th>MSE</th>
<th>PSNR</th>
</tr>
</thead>
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<tr>
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<td>Image</td>
<td>Fourier</td>
<td>DCT</td>
</tr>
<tr>
<td>1%</td>
<td>0.2145</td>
<td>0.8287</td>
<td>0.4979</td>
</tr>
<tr>
<td>5%</td>
<td>0.2510</td>
<td>0.9517</td>
<td>0.9669</td>
</tr>
<tr>
<td>10%</td>
<td>0.3044</td>
<td>0.9859</td>
<td>0.9863</td>
</tr>
<tr>
<td>20%</td>
<td>0.4128</td>
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<td>0.9958</td>
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<tr>
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<td>0.5270</td>
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<td>0.9981</td>
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<tr>
<td>40%</td>
<td>0.7754</td>
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<td>0.9990</td>
</tr>
<tr>
<td>50%</td>
<td>0.8967</td>
<td>0.9993</td>
<td>0.9995</td>
</tr>
<tr>
<td></td>
<td>MSE</td>
<td>Wavelet</td>
<td></td>
</tr>
<tr>
<td>1%</td>
<td>0.0264</td>
<td>0.0005</td>
<td>0.0004</td>
</tr>
<tr>
<td>5%</td>
<td>0.0138</td>
<td>0.0001</td>
<td>0.0000</td>
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<tr>
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<td>0.0065</td>
<td>0.0000</td>
<td>0.0000</td>
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<tr>
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<td>0.0000</td>
</tr>
<tr>
<td>30%</td>
<td>0.0002</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>40%</td>
<td>0.0001</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>50%</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td>PSNR</td>
<td>Wavelet</td>
<td></td>
</tr>
<tr>
<td>1%</td>
<td>15.7817</td>
<td>33.1738</td>
<td>34.3422</td>
</tr>
<tr>
<td>5%</td>
<td>18.6018</td>
<td>41.0798</td>
<td>44.1678</td>
</tr>
<tr>
<td>10%</td>
<td>21.8868</td>
<td>47.0966</td>
<td>48.1792</td>
</tr>
<tr>
<td>20%</td>
<td>28.8887</td>
<td>53.2489</td>
<td>53.2216</td>
</tr>
<tr>
<td>30%</td>
<td>36.7478</td>
<td>56.1309</td>
<td>56.6673</td>
</tr>
<tr>
<td>40%</td>
<td>40.3613</td>
<td>58.3794</td>
<td>59.6111</td>
</tr>
<tr>
<td>50%</td>
<td>43.3526</td>
<td>60.6063</td>
<td>62.5907</td>
</tr>
</tbody>
</table>
Figure 3.6: Image recovery of Wrist using 1%, 5%, 10%, 20% coefficients in different transform domains

3.5.6 Knee

Results for experiments on knee image show that only 5% of wavelet coefficients can recover a good quality image. However, image, Fourier and DCT has recovered good quality image at 10% sparsity.
Table 3.6: Sparsity in Knee MRI

<table>
<thead>
<tr>
<th>Sparsity</th>
<th>MSSIM</th>
<th>MSE</th>
<th>PSNR</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Image</td>
<td>Fourier</td>
<td>DCT</td>
</tr>
<tr>
<td>1%</td>
<td>0.0478</td>
<td>0.7237</td>
<td>0.4480</td>
</tr>
<tr>
<td>5%</td>
<td>0.0723</td>
<td>0.8448</td>
<td>0.8943</td>
</tr>
<tr>
<td>10%</td>
<td>0.0995</td>
<td>0.9270</td>
<td>0.9544</td>
</tr>
<tr>
<td>20%</td>
<td>0.1721</td>
<td>0.9873</td>
<td>0.9880</td>
</tr>
<tr>
<td>30%</td>
<td>0.2982</td>
<td>0.9969</td>
<td>0.9961</td>
</tr>
<tr>
<td>40%</td>
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<td>0.9985</td>
<td>0.9985</td>
</tr>
<tr>
<td>50%</td>
<td>0.5321</td>
<td>0.9992</td>
<td>0.9994</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.0030</td>
<td>0.0005</td>
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<td></td>
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<tr>
<td>1%</td>
<td>13.3959</td>
<td>30.1200</td>
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<tr>
<td>5%</td>
<td>14.0260</td>
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<td>37.1151</td>
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<tr>
<td>10%</td>
<td>14.8026</td>
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<td>20%</td>
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<td>20.6915</td>
<td>55.9345</td>
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<td>50%</td>
<td>23.2320</td>
<td>58.6619</td>
<td>60.1792</td>
</tr>
</tbody>
</table>
3.5.7 Summarizing Results for Transform Sparsity

Experiment showed that most of the images could not be recovered when sparsified in image domain directly. Whereas, wavelet sparsified MR images most and for all kinds. DCT and Fourier showed a moderated recovery. Fig.3.8 shows the average MSSIM for all kinds of images at different sparsity levels. Average values are used to analyze and find a general domain which will work for all kinds of images rather than MR-type specific transform. Comparison between different domains and quality indices was performed to define a generic
domain as well as a suitable quality index for further experiments.

Figure 3.8: Average MSSIM for reconstructed MR Images using 1%, 5%, 10%, 20%, 30%, 40%, 50% coefficients in Image, Fourier, DCT, Wavelet domain

Fig.3.9 shows the comparison in terms of SNR here DCT exceed Wavelet but at the cost of visible errors. Due, to under-sampling SNR gives confusing values which do not represent actual image quality very well.

Figure 3.9: Average PSNR for reconstructed MR Images using 1%, 5%, 10%, 20%, 30%, 40%, 50% coefficients in Image, Fourier, DCT, Wavelet domain

MSE has hardly gavin any error values as the graph in Fig.3.10 shows flat
lines on zero. Thus, MSE failed to quantify image quality

![Graph showing MSE vs Sparsity Level]

Figure 3.10: Average MSE for reconstructed MR Images using 1%, 5%, 10%, 20%, 30%, 40%, 50% coefficients in Image, Fourier, DCT, Wavelet domain

From these experiments it can be concluded that wavelet suits best to all kinds of MR images. Also, only MSSIM will be used to represent image quality as it has given most accurate quantification.

### 3.6 How Sparsity Varies Within MRI?

Fig.3.11 shows Fourier transform of brain. It can be seen that most of the Fourier data lies near origin while a large number of Fourier coefficients are zero or nearly zero. Areas near borders have low energy while origin contains most of the image energy. In others words, it can be said that sparsity varies in different sections or areas e.g. borders have less coefficients in comparison to origin. Sparsity is distributed non-uniformly in the image. Can this non-uniform nature of sparsity be exploited somehow to achieve even sparser images without having any effect on image quality?
Fig. 3.12 shows the energy distribution of k-space in Fourier domain. MR images read out samples along one direction i.e. vertical or horizontal. The energy of K-Space was summed up in direction of columns. The central peak and low energy ends allow an easy division when it comes to 2D Cartesian imaging.

Figure 3.12: K-Space Energy Distribution Brain MRI
3.7 Methodology for Experimenting Local Sparsity

To experiment on this apparent and visible non-uniform sparsity of MR images, K-Space was divided into non-overlapping sub regions and the partial Fourier operators will become $f_1, f_2...f_t$.

Figure 3.13: Illustration of estimation non-uniform sparsity level in MR images where 3 local regions are used with Wavelets as sparsifying transform domain

Fourier $F$ has total $N$ coefficients same as image $I$. So, sizes of $f_1, f_2...f_t$
are $n_1, n_2...n_t$ respectively such that $n_1 + n_2...n_t = N$. These partial Fourier’s were converted into their respective images $I_1, I_2...I_t$ and transform operators $\varrho_1, \varrho_2...\varrho_t$ were calculated. Each transform matrix was than sparsified with a different ratio. Only $s_j = k_j/n_j$ largest coefficients were taken from $\varrho_j$ while rests of the coefficients were discarded. Images were reconstructed again and merged into one. The whole process is explained figuratively in Fig.3.13.

3.7.1 Non-uniform Sparsity - Experiments with different Local Sparsity Levels

MR images are non-uniformly sparse in their Fourier domains. This non uniform nature of sparsity can be exploited for a better image quality even with lesser samples, thus making the whole process more efficient. Most of the data is centered on origin while the areas near the borders have fewer data values. The experiments are based on the idea that instead of using one global sparsity constraint, different local sparsity constraints corresponding to local sparsity levels in that particular region/area are applied. A series of experiments were conducted for each image. To analyze, how non uniform nature of sparsity can be better utilized for sparsifying the overall image. Results were then compared with previous set of experiments with a global threshold or global sparsity level. Experiments were done on images of brain, angiography, heart, spine, knee and wrist. Image size of 512x512 was used. From fully sampled images 5%, 10% and 20% largest coefficients were taken and rest of them were discarded. Image was reconstructed again to estimate the global sparsity level $S$. On same images another set of experiments was conducted. Fourier was divided into three different parts $f_1, f_2$ and $f_3$ of sizes 156x512,
200x512, 156x512 respectively as shown in Fig.3.14. The technique which is used to define these sizes is explained in detail in next chapter. The Fourier was converted into images $I_1$, $I_2$ and $I_3$. From the wavelets of these images only $s1\%$ coefficients from $w_1$, $s2\%$ from $w_2$, and $s3\%$ from $w_3$ were taken and a combined image was reconstructed. Results of global sparsity level $S$ were compared with non-uniform sparsity $s1$, $s2$ and $s3$.

![Figure 3.14: Image on left shows the global sparsity level. Images on right shows local sparsity levels](image)

The global sparsity levels of brain, angiography, spine, heart, wrist and knee MRIs are compared with local sparsity levels. The results showed that by implying local sparsity levels, the image can be further sparsified. Almost 30% of the global sparse coefficients can be further discarded to recover even better image. The results are summarized in Table. 3.7. The quality of image with a global sparsity level $S$ from an object of size $N$ is equal to the image quality obtained with local sparsity levels $s1$, $s2$ and $s3$ for object sizes $n1$, $n2$ and $n3$ as graph is shown in Fig.3.15.
Mathematically, if there are $N$ total coefficients with sub-partitions such that $n_1 + n_2 + n_3 = N$. Then by taking $S\%$ largest coefficients of $N$ to get a high quality image recovery will be equivalent to taking $s_1\%$ from $n_1$, $s_2\%$ from $n_2$ and $s_3\%$ from $n_3$. Practically it was observed that the results were improved when image was sparsified using local constraints. The quality of the recovered image was same for local sparsity constraint applied on different local areas and global sparsity constraint. This result was obtained when the sparsity of local area containing the origin $s_2$ was same as global sparsity constraint $S$, while the sparsity of other local areas $s_1$ and $s_3$ was half the global sparsity $S$.

$$s_2 = Ss_1 = s_3 = S/2$$ (3.4)
Table 3.7: Comparison of Global and Local Sparsity Constraint

<table>
<thead>
<tr>
<th>Area under test</th>
<th>MSSIM</th>
<th>Global Sparsity</th>
<th>Local Sparsity levels</th>
<th>MSSIM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Sparsity S</td>
<td>s1,s2,s3</td>
<td></td>
</tr>
<tr>
<td>Brain</td>
<td></td>
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<td>0.8711,1%,5%,1%</td>
<td>0.8756</td>
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<td>10%</td>
<td>0.9223,5%,10%,1%</td>
<td>0.9118</td>
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<tr>
<td></td>
<td></td>
<td>20%</td>
<td>0.9647,1%,20%,1%</td>
<td>0.9323</td>
</tr>
<tr>
<td></td>
<td></td>
<td>5%</td>
<td>0.8767,1%,5%,1%</td>
<td>0.88</td>
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<tr>
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<td></td>
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<td>0.9294,5%,10%,5%</td>
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<td></td>
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<td>0.9708,1%,20%,1%</td>
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<td></td>
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<td>0.9778,1%,5%,1%</td>
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<td>0.9892,5%,10%,5%</td>
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<td>20%</td>
<td>0.9958,5%,20%,5%</td>
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<td>5%</td>
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<td>20%</td>
<td>0.9946,5%,20%,5%</td>
<td>0.9932</td>
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<td></td>
<td></td>
<td>5%</td>
<td>0.9096,1%,5%,1%</td>
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<td></td>
<td>10%</td>
<td>0.9521,5%,10%,5%</td>
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<td></td>
<td></td>
<td>20%</td>
<td>0.9814,5%,20%,5%</td>
<td>0.9814</td>
</tr>
</tbody>
</table>
3.8 Conclusion

Based on the experimental data, it can be concluded that MR images show transform sparsity. For all different types of MR images, Wavelets showed the best results. Thus, it sparsifies MR images better than DCT, Fourier and Image domain. In sparse data, if under-sampling is done such that no significant coefficients are lost than resultant images show no visible artefacts. MSSIM quantify the image quality very well and is suitable to compare results when data is under-sampled and original signal is not noisy itself. SNR does not work well for quantifying the quality of under-sampled data. SNR is a suitable measure when dealing with fully sampled but noisy signals and improved SNR means de-noising. A global sparsity level of 10% is suitable for a good quality reconstruction for all types of MR images. Using multiple local sparsity levels, images were sparsified 30% more than global level S without affecting image quality.

All these findings will be used as the basis for the work presented in next chapters. Chapter 4 will present a de-noising technique based on these points.
Chapter 4

DE-NOISING MR IMAGES

4.1 Introduction

In last chapter, it was developed that MR images do not only exhibit sparsity but their sparsity takes a certain predictable shape which is common for all kinds of images. That region based localised sparsity can be used to de-noise MR images from random thermal noise. This chapter present a simple framework to exploit sparsity of MR images for image de-noising. As, noise in MR images tends to change its shape in distribution based on contrast level and signal itself, the proposed method is independent of noise shape and type.

4.2 Noise in MRI

MRI machines reads signals from RF coils and captures data in frequency domain. These readout have two components for each sample, real and imaginary.

\[ \text{Sig}(j) = \text{Sig}_{\text{real}}(j) + \iota \text{Sig}_{\text{imaginary}}(j) \]  

(4.1)

Here \( \text{Sig} \) is the recorded signal at location \( j \) in K-space. While, \( \text{Sig}_{\text{real}} \) and \( \text{Sig}_{\text{imaginary}} \) are the real and imaginary components of the signal and \( \iota = \]
Due to physical factors and patient’s body temperature, thermal noise is introduced in the signal which is additive white Gaussian noise. This AWGN affect both real and imaginary component of the signal.

\[
\text{Sig}(j) = (\text{Sig}_{\text{real}}(j) + \text{Noise}(j)) + i(\text{Sig}_{\text{imaginary}}(j) + \text{Noise}(j)) \tag{4.2}
\]

When data is in complex form Gaussian noise corrupts both real and imaginary components. The distribution of Gaussian noise for any random variable \(x\) mean \(\mu\) and variance \(\sigma^2\) can be described as

\[
\text{pdf}(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \text{ where } x \in (-\infty, \infty) \tag{4.3a}
\]

\[
\text{cdf}(x) = \frac{1}{2}[1 + \text{erf}((x - \mu)/\sqrt{2\sigma^2})] \text{ where } x \in (-\infty, \infty) \tag{4.3b}
\]

where \(\text{erf}\) is Gauss Error Function.

This distribution shows a bell shaped distribution with a peak in center as shown in Fig. 4.1. This noise is easy to remove and handle.

![Figure 4.1: Gaussian/Normal Distribution](image)

(a) Probability density function (b) Cumulative distribution function

However, this raw data is not available in most of the cases. MRI frequency data is converted into images using Discrete Fourier Transform (DFT). Fourier transform transfer the noise into image components without effecting its shape.
If $Y$ is the inverse Fourier than any element $i$ with AWGN can be stated as

$$Y(j) = (Y_{\text{real}}(j) + \text{Noise}(j)) + i(Y_{\text{imaginary}}(j) + \text{Noise}(j))$$ \hspace{1cm} (4.4)

In next step, magnitude images are calculated and the complex data is discarded.

$$m(j) = |y(j)|$$ \hspace{1cm} (4.5)

Now for each pixel $j$ $m(j)$ is combination of noise and real signal. This process changes the shape of noise distribution and make it Rician Distribution which is signal dependent.

$$pdf(x) = I_0(xs/\sigma^2)(x/\sigma^2)e^{-(x^2+s^2/2\sigma^2)} \hspace{0.5cm} \text{where} \hspace{0.5cm} x \in [0, \infty)$$ \hspace{1cm} (4.6a)

Here $I_0$ zeroth order Bessel function of first kind, and $s$ is non-centrality parameter. The shape of probability density function is shown in Fig. 4.2.

$$cdf x = 1 - Q_1(s/\sigma, x/\sigma) \hspace{0.5cm} \text{where} \hspace{0.5cm} x \in [0, \infty)$$ \hspace{1cm} (4.6b)

Here $Q_1$ is the Marcum Q-function.

Figure 4.2: Rician Distribution (a) Probability density function (b) Cumulative distribution function

The signal dependent noise is hard to predict and remove but this is the final form of MR image data and in most cases only magnitude images are
available. Noise removal is not only difficult in this form but also very crucial for most of MRI application. Furthermore, noise vary spatially in magnitude images. In high contrast or high magnitude images it tend to take shape of Gaussian distribution for low contrast images Rician distribution tends to shape like Rayleigh Distribution because $s$ becomes zero [21] as shown in Fig. 4.3.

\[
\text{pdf}(x) = \frac{x}{\sigma^2}e^{-x^2/2\sigma^2} \text{ where } x \in [0, \infty) \tag{4.7a}
\]

\[
\text{cdf}(x) = 1 - e^{-x^2/2\sigma^2} \text{ where } x \in [0, \infty) \tag{4.7b}
\]

![Figure 4.3: Rayleigh Distribution (a)Probability density function (b) Cumulative distribution function](image)

4.3 Why Local Sparsity Constraints Work Better?

Comparison of experimental results conducted for local and global sparsity level (as presented in last chapter) showed that global sparsity constraint is not an optimal way to sparsify MR images. These images can be sparsified even more if appropriate coefficients are chosen in local regions. Question arises that how is it even possible? Let say if an image needs 10 important coefficients to represent it, how can we reduce this number by simply dividing the image?
The image is still same than why the number of coefficients that are needed to represent this image are reduced? All these answers lie in the idea that the optimal method for sparsifying images would be the one which maximize the chance that only the right coefficients will be selected. Local sparsity constraints increase the probability of picking the important coefficients thus, sparsify images better than global sparsity constraints.

Figure 4.4: Understanding how local sparsity constraints work better

Fig.4.4 shows two regions, suppose the values within the blue circle are important coefficients but magnitude of these coefficients is lesser than the coefficients which are inside red circle. The red circle may have some coefficients which are noise but have high magnitude. The coefficients inside blue circle could not be selected until all coefficients inside red circle irrespective of their importance are selected. Thus a global sparsity constraint that picks largest coefficients out of full image reduces the probability of selecting the important
coefficients inside blue circle because they have lesser magnitude than some other coefficients. Instead of applying a global constraint, if a local constraint is applied on the region inside green rectangle only. This local constraint will pick the largest coefficients within local region but not from overall image thus increasing the probability of picking the important coefficients of that region. This ensures that the coefficients with less magnitude do not have to compete with all the coefficients of the image for selection. Exploiting the non-uniform nature and introducing local constraints increase the probability that all the important coefficients from different local regions will be selected. Mathematically this can be explained by modifying the definition of signal $Y$, now $Y$ will become set of $t$ vectors

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_j \\ \vdots \\ y_t \end{bmatrix}$$

(4.8)

and

$$y_j(i) = \begin{cases} 0 & i \notin g_j \\ \text{val} & i \in g_j \end{cases}$$

(4.9)

where $y_j$ is of size $n_j$ and sparsity is the ratio between the size of local region and significant coefficient in it $s_j = k_j/n_j$. Now $Y$ will be $S_L$ sparse such that $S_L = (k_1/n_1) + (k_2/n_2) + \ldots + (k_t/n_t)$ and $N = n_1 + n_2 + \ldots + n_t$.

Theorem 4.3.1. If local constraints sparsifies images better that means $S_L < S_g$. However, the overall sparsity of image is constrained by following two conditions. Firstly, $K$ are the minimum required coefficients to reconstruct image $I$. A diagnostic quality image cannot be reconstructed without at-least $K$ coefficients that means $\sum_{j=1}^{t} k_j = K$ also $\sum_{j=1}^{t} n_j = N$.

Proof.

$$S_g = K/N$$

(4.10)
and
\[ N = n_1 + n_2 + \ldots + n_t \]  
(4.11)

\( S_g \) will become
\[ S_g = K/(n_1 + n_2 + \ldots + n_t) \]  
(4.12)

However, while using global sparsity, the local sparsity level of each region is unknown. That is why \( K \) cannot be replaced with local levels which makes \( S_g \)
\[ S_g = (K/n_1) + (K/n_2) + \ldots + (K/n_t) \]  
(4.13)

where as
\[ S_L = (k_1/n_1) + (k_2/n_2) + \ldots + (k_t/n_t) \]  
(4.14)

From rules of inequalities of Fractions, it is known that if two fractions have same denominators than the one with smaller numerator is the smaller fraction/ratio. Thus for any region \( j \)
\[ k_j/n_j < K/n_j \]  
(4.15a)

and
\[ \sum_{j=1}^{t} k_j/n_j \ll \sum_{j=1}^{t} K/n_j \]  
(4.15b)

Hence proved
\[ S_L \ll S_g \]  
(4.15c)

\[ \square \]

4.4 De-noising MR Images using Local Sparsity constraints

When images holds the sparsity condition but measured signals has noise, in that case MR signals can be represented as
\[ Y(i) = \begin{cases} \gamma & i \notin \varnothing \\ \text{val} + \gamma & i \in \varnothing \end{cases} \]  
(4.16)

here \( \gamma \) is the noise level at any spatial location. Due to sparsity, \( Y \) has only \( S = K/N \) significant coefficients while rest are zero or nearly zero. If \( i \) is a non-significant coefficient and its value can be discarded than from sparsity point of view it only holds noise. Whereas, if \( i \) is a significant coefficient it
hold coefficient value with added noise. From this sparse condition it can be concluded that \( \Gamma = N - S \) percent coefficients are just noise and can be discarded or replaced by zero. Also lesser the value of \( S \) means higher value of \( \Gamma \) as \( N \) is a constant size of any image. \( \Gamma \) with a larger value means more coefficient can be discarded and noise can be reduced further. Thus, replacing \( S \) with \( S_L \) as it is less than \( S_g \).

\[
\Gamma = N - S_L \tag{4.17}
\]

\( \Gamma \) is the percent of coefficients which are pure noise and have no-significant value. The higher value of \( \Gamma \) means more coefficients can be discarded and less noise. Using, local energy level estimation images were sparsified better thus making \( S_L \) a lesser value and a more useful measure in terms of de-noising.

### 4.5 Methodology

This section propose a novel method to de-noise MR images based on the fact that MR images exhibits sparsity. Sparsity is previously used in literature of MRI for under-sampled data [13]. In under-sampling we have missing information but when image is fully-sampled and is corrupted by noise, it is needed to somehow extract only information bits and discarding the rest. The proposed method works on transform sparsity of MR images. This method basically reduce the number of coefficients that are used to represent image based on image sparsity information. As, it does not partially change or modify any values. It will either select a coefficient value or will discard it completely. Thus, it can be used in combination with other noise removal methods which estimate noise and modifies the data. This will further enhance the quality of
resultant image.

4.5.1 Prerequisites

- Generate regional map and find suitable threshold levels using a reference image such that the resolution of reference image is same as images under experimentation.
- Finding sparsifying transform.
- Find sparsity ratios for each region.

4.5.2 Input

- Noisy image \( I \).
- Threshold vector \( \tau \) and respective Sparsity vector \( S \).
- Transform operator \( \alpha \).

4.5.3 Algorithm

- Transform \( I \) into \( \omega \) using transform operator \( \alpha \).
- Generate a region/sparsity map of input image \( I \) based on threshold vector \( \tau \) such that each element of \( \tau \) is used to generate a sub-region in transform domain \( \omega \).
- Select \( S_i \) percent highest values from \( i^{th} \) region and discard rest of values.
- Regenerate \( I \) from \( \omega \).

4.5.4 Output

- Output image with reduced noise levels.
4.6 Experimental Results

As concluded in last chapter, Wavelets sparsify MR images very well. As, images are fully sampled and sampling is done in Fourier domain. For these experiments the regional sparsity of MR images is analysed in Wavelets and the regions are also defined in Wavelet domain unlike last chapter where the regions were defined in Fourier domain based on Energy distribution of Fourier. Fig. 4.5 shows 1-D energy distribution of MR image in wavelet domain. It shows that energy decreases along x-axis. Unlike Fourier there is no energy peak in center rather, the highest energy coefficients resides near start.

![Figure 4.5: MR image 1-D energy distribution in Wavelets](image)

4.6.1 Wavelets based regional Sparsity

Reference images were used to analyse the regional sparsity of MR images. In last chapter, results for Global sparsity for Wavelets was represented. Using 10% Wavelet coefficients, a good quality image was recovered. In this section
results are presented for Region based MR reconstruction using Wavelets. 80 images were used and divided into six categories based on MRI type. Six different kinds of reference images were used with different region threshold and sparsity levels. Each image has a resolution of 512x512. Images used were fully-sampled and noiseless.

**Wavelets with 2 sub-regions**

In first phase, experiments were done with single threshold level which was used to divide Wavelet into two regions. While choosing a threshold level it is critical to keep in mind that it should be a moderate value. Three different values were used for dividing wavelets of images with different sparsity levels and the results are presented in tables below.

Table 4.1: MSSIM for Reconstructed image using single threshold level for sub-regions and 10% sparsity level

<table>
<thead>
<tr>
<th>Area Under test</th>
<th>Region Size</th>
<th>Sparsity level: 10%-10%</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>150-rest</td>
<td>200-rest</td>
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<tr>
<td>Brain</td>
<td>0.8919</td>
<td>0.8973</td>
</tr>
<tr>
<td>Angiography</td>
<td>0.8997</td>
<td>0.8973</td>
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<td>Spine</td>
<td>0.9792</td>
<td>0.9815</td>
</tr>
<tr>
<td>Heart</td>
<td>0.9597</td>
<td>0.9657</td>
</tr>
<tr>
<td>Wrist</td>
<td>0.9782</td>
<td>0.9784</td>
</tr>
<tr>
<td>Knee</td>
<td>0.9214</td>
<td>0.9210</td>
</tr>
</tbody>
</table>

Table 4.1 shows averaged MSSIM results for Brain, Angiography, Spine, Heart, Wrist and Knee. Wavelets are divided in two sub-regions with varying sizes for each set of experiments. The partitions used were 1-D e.g. vertical or horizontal. As, it was concluded in last chapter that 10% Wavelet coefficients
are enough for representing any MR image. For initial experiments 10\# sparsity level was used for each region. Only MSSIM is used as we are dealing with incomplete data coefficients and SNR tends to give confusing results in this case. As, Wavelet energy distribution of MRI was shown in Fig.4.5. It can be seen that the energy peak is in very start. When the first region has very small size (150 columns) the quality index of recovered image is slightly low (refer Table 4.1) but for a moderate size of (200 or 250 columns) it showed increased quality with same sparsity levels. This is due to the reason that sparsity level is a ratio and it varies as the size of region varies. In effect changing image quality as region size varies.

Table 4.2: MSSIM for reconstructed image using single threshold level for sub-regions and 10\% and 5\% sparsity level

<table>
<thead>
<tr>
<th>Area Under test</th>
<th>Region Size</th>
<th>150-rest</th>
<th>200-rest</th>
<th>250-rest</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brain</td>
<td>0.8733</td>
<td>0.8843</td>
<td>0.8883</td>
<td></td>
</tr>
<tr>
<td>Angiography</td>
<td>0.8825</td>
<td>0.8866</td>
<td>0.8905</td>
<td></td>
</tr>
<tr>
<td>Spine</td>
<td>0.9766</td>
<td>0.9797</td>
<td>0.9811</td>
<td></td>
</tr>
<tr>
<td>Heart</td>
<td>0.9566</td>
<td>0.9640</td>
<td>0.9648</td>
<td></td>
</tr>
<tr>
<td>Wrist</td>
<td>0.9748</td>
<td>0.9761</td>
<td>0.9770</td>
<td></td>
</tr>
<tr>
<td>Knee</td>
<td>0.9112</td>
<td>0.9161</td>
<td>0.9133</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.2 shows results for same experimental settings but with different sparsity levels. the sparsity on high energy end is 10\% while on the low energy end it is 5\%. While the sparsity level is reduced to half for low energy end there is only minor difference in image quality. Localise sparsity constraints allows us the liberty of choosing different sparsity levels within an image and as results shows that we can still get a good reconstruction while reducing
coefficient to half. There is a trade-off between number of condiments and quality of image. However, in the required setting it is better to choose as less coefficients as possible, specially when the effect on image quality is negligible.

Table 4.3: MSSIM for reconstructed image using single threshold level for sub-regions and 10% and 3% sparsity level

<table>
<thead>
<tr>
<th>Sparsity level:10%-3%</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Area Under test</strong></td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>Brain</td>
</tr>
<tr>
<td>Angiography</td>
</tr>
<tr>
<td>Spine</td>
</tr>
<tr>
<td>Heart</td>
</tr>
<tr>
<td>Wrist</td>
</tr>
<tr>
<td>Knee</td>
</tr>
</tbody>
</table>

Same set of experiment was repeated but the sparsity level was further decreased for low energy end just 3%) while keeping same sparsity level at high energy end and the results are given in Table 4.3.

**Wavelets with 3 sub-regions**

In second phase experimental settings were slightly changed and images were divided into three regions based on energy thresholding. Two different partition sizes are used with different sparsity levels. As, last set of experiment shows that for a 512X512 image high energy end of size 250 columns gives good results. Thus, in first settings high energy end was kept same while low energy end was further sub-divided. The idea was to sparsify images further if possible. In second set of experiments high energy region size was increased while using same sparsity levels. The results are given in Table 4.4.
Table 4.4: MSSIM for reconstructed image using two threshold levels for sub-regions and 13%, 3%, 2% sparsity level

<table>
<thead>
<tr>
<th>Area Under test</th>
<th>Region Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>250-150-112</td>
<td>300-100-112</td>
</tr>
<tr>
<td>Brain</td>
<td>0.8834</td>
</tr>
<tr>
<td>Angiography</td>
<td>0.8861</td>
</tr>
<tr>
<td>Spine</td>
<td>0.9804</td>
</tr>
<tr>
<td>Heart</td>
<td>0.9645</td>
</tr>
<tr>
<td>Wrist</td>
<td>0.9760</td>
</tr>
<tr>
<td>Knee</td>
<td>0.9123</td>
</tr>
</tbody>
</table>

Images reconstructed using three sub-regions showed better quality with lesser samples. To test these settings further another set of experiment is done with increased sparsity level (13%) in high energy region. The results showed slight improvement but at the cost of increased sample set refer Table 4.5.

Table 4.5: MSSIM for reconstructed image using two threshold levels for sub-regions and 13%, 3%, 2% sparsity level

<table>
<thead>
<tr>
<th>Area Under test</th>
<th>Region Size</th>
</tr>
</thead>
<tbody>
<tr>
<td>250-150-112</td>
<td>300-100-112</td>
</tr>
<tr>
<td>Brain</td>
<td>0.9032</td>
</tr>
<tr>
<td>Angiography</td>
<td>0.9063</td>
</tr>
<tr>
<td>Spine</td>
<td>0.9852</td>
</tr>
<tr>
<td>Heart</td>
<td>0.9742</td>
</tr>
<tr>
<td>Wrist</td>
<td>0.9818</td>
</tr>
<tr>
<td>Knee</td>
<td>0.9296</td>
</tr>
</tbody>
</table>

Number of sub-regions were kept limited to three because these images are reference images and these results and findings are to be used in further experiments. Using too many sub-regions means making image specific findings
which can not be used for other images. Thus, keeping all the parameters and setting generic is important.

4.6.2 De-noising using Local Sparsity Constraints

All the experiments that are presented in previous section are used for MR de-noising. The experiments helped in understanding the sparsity of MR images in Wavelets and helped in developing some generic key features which can be used for image de-noising. The basic idea is to select limited number of coefficients and to preserve the over-all energy shape. As, energy distribution shows same kind of curve for all different kinds of MR images. In Fourier it shows a high energy peak in center and low energy regions on both ends while in Wavelets it shows high energy peak in start and low energy region afterwards. All MR images roughly maintain this shape. Thus, it can be used as a generic feature and can be used for image de-noising. Fig. 4.6 illustrates the basic idea of preserving energy distribution shape using sparsity features.
Figure 4.6: Illustration how local sparsity levels can be used to recover overall energy shape and can be helpful for de-noising MRI. First row shows original noiseless image where (a) Fourier K-Space (b) Energy distribution in Fourier domain (c) MR image. Second row shows same image with AWGN (d)(e)(f) are noisy K-Space, energy distribution and image respectively. (g)(h)(i) shows effect of sparsifying noisy brain MRI with single sparsity level and (j)(k)(l) depicts the use of local constraints in preserving the overall energy trend of MRI.
Different kinds of noisy images were used with image and experimental results of previous sections were used as reference point. For any image resolution, reference image should have same resolution. Experiments were done on two sets of images 448x448 and 512x512. Both Fourier and Wavelets were used as sparsifying domains. Firstly noisy image was sparsified using one global level based on experimental results shown in last chapter. Later for Local regions, 3 sub-regions were used for both Wavelets and Fourier. To quantify the results both MSSIM and SNR is used. As, we are dealing with noisy data SNR gives an estimation of de-nosing. However, incomplete data set effects the results but both results are presented for better understanding of the proposed method. AWGN with different levels of \( \sigma \) was used and added to K-space. That K-space was then converted into magnitude images and those images were used for experiments.

Table 4.6: MSSIM and PSNR for de-noised Image where noise ratio for AWGN is \( \sigma = 10 \)

<table>
<thead>
<tr>
<th>Noisy Image</th>
<th>MSSIM</th>
<th>PSNR</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Brain</td>
<td>Angio</td>
</tr>
<tr>
<td>Fourier</td>
<td>0.6376</td>
<td>0.5747</td>
</tr>
<tr>
<td>Localised Fourier</td>
<td>0.6082</td>
<td>0.6472</td>
</tr>
<tr>
<td>Wavelets</td>
<td>0.7342</td>
<td>0.6849</td>
</tr>
<tr>
<td>Localised Wavelets</td>
<td>0.7561</td>
<td>0.7389</td>
</tr>
<tr>
<td>Noisy Image</td>
<td>23.1</td>
<td>22.5</td>
</tr>
<tr>
<td>Fourier</td>
<td>23.0</td>
<td>22.9</td>
</tr>
<tr>
<td>Localised Fourier</td>
<td>21.9</td>
<td>22.4</td>
</tr>
<tr>
<td>Wavelets</td>
<td>23.4</td>
<td>22.9</td>
</tr>
<tr>
<td>Localised Wavelets</td>
<td>24.0</td>
<td>25.4</td>
</tr>
</tbody>
</table>
Table 4.6 shows MSSIM and PSNR for reconstructed images. All the results were averaged out based on image type. The noise level for this set of experiments was $\sigma = 10$. All images showed improved quality when Wavelets are used as their sparse domain. Localise wavelets showed further improvement. Images like spine and wrist are the ones which were most effected by noise. Yet, all showed an improvement.

Table 4.7: MSSIM and PSNR for de-noised Image where noise ratio for AWGN is $\sigma = 20$

<table>
<thead>
<tr>
<th></th>
<th>MSSIM</th>
<th>PSNR</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Brain</td>
<td>Angio</td>
</tr>
<tr>
<td>Noisy Image</td>
<td>0.4230</td>
<td>0.3423</td>
</tr>
<tr>
<td>Fourier</td>
<td>0.5358</td>
<td>0.4799</td>
</tr>
<tr>
<td>Localised Fourier</td>
<td>0.4896</td>
<td>0.4831</td>
</tr>
<tr>
<td>Wavelets</td>
<td>0.5268</td>
<td>0.4542</td>
</tr>
<tr>
<td>Localised Wavelets</td>
<td>0.5725</td>
<td>0.5131</td>
</tr>
<tr>
<td>Noisy Image</td>
<td>17.9</td>
<td>17.2</td>
</tr>
<tr>
<td>Fourier</td>
<td>18.2</td>
<td>17.6</td>
</tr>
<tr>
<td>Localised Fourier</td>
<td>17.9</td>
<td>17.5</td>
</tr>
<tr>
<td>Wavelets</td>
<td>18.2</td>
<td>17.5</td>
</tr>
<tr>
<td>Localised Wavelets</td>
<td>19.1</td>
<td>18.9</td>
</tr>
</tbody>
</table>

Table 4.7 shows recovered image quality when noise level is $\sigma = 20$. Rest of the experimental settings and parameters remain same. As, noise level increase the overall quality decreased but the suggested method showed improvement in quality. for the sake of experiments a very high level of noise is used with $\sigma = 50$ and results are presented in Table 4.8. Due to very high noise as well as incomplete data after sparsification resulted in confusing results for SNR.
Table 4.8: MSSIM and PSNR for de-noised Image where noise ratio for AWGN is $\sigma = 50$

<table>
<thead>
<tr>
<th></th>
<th>MSSIM</th>
<th></th>
<th>PSNR</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Brain</td>
<td>Angio</td>
<td>Spine</td>
<td>Heart</td>
</tr>
<tr>
<td>Noisy Image</td>
<td>0.2170</td>
<td>0.1583</td>
<td>0.0457</td>
<td>0.1424</td>
</tr>
<tr>
<td>Fourier</td>
<td>0.2887</td>
<td>0.2356</td>
<td>0.0649</td>
<td>0.2567</td>
</tr>
<tr>
<td>columns Fourier</td>
<td>0.3007</td>
<td>0.2592</td>
<td>0.0725</td>
<td>0.2928</td>
</tr>
<tr>
<td>Wavelets</td>
<td>0.2843</td>
<td>0.2278</td>
<td>0.0659</td>
<td>0.2417</td>
</tr>
<tr>
<td>Localised Wavelets</td>
<td>0.3210</td>
<td>0.2556</td>
<td>0.0723</td>
<td>0.2823</td>
</tr>
</tbody>
</table>

4.6.3 Using Wiener and Gaussian Filter

To further test the method, it was combined with other noise removal techniques. Simple linear filtering was applied for this purpose. Two kinds of filters were used Gaussian and Wiener. These are low pass filters for additive noise. Wiener filter works on each image pixel based on local neighbors. Firstly, images were sparsified using previously suggested method using Wavelets. Later the filters were applied. Later the filters were applied without sparsifying the data and the results are compared in both cases. Fig. 4.7 shows the trends in average MSSIM of the reconstructed images. Three different noise levels were used where $\sigma = 10, 15 and 20$. MSSIM showed improved quality when data was sparsified. The graphs compares the results with and without sparsifying data in six kinds of MRI.
Figure 4.7: MSSIM for De-noising methods. First row is for noise level $\sigma = 10$ where (a) is quality index for sparsifying the noisy data (b) Sparsification with Gaussian Filter (c) Sparsification with Wiener Filter. Second row is for noise level $\sigma = 15$ and (d), (e), (f) are noisy data, Gaussian and Wiener filtered data respectively. Third row is for for noise level $\sigma = 20$.

SNR was also calculated and compared for these experiments. It was used to verify the results generated by MSSIM. Fig. 4.8 summarise the results and shows a better quality reconstruction when data was sparsified.
Figure 4.8: PSNR for De-noising methods. First row is for noise level $\sigma = 10$ where (a) is quality index for sparsifying the noisy data (b) Sparsification with Gaussian Filter (c) Sparsification with Wiener Filter. Second row is for noise level $\sigma = 15$ and (d), (e), (f) are noisy data, Gaussian and Wiener filtered data respectively. Third row is for for noise level $\sigma = 20$.

4.7 Conclusion

The proposed method use sparsity information of MR images for reducing the number of coefficients and in effect reducing the noise. This method does not try to replace previous methods which are proposed in literature. It tries to improve and enhance previous methods and can be used in combination with
any other noise removal methods.

Next chapter will use local sparsity constraints for Rapid MRI.
Chapter 5

Locally Sparsified Compressive Sensing

5.1 Introduction

This chapter deals with rapid MRI. MRI is a slow process and it is crucial to speed it up. Compressive Sensing suggests that a sparse signal should be measured at a sampling rate much lesser than the traditional Nyquist sampling rate. In MRI, less sampling means rapid imaging. This chapter will present a novel framework for rapid MRI where Compressive Sensing is combined with non-uniform sparsity of MR images to achieve a good quality image with small sample set.

5.2 Under-Determined System

All the real-time analog signals are continuous. These analog signals cannot be directly used in digital systems due to their continuity. The analog signals are digitized and converted to discrete signals for digital processing. A sensing operator or matrix is used e.g. Fourier during this conversion. Later, original signal can be reconstructed using the known sensing operator and digitised
signal. This process can be expressed as a linear system.

\[ Ax = b \]  

(5.1)

Where \( x \) is original signal, \( A \) is the sensing matrix and \( b \) is the digitized output of \( x \). It is the property of linear system that if there are \( m \) measurements and \( n \) unknowns, then to solve the linear system \( x \), \( m \) should be greater that \( n \), this is an ideal case when system is over determined and exact solution of \( x \) can be found [26]. In case of under sampling, the system is under-determined. As measurements or samples \( m \) are less than \( n \), the exact solution of \( x \) cannot be determined rather a sub-plane is the solution. The estimated solution is obtained by guessing \( x \) from that sub-plane.

### 5.2.1 L2 Norm of signal

Traditionally least square method or L2 norm is used for estimating the solution for \( x \).

\[ x^* = \arg\min_{x \rightarrow Ax = b} \| x \|_2 = \sqrt{\sum_{i=0}^{n} (x_i)^2} \]  

(5.2)

Where, \( x^* \) is the estimated solution from the sub-plane and has minimum energy value. As, L2 norm represents the energy of the signal, least square or minimum L2 norm means finding \( x^* \) with lowest possible energy on the sub-plane such that \( x^* \) satisfies \( Ax^* = b \). Low energy means low noise and better recovery. So, among all possible \( x \) on the sub-plane least square method picks the one with lowest energy level thus minimizing the noise as well. L2 norm does not work in all cases. For example when working in Fourier domain,
under-sampled or under-determined system means partial Fourier. If L2 norm is applied to minimize energy, it will simply set all unknown coefficients to zero and this solution will give least energy. This implies merely taking partial Fourier for reconstruction and ignoring rest of the coefficients by setting them to zero. In other words, minimizing L2 norm of partial Fourier will return back partial Fourier. Thus L2 norm will not work in this scenario.

![Signal recovery using Least Squares or L2](image)

Figure 5.1: Signal recovery using Least Squares or L2

### 5.2.2 L1 minimisation

Suppose $x$ be sparse and it has few important coefficients while most of its data is blank, static or boring. In an under-determined system, when $x$ is $S$-sparse a better inference can be made because the estimation will not consider the whole sub plane rather it will consider only those possible solution which are $S$ sparse and reject others. So the candidate set of possible solutions is much smaller due to sparsity and should lead to better estimation.

If $x$ is sparse and under-sampled signal, exact recovery or finding the best
solution is possible by minimizing the sparsity of $x$. L0 norm is the sparsifying norm of the signal which means minimum L0 norm will return sparsest solution for $x$. However, finding L0 norm is NP hard problem. Substituting L0 with L1 norm for sparse signal recovery gave good results [13]. L1 minimization of a signal can be written as

$$
x^* = \underset{x \rightarrow Ax=b}{\text{argmin}} |x|_1 = \sum_{i=0}^{n} |x|_i
given \quad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^{m},
$$

(5.3)

![Figure 5.2: Signal recovery using L1 minimisation](image)

### 5.3 Compressive Sensing

Sampling process converts analog signals, which are continuous in time domain, into discrete digital signals. These discrete signals are quantized. Analog signals can be approximated later from these discrete values. Traditionally Nyquist sampling rate is used for digitisation process. In MRI, the image samples are stored in a 2D Fourier matrix or K-space to generate the images.
K-space can be expressed as

\[ y = F.x \] (5.4)

where \( F \) is Fourier operator, \( y \) is received signal and \( x \) is the required image.

Compressive sampling suggests that k-space should be partially scanned to store only adaptively chosen coefficients [3]. The under sampled k-space can be formulated as

\[ y = F_u.x \] (5.5)

\( F_u \) is partially sampled sensing matrix. Due to under-sampling, \( x \) cannot be regenerated exactly. The system is under-determined as it violates the Nyquist rate.

5.3.1 Sparsity

Each signal detected on the sensors contains specific information about the tissue or organ of human body. This information or the sparsity determines the compressibility in the image. As, all of the information is not important for image generation. Most of the images can be compressed without deteriorating the image quality.

Compressive Sensing states that required measurements for regenerating the sparse signals are far lesser than what traditionally used. This is due to the fact that sparse signals are highly compressible and most of the coefficients are zero or not significant. Usually signals are captured and then compressed by discarding non-significant coefficients and to make these signals memory and transmission efficient. CS states that the required samples or measurements
should be according to the compressed size or Sparsity $S$ of Signal. Why measure all the coefficients when they will be discarded later? Medical images are sparse and this property can be used to reduce the sample set. Hence, the image generation process can be sped up. Conventionally, an under-sampled system cannot be recovered exactly. It can just be estimated and chance of error is really high. However, with the condition of sparsity under-determined system becomes a special case. CS allows another degree of freedom as it requires sparsity in any orthonormal transform domain.

\[ a = \Psi x \]  
\[ x = a \Psi^T \]  

$\Psi$ be the sparse transform domain and $a$ is transform of image $x$. Thus, substituting $a$ by $x$ in (5.5)

\[ y = F_u.a \Psi^T \]  

5.3.2 L1 minimisation

As $x$ is sparse, the best solution will be to find the sparsest solution for $a$. This can be achieved by minimizing the sparsifying norm (L0 norm) of the signal.

\[ \min \|a\|_0 \text{ s.t. } \|y - F_u.a \Psi^T\| < \epsilon \]  

where $\epsilon$ is noise and $\|a\|_0 = \sum_{i=1}^{N} |x_i|^0$. However, solving L0 norm is NP hard problem. CS proposes that minimized L1 norm ($\|\tilde{x}\|_1 := \sum_{i=1}^{N} |\tilde{x}_i|$) can be used to obtain the sparsest solution of $x$, if it holds certain conditions.
Minimizing L1 norm is a convex optimization problem and there are many efficient algorithms to find it [172]

\[
\min \|a\|_1 \text{ s.t. } \|y - F_u.a.\Psi^T\| < \epsilon
\]  

(5.9)

Compressive Sensing suggests that a signal of size \(N\) with a sparsity level \(S\), can be reconstructed by taking \(O(S\log(N/S))\) random samples, where the expected error is no more than the traditional methods [4]. In other words, \(S\) sparse signal of size \(N\), can be recovered almost exactly by collecting \(K\) random samples where

\[
K > S\log N.
\]  

(5.10)

Using a sample set of size \(K\) instead of \(N\) makes the acquisition process much faster. Compressive sensing states that L1 minimisation can reconstruct an image from under sample data, given that image is sparse enough and noise is incoherent. The study shows that equi-spaced samples cause coherent noise and the original signal is unrecoverable. So, \(K\) samples must be randomly selected [3]. As it generates in-coherent noise and allows the sparse coefficients to stand out. Later, a non-linear recovery method can be used to recover the signal. The literature shows that two to five samples for each nonzero term are required and a non linear reconstruction method (L1 minimization) can recover the exact signal [4].
5.4 Non-Uniform Sparsity

MR machines sense data in Fourier domain. However, CS requires sparsity in any orthonormal transform domain. Sparsity level $S$ in any transform domain $\Psi$ is the amount of coefficients required for diagnostic-quality image reconstruction. Compressive sensing use one global sparsity level $S$ for the whole image and the amount of sampling $K$ is directly dependent of $S$.

In Frequency domain, origin of k-space is highly concentrated in terms of energy whereas the energy level decreases as the distance from origin increases. MRI coefficients can be divided and their significance can be determined based on their spatial locations. This property of K-space has been used previously by many researchers for different purposes in parallel imaging e.g. to improve
estimation of missing harmonics, to generate randomization [173, 174].

In previous chapter, it was established that MR images are non-uniformly sparse; low frequency areas contain most of the image energy whereas high frequency areas have relatively less energy. Multiple local sparsity constraints were applied based on local sparsity levels. Thus, allowing different sparsity levels for different regions. It works better because all the coefficients are competing only within their local regions and this improves their chances of selection. Global CS is bound to have the same sparsity rate in all the regions as there is only one sparsity constraint and sparsities of different regions are not differentiable. However in my technique of CS with local constraint, each sub-region can have different sparsity level.

5.5 Locally Sparsified Compressive Sensing

CS requires images to be sparse. The sparsity condition can hold in any domain. MR images sense data in Fourier. However, for image reconstruction CS allow to choose any orthonormal transform which can better sparsify the images. Any $S$-Sparse image in a transform domain $\Psi$ can be reconstructed using $K$ random samples where $K$ is dependent on $S$ as shown in Eq. (5.10). In any transform domain, lesser $S$ means lesser measurements. Simple CS uses a global sparsity level $S$ for whole image whereas my proposed method Locally Sparsified CS or Local CS suggests that dividing K-space into sub-regions and using multiple local constraints can sparsify images better. Thus, allowing fewer measurements.

The energy distribution in MR images is non-uniform. Some regions are
highly sparse while others are not. The origin of K-space is packed with energy. On the other hand, that the outer regions are highly sparse. Thus, localizing the sparsity levels and sampling patterns allow a better image recovery.

Theoretically it works better because significant coefficient are targeted more closely. It narrows down sparsity level of each region within K-space and make sure that specific number of samples get picked from that particular region. Thus, allowing a better chance of selection for all the significant coefficients.

Let’s assume $S$ sparse K-space with total size $N$ is sub-divided into $t$ local regions where each region has size $n_i$ and sparsity level $s_i$. Mathematically, its can be stated as

$$\Psi = \begin{bmatrix}
\psi_1 = \psi_1, \psi_2...\psi_{n_1} \\
\vdots \\
\psi_j = \psi_1, \psi_2...\psi_{n_j} \\
\vdots \\
\psi_t = \psi_1, \psi_2...\psi_{n_t}
\end{bmatrix}$$

(5.11)

$\psi$ is sub-division of $\Psi$ into $t$ local regions of sizes $n_1, n_2...n_t$ with sparsity levels $s_1, s_2...s_t$ respectively.

From Eq.(5.10) the sampling rate $K$ for simple CS should be greater than $S\log N$. Based on this equation sampling rate for locally sparsified compressive sensing can be written as

$$K > s_1\log n_1 + s_2\log n_2... + s_t\log n_t$$

(5.12)

**Theorem 5.5.1.** If $x$ is $S$-sparse than dividing the sparsity $S$ into multiple local constraints will require fewer samples such that $s_1\log n_1 + s_2\log n_2... + s_t\log n_t < S\log N$
Proof. If

\[ N = n_1 + n_2 \ldots + n_t \]  \hspace{1cm} (5.13)

and

\[ S = s_1 + s_2 \ldots + s_t \]  \hspace{1cm} (5.14)

By Using logarithmic and power rules on Eq. (5.10)

\[ S\log N = \log N^S \]  \hspace{1cm} (5.15a)

Substituting \( S \) from Eq.(5.14)

\[ = \log N^{(s_1+s_2+\ldots+s_t)} \]  \hspace{1cm} (5.15b)

\[ = \log(N^{s_1}N^{s_2} \ldots N^{s_t}) \]  \hspace{1cm} (5.15c)

Now using same logarithmic and power rules on Eq. (5.18)

\[ s_1\log n_1 + s_2\log n_2 + \ldots + s_t\log n_t \]  \hspace{1cm} (5.16a)

\[ = \log n_1^{s_1} + \log n_2^{s_2} + \ldots + \log n_t^{s_t} \]  \hspace{1cm} (5.16b)

\[ = \log(n_1^{s_1}n_2^{s_2} \ldots n_t^{s_t}) \]  \hspace{1cm} (5.16c)

Hence from Eq.(5.15c) and Eq.(5.16c) it can be concluded that local constraints work better than one global constraint because \( \log(N^{s_1}N^{s_2} \ldots N^{s_t}) > \log(n_1^{s_1}n_2^{s_2} \ldots n_t^{s_t}) \). This proves that \( s_1\log n_1 + s_2\log n_2 + \ldots + s_t\log n_t < S\log N \) and \( K_{LCS} < K_{CS} \)

**Numerical Example using Shepp-Logan phantom** 5.5.1. For further explanation Shepp-Logan phantom was used to assign some numerical values for above equations. Shepp-Logan phantom of size 512x512 is used with 5% global sparsity level.

\[ N = 512 \times 512 = 262144 \]

\[ S = .05 \times 512 \times 512 = 13107 \]  and

\[ t = 3 \]

which means k-space is divided in 3 sub-regions and \( s_1, s_2 \) and \( s_3 \) are 5%, 5%, 5% respectively and their overall sum is equal to \( S \).

The sizes of three regions are

\[ n_1 = 156 \times 512 \]

\[ b_2 = 200 \times 512 \]
\[ b_3 = 156 \times 512 \]

respectively and the \( n_1 + n_2 + n_3 = N \) which is
\[ 79872 + 102400 + 79872 = 262144 \]
and
\[ s_1 = .05 \times 79872 = 3993.6 \]
\[ s_2 = .05 \times 102400 = 5120 \]
\[ s_3 = .05 \times 79872 = 3993.6 \]
also
\[ s_1 + s_2 + s_3 = S \] which is
\[ 3993.6 + 5120 + 3993.6 = 13107.2 \]

Proof. Now Substituting these numerical values in Eq.(5.10) and finding \( K \).
\[ K > \log N \]
\[ K > 13107 \times \log(262144) \]
\[ K > 13107 \times (5.4) \]
\[ K > 71021.8 \]
For simple CS required value of \( K \) should be greater than 71021.8 for this
given data now Substituting these values in Eq.(5.18)
\[ K > s_1 \log n_1 + s_2 \log n_2 + s_3 \log n_3 \]
\[ K > 3993.6 \times \log(79872) + 5120 \times \log(102400) + 3993.6 \times \log(79872) \]
\[ K > (3993.6 \times 4.9) + (5120 \times 5.01) + (3993.6 \times 4.9) \]
\[ K > 64788.48 \]

Fig.5.4 shows reconstructed images have same quality while the required

![Image](image_url)
value of $K$ for local CS is much lesser than simple or global CS. Next section will define basis for applying Local CS and a complete algorithm for image reconstruction.

5.6 Defining Sub-regions

The problem is how to define the size and number of sub-divisions prior to image acquisition process. Wavelets sparsify MR images better. Hence, using $\psi = \text{wavelets}$, experimental results show good image recovery with lesser coefficients. I have used Sparsity/energy distribution to define total number of local regions $t$ and their respective sizes $n_1, n_2, ... n_t$. However, $t$ can neither be too large nor be too small. As large value of $t$ will generate total random sampling effect and will increase overhead while smaller $t$ will work as Global CS.

5.6.1 Total number of sub-regions

Fig.5.5 shows the energy distribution of k-space in horizontal and vertical directions. It can be seen that in any direction the distribution is same i.e. high energy is confined in the middle while the ends or edges have low energy. While choosing the value for $t$ (total number of sub-regions), it is important that value should neither be very small nor too big. A small value of $t$ will not allow exploitation of the non-uniform nature of sparsity in k-space as it would be similar to a global region and global sparsity constraint. On the other hand, a large value of $t$ means having too many local regions. Sampling each sub-region separately is equivalent to randomly sampled k-space, therefore any large value of $t$ will result in increased sampling overhead rather than making
it efficient [13].

Figure 5.5: 1-D Energy distributions for K-space (a)Energy distribution column-wise (Vertical Direction) (b)Energy distribution row-wise (Horizontal Direction)

Non-uniform 1D sparsity or energy distribution in Fig.5.5 can be divided into 3 visible and most prominent subsections based on amount of energy/sparsity values.

1. Low energy/sparsity area before the peak.

2. Peak section or high energy area in the middle.

3. Low energy/sparsity area after the peak.

Based on this information it can be safely concluded that $t = 3$.

5.6.2 Determining the size of sub-regions

If the size of the high energy area which lies near origin can be determined, the upper and lower sub-regions will be defined automatically. Defining the middle region is critical as this region contains high energy values and any
wrong estimation can result in high noise levels. Based on required field of view (FOV) and resolution, k-space origin can be calculated. Let \( x_0 \) and \( y_0 \) be the origin coordinates. The energy decreases gradually as you move away from the origin. The idea is to calculate the high energy region based on its distance from origin. However, sub-regions will be defined in one dimension (either vertical or horizontal). One dimension Euclidean distance is simply the distance between two rows (in horizontal direction) or two columns (in vertical direction). Therefore, the distance \( \Delta \) which is required to segment the image can be stated as

\[
\Delta = \sqrt{(x_0 - x_1)^2} = |x_0 - x_1| \quad (5.17a)
\]

\[
\Delta = \sqrt{(y_0 - y_1)^2} = |y_0 - y_1| \quad (5.17b)
\]

Based on FOV and image resolution, \( \Delta \) can vary. PSF (Point Spread Function) is used to assure that appropriate value of \( \Delta \) is being selected. PSF is a technique to analyze noise or interference based on the image reference point.

\[
PSF(i; j) = x^*_j F^*_u F_u x_i 
\]

(5.18)

Where \( x \) be the vector which is all zeros except 1 at the \( i^{th} \) location, \( i \) be the reference pixel and \( j \) is the pixel where interference of \( i \) will be analyzed. For a fully sampled image, \( PSF(i; j) = 0 \), given that \( i \neq j \). Multiple values of \( \Delta \) can be chosen, partial Fourier will be taken based on different values of \( \Delta \) and image will be generated. PSF will analyze the contribution of each intensity pixels on the resultant noise. Once \( \Delta \) is determined it can be set for
future use without having any prior image information just based on required resolution. Furthermore, this method will define sub-regions prior to image acquisition.

Figure 5.6: (a) shows the region with $\Delta = 50$, (b) has $\Delta = 100$ and (c) has $\Delta = 200$ while (e),(f),(g) are their respective PSF.

We have used a fully sampled reference image to define sub-regions. Once sub-regions were determined for the reference image, they can be used in future for all the images with same FOV. If appropriate size can be defined for the central high energy peak (determining the starting and ending coordinates of high energy area) than K-space can easily be divided in 3 energy/sparsity regions. From a fully sampled K-space, only a block or subset of values around origin was taken using this algorithm and image was generated. PSF was calculated for this partial data. Block size which resulted in best PSF was used to define sub-regions for all the images with same resolution as shown in Fig.5.6. It shows that for a 512x512 resolution image when the $\Delta$ is equal to 50 and 100, the energy spike in center is not differentiable because the level of interference is so high, while for $\Delta = 200$ the central spike can be seen as the
interference level is low. Therefore, the selected value of $\Delta = 200$ for image resolution 512x512.

5.6.3 Algorithm for Local CS

Prerequisites As, previously developed different sub-regions in MR images hold different sparsity level. For implementation of Local CS, size and sparsity level of each sub-region was estimated prior to image acquisition.

- Number of sub-regions $t = 3$
- $\Delta$ is the size of middle region which was estimated using PSF.
- Local sparsity levels ($s_1, s_2$ and $s_3$) in transform domain ($\Psi$) must be known. For estimating the sparsity level same method was used as given in [13]. However, for our method we have implemented it on local regions rather than whole image. From each sub-region $s_i\%$ largest coefficients were chosen while rest were discarded given that a good quality image was reconstructed using only $s_i\%$ coefficients.

INPUTS:

- $y$ - k-space.
- sparsity levels $s_1, s_2...s_t$ for $t$ regions and their respective sizes $n_1, n_2...n_t$.
- $\Psi$ - Sparse transform.
- $\epsilon$ - Constant for data consistency constraint used in compressive sensing MRI.

OUTPUTS:

- $x$ - Regenerated image.
5.7 Methodology

In order to determine the sparsifying transform, we have used four different transform domains i.e. Discrete Cosine Transform (DCT), Wavelet and Fourier and Image domain. From fully sampled image, only largest coefficients were taken in fixed percentages while the rest of them were discarded. The images were reconstructed later to determine the suitable sparsity transform $\Psi$. The results were compared with the fully sampled image, recorded in terms of image quality and were presented in last chapter. For implementing Local CS, I have divided each image into non-overlapping sub-regions using PSF and determined local sparsity constraint $s_1, s_2, ..., s$ for each sub-region. Then a local Fourier Operator $F_{u_i}$ was generated for each region $i$ such that it satisfies the sparsity constraint $s_i$. These Fourier operators are later combined as $F_u$ to form a Global Fourier operator of the image. Images were reconstructed using Partial Fourier $F_u$, selected transform domain $\Psi$ and L1 minimisation. Later Global CS was applied using variable density sampling and L1 minimisation. The

- $MSSIM$ - Quality Index.

**ALGORITHM:**

- Generate $F_{u_k}$ under-sampled Fourier Operator for a region of k-space with size $n_k$ such that it has $s_k$ random sample.
- Generate Fourier operator $F_u$ for whole K-space by combining Fourier operators of all the regions.
- Minimize $\min \|\Psi a\|_1 \text{ s.t. } \|y - F_u a\| < \epsilon$
- Calculate $MSSIM$ for recovered image $x$. 
Process for Locally Sparsified Compressive Sensing is illustrated in Fig. 5.7.

Figure 5.7: Illustration of the process for Locally Sparsified Compressive Sensing which reconstructs image using L1-minimisation from partial Fourier data with local regions and sparse transform domain. Local regions are defined using Point Spread Function (PSF) and image energy distribution such that generated noise is incoherent.

5.8 Experimental evaluation

5.8.1 Sparsity in Local Regions

Experiments were done to analyze the nature of sparsity within different regions of image. 56 different images of brain, heart, angiography, knee, spine and wrist were used. All experiments were done using MATLAB. All images
were of size 448x448. In step one, images were reconstructed using a Global CS. A global sparsity constraint of 10% was applied with variable density random sampling and L1 minimisation. Average MSSIM for Compressive Sensing MRI was 0.9603. Later, Images were divided into different sections (top, bottom, left, right) and series of experiments were done for each region separately to determine how local sparsity levels vary within that region. The final Quality index was calculated by averaging out all 56 resultant values. Results were later compared with global CS. For each sub-region of K-space i.e top, left, right, bottom; 7 different sparsity levels and 2 different region sizes were used to analyse the sparsity variation within K-space itself.

For the top margin, experiments were done on top 100 and 200 rows. In these experiments only the top margin was under sampled differently, rest of the k-space was under sampled same as for the Global CS. The results are shown in Table 5.1.

<table>
<thead>
<tr>
<th>Sparsity</th>
<th>Margin size</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(100x448)</td>
</tr>
<tr>
<td>0%</td>
<td>0.9531</td>
</tr>
<tr>
<td>10%</td>
<td>0.9548</td>
</tr>
<tr>
<td>20%</td>
<td>0.9574</td>
</tr>
<tr>
<td>30%</td>
<td>0.9580</td>
</tr>
<tr>
<td>40%</td>
<td>0.9596</td>
</tr>
<tr>
<td>50%</td>
<td>0.9603</td>
</tr>
<tr>
<td>60%</td>
<td>0.9604</td>
</tr>
</tbody>
</table>

MSSIM=0.9 shows no critical loss of information. Even with 0% sparsity in top margin of size 100, the MSSIM of recovered images was very good. However, these results are greatly dependent on sub-region size. The quality
decreased as the margin sizes was increased to 200. However, with increased sparsity level, good quality results were achieved again. Similar results were found for Bottom, Left and Right margins. Same method was used with 7 sparsity levels and 2 different sizes. Under-sampling was done in one region at a time to analyse the effect of each region separately. Results are shown in Table 5.2.

Table 5.2: MSSIM for bottom, left, right Margin

<table>
<thead>
<tr>
<th>Sparsity</th>
<th>Bottom Margin (100x448)</th>
<th>Bottom Margin (200x448)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>0.9536</td>
<td>0.9053</td>
</tr>
<tr>
<td>10%</td>
<td>0.9551</td>
<td>0.9257</td>
</tr>
<tr>
<td>20%</td>
<td>0.9572</td>
<td>0.9355</td>
</tr>
<tr>
<td>30%</td>
<td>0.9586</td>
<td>0.9426</td>
</tr>
<tr>
<td>40%</td>
<td>0.9595</td>
<td>0.9494</td>
</tr>
<tr>
<td>50%</td>
<td>0.9605</td>
<td>0.9558</td>
</tr>
<tr>
<td>60%</td>
<td>0.9612</td>
<td>0.9604</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Sparsity</th>
<th>Left Margin (448x100)</th>
<th>Left Margin (448x200)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>0.9554</td>
<td>0.8982</td>
</tr>
<tr>
<td>10%</td>
<td>0.9538</td>
<td>0.9170</td>
</tr>
<tr>
<td>20%</td>
<td>0.9554</td>
<td>0.9385</td>
</tr>
<tr>
<td>30%</td>
<td>0.9580</td>
<td>0.9452</td>
</tr>
<tr>
<td>40%</td>
<td>0.9612</td>
<td>0.9516</td>
</tr>
<tr>
<td>50%</td>
<td>0.9619</td>
<td>0.9568</td>
</tr>
<tr>
<td>60%</td>
<td>0.9628</td>
<td>0.9615</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Sparsity</th>
<th>Right Margin (448x100)</th>
<th>Right Margin (448x200)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>0.9585</td>
<td>0.9040</td>
</tr>
<tr>
<td>10%</td>
<td>0.9602</td>
<td>0.9264</td>
</tr>
<tr>
<td>20%</td>
<td>0.9604</td>
<td>0.9389</td>
</tr>
<tr>
<td>30%</td>
<td>0.9618</td>
<td>0.9460</td>
</tr>
<tr>
<td>40%</td>
<td>0.9624</td>
<td>0.9545</td>
</tr>
<tr>
<td>50%</td>
<td>0.9639</td>
<td>0.9599</td>
</tr>
<tr>
<td>60%</td>
<td>0.9647</td>
<td>0.9632</td>
</tr>
</tbody>
</table>
Later all four margins were under sampled simultaneously and the results are accumulated in Table 5.3.

Table 5.3: MSSIM for all margins

<table>
<thead>
<tr>
<th>Sparsity</th>
<th>Central block size</th>
<th>248x248</th>
<th>48x48</th>
</tr>
</thead>
<tbody>
<tr>
<td>0%</td>
<td>0.9304</td>
<td>0.6656</td>
<td></td>
</tr>
<tr>
<td>10%</td>
<td>0.9381</td>
<td>0.7302</td>
<td></td>
</tr>
<tr>
<td>20%</td>
<td>0.9509</td>
<td>0.798</td>
<td></td>
</tr>
<tr>
<td>30%</td>
<td>0.9546</td>
<td>0.8534</td>
<td></td>
</tr>
<tr>
<td>40%</td>
<td>0.9589</td>
<td>0.8948</td>
<td></td>
</tr>
<tr>
<td>50%</td>
<td>0.9622</td>
<td>0.9272</td>
<td></td>
</tr>
<tr>
<td>60%</td>
<td>0.9653</td>
<td>0.9492</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 5.8 shows trends of Global CS and quality comparison with image reconstruction using local Sparsity constraints applied in all margins. Global CS used 10% sparsity level in all regions while local CS use reduced sparsity level near boarders. Results showed good recovery with size 100 while with too large margin size like 200 quality was degraded.

![Figure 5.8: Trend of MSSIM for all margins](image-url)
5.8.2 Locally Sparsified Compressive Sensing

A systematic and structured approach was further applied on a different size of image (512x512) to make sure that results are consistent and local constraints can actually reduce sample set without degrading image quality. Rather than defining local regions manually, PSF was used. For an image of size 512x512, a good PSF was achieved at block size of 200 and parameters for local CS were settled as follows $t = 3$, $n_1 = 156 \times 512$, $n_2 = 200 \times 512$, $n_3 = 156 \times 512$. Sparsity level for Global CS was kept same as in previous experiment i.e. 10%. For Local CS the origin or middle region $l_2$ was kept unchanged with sparsity level $s_2 = 10\%$ while for $s_1$ and $s_3$, sparsity level was reduced by half $s_1 = 5\%$, $s_3 = 5\%$. Results are summarised in Table 5.4.

<table>
<thead>
<tr>
<th>Area Under Test</th>
<th>MSSIM</th>
<th>No. of Sample</th>
<th>% sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brain CS</td>
<td>0.9410</td>
<td>102912</td>
<td>40%</td>
</tr>
<tr>
<td>Local CS</td>
<td>0.9406</td>
<td>75900</td>
<td>20%,40%,20%</td>
</tr>
<tr>
<td>Angio CS</td>
<td>0.9561</td>
<td>102912</td>
<td>40%</td>
</tr>
<tr>
<td>Local CS</td>
<td>0.9454</td>
<td>75900</td>
<td>20%,40%,20%</td>
</tr>
<tr>
<td>Spine CS</td>
<td>0.8780</td>
<td>102912</td>
<td>40%</td>
</tr>
<tr>
<td>Local CS</td>
<td>0.9881</td>
<td>75900</td>
<td>20%,40%,20%</td>
</tr>
<tr>
<td>Heart CS</td>
<td>0.9472</td>
<td>102912</td>
<td>40%</td>
</tr>
<tr>
<td>Local CS</td>
<td>0.9820</td>
<td>75900</td>
<td>20%,40%,20%</td>
</tr>
<tr>
<td>Wrist CS</td>
<td>0.9942</td>
<td>102912</td>
<td>40%</td>
</tr>
<tr>
<td>Local CS</td>
<td>0.9932</td>
<td>75900</td>
<td>20%,40%,20%</td>
</tr>
<tr>
<td>Knee CS</td>
<td>0.9235</td>
<td>102912</td>
<td>40%</td>
</tr>
<tr>
<td>Local CS</td>
<td>0.9460</td>
<td>75900</td>
<td>20%,40%,20%</td>
</tr>
</tbody>
</table>

CS suggests taking samples roughly four times of sparsity level [13]. Thus, global CS required 40% samples of 512x512 while for local CS, 40% of 200x512 (size of middle region) and the amount of coefficients was roughly reduced to half for outer regions based on sparsity experiments of reference image. 20-23%
of 156x512 (size of outer regions) samples were taken.

Fig. 5.9 shows the comparison of these methods in term of required measurements/samples. Local CS showed good recovery even for some images it gave better results than global CS.

Fig. 5.10 shows quality comparison between both techniques for different kinds of images. Local CS reduces sample set approximately 30%. Thus, making process time efficient.
5.9 Overlapping Regions

In previous section i have developed and used Locally Sparsified Compressive Sensing using disjoint sub-regions. Local regions were defined based on energy distribution and required resolution using PSF. Sparsity constraints were defined for each region using a fully sampled reference image. To further extend previous work, Local CS was implemented using overlapping regions and the results are analyzed and presented here. Six different MR images of a human body i.e. Brain, Angiography, Spine, Heart, Wrist and Knee were used having image size of 512 x 512. The images were selected on the basis of complexity of inner organs and tissue densities so that, this method can be analyzed in detail. MR images were initially recovered using the Global CS [13].

After defining local regions and constraints, local CS was implemented with disjoint regions. The effect of overlapping regions was investigated by varying the amount of overlapped area between regions. Three different sizes of overlapping area (50 rows, 100 rows and 200 rows) were used to study the results. The results of all the applied techniques were compared in terms of quality and number of samples and summarized in Tabel 5.5.
Table 5.5: Disjoint Vs. Overlapping Local CS (Quality index for Brain MRI using different sparsity levels)

<table>
<thead>
<tr>
<th>Local CS</th>
<th>MSSIM</th>
<th>Disjoint overlapping size 50</th>
<th>Local CS overlapping size 100</th>
<th>Local CS overlapping size 200</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Brain</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5%</td>
<td>0.871</td>
<td>0.8766</td>
<td>0.7745</td>
<td>0.7725</td>
</tr>
<tr>
<td></td>
<td>3%</td>
<td>0.9126</td>
<td>0.8564</td>
<td>0.8546</td>
</tr>
<tr>
<td>10%</td>
<td>0.941</td>
<td>0.9741</td>
<td>0.9708</td>
<td>0.9674</td>
</tr>
<tr>
<td>20%</td>
<td>0.994</td>
<td>0.9963</td>
<td>0.9945</td>
<td>0.9915</td>
</tr>
<tr>
<td><strong>Arteriography</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5%</td>
<td>0.876</td>
<td>0.8880</td>
<td>0.8166</td>
<td>0.8147</td>
</tr>
<tr>
<td></td>
<td>3%</td>
<td>0.9398</td>
<td>0.9403</td>
<td>0.9366</td>
</tr>
<tr>
<td>10%</td>
<td>0.948</td>
<td>0.9749</td>
<td>0.9716</td>
<td>0.9677</td>
</tr>
<tr>
<td>20%</td>
<td>0.979</td>
<td>0.9967</td>
<td>0.9952</td>
<td>0.9928</td>
</tr>
<tr>
<td><strong>Spine</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5%</td>
<td>0.978</td>
<td>0.9786</td>
<td>0.9314</td>
<td>0.9301</td>
</tr>
<tr>
<td></td>
<td>3%</td>
<td>0.9917</td>
<td>0.9933</td>
<td>0.9917</td>
</tr>
<tr>
<td>10%</td>
<td>0.994</td>
<td>0.9969</td>
<td>0.9964</td>
<td>0.9952</td>
</tr>
<tr>
<td>20%</td>
<td>0.998</td>
<td>0.9998</td>
<td>0.9996</td>
<td>0.9991</td>
</tr>
<tr>
<td><strong>Heart</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5%</td>
<td>0.961</td>
<td>0.9623</td>
<td>0.8955</td>
<td>0.8955</td>
</tr>
<tr>
<td></td>
<td>3%</td>
<td>0.9834</td>
<td>0.9875</td>
<td>0.9874</td>
</tr>
<tr>
<td>10%</td>
<td>0.985</td>
<td>0.9825</td>
<td>0.9796</td>
<td>0.9794</td>
</tr>
<tr>
<td>20%</td>
<td>0.992</td>
<td>0.9954</td>
<td>0.9955</td>
<td>0.9952</td>
</tr>
<tr>
<td><strong>Wrist</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5%</td>
<td>0.973</td>
<td>0.9735</td>
<td>0.9486</td>
<td>0.9466</td>
</tr>
<tr>
<td></td>
<td>3%</td>
<td>0.9946</td>
<td>0.9960</td>
<td>0.9945</td>
</tr>
<tr>
<td>10%</td>
<td>0.986</td>
<td>0.9886</td>
<td>0.9892</td>
<td>0.9874</td>
</tr>
<tr>
<td>20%</td>
<td>0.993</td>
<td>0.9952</td>
<td>0.9954</td>
<td>0.9952</td>
</tr>
<tr>
<td><strong>Knee</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5%</td>
<td>0.908</td>
<td>0.9088</td>
<td>0.8495</td>
<td>0.8494</td>
</tr>
<tr>
<td></td>
<td>3%</td>
<td>0.9513</td>
<td>0.9526</td>
<td>0.9526</td>
</tr>
<tr>
<td>10%</td>
<td>0.954</td>
<td>0.9542</td>
<td>0.9706</td>
<td>0.9706</td>
</tr>
<tr>
<td>20%</td>
<td>0.991</td>
<td>0.9805</td>
<td>0.9805</td>
<td>0.9805</td>
</tr>
</tbody>
</table>
The results conclude that maximum quality of reconstructed image is achieved when disjoint CS is used or when size of overlapping region is very small. It could be approximately viewed as the disjoint CS. The minor improvement in quality in case of small overlapping region can be attributed due to increase in over all sampling rate. The quality of image gradually decreases when the size of overlapping region (or the number of rows) increases as shown in Fig. 5.11.

![Figure 5.11: Average Quality index for different methods](image)

With increased size of overlapping area, most of the coefficients become common between the sub-regions, and the chances increase that same coefficients will get picked among all the regions. The extreme case for overlapping regions is when the overlapping region is equal to image size which means all the sub-sections are exactly same. The sampling in this case is like global CS, however the sampling rate will be 3-4 times of the original sampling because all the regions will be sampled separately but exactly same coefficients will be
selected from each region. As shown in Fig. 5.12, the highest image quality in all kinds of images was achieved with either due to local CS with disjoint sections or local CS with overlapping region of size 50. As the size was increased the quality started decreasing.

![Figure 5.12: Quality index MSSIM for different methods](image)

Overlapping regions cause duplication of samples which decrease image quality as well as increase sampling overhead as shown in Fig. 5.13. Total number of samples increase as the size of overlapping regions increase. The lowest sampling rate was achieved when disjoint sections were used. Thus it can be safely concluded that local CS with disjoint regions can recover a good quality image with minimum sampling overhead. This increases the time-efficiency with a better image quality.
5.10 Testing Local CS in different sparse domains

To further test the technique Local CS was applied on different level of sparse domains. Previously, local CS was tested on wavelets as it tends to sparsify MR images very well. Local CS was tested on less sparse domains to understand its true potential and the results were compared with Global CS. Three basic domains were used i.e. Discrete Cosine Transform (DCT), sensing domain itself which is Fourier and Finite Difference (FD). Images of six different body tissues were used same as previous experiments brain, angiography, heart, spine, knee and wrist. DCT tends to sparsify images up to 20% which means image can be reconstructed using only 20% coefficients of whole sample set. Sparsity level for Fourier is between 15-20% depending on image type while Finite Difference was unable to sparsify image even with 60% coefficients. The results are summarised in Table 5.6.
Table 5.6: Comparison between CS and Local CS using Finite Difference Transform

<table>
<thead>
<tr>
<th>Area Under Test</th>
<th>DCT</th>
<th>FD</th>
<th>Fourier</th>
<th>DCT</th>
<th>FD</th>
<th>Fourier</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Global CS</td>
<td>Local CS</td>
<td>Global CS</td>
<td>Local CS</td>
<td>Global CS</td>
<td>Local CS</td>
</tr>
<tr>
<td>Brain</td>
<td>0.9241</td>
<td>0.9424</td>
<td>0.5653</td>
<td>-0.1124</td>
<td>0.9453</td>
<td>0.9350</td>
</tr>
<tr>
<td>Angio</td>
<td>0.9522</td>
<td>0.9471</td>
<td>0.5465</td>
<td>-0.1261</td>
<td>0.9512</td>
<td>0.9443</td>
</tr>
<tr>
<td>Spine</td>
<td>0.8832</td>
<td>0.9890</td>
<td>0.2269</td>
<td>0.2172</td>
<td>0.9387</td>
<td>0.9882</td>
</tr>
<tr>
<td>Heart</td>
<td>0.9720</td>
<td>0.9822</td>
<td>0.3961</td>
<td>-0.0952</td>
<td>0.9311</td>
<td>0.9801</td>
</tr>
<tr>
<td>Wrist</td>
<td>0.9943</td>
<td>0.9940</td>
<td>0.2327</td>
<td>-0.0932</td>
<td>0.9934</td>
<td>0.9923</td>
</tr>
<tr>
<td>Knee</td>
<td>0.9286</td>
<td>0.9468</td>
<td>0.3301</td>
<td>-0.0412</td>
<td>0.9388</td>
<td>0.9439</td>
</tr>
</tbody>
</table>

Local CS performs better in DCT and Fourier because these domains sparsify images reasonably while for FD local CS performs very poorly. FD doesn’t sparsify image that is why bigger sample set means better quality. Whereas, Local CS uses smaller sample set than simple CS and leads to a decreased image quality.

![Figure 5.14: Average Quality in different domains Vs. Sparsity](image)

Figure 5.14: Average Quality in different domains Vs. Sparsity

Fig. 5.14 shows average quality index for local and global CS vs. sparsity level of different sparse domains. Local CS works better than global CS in any sparse domain. If domain is not sparse like FD in that case both methods
show poor quality but global CS is a little better as it uses a larger sample set.

5.11 Conclusion

In this Chapter, we have used non-uniform sparsity of MR images to make the imaging process more efficient in terms of time. Varying sparsity can result in varying sampling rate. There is no need to sample low and highly sparse area with the same amount and local CS allows this by applying independent constraints and sampling rates within image sub-regions. Results show that a good quality image can be generated using local constraints. Local CS can reduce the sample set up to 30% while sub-regions are easily identifiable based on sparsity distribution and PSF. This presented framework is named as Locally Sparsified CS.

Local CS is a specialised case of simple CS. It was explored further in terms of sparsity and co-dependent sub-regions as well as justified mathematically. Results show Local Sparsity constraints works better when used with independent disjoint sub-regions. Overlapping regions tends to increase sampling duplication which causes sampling overhead and degrade image quality. As the size of overlapping region increases, the quality starts decreasing. Locally Sparsified Compressive Sensing with disjoint sub-regions allows a very good quality image recovery without any sampling overhead. It works better than global Compressive Sensing as well as Compressive Sensing with overlapping regions. Furthermore, Local CS works better than simple CS under any domain which is sparse and a limited number of coefficients hold the actual image energy. Whereas, in domains like finite differences where there are very few
non-significant coefficients it fails to reconstruct images.

Next chapter will extend the algorithm of Local CS for improving image quality and dynamic MRI.
Chapter 6

Application and Extension of Locally Sparsified CS

6.1 Introduction

This chapter extends the basic algorithm of Locally Sparsified CS as presented in previous chapter. The basic framework is used for two applications. Firstly, to enhance image quality and reducing artefacts caused by under-sampling. Under-sampling in different frequency region cause different kind of noise and Local CS allows region based sampling and sparsity. This property of Local regions is exploited to reduce noise and the results are presented in this chapter.

Secondly, it is modified for Dynamic MRI. Dynamic MRI is used to generate an image sequence over time rather than one single image. The additional images provide more information about targeted area when used in a sequence, which is used for different applications e.g. speech therapy, watching movement in larynx over time. However, due to slow imaging speed only a limited number of samples can be acquired in a frame. So that, next frame can be captured within a short time of previous frame. This chapter will define a framework to use Locally Sparsified Compressive Sensing for dynamic MRI.
6.2 Locally Sparsified CS for improving image Quality

Compressive Sensing (CS) works on compressibility of medical images. Medical images are really sparse and can be generated using a small amount of coefficients while discarding other non-significant coefficients. CS-MRI uses this property to reduce required measurements thus resulting in faster image acquisition. However, violation of Nyquist rate can result in noise like artefacts. There is a compromise between speed and quality of image. Lesser measurements means faster acquisition but degraded quality.

Energy levels vary significantly within image. K-space center is the high energy region and contains most of image energy while away from origin energy levels are relatively low. Under-sampling in high energy region is the reason for most of the energy loss and image noise. Local CS allows independent sub-regions and different sampling rates within an image. Increasing sampling rate just in high energy regions will result in a better quality image without increasing sampling rate for whole image. On the other hand, low energy areas can be under-sampled further based on energy level in those areas. This chapter defines a framework to use local CS for improved image quality.

6.2.1 Artefacts due to Under-sampled K-space

Due to under-sampling, image quality is compromised and noise is generated. Violation of Nyquist rate results in wrap around artifacts as shown in Fig.6.1. This wrap around effect is created due to sampling rate less than twice the highest frequency which leads to misinterpretation of high-frequency signal as
of low frequency and it gives a wrap around effect [175].

![Figure 6.1](image)

**Figure 6.1:** Wrap around artefact for Spine image (a) Fully sampled Image (b) Noise due to under-sampling (wrap around artefact)

Under-sampling might also cause ringing artifact near sharp edges. It shows circular rings like structures near edges which are not part of the real image. This occurs due to under-sampling in high frequency regions [176] as shown in Fig. 6.2.
6.2.2 Proposed Methodology

The propose method uses multiple local constraints to achieve better image quality without varying sampling rate. Locally sparsified regions enable individual sparsity constrains and sampling rates. This approach is used to
increase the sampling amount in high energy areas such that effects of inadequate sampling can be reduced while decreasing sampling amount in low energy region. This will improve image quality without affecting image acquisition speed. Fig. 6.3, shows 1-D energy distribution of Spine in Fourier domain. Energy levels vary in different sections, there is a high energy peak in the middle which contains most of the energy and causes most of the artefacts. High sampling rate can be chosen in this area while low sampling rate in low energy areas.

![Figure 6.3: K-Space Energy distribution for Spine](image)

K-space shows similar energy distribution for different kinds of images. Local regions and constraints were defined for a fully sampled reference image and were later used for all the images. For localising K-space, Point Spread Function (PSF) was used. It measures interference levels based on a reference point. High energy peak in the middle is the focus point in terms of quality. By identifying the central region using PSF, K-space will automatically be divided into one high energy region and two low energy boundary regions. To determine the sampling rate for local regions same method was applied as for Global CS [13] that is only $samp_1, samp_2, \ldots samp_k$ coefficients were taken from sub-regions $\psi_1, \psi_2, \ldots \psi_k$ respectively and rest of the coefficients were discarded.
Then generated image was compared with fully sampled image, for all $\psi_i$, a sampling rate $samp_i$ exist such that it will generate a diagnostic quality image using only $samp_i$ coefficients from sub-region $\psi_i$. Rates which are defined for reference image were later used for all kinds of other images. Overall K-space sampling rate can be stated as

$$\delta = \sum_{i=1}^{n} samp_i$$

(6.1)

which is sum of sampling rates of all regions.

ALGORITHM:

- Divide K-space into sub-regions using PSF such that the middle energy peak is preserved.
- Generate $F_{nk}$ partial Fourier for each sub-region such that $samp_k$ random sample are taken from region $n_k$.
- Combine it into $F_n$.
- Minimize $\min \| \Psi m \|_1 \text{ s.t. } \| y - F_a m \| < \epsilon$

Later, same amount of sampling rate $\delta$ was used with Global compressive sensing. L1 minimization and probability density sampling is used with Wavelets as sparse transform domain for image reconstruction [13]. MSSIM is used to compare and quantify visible difference between both images.
6.2.3 Experimental Evaluation and Discussion

For experiments and a better understanding of the proposed techniques, six different kinds of images were used i.e., brain, angiography, heart, spine, knee and wrists. As, contrast level varies with tissue density that is why all these different kinds of hard and soft tissue images were used for experiments. Image resolution was 512x512. Firstly, simple CS was applied on all the images. Later, images were sub-divided into three local regions of sizes 156x512, 200x512 and 156x512 respectively and different sampling rate was used for each region based on reference image. For quality comparison MSSIM was calculated and results were analyzed and summarized in Table 6.1.

Table 6.1: Quality Comparison of Globally and Locally Sparsified Compressed Sensing

<table>
<thead>
<tr>
<th>Area Under Test</th>
<th>MSSIM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Globally Sparsified</td>
</tr>
<tr>
<td>Brain</td>
<td>0.9120</td>
</tr>
<tr>
<td>Angiography</td>
<td>0.9331</td>
</tr>
<tr>
<td>Spine</td>
<td>0.7384</td>
</tr>
<tr>
<td>Heart</td>
<td>0.8793</td>
</tr>
<tr>
<td>Wrist</td>
<td>0.9823</td>
</tr>
<tr>
<td>Knee</td>
<td>0.8959</td>
</tr>
</tbody>
</table>

For all kinds of different images, experiments showed a better quality reconstruction with Local CS. Locally sparsified CS has distributed samples into sub-regions such that sum of all local sampling rates $\delta = S$. Localisation of samples is based on the local energy levels and its impact on overall noise as shown in Fig.6.4.
Table 6.2 shows distribution and amount of sampling used in each method. The sampling amount was reduced in boundary areas and extra samples were taken near origin. Global CS has used 29% samples from overall image of size 512x512 while Local CS has sub-divided this percentage and used 40% samples from the middle high energy peak area of size 200X512 while 21-23% from border area of size 156x512. Rather than using same sampling rate in whole image, sampling distribution was designed according to image energy. A better quality and less noisy image reconstruction was achieved without any increase in overall sampling measurements. Fig. 6.5, shows reconstructed brain image.

The graph in Fig. 6.6, shows quality comparison between both methods for different types of MR images. All images, either hard or soft tissues, showed
better and improved recovery when local constraints were used without slowing acquisition process.

6.3 Locally Sparsified CS in Dynamic MRI

In last chapter has explained basic algorithm for Locally Sparsified CS, It differs from all the recent work because it a generic method which changes the way of understanding sparsity. It can be merged with any or all kinds
of CS based methods by localising the sparsity within image rather than just considering a global sparsity threshold. In this chapter I will use Local CS for dynamic MRI. On simple individual images as done in [13,177] but with local sparsity constraints and regions. Also I will define a framework for applying local constraints on difference images to exploit the non-uniform sparsity in time domain. This method is different from previously difference image based method for CS [178–181] because they do not consider non-uniform sparsity of images in Fourier domain.

6.3.1 Dynamic MRI

In dynamic MRI spatial and temporal resolutions both matter and increasing one means effecting other because a high resolution image in spatial domain (K-Space) means bigger sample set which will decrease temporal resolution because frame are further apart in temporal domain. However, closely coupled frames over times mean each frame has only limited and small amount of time, which will affect spatial resolution. To overcome this trade-off between spatial and temporal resolution, a range of methods have been suggested based on sampling techniques and image reconstruction methods.

A category of reconstruction methods used for dynamic MRI uses concept of data sharing between frames and assume that the targeted object differs slowly. So, most of the data among two consecutive frames remains same. Data interpolation [182], motion estimation [183] and other such techniques are used to reconstruct images from under-sampled data. Keyhole is one of the earliest methods in this category [184,185]. The methods which fall under this technique use two fully sampled reference frames one in start and one at end.
The under-sampled frames in between are reconstructed using different data filling and estimation methods using reference frames [186–188]. Another set of methods used for dynamic MRI is reduced-encoding imaging by generalized-series reconstruction (RIGR) [189,190]. It reconstructs under-sampled data by analysing data in image domain and estimating change and motion for under-sampled frames [191,192].

Recently Compressive Sensing based methods are also used for dynamic MRI [177] which exploits sparsity in spatial and temporal domains. CS allows under-sampling in spatial-temporal domain without degrading image quality and resolution. Some CS based methods use sparsity of individual frames and reconstruct frames independently [177], while others use CS and L1 minimisation on overall Kt-Space [183,193]. The methods in this category read frames of dynamic MRI as 3D image where the time is the third dimension and try to minimise L1 norm of 3D signal as one entity [194]. Another category of methods is related to difference imaging. These methods assume that most of the content or data between two consecutive frames is same. Thus the difference of those frames will be approximately zero except few coefficients this sparsity make it ideal for CS [178–181].

6.3.2 Frame based Locally Sparsified CS

In this section Locally Sparsified CS is implemented on dynamic MRI. In two dimensional dynamic MRI, time is represented by third dimension which changes K-space into k-t space. Whereas, the coordinate system will become $k_x, k_y, t$. $x, y$ are the spatial coordinates at any time frame $t$. In a dynamic
data-set, each frame exhibits same properties as a 2-D MR image when considered independently. Thus, if taken independently, each frame can be handled as any other kind of 2-D MR imaging. Three key-points to implement Locally Sparsified CS are sparsity in some fixed basis, shape of energy distribution and random under-sampling for incoherent noise. The energy distribution is non-uniform within a frame but shows a uniform or repetitive behavior when analysed in time domain Fig.6.7. After a limited time it shows a high energy peak which represents image energy of frame at that specific time instant \( t \) and is approximately blank or zero otherwise. Thus, in this specific setting applying Locally Sparsified CS is as applying it on individual and independent frames. Where each frame will be non-uniformly under-sampled based on its own energy distribution and independent of its relation with other frames. This will be called frame by frame implementation of Locally Sparsified CS.

Figure 6.7: Continuous energy distribution for dynamic heart MRI over time axis
In dynamic MRI, multiple frames are acquired over time thus, making K-Space a k-t space.

\[
Y = \begin{bmatrix}
    y_1 \\
    \vdots \\
    y_t \\
    \vdots \\
    y_T
\end{bmatrix}
\]  

(6.2)

The Signal equation for any frame at time \( t \) will as follows

\[
y_t = F_{ut}.x_t
\]  

(6.3)

here \( x_t \) is the original object and \( y_t \) is acquired signal for frame \( t \) and \( t \in T = \{1, 2...T\} \). for any frame signal \( y_t \) has size \( N \) and \( F_{ut} \) is under-sampled Fourier operator for that respective frame. By introducing the sparse transform domain for signal \( y_t \) equation will now become

\[
y_t = F_{ut}.a_t.\Psi^T_t
\]  

(6.4)

where \( a_t \) is the transformed version of object \( x_t \) in sparse transform domain \( \Psi^T_t \). The localise version of this signal at \( r^{th} \) region of frame \( t \) will be

\[
y_{tr} = F_{utr}.a_{tr}.\Psi^T_{tr}
\]  

(6.5)

and overall signal will be

6.3.3 Methodology for Frame based Locally Sparsified CS

As, established in Chapter 3, wavelets sparsify MR images better. For implementing Local CS on dynamic MRI, Wavelets are used with local sparsity constraints. These constraints were determined based on reference image.
Whereas, region sizes were determined using Point Spread Function based on required FOV and resolution. Each frame was under-sampled using region based sparsity constraints and later reconstructed using L1 minimisation.

**Algorithm:** For all $t \in T = \{1, 2...T\}$

- Generate $F_{ut}$ under-sampled region based Fourier Operator for frame $t$.
- Minimize $min \| \Psi_t a_t \|_1$ s.t. $\| y_t - F_{ut} a_t \| < \epsilon$

### 6.3.4 Locally Sparsified CS on Block of Frames

Applying Local CS on each frame is time taking process. The optimisation problem of minimising L1 norm is extensive in it self. For dynamic MRI, it takes $T$ times the processing time of single image. Also, it only exploits sparsity within a frame or in K-space of a frame. However, the sparsity of images over time is not considered in this method. To analyse the sparsity of images over time. The Fourier signal in 2-D in K-Space which is continuous over time shows a repetitive energy distribution. To, better understand sparsity in Dynamic MRI verse time a concatenated version of Fourier is implemented. This will combine and concatenate signals in K-space and the 3rd- dimension (as used in pervious section) will no longer be needed and signal will become 2-D. This will allow experimentation on sparsity in both temporal n spatial domains. The modified signal became

$$Y_A = F_{uA} x_A$$

(6.6)

$x_A$ is an object with continuous change, generating series of signals $y_A$ which is continuous over time. Also, in this approach all the frames are combined
in K-Space rather than k-t space. However, concatenated signal in Fourier domain will result in multiple energy peaks as shown in Fig.6.8

![Figure 6.8: Stacked K-Space for dynamic heart MRI](image)

This is same as the case discussed in previous section where Fourier domain shows repetitive and uniform energy patterns Fig.6.7. To resolve this issue of uniform energy distribution over multiple frames, firstly frame by frame Locally Sparsified CS was applied and images were recovered. Later, concatenation is done in the image domain which is orthonormal basis and a combined Fourier transform is calculated. For any frame \( y_t \), \( I_t \) is the recovered image which is reconstructed using under-sampled Fourier Signal.

\( I_t \) was calculated for each frame separately using Locally Sparsified CS. All the images were combined in a 2-D square matrix \( I_A \) of size \( N \times N \). If \( \lambda \) is the total number of coefficients/elements in all frames then \( \lambda = T \times n \times n \) such that \( T \) is the number of total frames and \( n \times n \) is the size of a single frame. Furthermore, number of total elements in a 2-D array of size \( N \times N \) is \( N^2 \). In other words, number of total elements in a set of all image frames is \( N^2 = \lambda \) that is
\[ N = \left\lceil \sqrt{T \times (n \times n)} \right\rceil \quad \text{such that} \quad N \parallel n \quad (6.7) \]

Here \( N \) is rounded up to the nearest value of \( \sqrt{\lambda} \) such that \( N \) is fully divisible by \( n \). This condition makes sure that the size of combined matrix \( I_A \) can hold a complete image frame in consecutive locations.

\[ I_A = \begin{bmatrix}
I_1 & I_2 & \ldots & \ldots \\
\vdots & \vdots & \ddots & \vdots \\
\ldots & \ldots & \ldots & I_t
\end{bmatrix} \quad (6.8) \]

Concatenation is done such that each frame \( I_t \) was mapped into \( \hat{I}_t \) which is of size \( N \times N \). Later, \( \hat{I}_t \) is divided in \( t \) sub-sections of size \( n \times n \) such that all sub-sections are zero filled except for \( t^{th} \) section

\[ \hat{I}_t = \begin{cases} 
0 & t \neq \hat{t} \\
I_t & t = \hat{t}
\end{cases} \quad (6.9) \]

Thus, making \( I_A \) the sum of all zero padded matrices \( \hat{I}_t \)

\[ I_A = \sum_{t=1}^{T} \hat{I}_t \quad (6.10) \]

The expanded form of resultant image will be like this

\[ I_A = \begin{bmatrix}
I_1(1,1) & \ldots & I_1(1,n) & I_2(1,1) & \ldots & I_k(1,n) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
I_t(1,1) & \ldots & I_t(1,n) & I_2(n,1) & \ldots & I_k(n,n) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
I_t(1,1) & \ldots & I_t(n,1) & \ldots & \ldots & \ldots & \ldots & I_T(n,n)
\end{bmatrix} \quad (6.11) \]

The combined image was later converted back into Fourier which showed concentrated energy with in a single peak as shown in Fig.6.9.
Figure 6.9: (a) shows concatenation of Larynx data-set in image domain and (b) shows its respective Fourier Transform which single energy peak.

Local constraints were enforced again on the combined image and reconstructed again by minimising L1-norm jointly for whole batch. The quality was slightly improved in this case.

6.3.5 Methodology for Block Local CS

For implementation of Local CS in Block Formate, frame by frame reconstruction was done using local constraints with Wavelets as described in previous
section. Furthermore, all reconstructed images were combined and local constraints were enforced jointly on whole block and images were reconstructed again using L1 minimisation.

**ALGORITHM:**

- Calculate $N$ using frame size of $n \times n$ and number of total frames $T$.
- For all $t \in T = \{1, 2...T\}$
  - Generate $F_{ut_t}$ under-sampled region based Fourier Operator for frame $t$.
  - Minimise $min \| \Psi_t a_t \|_1$ s.t. $\| y_t - F_{ut_t} a_t \| < \epsilon$ and reconstruct $I_t$.
  - Convert $I_t$ of size $n \times n$ into $\hat{I}_t$ of size $\frac{N}{n} \times \frac{N}{n}$ by zero filling
- Sum $\hat{I}_t$ for all $t \in T = \{1, 2...T\}$ and construct $I_A$
- Enforce local sparsity constraints on $I_A$ and apply joint L1-minimisation.

### 6.3.6 Locally Sparsified CS on Difference Images

The reconstruction quality was only a little better in Block-LCS. To further improve the enhance the technique, non-uniform sparsity of difference images were analysed. As, the the motion between two consecutive frames is very little thus, making difference image sparser than original frame image it-self. Difference images of dynamic NRI were previously used in many different ways. It includes reconstruction from simple L1-minimisation [180], using it for repaid reconstruction in online and real-time systems where already some K-Space is known and image construction should be as quick as acquisition [178]. Also, basic conditions of CS was modified using difference images to modify CS
with already known support [179]. However, this method will exploit the non-uniform sparsity of difference image and difference images are reconstructed rather than original. K-space of any difference image between two consecutive frames can be defined as

\[
\hat{y}_{t-1} = F^{-1}\hat{x}_{t-1}
\]  

(6.12a)

where \(\hat{x}_{t-1}\) is the reconstructed image for previous frame using L1-minisation, \(F^{-1}\) is Inverse Fourier transform and \(\hat{y}_{t-1}\) is Fourier inverse or K-space for recovered image of previous frame.

\[
\Delta y_{t} = \hat{y}_{t-1} - y_{t}
\]  

(6.12b)

\(\Delta y_{t}\) is the difference of two consecutive frame \(t - 1\) and \(t\) in Fourier domain where \(y_{t}\) is under-sampled Fourier signal for \(t^{th}\) frame where change in any coefficient \(i\) can be defined as

\[
\Delta i_{t} = \hat{y}_{t-1}(i) - y_{t}(i)
\]  

(6.12c)

As, amount of change between two consecutive frames is limited which means a lot of coefficients will cancel out and change will be zero or approximately zero Fig.6.10. Thus, the difference of two frames will be sparser than the the original frame itself, and can be reconstructed using even fewer samples.
Figure 6.10: Dynamic Heart MRI (a) and (b) show two consecutive image frames and (c) is the difference in image domain. (d),(e) and (f) show K-Space of the frames and difference image respectively. (g),(h),(i) are the K-space energy graphs in $K_y$ direction for all 3 images.

6.3.7 Transform Sparsity in Difference Images

As, CS requires to hold the transform sparsity in sensing domain it self or any orthonormal domain. Local CS however, works on sparsity of local regions within image. In our previous work, transform sparsity of MR images based on local regions was analysed. In this section we will analyse transform sparsity
of Difference images in different domains. To estimate the local sparsity in transform domains. K-space for difference image $\Delta y_t$ for any frame $t$ was calculated. It was sub-divided in regions using Point spread Function (PSF), energy distribution and image resolution as defined in last chapter. The sub-divided K-space was transformed into four domains (DCT, Wavelets, Finite Difference, Image domain) and energy decay was estimated for each region separately. Energy decay was estimated by sorting out values of coefficients. An abrupt decay shows that all the energy is confined within limited number of coefficients. Finite difference and wavelet domain showed high sparsity for all three regions while DCT, image and Fourier domain were not able to sparsify images properly. An example of dynamic heart MRI and its energy decay is shown in Fig.6.11.
Figure 6.11: Energy decay of difference image of Dynamic Heart MRI using different transform domains is shown. Image is sub-divided in three disjoint horizontal region based on its resolution and energy distribution referred as top, bottom and middle regions of image. (a),(b),(c) show energy decay in image domain for top, middle and bottom region respectively. Second row: Energy decay in Fourier Domain. Third row: Energy decay in Discrete Cosine Transform (DCT). Fourth row: Energy decay in Finite Difference domain (difference between consecutive pixels in image). Last row: Energy decay in Wavelets.
Experiments based on energy decay showed that Finite Difference and wavelets saprsifies difference images better. To further compare both domains and to understand the sparsity, images were reconstructed using only $s_i\%$ coefficients form $i^{th}$ region. Rest of the coefficient were discarded and filled with zero. The results showed that the quality of images which were reconstructed using wavelets is better than Finite Difference Fig.6.12.

Figure 6.12: Finite Difference (F.D) and Wavelets were applied on six difference images of dynamic MRI for Larynx. Each transform was applied using 3 different sparsity constraints while each image was sub-divided in 3 regions. First set of constraints was 1%,5%,1% coefficients form top,middle and bottom regions respectively. Other two were 3%,5%,3% and 3%,8%,3%. Wavelets achieve a good quality image from 3%,5%,3% coefficients which was also approximately equal to the quality which F.D recovered using 3%,8%,3% coefficients.

Concluding from these results, wavelets are the better choice to use as
sparsifying transform in difference images. The methodology is defined in the next section.

6.3.8 Methodology for Local CS on difference Images

For implementation of Local CS on difference images, sparsity of images were analysed and wavelets selected to use as sparsifying transform. Sub-regions were estimated based on reference image. First image was captured fully-sampled and used for estimation of regions and constraints. Later the difference images were sampled only with very small set of coefficients. Instead of reconstructing whole image, difference image was reconstructed using L1 minimisation. The recovered image was added in image from previous frame. Thus, recovering complete image for current frame.

**ALGORITHM:**

- Calculate sub-regions and their sizes using image resolution and energy distribution of reference image.

- Capture fully sampled first frame $\hat{y}_t$ and reconstruct $\hat{I}_t$ using inverse Fourier.

- For all $t \in T = \{2, 3 \ldots T\}$
  - Generate $F_{ut_r}$ under-sampled region based Fourier Operator for frame $t$.
  - Capture under-sampled signal $y_t$.
  - Find Fourier K-space for difference image $\Delta y_t = \hat{y}_{t-1} - y_t$. 
Minimise $\min \| \Psi_t a_t \|_1$ s.t. $\| \Delta y_t - F_{at} a_t \| < \epsilon$ using $\Psi$ as wavelets and reconstruct $I_t$.

Reconstruct image frame $\hat{I}_t$ where $\hat{I}_t = \hat{I}_{t-1} + I_t$

6.3.9 Experimental Results and Discussion

Experiments are conducted on 5-Data set, 2 of which are the Cardiac sequence MRI with resolution 128x128 and 256x256 with 16 frames in each set. While other two are Larynx data-set of size 256x256 each frame with total 10 frames. Last sequence was a simulated heart Phantom of resolution 256x256 with 16 frames. All the above mentioned techniques were applied and results are summarised in graphs based on image type Fig. 6.13.

![Graph](image)

Figure 6.13: Quality comparison for Dynamic Heart MRI using different methods
Figure 6.14: Quality comparison for Dynamic Larynx MRI using different methods

Figure 6.15: Quality comparison for simulated Heart Phantom using different methods
Global CS was implemented using 40% samples while Local CS was able to recover almost same quality of images with only 28% samples. Block CS used same sample set as Local CS but has improved quality slightly. Whereas, Difference CS was able to achieve a better image quality with only 25% samples. A comparison graph is shown in Fig.6.16 which shows Average quality index for all frames in all different images sequence and sampling ratio for each technique.

![Figure 6.16: Average Quality vs. Sampling rate for CS, LCS, Block CS and Difference CS in dynamic MRI](image)

6.4 Conclusion

This work has implemented Locally Sparsified CS for de-nosing MRI and handling warp around and ringing artefacts which occur due to under-sampling. Local CS allows multiple sampling rates within different regions of K-space. By using this property of Local CS, noise was reduced effectively. Local CS was also applied on dynamic MRI using three different approaches. Global CS reconstructed a better image quality than Local CS frame by frame and
Block implementation. The quality difference was minute. However, CS has used 12% more sample than Local CS. The implementation of Local CS on difference images were able to recover a better image quality than CS with even fewer samples.
Chapter 7
Conclusion and Future Work

In this research, the main focus was the sparsity of MR images, finding the key-features of sparsity of MRI and using it for better imaging in terms of speed and quality.

Figure 7.1: Different kinds of MRI that are used in this research. (a) Brain (b) Angiography (c) Spine (d) Heart (e) Upper limbs i.e Wrist (f) Lower Limbs i.e. Knee
After literature survey the first phase of experimental work was to develop a data-set of different kinds of MR images with different resolution. A data set of 100 images divided in six categories is used as shown in Fig. 7.1.

7.1 Transform Sparsity

Later, this data set was used to analyse the sparsity of MR images. Experiments were done on different kinds of images so that MRI can be analysed on more generic terms rather than any specific kind. Different sparsity levels (1%, 5%, 10%, 20%, 30%, 40% and 50%) were used to find a level that is best suited for all types of images. All the experiments were done in four domains (Image, Fourier, Discrete Cosine Transform and Wavelets). MR images are sparse in their sensing domain (Fourier) but they exhibits transform sparsity more.

Image, Fourier, DCT and Wavelets were tested. Experimental data showed that DCT and Wavelets showed good results. Fig. 7.2 shows an energy decay graph of brain MRI for both DCT and Wavelet. All coefficients were sorted out in descending order. A rapid decay means more sparsity. As, it can be seen that Wavelets has an edge over DCT. Results showed that Wavelets require only 10% wavelet coefficients to represent any MR image while rest can be discarded.
For all the research it was crucial to determine a suitable index to quantify the image quality. Three measures were tested MSE, SNR and MSSIM. MSSIM works on principle of human visions and worked well in case of both noisy and incomplete data. SNR showed confusing results in case of under-sampling or incomplete image data, by exceeding PSNR for DCT while visibly images are better in Wavelets. Whereas, MSE failed to quantify the results in both cases. Based on the results it was concluded that MSSIM is suitable for measuring visible errors when fully sampled original image is available and images are under-sampled. SNR is good to verify the results of MSSIM in case of noisy data and for noise reduction method because an increased SNR means image was de-noised. However, in case of rapid MRI and under-sampled data SNR is not suitable refer Fig. 7.3.
7.2 Local Sparsity

MRI has non-uniform nature of sparsity. The energy distribution is highly dense near origin while as you move away from origin, energy content is very low and nearly zero. This feature of non-uniformity can be exploited as sparsity distribution remains same for all kinds of images. Energy decay graphs show difference in energy levels for different regions Fig. 7.4. Localising sparsity means allowing different sampling rates and sparsity levels within an image. For experiments, high energy peak area was sampled same as global level while reduced sparsity levels were used for low energy regions. For all kinds of images, with 5%, 10%, 5% wavelet coefficients the local constraints performed equally well as 10% global level. Using multiple local sparsity levels, images were sparsified 30% more than global level without affecting image quality.
7.3 De-noising MRI

MRI is effected by thermal noise caused by patient’s body heat. MR machines samples data in frequency domain. The data is captured in complex form and noise affect both real and imaginary components of data. Using Fourier Transform this data is converted into images and magnitude images are calculated from complex data. During this process, additive Gaussian noise converts into signal dependent Rician noise which is hard to remove. A method is proposed in Chapter 4 to use local sparsity constraints for noise removal. The proposed method is independent of type or distribution of noise. The basic idea behind the method is to maximise energy and minimise number of coefficients and substituting as many coefficients with zero as possible. This was done using the sparsity constraints. A reference image was used to generate a sparsity map by simple threshold method. From a noisy image, each region is substituted with zero such that it fulfils the sparsity constraint and only the highest coefficients are selected. As, the method uses a reference image thus number of sub-regions can neither be too high nor too low. Low means single region and it will fail to preserve the shape of energy distribution as well as will not sparsify images optimally. Too many local regions will make the sparsity information image specific. For keeping it generic so that it can be used for all different images experiments were done with two and three sub-regions. The proposed technique can be used in combination with other noise removal methods as it does not change or interpolate data values. It either selects a coefficient or discard it completely. Firstly, the proposed method was applied on noisy data later it was used in combination with Linear filtering methods for noise
removal where Wiener and Gaussian filters are used. The experiments were repeated for different noise levels and results showed improved MSSIM and SNR Fig. 7.5.

![Figure 7.5: Average results of de-noising methods for different kinds of MR images. (a) MSSIM (b) PSNR](image)

### 7.4 Locally Sparsified Compressive Sensing

Compressive Sensing (CS) suggests that samples should be measured based on compressed size or sparsity level of image rather than its overall size. Compressive sensing use sparsity for compressed signals. A method is proposed to use local sparsity of MR images with compressed sensing called Locally Sparsified Compressive Sensing. It is an enhanced form of simple CS. It allows multiple sampling rates within image which results in Rapid imaging. Unlike, noise removal method proposed in chapter 4 all the sparsity constraints and regions must be defined before image acquisition. Thus, the shape of sub-region depends on sampling method and trajectory. This work has been done with 2-D Cartesian sampling and 1-D rectangular regions were defined. For defining number of regions, energy distribution was used and their size was
determined using Point Spread Function. Experiments were done and com-
pared with simple CS and results showed sample set has decreased up to 30%.
This non-uniform sparsity can be utilised to make the process more efficient in
terms of time. Varying sparsity can result in varying sampling rate. There is
no need to sample low and highly sparse area with the same amount and local
CS allows this by applying independent constraints and sampling rates within
image sub-regions. Thus, a simple and structured approach can enhance CS
further. It was also extended and used for improving quality of Rapid MR and
dealing with noise caused by under-sampling as well as for dynamic MRI.

7.5 Future Work

The non uniform nature of sparsity in MR images has been analysed in this
work. This work can be extended in following directions. This research did not
focus on finding optimal sparsity transform for MRI. Four basic transform were
tested and Wavelets worked better among all. More transform domains can
be tested and used to sparsify MR images further. Moreover, multiple sparsity
domains can be used in combination rather than just using a single domain.
The work has been implemented on offline data set. It can be extended for
different real time clinical settings. The data set used for this research is in
2-D except for the dynamic MRI. All the work can be extended to higher di-
mensions. Proposed noise removal method is very basic. It can be extended
for other sparse domains. Current work used 1-D division of image. For fu-
ture work this technique can be extended with multi-dimensional sub-regions.
These sub-regions can be of different shapes based on energy distribution e.g.
circular, square and can be dynamic rather than of fixed size. The optimal way to define regions for noise removal is yet to be explored. Currently it was implemented alone and with Gaussian and Wiener Filtering. It can be extended and combined with other more complex noise removal methods.

Locally Sparsified CS was tested for 2D and 3D cartesian. It can be used for other sampling trajectories e.g. Radial, Spiral. L1 minisation is used in current work but there are other non-linear methods which can be used to improve imaging quality or processing speed. LSCS can be merged with other advance techniques based on CS as it a basic method which modifies the concept of sparsity in CS. In dynamic MRI, L1 minimsation results in slow and extensive processing. As, it work on multiple image frames and minimising L1-norm for all the frames is costly in terms of processing time. It can be extended and used with a better non linear method which can make it faster in terms of processing. This will allow to use dynamic MRI in real time systems e.g. speech therapy. Faster version of this technique can be implemented in hospitals and the faster processing will ensure the results are as quicker as real time processing and image capturing.

This work is only limited to MRI and can be extended for other medical imaging techniques. It can also be implemented on natural images or any sparse signals where energy distribution is sparse and its shape is predictable. In this work sparsity features are purely used in term of improving software of MR machines and image quality. However, sparsity summarise data and it can be used for identifying medical features of images e.g. certain shapes or spikes in data under different domains.
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