$\mathcal{H}_\infty$ OBSERVER-BASED CONTROL FOR DISCRETE-TIME ONE-SIDED LIPSCHITZ SYSTEMS WITH UNKNOWN INPUTS.

M. BENALLOUCH*, M. BOUTAYEB†, AND H. TRINH‡

Abstract. This paper investigates the problem of robust observer-based stabilization for a class of one-sided nonlinear discrete-time systems subjected to unknown inputs. We propose a simple simultaneous state and input estimator. A nonlinear controller is then proposed to compensate the effects of unknown inputs and to ensure asymptotic stability in closed loop. Several mathematical artifacts are used to deduce stability conditions expressed in terms of LMIs. To show high performances of the proposed technique, a relevant example is provided with comparisons to recent results.

Key words. $\mathcal{H}_\infty$ observer-based controller, One-sided Lipschitz condition, Quadratic inner-boundedness, unknown input estimation, Linear matrix inequalities, Discrete-time nonlinear systems.

1. Introduction.

During the last decades, tremendous research activities have been focused on observers and controllers design for dynamical systems especially in the case where such systems are modeled with unknown inputs. Several theoretical results with interesting applications were established in the literature, in particular for nonlinear systems, we refer the reader to [1]-[2] and to the references therein.

Before we state the aim of this paper, we would like to mention some of the interesting works dealing with observer design for nonlinear systems. As there exists no universal approach, state observer design, in particular for nonlinear systems, is still a challenging and open problem. It is worth noticing that most of the existing results concern continuous time systems [3], [4] and [5] with only a few extensions to discrete-time systems [6], [7], [8]. Beside the famous extended Kalman filter, we distinguish a simple and useful nonlinear state observer that is based on the solution of a Riccati-like equation and the Lipschitz condition, we refer the reader to the pioneering work in [9] and some extensions in [10]-[11]. In order to enlarge the domain of attraction and the class of nonlinear systems that can be considered, a useful and more general condition was recently introduced for observers design, that is the one-sided Lipschitz condition. Interesting works on state observer design for this type of nonlinear systems were recently developed in [12]-[13]. For example, Hu in [12] and [14], proposed sufficient conditions for the existence of state observers for a class of one-sided Lipschitz nonlinear systems. The one-sided Lipschitz condition was introduced to estimate the influence of the nonlinear functions vector on the observer and to show inherent advantages with respect to the conservativeness induced by the classical Lipschitz condition. In [15], the problem of designing reduced-order observer for the same class of systems was discussed. The existence condition of the observer was further discussed by Zhao et al. [16]. Furthermore, one can mention the work of Abbaszadeh and Marquez in [17], in which a new condition known as the quadratically inner bounded property was introduced. In that paper, the advantages of the one-sided Lipschitz formulation in control and observation theory were acknowledged and the stability conditions were expressed in terms of Linear Matrix Inequalities (LMIs).

*Lab. Ampère UMR CNRS 5005, Dpt EEA, Ecole Centrale de Lyon, Université de Lyon, Ecully, France

†Centre de Recherche en Automatique de Nancy (CRAN-UMR 7039), Nancy University-CNRS, IUT de Longwy, 54400 Cosnes et Romain, France

‡School of Engineering, Deakin University, Geelong 3217, Australia.

More recently, this approach was extended to discrete-time systems [18] and [13].

The problem of observer-based controller design for nonlinear systems has also its share of attention from the automatic control community [19], [20], [21], [22] and [23]. This is motivated by the fact that state estimation is not only important for the purpose of diagnosis and/or monitoring industrial systems but also for control design. In fact, introducing an observer in the closed loop control structure can show different sources of conservatism such as the non-convexity of the bilinear matrix inequality (BMI) formulations. This non-convexity is generally due to the coupling between the unknown matrices of the observer and controller, and Lyapunov matrices. The available solutions to this problem generally involve iterative linear matrix inequality conditions [24]; constrained convex optimization conditions involving the satisfaction of some equality constraints along with an LMI condition [21]; the feasibility of different dependent LMI conditions simultaneously [25]-[26]. A different solution was recently presented in [23] which involved using the Young inequality in a more appropriate manner, the observer and controller gains were computed simultaneously by solving only one linear matrix inequality. Another interesting result was presented in [22] for the case of observer-based output feedback controllers for nonlinear uncertain systems.

The presence of unknown inputs in the system is a severe restriction that can compromise the stability and even deteriorate the performance. Their effect can be more intensified if they affect nonlinearly the system. One of the most successful unknown inputs observer (UIO) design techniques is the use of the decoupling principle in which the estimation error dynamics is completely separated from the unknown inputs. This approach was developed by Wang et al. [27], which proposed a procedure to design a reduced order UIO structure for linear systems with unknown inputs. This pioneering work was followed by several approaches for designing UIOs. For instance, in [28] the authors dealt with the geometric approach, in [29] the authors considered the inversion algorithm. In [30] the singular value decomposition technique was proposed. Besides, concerning the works on algebraic approaches the reader is referred, for instance, to [31]-[32]. The UIO application in fault tolerant control (FTC) procedures has also attracted many researchers (see for instance [33], [34], [35]). In fact, the FTC procedures can be treated using the observer-based controller technique in the presence of unknown inputs. For this purpose, a fixed state feedback controller is designed so that stability and some $\mathcal{H}_\infty$ performances are guaranteed for both fault free and faulty configuration of a class of nonlinear systems. In what follows, let us mention some basic results on active FTC approaches. Indeed, an interesting work developed by Kabore and Wang [33] proposed a simultaneous state and fault estimator based FTC. The considered systems, under some structural conditions, were transformed into special interconnected canonical forms. Those were explored and used to deduce sufficient conditions for fault estimation and stability in closed loop. More recently, the authors in [34] proposed a passive actuator FTC approach for a class of nonlinear systems using the fact that the safe nominal system is locally uniformly asymptotically stable. The Lyapunov state feedback controller proposed there needs the complete knowledge of the state vector as well as the upper bound function of faults. Finally, a simultaneous state and fault estimator to compensate and construct an FTC was proposed by Jiang et al. [35] for a class of nonlinear systems containing additive faults. In addition, the controllers construction is divided into two parts, the first one compensates the fault effects while the second part stabilizes the closed loop system through arbitrary nonlinear functions fixed by the user.

In this paper, we investigate a class of nonlinear discrete-time systems with unknown inputs to provide a constructive approach for $\mathcal{H}_\infty$ observer-based control design. In order to put our work with respect to the existing results, we summarize the main contributions by the following points: Firstly, we consider a general class of one-sided Lipschitz systems in discrete-time where the nonlinearity depends on both the states and the unknown inputs. Secondly, the one-sided Lipschitz and quadratically inner-bounded conditions are introduced to provide nonrestrictive sufficient conditions. Thirdly, a particular Lyapunov function and an observer-based controller both dependent on the nonlinearities of the system are proposed. Furthermore, unlike some existing works, a matching condition to cancel out the effects of faults is not required. Finally, some mathematical artifacts are used to provide feasible and less conservative LMI stability conditions. These are the reasons why our proposed approach works for a larger class of dynamical systems. With the goal of showing the high performance of the proposed results, we consider a relevant numerical example.

Notations. In a matrix, the notation $(\star)$ is used for the blocks induced by symmetry. $\|x\| = \sqrt{x^T x}$ is the Euclidean norm. $\|x\|_{\ell_p^n} = \left(\sum_{k=0}^{\infty} \|x(k)\|^p\right)^{1/p}$ represents the ℓ_p^n norm of the vector function $x(\cdot) : \mathbb{N} \to \mathbb{R}^n$. The set ℓ_p^n is the Lebesgue space defined by $\ell_p^n = \{x(\cdot) : \mathbb{N} \to \mathbb{R}^n : \|x\|_{\ell_p^n} < +\infty\}$.

The next well-known lemma [36] is useful in the paper.

Lemma 1. *[36]. For any given matrices Ψ and Φ, and a positive definite matrix Σ with compatible dimension, one has*

$$\forall \epsilon \in \mathbb{R},\ \epsilon > 0,\ \Psi\Phi^T + \Phi\Psi^T \leqslant \epsilon\Phi\Sigma\Phi^T + \frac{1}{\epsilon}\Psi\Sigma^{-1}\Psi^T.$$

2. Problem statement.

Let us consider the following nonlinear system

$$\begin{cases} x(k+1) = A_x x(k) + E_x f(k) + B_x u(k) + F_x g(x(k), f(k)) + D_x w(k) \\ y(k) = C_y x(k) + D_y w(k) \end{cases} \tag{2.1}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$ and $y(k) \in \mathbb{R}^p$ denotes respectively the state, the input vector and the linear output. $f(k) \in \mathbb{R}^r$ is the unknown actuator fault vector which changes unexpectedly when a fault occurs and is treated here as an unknown input. $w(k) \in \mathbb{R}^t$ is the disturbances signal which belongs to $\ell_2[0, \infty]$. A_x, B_x, E_x, F_x, C_y, D_x and D_y are constant matrices of adequate dimensions. $g : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^q$ is a real nonlinear vector field.

First, for doing so we introduce a simple and equivalent form to (2.1) to estimate both the state and unknown input using H_∞ state estimation techniques.
As $f(k)$ is unknown, let us introduce the artificial model $f(k+1) = f(k) + w_f(k)$, the system (2.1) is equivalent to

$$\begin{cases} x(k+1) = A_x x(k) + E_x f(k) + B_x u(k) + h(x(k), f(k)) + D_x w(k) \\ f(k+1) = f(k) + w_f(k) \\ y(k) = C_y x(k) + D_y w(k) \end{cases} \tag{2.2}$$

$w_f \in \ell_2^r$ may be considered as a vector of disturbances and $h(x, f) = F_x g(x, f)$.

The second part of this paper is dedicated to control design that ensures some H_∞ performances. Indeed, what we provide here is a constructive approach to minimize the unknown input effects and at the same time to ensure asymptotic stability. Owing to some mathematical artifacts, we deduce simple LMI stability conditions. The given system (2.2) is equivalent to

$$\begin{cases} \xi(k+1) = A\xi(k) + Bu(k) + l(\xi(k)) + W_1 w_f(k) + W_2 w(k) \\ y(k) = C\xi(k) + D_y w(k)\ ; \quad \xi \in \mathbb{R}^{o=n+r} \end{cases} \tag{2.3}$$

where $l(x,f) = Fg(\xi) = \begin{bmatrix} h(x,f)^T & 0 \end{bmatrix}^T$ and

$$\xi(k)=\begin{bmatrix} x(k) \\ f(k) \end{bmatrix}, A=\begin{bmatrix} A_x & E_x \\ 0 & I_r \end{bmatrix}, B=\begin{bmatrix} B_x \\ 0 \end{bmatrix}, F=\begin{bmatrix} F_x \\ 0 \end{bmatrix}, W_1=\begin{bmatrix} 0 \\ I_r \end{bmatrix}, W_2=\begin{bmatrix} D_x \\ 0 \end{bmatrix}, C^T=\begin{bmatrix} C_y^T \\ 0 \end{bmatrix}.$$

The following assumption is made throughout this paper.

Assumption 1.

1. *[12] $l : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^{n+r}$ is one-sided Lipschitz with respect to x and f. i.e.*

$$\left\langle l(x,f) - l(\hat{x},\hat{f}), \begin{bmatrix} x-\hat{x} \\ f-\hat{f} \end{bmatrix} \right\rangle \leqslant \rho \left\| \begin{bmatrix} x-\hat{x} \\ f-\hat{f} \end{bmatrix} \right\|^2, \quad \forall\ x,\hat{x} \in \mathbb{R}^{\mathrm{n}},\ f,\hat{f} \in \mathbb{R}^{\mathrm{r}} \tag{2.4}$$

   The constant ρ is the so-called one-sided Lipschitz constant.

2. *[17] l is quadratically inner-bounded with respect to x and f. i.e.*

$$\left\| l(x,f) - l(\hat{x},\hat{f}) \right\|^2 \leqslant \beta \left\| \begin{bmatrix} x-\hat{x} \\ f-\hat{f} \end{bmatrix} \right\|^2 + \gamma \left\langle \begin{bmatrix} x-\hat{x} \\ f-\hat{f} \end{bmatrix}, l(x,f) - l(\hat{x},\hat{f}) \right\rangle. \tag{2.5}$$

   where β and γ are real scalars.

Unlike the well-known Lipschitz condition, the constants ρ, β and γ can be positive, negative or zero. In addition, if the function l is Lipschitz, then it also one-sided Lipschitz and quadratically inner-bounded, but the converse is not true (see [17]). In addition, in the observer synthesis of one-sided Lipschitz systems, the concept of quadratic inner-boundedness of l (see [17]) is useful.

The aim of this paper is to investigate the problem of robust observer-based stabilization for a class of one-sided nonlinear discrete-time systems in the presence of unknown inputs. The contributions of this paper and the improvements with respect to existing results can be summarized by the following points:

- We consider a general class of nonlinear discrete-time systems where the nonlinearity depends on both the states and the unknown inputs. In this sense, this contribution can be then considered as an extension of the papers [23] and [35].
- The one-sided Lipschitz and quadratically inner-bounded conditions are introduced to provide nonrestrictive sufficient conditions.
- The well-known matching conditions $\mathrm{Im}(E_x) \subseteq \mathrm{Im}(B_x)$ and $\mathrm{rank}(C_y E_x) = \mathrm{rank}(E_x)$ to cancel completely the unknown inputs effects are not required, contrarily to [35].
- The upper bounds on the unknown inputs and their derivatives are not needed nor used for the observer and controller design.

- Only one step is needed to solve the observer-based stabilization problem. As a consequence, unlike some existing works in the literature (see for instance [21], [6]), the problem is stated in the form of one LMI condition without any additional restrictive conditions, such as equality constraints or a priori choice of the Lyapunov matrix.
- A particular Lyapunov function and a controller law both dependent on the nonlinearities of the system are proposed.
- Through the simulations, we obtain high performances of the proposed approach under severe conditions (small disturbance attenuation level and increased frequency of the unknown input signal).

These are the reasons why our proposed approach represents a useful degree of freedom and works for a large class of dynamical systems.

3. $\mathcal{H}_\infty$ nonlinear unknown input observer.

The goal of this section is to design an asymptotic nonlinear observer to estimate simultaneously the state vector and the unknown inputs. This observer has the following form

$$\begin{cases} \hat{x}(k+1) = A_x\hat{x}(k) + E_x\hat{f}(k) + B_xu(k) + h(\hat{x}(k),\hat{f}(k)) + K_1(y(k)-\hat{y}(k)) \\ \hat{f}(k+1) = \hat{f}(k) + K_2(y(k)-\hat{y}(k)) \\ \hat{y}(k) = C_y\hat{x}(k) \end{cases} \tag{3.1}$$

such that $\hat{\xi} = \begin{bmatrix} \hat{x}^T & \hat{f}^T \end{bmatrix}^T$ is an asymptotic estimate of ξ, K_1 and K_2 are the matrix gains to be computed. For observer design, we need the following assumption.

Assumption 2. *The pair (A, C) is assumed to be detectable.*

Remark 1. *The assumption 2 is expressed in terms of the matrices of system* (2.3). *It can equivalently be formulated in terms of the matrices of system* (2.1) *as follows*

1. *the pair (A_x, C_y) is detectable,*
2. $\operatorname{rank} E_x = r$,
3. $\operatorname{rank}\begin{bmatrix} A_3 & A_4 - I \\ C_1 & C_2 \end{bmatrix} = n$, *where the real matrices A_3, A_4, C_1 and C_2 are obtained by using two arbitrary non singular matrices Ω_1 and Ω_2 as follows*

$$\Omega_1 A_x \Omega_1^{-1} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad C_y\Omega_1^{-1} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \quad -\Omega_1 E_x \Omega_2 = \begin{bmatrix} I_r \\ 0 \end{bmatrix}. \tag{3.2}$$

Proof of remark 1.
A simple reasoning shows that

$$\begin{aligned} \operatorname{rank}\begin{bmatrix} sI_{n+r} - A \\ C \end{bmatrix} &= \operatorname{rank}\begin{bmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} sI_n - A_x & -E_x \\ 0 & s_1 I_r \\ C_y & 0 \end{bmatrix}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \\ &= \operatorname{rank}\begin{bmatrix} s_1 I_r & 0 \\ -E_x & sI_n - A_x \\ 0 & C_y \end{bmatrix}, \quad \text{with} \quad s_1 = s - 1, \end{aligned} \tag{3.3}$$

If $s_1 = s - 1 \neq 0$ and from the detectability of the pair (A, C) and by the fact that (3.3) *is a triangular matrix, we have*

$$\operatorname{rank}\begin{bmatrix} sI_{n+r} - A \\ C \end{bmatrix} < n + r \iff \operatorname{rank}\begin{bmatrix} sI_n - A_x \\ C_y \end{bmatrix} < n$$

Therefore, the pair (A_x, C_y) is then detectable.

On the other hand, if $s_1 = s - 1 = 0$, we have

$$\begin{bmatrix} s_1 I_r & 0 \\ -E_x & sI_n - A_x \\ 0 & C_y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -E_x & I_n - A_x \\ 0 & C_y \end{bmatrix} \text{ with } \operatorname{rank}(E_x) = r. \tag{3.4}$$

Let introduce the arbitrary non singular matrices Ω_1 and Ω_2 defined in (3.2) so that

$$\begin{aligned} \begin{bmatrix} \Omega_1 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} -E_x & I_n - A_x \\ 0 & C_y \end{bmatrix} \begin{bmatrix} \Omega_2 & 0 \\ 0 & \Omega_1^{-1} \end{bmatrix} &= \begin{bmatrix} -\Omega_1 E_x \Omega_2 & I - \Omega_1 A_x \Omega_1^{-1} \\ 0 & C_y \Omega_1^{-1} \end{bmatrix} \\ &= \begin{bmatrix} I_r & I - A_1 & -A_2 \\ 0 & -A_3 & I - A_4 \\ 0 & C_1 & C_2 \end{bmatrix} \end{aligned} \tag{3.5}$$

For $s = 1$, by the fact that (3.5) is a triangular matrix, we have

$$\operatorname{rank} \begin{bmatrix} I_r & I - A_1 & -A_2 \\ 0 & -A_3 & I - A_4 \\ 0 & C_1 & C_2 \end{bmatrix} = n + r \iff \operatorname{rank} \begin{bmatrix} -A_3 & I - A_4 \\ C_1 & C_2 \end{bmatrix} = n.$$

finally, the item 3 of Remark 1 is also verified. This ends the proof.

Let $e = \xi - \hat{\xi}$ and $K = \begin{bmatrix} K_1^T & K_2^T \end{bmatrix}^T$. Then from the observer (3.1) and the system (2.2), the state estimation error is described by

$$e(k+1) = (A - KC)e + \Delta l_k + W_1 w_f + (W_2 - KD_y)w. \tag{3.6}$$

where $\Delta l_k = F \Delta g_k = \mathcal{E}_L^T \Delta h_k$, $\Delta g_k = g(\xi) - g(\hat{\xi})$, $\Delta h_k = h(\xi) - h(\hat{\xi})$ and $\mathcal{E}_L = \begin{bmatrix} I_n & 0_{\mathbb{R}^{n \times r}} \end{bmatrix}$.

Taking into account the presence of the disturbances w_f and w, the aim is to minimize the ℓ_2 gain between the noise input $\mathbf{w} = \begin{bmatrix} w_f^T & w^T \end{bmatrix}^T$ and the estimation error e. By extension of the linear case, the ℓ_2 gain is called "$\mathcal{H}_\infty$ norm" (see [37]). More precisely, given the system (2.2) and the observer (3.1), the problem of $\mathcal{H}_\infty$ filter design is to determine matrices K_1 and K_2 in (3.1) so that

$$\lim_{k \to \infty} e(k) = 0 \ \text{ for } \ w_f = 0; \ w = 0 \tag{3.7}$$

$$\|e(k)\|_{\ell_2} < \sqrt{\lambda \left\| \begin{bmatrix} w_f(k) \\ w(k) \end{bmatrix} \right\|_{\ell_2}^2 + \kappa_0 \|e(0)\|^2}, \quad \text{for all } \begin{bmatrix} w_f \\ w \end{bmatrix} \neq 0 \tag{3.8}$$

where $\lambda > 0$ is the disturbance attenuation level to be minimized. Notice that where $w_f = 0$ this represents the case of constant inputs. The problem of $\mathcal{H}_\infty$ filtering design (3.7)-(3.8) can be reduced to finding a Lyapunov function $V_k(e) > 0$ so that

$$\mathcal{V} \triangleq \Delta V + e^T e - \lambda w_f^T w_f - \lambda w^T w < 0. \tag{3.9}$$

where $\Delta V = V_{k+1} - V_k$.

A simple reasoning shows that (3.9) implies the two conditions (3.7) and (3.8) (see appendix 8.2).

In the following theorem, sufficient conditions ensuring (3.9) are provided with an optimal disturbance attenuation level λ.

THEOREM 1. *Under the assumption 1, the system* (3.1) *is an asymptotic observer for system* (2.2) *if there exist scalars* $\lambda > 0$, $\alpha > 0$, $\mu_1 > 0$, $\mu_2 > 0$, *and* $\epsilon > 0$ *and matrices* $P = P^T > \alpha I_{n+r}$, $Q = Q^T > 0$, S *and* $\mathcal{X}$, *that solve the following convex optimization problem*

$$\min(\lambda) \quad \textit{subject to} \quad \mathbb{N} < 0 \tag{3.10}$$

and

$$\begin{bmatrix} P & S \\ S^T & Q \end{bmatrix} > 0 \tag{3.11}$$

where $\mathbb{N}$ *is given by*

$$\mathbb{N}=\begin{bmatrix} \mathbb{N}_{11} & \eta\mathbb{N}_{12}\mathcal{E}_L^T+\mathbb{S} & 0 & \eta\mathbb{N}_{12}W_1 & 0 & \mathbb{N}_{12} & \mathbb{N}_{12} & 0 & 0 \\ \star & \mathbb{N}_{22} & \mathcal{E}_L\mathbb{N}_{23} & \eta\mathcal{E}_L PW_1 & \eta\mathcal{E}_L\mathbb{N}_{15}^T & 0 & 0 & 0 & 0 \\ \star & \star & \mathbb{N}_{33} & \mathbb{N}_{23}^T W_1 & 0 & 0 & 0 & \mathbb{N}_{23}^T & 0 \\ \star & \star & \star & \mathbb{N}_{44} & \eta W_1^T\mathbb{N}_{15}^T & 0 & 0 & 0 & 0 \\ \star & \star & \star & \star & -\lambda I_t & \mathbb{N}_{15} & 0 & 0 & \mathbb{N}_{15} \\ \star & \star & \star & \star & \star & -\eta^{-1}P & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -\epsilon P & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\tau_1 P & 0 \\ \star & \star & \star & \star & \star & \star & \star & \star & \frac{-1}{\epsilon}P \end{bmatrix} \tag{3.12}$$

with

$$\begin{cases} \eta = 1 + 2(|\rho| + |\beta|) \\ \tau_1 = \frac{\epsilon}{1+\epsilon^2} \\ \delta = \mu_1\rho + \mu_2\beta \\ \mathbb{S} = (\mu_2\gamma - \mu_1)\mathcal{E}_L^T - S \\ \mathbb{N}_{11} = -P + (2\delta + 1)I_{n+r} \\ \mathbb{N}_{12} = A^T P - C^T\mathcal{X} \\ \mathbb{N}_{15} = W_2^T P - D_y^T\mathcal{X} \\ \mathbb{N}_{22} = \eta\mathcal{E}_L P\mathcal{E}_L^T - Q - 2\mu_2\mathcal{E}_L\mathcal{E}_L^T \\ \mathbb{N}_{23} = S + \alpha(\gamma - 1)\mathcal{E}_L^T \\ \mathbb{N}_{33} = Q - 2\alpha\mathcal{E}_L\mathcal{E}_L^T \\ \mathbb{N}_{44} = \eta W_1^T PW_1 - \lambda I_r \end{cases} \tag{3.13}$$

Then, the gain for observer is given by $K = P^{-1}\mathcal{X}^T$.
Proof. Let us consider the quadratic Lyapunov function

$$V_k = \begin{bmatrix} e(k) \\ \Delta h_k \end{bmatrix}^T \begin{bmatrix} P & S \\ S^T & Q \end{bmatrix} \begin{bmatrix} e(k) \\ \Delta h_k \end{bmatrix} \tag{3.14}$$

where Δh_k defined in (3.6) and $\begin{bmatrix} P & S \\ S^T & Q \end{bmatrix} > 0$. Moreover, the variation $\Delta V = V_{k+1} - V_k$ of this Lyapunov function is given by

$$\Delta V = e^T(k+1)Pe(k+1) - e^T(k)Pe(k) - \Delta h_k^T Q \Delta h_k + \Delta h_{k+1}^T Q \Delta h_{k+1} \\ + 2e^T(k+1)S\Delta h_{k+1} - 2e^T(k)S\Delta h_k. \quad (3.15)$$

The one-sided Lipschitz and the quadratically inner-bounded conditions (2.4) and (2.5) give the following inequality

$$\begin{cases} \mu_1 \rho \|e\|^2 - \mu_1 e^T \Delta l_k \geqslant 0 \\ \mu_2 \beta \|e\|^2 + \mu_2 \gamma e^T \Delta l_k - \mu_2 \|\Delta l_k\|^2 \geqslant 0 \end{cases} \quad (3.16)$$

where μ_1 and μ_2 are arbitrary strictly positive scalars.

The following inequality is obtained by adding the left hand side of (3.16) to (3.15)

$$\Delta V \leqslant e^T(k+1)Pe(k+1) + e^T(k)(-P + 2\delta I)e(k) + 2e^T(k)\mathbb{S}\Delta h_k \\ + 2e^T(k+1)S\Delta h_{k+1} - \Delta h_k^T(Q + 2\mu_2 \mathcal{E}_L \mathcal{E}_L^T)\Delta h_k + \Delta h_{k+1}^T Q \Delta h_{k+1} \quad (3.17)$$

where δ and $\mathbb{S}$ are given by (3.13).

On the other hand, using the one-sided Lipschitz and the inner-bounded conditions (2.4) and (2.5) with the fact that $P > \alpha I_n$, it follows that

$$|\rho|\, e^T(k+1)Pe(k+1) - \alpha e^T(k+1)\Delta l_{k+1} \\ > \alpha \Big(|\rho|\, \|e(k+1)\|^2 - e^T(k+1)\Delta l_{k+1} \Big) \geqslant 0 \quad (3.18)$$

$$|\beta|\, e^T(k+1)Pe(k+1) + \alpha\gamma e^T(k+1)\Delta l_{k+1} - \alpha \Delta l_{k+1}^T \Delta l_{k+1} \\ > \alpha \Big(|\beta|\, \|e(k+1)\|^2 + \gamma e^T(k+1)\Delta l_{k+1} - \Delta l_{k+1}^T \Delta l_{k+1} \Big) \geqslant 0. \quad (3.19)$$

Thus, by adding the left terms in inequalities (3.18) and (3.19) to (3.17), we get

$$\Delta V \leqslant \eta e^T(k+1)Pe(k+1) + e^T(k)(-P + 2\delta I)e(k) + 2e^T(k)\mathbb{S}\Delta h_k \\ - \Delta h_k^T(Q + 2\mu_2 \mathcal{E}_L \mathcal{E}_L^T)\Delta h_k + 2e^T(k+1)\mathbb{N}_{23}\Delta h_{k+1} + \Delta h_{k+1}^T \mathbb{N}_{33} \Delta h_{k+1} \quad (3.20)$$

where η, $\mathbb{N}_{23}$ and $\mathbb{N}_{33}$ are given by (3.13).

Using the dynamics of the estimation error (3.6) and based on the Lyapunov stability theory, the convergence of the estimation error is guaranteed, as long as $\mathcal{V}$ is negative definite, which holds if

$$\xi^T \mathcal{N} \xi < 0 \quad (3.21)$$

where

$$\xi^T(k) = \begin{bmatrix} e^T(k) & \Delta h_K^T & \Delta h_{K+1}^T & w_f^T(k) & w^T(k) \end{bmatrix} \quad (3.22)$$

and

$$\mathcal{N} = \begin{bmatrix} \mathbb{N}_{11} + \eta \mathcal{N}_{12} P^{-1} \mathcal{N}_{12}^T & \eta \mathcal{N}_{12} \mathcal{E}_L^T + \mathbb{S} & \mathcal{N}_{12} P^{-1} \mathbb{N}_{23} & \eta \mathcal{N}_{12} W_1 & \eta \mathcal{N}_{12} P^{-1} \mathcal{N}_{15}^T \\ \star & \mathbb{N}_{22} & \mathcal{E}_L \mathbb{N}_{23} & \eta \mathcal{E}_L P W_1 & \eta \mathcal{E}_L \mathcal{N}_{15}^T \\ \star & \star & \mathbb{N}_{33} & \mathbb{N}_{23}^T W_1 & \mathbb{N}_{23}^T P^{-1} \mathcal{N}_{15}^T \\ \star & \star & \star & \mathbb{N}_{44} & \eta W_1^T \mathcal{N}_{15}^T \\ \star & \star & \star & \star & \eta \mathcal{N}_{15} P^{-1} \mathcal{N}_{15}^T - \lambda I \end{bmatrix} \quad (3.23)$$

with $\mathcal{N}_{12} = (A - KC)^T P$ and $\mathcal{N}_{15} = (W_2 - KD_y)^T P$.

The BMI problem of (3.21)-(3.23) is not convex and the existing LMI computational procedures can not be applied. To linearize the BMI problem (3.21)-(3.23), we proceed in three steps.

The first step concerns matrices $\mathcal{N}_{12}P^{-1}\mathbb{N}_{23}$ and $\mathbb{N}_{23}^T P^{-1}\mathcal{N}_{15}^T$. Since we have P, $\mathcal{N}_{12}$, $\mathcal{N}_{15}$ and $\mathbb{N}_{23}$ acting separately in (3.23), and $\mathcal{N}_{12}$, $\mathcal{N}_{15}$ and $\mathbb{N}_{23}$ are a function of K, P and S, it is necessary to rewrite $\mathcal{N}_{12}P^{-1}\mathbb{N}_{23}$ and $\mathbb{N}_{23}^T P^{-1}\mathcal{N}_{15}^T$ under a form so that $\mathcal{N}_{12}$, $\mathcal{N}_{15}$ and $\mathbb{N}_{23}$ appear separately in this equation. Notice that the matrix $\mathcal{N}$ can be rewritten as follows

$$\mathcal{N} = \overline{\mathcal{N}}_1 + \underbrace{\Psi\Phi^T + \Phi\Psi^T}_{\overline{\mathcal{N}}_2} \tag{3.24}$$

where

$$\overline{\mathcal{N}}_1 = \begin{bmatrix} \mathbb{N}_{11} + \eta\mathcal{N}_{12}P^{-1}\mathcal{N}_{12}^T & \eta\mathcal{N}_{12}\mathcal{E}_L^T + \mathbb{S} & \mathbf{0} & \eta\mathcal{N}_{12}W_1 & \eta\mathcal{N}_{12}P^{-1}\mathcal{N}_{15}^T \\ \star & \mathbb{N}_{22} & \mathcal{E}_L\mathbb{N}_{23} & \eta\mathcal{E}_L P W_1 & \eta\mathcal{E}_L\mathcal{N}_{15}^T \\ \star & \star & \mathbb{N}_{33} & \mathbb{N}_{23}^T W_1 & \mathbf{0} \\ \star & \star & \star & \mathbb{N}_{44} & \eta W_1^T\mathcal{N}_{15}^T \\ \star & \star & \star & \star & \eta\mathcal{N}_{15}P^{-1}\mathcal{N}_{15}^T - \lambda I \end{bmatrix}$$

and

$$\Psi^T = \begin{bmatrix} \mathcal{N}_{12}^T & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbb{N}_{23} & 0 & 0 \end{bmatrix}, \ \Phi^T = \begin{bmatrix} 0 & 0 & P^{-1}\mathbb{N}_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & P^{-1}\mathcal{N}_{15}^T \end{bmatrix}.$$

In the second step, we use the lemma 1 with $\Sigma = P$ to obtain an upper bound of the term $\Psi\Phi^T + \Phi\Psi^T$ in (3.24) and the following inequality holds for any scalar $\epsilon > 0$

$$\Psi\Phi^T + \Phi\Psi^T \leqslant \epsilon \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \mathbb{N}_{23}^T & 0 \\ 0 & 0 \\ 0 & \mathcal{N}_{15} \end{bmatrix} P^{-1} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \mathbb{N}_{23}^T & 0 \\ 0 & 0 \\ 0 & \mathcal{N}_{15} \end{bmatrix}^T + \frac{1}{\epsilon} \begin{bmatrix} \mathcal{N}_{12} & 0 \\ 0 & 0 \\ 0 & \mathbb{N}_{23}^T \\ 0 & 0 \\ 0 & 0 \end{bmatrix} P^{-1} \begin{bmatrix} \mathcal{N}_{12} & 0 \\ 0 & 0 \\ 0 & \mathbb{N}_{23}^T \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^T \tag{3.25}$$

Substituting (3.25) into (3.24), we get:

$$\mathcal{N} \leqslant \overline{\mathcal{N}}_1 + \epsilon \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \mathbb{N}_{23}^T & 0 \\ 0 & 0 \\ 0 & \mathcal{N}_{15} \end{bmatrix} P^{-1} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \mathbb{N}_{23}^T & 0 \\ 0 & 0 \\ 0 & \mathcal{N}_{15} \end{bmatrix}^T + \frac{1}{\epsilon} \begin{bmatrix} \mathcal{N}_{12} & 0 \\ 0 & 0 \\ 0 & \mathbb{N}_{23}^T \\ 0 & 0 \\ 0 & 0 \end{bmatrix} P^{-1} \begin{bmatrix} \mathcal{N}_{12} & 0 \\ 0 & 0 \\ 0 & \mathbb{N}_{23}^T \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^T \tag{3.26}$$

This implies that if the linear matrix inequality

$$\mathbf{N} = \begin{bmatrix} \mathbb{N}_{11} + \Upsilon_1^T\Pi\Upsilon_1 & \eta\mathcal{N}_{12}\mathcal{E}_L^T + \mathbb{S} & 0 & \eta\mathcal{N}_{12}W_1 & \Upsilon_1^T\Pi\Upsilon_5 \\ \star & \mathbb{N}_{22} & \mathcal{E}_L\mathbb{N}_{23} & \eta\mathcal{E}_L P W_1 & \eta\mathcal{E}_L\mathcal{N}_{15}^T \\ \star & \star & \mathbb{N}_{33} + \Upsilon_3^T\Pi\Upsilon_3 & \mathbb{N}_{23}^T W_1 & 0 \\ \star & \star & \star & \mathbb{N}_{44} & \eta W_1^T\mathcal{N}_{15}^T \\ \star & \star & \star & \star & \Upsilon_5^T\Pi\Upsilon_5 - \lambda I \end{bmatrix} < 0 \tag{3.27}$$

is verified then $\Delta V < 0$, where

$$\begin{cases} \Upsilon_1^T = \begin{bmatrix} \mathcal{N}_{12} & \mathcal{N}_{12} & 0 & 0 \end{bmatrix} \\ \Upsilon_3^T = \begin{bmatrix} 0 & 0 & \mathbb{N}_{23}^T & 0 \end{bmatrix} \\ \Upsilon_5^T = \begin{bmatrix} \mathcal{N}_{15} & 0 & 0 & \mathcal{N}_{15} \end{bmatrix} \\ \Pi = \text{bdiag}(\eta P^{-1}, \epsilon^{-1} P^{-1}, \frac{1+\epsilon^2}{\epsilon} P^{-1}, \epsilon P^{-1}) \end{cases}$$

Now, notice that

$$\mathbf{N} = \mathbf{N}_1 + \Upsilon \Pi \Upsilon^T < 0 \tag{3.28}$$

where

$$\Upsilon^T = \begin{bmatrix} \Upsilon_1 & 0 & \Upsilon_3 & 0 & \Upsilon_5 \end{bmatrix}$$

and

$$\mathbf{N}_1 = \begin{bmatrix} \mathbb{N}_{11} & \eta \mathcal{N}_{12} \mathcal{E}_L^T + \mathbb{S} & 0 & \eta \mathcal{N}_{12} W_1 & 0 \\ \star & \mathbb{N}_{22} & \mathcal{E}_L \mathbb{N}_{23} & \eta \mathcal{E}_L P W_1 & \eta \mathcal{E}_L \mathcal{N}_{15}^T \\ \star & \star & \mathbb{N}_{33} & \mathbb{N}_{23}^T W_1 & 0 \\ \star & \star & \star & \mathbb{N}_{44} & \eta W_1^T \mathcal{N}_{15}^T \\ \star & \star & \star & \star & -\lambda I \end{bmatrix}$$

Finally, using the Schur lemma [38] and the notation $X = K^T P$, the inequality in (3.28) and the LMI in (3.10) are equivalent, which completes the proof. □

4. $\mathcal{H}_\infty$ unknown input observer-based control.

In this section, we study and deduce sufficient conditions under which the discrete-time one-sided Lipschitz nonlinear system (2.1) is asymptotically stable under the action of unknown input observer-based nonlinear feedback. To achieve this goal, we first introduce two assumptions.

ASSUMPTION 3.

1. *The pair (A_x, B_x) is assumed to be stabilizable.*
2. *In the sequel, without loss of generality, we consider the class of nonlinear functions that satisfy $l(0,0) = 0$. Applying the one-sided Lipschitz and the quadratically inner-bounded conditions* (2.4) *and* (2.5) *on the system* (2.1), *we get*

$$\begin{cases} \langle l(x,f), \xi \rangle \leqslant \rho \left\| \xi \right\|^2 \\ \left\| l(x,f) \right\|^2 \leqslant \beta \left\| \xi \right\|^2 + \gamma \langle \xi, l(x,f) \rangle \end{cases} \tag{4.1}$$

where $\xi = \begin{bmatrix} x^T & f^T \end{bmatrix}^T$.

Since our objective is to develop a state observer-based controller, then only the estimates of x and f, are available for feedback. The controller of (2.1) we propose here is given by

$$u = u_x + u_f + u_g \tag{4.2}$$

where $u_x = -L_1 \hat{x}$, $u_f = -L_2 \hat{f}$ and $u_g = -L_3 g(\hat{x}, \hat{f})$.
In the following, owing to several linearization techniques, we give a synthesis method of the controller which consists in finding the gains L_1, L_2, L_3, K_1 and K_2 through an LMI approach.

The closed-loop system generated by system (2.1) with the control law (4.2) is given by

$$x(k+1) = (A_x - B_x L_1)x + B_x L e + (E_x - B_x L_2)f + B_x L_3 \Delta g_k + (F_x - B_x L_3)g(\xi) + D_x w \tag{4.3}$$

with $L = [\, L_1 \;\; L_2 \,]$ and Δg_k defined in (3.6).

Taking the presence of the disturbance $\begin{bmatrix} f^T & w_f^T & w^T \end{bmatrix}^T$ into account, the aim is to minimize the ℓ_2 gain from the noise input $\begin{bmatrix} f^T & w_f^T & w^T \end{bmatrix}^T$ to the vector $[\, e^T \;\; x^T \,]^T$. This problem is called in the sequel as the $\mathcal{H}_\infty$ observer-based control design (see [37]). More precisely, given the system (2.1) and the dynamical equation (3.6), the problem of $\mathcal{H}_\infty$ observer-based control design is to determine matrices K, L_1, L_2 and L_3 in (3.1) and (4.2) so that

$$\lim_{k\to\infty} \begin{bmatrix} e(k) \\ x(k) \end{bmatrix} = 0 \;\text{ for } \begin{bmatrix} f \\ w_f \\ w \end{bmatrix} = 0 \tag{4.4}$$

$$\left\| \begin{bmatrix} e(k) \\ x(k) \end{bmatrix} \right\|_{\ell_2} < \sqrt{\lambda \left\| \begin{bmatrix} f(k) \\ w_f(k) \\ w(k) \end{bmatrix} \right\|_{\ell_2}^2 + \kappa_0 \left\| \begin{bmatrix} e(0) \\ x(0) \end{bmatrix} \right\|^2}, \quad \text{for all } \begin{bmatrix} f \\ w_f \\ w \end{bmatrix} \neq 0 \tag{4.5}$$

where $\lambda > 0$ is the disturbance attenuation level to be minimized.

The $\mathcal{H}_\infty$ observer-based control design (4.4)-(4.5) can be reduced to find a Lyapunov function $V(e,x) > 0$ so that

$$\mathcal{V} \triangleq \Delta V(e,x) + e^T e + x^T x - \lambda(f^T f + w_f^T w_f + w^T w) < 0. \tag{4.6}$$

A simple reasoning shows that (4.6) implies the two conditions (4.4) and (4.5) (see appendix 8.2).

Now, we are in position to state and prove our result on the existence of observer-based controller (3.1) and (4.2) for system (2.1).

THEOREM 2. *Under the assumptions 1, 2 and 3, the $\mathcal{H}_\infty$ observer-based control design problem for system* (2.1), *observer* (3.1) *and feedback control* (4.2) *is achieved if there exist real scalars $\lambda > 0$, $\mu_1 > 0$, $\mu_2 > 0$, $\theta_1 > 0$, $\theta_2 > 0$, $\epsilon > 0$, $\alpha > 0$ and matrices $P = P^T > \alpha I_{n+r}$, $\Gamma = \Gamma^T$, S, Q, $\mathcal{X}$, $\mathcal{Y}$, L_2 and L_3 that solve the following convex optimization problem*

$$\min(\lambda) \;\; \textit{subject to} \;\; \mathbb{M} < 0 \tag{4.7}$$

and

$$\begin{bmatrix} P & S \\ S^T & Q \end{bmatrix} > 0, \tag{4.8}$$

where

$$
\mathbb{M}=\begin{bmatrix}
\mathbb{M}_{11} & 0 & \mathbb{S} & 0 & 0 & 0 & 0 & 0 & \mathbb{M}_{19} & \mathbb{M}_{12} & \mathbb{M}_{12} & \mathcal{E}_L^T & 0 & 0 & 0 & 0 \\
\star & -\Gamma & 0 & 0 & \mathbb{M}_{25} & 0 & 0 & 0 & \mathbb{H}_1 & 0 & 0 & 0 & \Gamma & 0 & 0 & 0 \\
\star & \star & \mathbb{M}_{33} & F^T\mathbb{M}_{23} & 0 & 0 & 0 & 0 & L_3^T B_x^T & F^T P & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \mathbb{M}_{44} & 0 & 0 & \mathbb{M}_{23}^T W_1 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{M}_{23}^T & 0 & 0 \\
\star & \star & \star & \star & \mathbb{M}_{55} & 0 & 0 & 0 & \mathcal{H}_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \mathbb{M}_{66} & 0 & 0 & \mathcal{H}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & -\lambda I_r & 0 & 0 & W_1^T P & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & -\lambda I_t & D_x^T & \mathbb{M}_{15} & 0 & 0 & 0 & 0 & \mathbb{M}_{15} & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & -\Gamma & 0 & 0 & 0 & 0 & 0 & 0 & B_x\mathcal{Y} \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \frac{-1}{\eta}P & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -\epsilon P & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -\epsilon\Gamma & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -\tau_3 I_n & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & -\tau_4 P & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \frac{-1}{\epsilon}P & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \frac{-1}{\epsilon}\Gamma
\end{bmatrix}
\tag{4.9}
$$

with

$$
\begin{cases}
\eta = 1 + 2(|\rho| + |\beta|) \\
\delta_1 = 2(\mu_1\rho + \mu_2\beta), \\
\delta_2 = 2(\theta_1\rho + \theta_2\beta), \;\; \delta_3 = \theta_2\gamma - \theta_1 \\
\tau_3 = \frac{1}{1+\delta_2}, \quad \tau_4 = \frac{\epsilon}{1+\epsilon^2} \\
\mathbb{S} = (\mu_2\gamma - \mu_1)F - SF_x \\
\mathbb{M}_{11} = -P + (\delta_1 + 1)I_{n+r} \\
\mathbb{M}_{12} = A^T P - C^T\mathcal{X} \\
\mathbb{M}_{15} = W_2^T P - D_y^T\mathcal{X} \\
\mathbb{M}_{19} = \mathcal{E}_r^T L_2^T B_x^T \\
\mathbb{M}_{23} = SF_x + \alpha(\gamma - 1)F \\
\mathbb{M}_{25} = \delta_3\Gamma F_x \\
\mathbb{M}_{33} = -F_x^T(Q + 2\mu_2 I_n)F_x \\
\mathbb{M}_{44} = F_x^T(Q - 2\alpha I_n)F_x \\
\mathbb{M}_{55} = -2\theta_2 F_x^T F_x \\
\mathbb{M}_{66} = (\delta_2 - \lambda)I_r \\
\mathbb{H}_1 = \Gamma A_x^T - \mathcal{Y}^T B_x^T \\
\mathcal{H}_2 = (E_x - B_x L_2)^T, \quad \mathcal{H}_3 = (F_x - B_x L_3)^T
\end{cases}
\tag{4.10}
$$

The gains K and L_1 are given by $K = P^{-1}\mathcal{X}^T$ and $L_1 = \mathcal{Y}\Gamma^{-1}$.

Proof. See the Appendix 8.1. ∎

Remark 2. *The optimization approach proposed in this paper relies on the inequality* (4.6) *which is later relaxed in the form of LMI* (4.7)*. In the relaxation process we restricted the search for λ to a particular class of Lyapunov functions in quadratic form. This comes with some loss of generality, potentially resulting in a limitation of the performance one can hope for. One possible solution for enhancing performance (i.e., obtaining a smaller λ) would be to optimize over a larger class of Lyapunov functions. However, in this general case, it is hard to get a convex optimization problem of finite dimension such as* (4.7)*.*

5. Comparative analysis and discussions.

This section provides an additional opportunity to clarify the contribution of this paper by comparing to some existing results in the literature.

5.1. On the LMI conditions under equality constraint. One of the classical methods in the literature provides a solution to the non-convexity problem of observer-based controllers on the expense of imposing an additional condition in the form of equality constraints. For example, in [21], in order to linearize their BMI, the authors have chosen to take some particular solutions: $R = I$ [21, eq.(8) Theorem 1] in the first place, and the additional strong equality constraint $RB_x = B_x\hat{R}$ [21, eq.(11) Theorem 2] in the second place, where $\hat{R}$ is an unknown arbitrary matrix. In addition, since in [21] the authors dealt with the continuous-time case, it is therefore essential, in the absence of any perturbations, that the following Lyapunov inequality should be verified

$$Y_c = (A_x - B_x L_1)^T R + R(A_x - B_x L_1) < 0, \quad [21, \text{Theorem } 2] \tag{5.1}$$

Now, let us consider the following example:

$$A_x = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}; \quad B_x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C_y = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{5.2}$$

If the equality constraint $RB_x = B_x\hat{R}$ [21, eq.(11) Theorem 2] holds, then we will get $B_x^{\perp} R B_x = 0$, where $B_x^{\perp} = \begin{bmatrix} 0 & 1 \end{bmatrix}$ is the orthogonal matrix of B_x. Let us now define $L_1 = \begin{bmatrix} l_1 & l_2 \end{bmatrix}$ and $R = \begin{bmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{bmatrix}$. Since $B_x^{\perp} R B_x = r_{12} = 0$, then the Lyapunov matrix R must be diagonal. By developing the equality (5.1), we obtain

$$Y_c = \begin{bmatrix} 2(1-l_1)r_{11} & r_{22} - (1+l_2)r_{11} \\ \star & 2r_{22} \end{bmatrix} < 0$$

and consequently, $2r_{22} < 0$, which contradicts the definition of $R > 0$. Therefore, the LMI [21, eq.(11) Theorem 2] is not solvable for this numerical example.

On the other hand, a simple reasoning shows that the discrete-time version of the inequality (5.1) is given by:

$$Y_d = (A_x - B_x L_1)^T R (A_x - B_x L_1) - R < 0, \tag{5.3}$$

By developing the equality (5.3), we get

$$\begin{bmatrix} (1-l_1)^2 r_{11} + r_{22} - r_{11} & -(1-l_1)(1+l_2)r_{11} + r_{22} \\ \star & (1+l_2)^2 r_{11} \end{bmatrix} < 0$$

and consequently, $(1+l_2)^2 r_{11} < 0$. In the same way as above, the condition $R > 0$ is not satisfied. Therefore, the discrete-time version (5.3) is also not solvable for this numerical example.

Now, it is possible to compare with the papers that address output feedback stabilization problems. For example, an elegant approach has been developed in [22] in which the authors use LMI technique to design: firstly, a robust observer for a class of uncertain Lipschitz nonlinear discrete-time systems. Secondly, a robust output feedback stabilization method with $\mathcal{H}_\infty$ performance for the same class of systems.

Besides, let us consider the Corollary 2 given in [22, eq.(60-63)]. In the first step, using the Schur lemma, and without any uncertainties, the inequality [22, eq.(62)] is equivalent to the following LMI:

$$\begin{bmatrix} \hat{H}^T\hat{H} - R + 2M_{13}R^{-1}M_{13}^T & I & 2M_{13} \\ \star & -\alpha I & 0 \\ \star & \star & 3R - 2\epsilon_1 I \end{bmatrix} < 0 \tag{5.4}$$

where $M_{13} = A_x^T R + C_y^T G^T B_x^T$, $\hat{H}$ is a known matrix, G is an arbitrary matrix, and $\alpha > 0$, $\epsilon_1 > 0$ are real scalars. The LMI (5.4) is verified if the following inequality is satisfied:

$$\hat{H}^T\hat{H} - R + 2M_{13}R^{-1}M_{13}^T = \hat{H}^T\hat{H} - R + 2(RA_x + B_xGC_y)^T R^{-1}(RA_x + B_xGC_y) < 0$$

Using the following change of variables $RB_xL_1 = B_x\hat{R}L_1 = B_xG$ [22, Corollary 2], it follows that

$$\hat{H}^T\hat{H} - R + 2(A_x + B_xL_1C_y)^T R(A_x + B_xL_1C_y) < 0. \tag{5.5}$$

It should be noted that (5.5) implies the inequality (5.3). So, in the same way, if the equality constraint $RB_x = B_x\hat{R}$ [22, eq.(61)] holds, then we will get $B_x^{\perp}RB_x = 0$ and $R = \begin{bmatrix} r_{11} & 0 \\ 0 & r_{22} \end{bmatrix}$. Let us now define $L_1 = l$. If we chose $\hat{H} = I$ and by developing the equality (5.5), we get

$$\begin{bmatrix} 1 + 2(1-l)^2 r_{11} - r_{11} + 2r_{22} & 2(l-1)r_{11} + 2r_{22} \\ \star & 1 + 2r_{11} + r_{22} \end{bmatrix} < 0$$

and consequently, $1 + 2r_{11} + r_{22} < 0$, which contradicts again the definition of $R > 0$. Therefore, the LMI [22, eq.(62)] is also not solvable for this numerical example. This example, despite the high quality and effectiveness of the works [21]-[22], shows the conservatism that may be associated with the introduction of the equality constraint in the feedback stabilization problem. In order to overcome this difficulty and avoid this critical equality constraint in the observer-based stabilization problem, we have performed a congruence transformation in addition to the Young's relation (see lemma 1) with an appropriate manner. This has allowed us to better reduce the conservatism as we will see later in two examples.

5.2. About the presence of unknown inputs. Now, as has previously been announced, our paper addresses principally the problem of observer-based control in the presence of unknown inputs. This is why we also suggest to make the comparison with [35]. Indeed, in [35] the authors proposed a simultaneous state and fault estimator to compensate and construct an FTC for a class of nonlinear systems containing additive faults. Those may represent actuator or process faults. In this paper, the control design is divided into two parts. The first one compensates the fault effects while the second part stabilizes the closed loop system through arbitrary nonlinear functions fixed by the user.

In order to compare with [35], the class of systems under consideration will be enlarged. We assume that the system (5.2) is affected by an unknown input signal f

as follows:

$$\begin{cases} x(k+1) = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{A_x} x(k) + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B_x} u(k) + \underbrace{\begin{bmatrix} 0 \\ 7 \end{bmatrix}}_{E_x} f(k) \\ y(k) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C_y} x(k). \end{cases} \tag{5.6}$$

It should be noted that the approach proposed in [35] is based on the matching conditions $\mathrm{Im}(E_x) \subseteq \mathrm{Im}(B_x)$ and $\mathrm{rank}(C_y E_x) = \mathrm{rank}(E_x)$ (see assumptions 1 and 3; and remarks 1, 2 and 6 in [35]). Therefore, if these conditions are not satisfied, the approach given in [35] can not be applied. In this sense, one can easily check that the example in (5.6) can not verify these conditions and then once again the approach proposed by [35] can not be applied to this numerical example. However, in comparison to [35], our method is less restrictive with respect to the dynamics (2.1). In particular:

- We introduce an artificial model $f(k+1) = f(k) + w_f(k)$ that allows to obtain only one LMI condition (after several linearization techniques) without any additional conditions such as $\mathrm{Im}(E_x) \subseteq \mathrm{Im}(B_x)$ and $\mathrm{rank}(C_y E_x) = \mathrm{rank}(E_x)$.
- No upper bounds on the fault variables nor on their time derivatives are needed for the observer and controller design.
- We consider, a general class of nonlinear systems where faults may intervene linearly as well as non-linearly into the model.

5.3. The contribution made by the nonlinearity. Recently, another interesting approach has been proposed in [23]. In this paper, the authors developed an observer-based $\mathcal{H}_\infty$ stabilization method for a class of Lipschitz nonlinear discrete-time systems. The advantage of this method consists of exploiting Young's inequality to provide only one set of strict LMI conditions without any supplementary equality constraint. Despite that this method provides a good degree of freedom, the class of systems treated by this method remains less general than the class treated in our paper. Indeed, our approach takes into account the presence of unknown inputs and in addition considers the concept of a one-sided Lipschitz condition which is an extension of its well-known Lipschitz counterpart. For example, the method presented in [23] can not treat effectively the nonlinear function $h = \begin{bmatrix} -x_1(x_1^2 + x_2^2) \\ -x_2(x_1^2 + x_2^2) \end{bmatrix}$ given in [13]-[17].

Apart from these novel aspects, another main feature of the present work is the use of a particular Lyapunov function in which the nonlinearity is taken into account through the matrices Q and S. Finally, a further interesting point is the integration of the nonlinearity and the unknown input estimation in the control law through the matrices L_3 and L_2 respectively. These initiatives allow to obtain less conservative sufficient conditions, than those established in [23], that ensure the stability of the considered systems.

6. NUMERICAL APPLICATION. In order to test the effectiveness of the robust observer-based control technique described in this paper, we revisit the example (5.6) with the nonlinearity $h = \left[-x_1(x_1^2 + x_2^2); -x_2(x_1^2 + x_2^2) \right]$. This example is reported in [17] and consists of the motion of a moving object model whose discrete-time equivalent (obtained using Euler first-order approximation with sampling time of $T_e = 0.005$ [sec]) is:

$$\begin{cases} x(k+1) = \delta_e A_x x(k) + T_e B_x u(k) + T_e E_x f(k) + \underbrace{T_e \begin{bmatrix} -x_1(x_1^2 + x_2^2) \\ -x_2(x_1^2 + x_2^2) \end{bmatrix}}_{h(x(t))} + D_x w(k) \\ \\ f(k+1) = f(k) + w_f(k) \\ y(k) = C_y x(k) + D_y w(k) \end{cases} \tag{6.1}$$

where $\delta_e = I_2 + T_e$. A_x, B_x, E_x and C_y are given by (5.6). Note that this system is globally one-sided Lipschitz with the one-sided Lipschitz constant $\rho = 0$. Also, the system is locally Lipschitz and on any set $D = \{x \in \mathbb{R}^2 : \ \|x\| \leq r\}$, the Lipschitz constant l is $3r^2$, i.e. the Lipschitz constant rapidly increases with the increase of r. Now, considering the set $\tilde{D} = \{x \in \mathbb{R}^2 : \ \|x\| \leq \tilde{r}\}$, consequently if

$$\tilde{r} = \min\left(\left(\frac{-\gamma}{4}\right)^{1/2}, \left(\beta + \frac{\gamma^2}{4}\right)^{1/4}\right), \qquad \gamma \leqslant \ 0, \ \ \beta \geqslant \frac{-\gamma^2}{4} \tag{6.2}$$

then the quadratically inner-bounded property of $h(x)$ is verified in $\tilde{D}$. As the system is globally one-sided Lipschitz, i.e., $D = \mathbb{R}^2$, $D \cap \tilde{D} = \tilde{D}$. Note that the region $\tilde{D}$ can be made arbitrarily large by choosing appropriate values for γ and β. And that to maximize $\tilde{r}$ in relation (6.2) for a given γ, the parameter β should satisfy $\left(\frac{-\gamma}{4}\right)^{1/2} \leqslant \left(\beta + \frac{\gamma^2}{4}\right)^{1/4}$, which is equivalent to $\beta \geqslant \frac{-3\gamma^2}{16}$. If we chose for example $\beta = -200$ and $\gamma = -141$, we get $r = 8$.
Without loss of generality, with respect to practical applications, a measurement noise is added in the simulations, the noise is a Gaussian distributed random signal with mean zero and standard deviation $\hat{\sigma} = 0.1$ and the corresponding matrices are $D_x = [\,0.3 \ \ 0.7\,]^T$ and $D_y = 0.1$.

We consider two cases. First the conditions $\mathrm{Im}(E_x) \subseteq \mathrm{Im}(B_x)$ and $\mathrm{rank}(C_y E_x) = \mathrm{rank}(E_x)$ hold as respectively in assumptions 1 and 3 (see remarks 1, 2 and 6) in [35] (see section 6.0.1). Second, these conditions are not satisfied (see section 6.0.2).

6.0.1. Case where $\mathrm{Im}(E_x) \subseteq \mathrm{Im}(B_x)$ **and** $\mathrm{rank}(C_y E_x) = \mathrm{rank}(E_x)$**.** We assume that the input $f(k)$ affects the dynamics of the system with the sinusoidal signal

$$f(k) = \begin{cases} 0.5 & k * T_e < 2 \text{ [sec]} \\ 0.3 + 0.3\cos(4.5 * T_e * k) & 2 \leqslant k * T_e \leqslant 9 \text{ [sec]} \\ 0 & k * T_e > 2 \text{ [sec]} \end{cases}$$

and that $E_x = [\,7 \ \ 0\,]^T$. Moreover, by using Matlab LMI toolbox, our LMI (4.7) is solvable by choosing $\mu_1 = 1$, $\mu_2 = 1$, $\theta_1 = 1$, $\theta_2 = 1$, $\epsilon = 1$, $\alpha = 1$, and we get the following controller and observer gains:

$$K_1 = \begin{bmatrix} 1.0619 \\ 1.8495 \end{bmatrix}, \ \ K_2 = 104, \ L_1 = \begin{bmatrix} 3.2752 & 10.8996 \end{bmatrix}, \ L_2 = 2.7663,$$

$L_3 = \begin{bmatrix} -0.1637 & -0.0795 \end{bmatrix}.$

The optimal value of the disturbance attenuation level obtained with theorem 2 is $\lambda = 1.2$. Simulations were processed including the input f with the sample time $T_e = 0.005$ [sec]. The simulation results are given in Figures 8.1.

6.0.2. Case where $\mathrm{Im}(E_x) \nsubseteq \mathrm{Im}(B_x)$ and $\mathrm{rank}(C_y E_x) < \mathrm{rank}(E_x)$. Now we modify the example such that the assumption 3 in [35] does not hold by choosing the matrix $E_x = [\,0 \;\; 7\,]^T$. We assume that the input $f(k)$ affects the dynamics with the sinusoidal signal.

$$f(k) = \begin{cases} 0.5 & k * T_e < 2 \text{ [sec]} \\ 0.3 + 0.7\cos(3 * T_e * k) & 2 \leqslant k * T_e \leqslant 9 \text{ [sec]} \\ 0 & k * T_e > 2 \text{ [sec]} \end{cases}$$

The LMI-based observer and controller gain matrices are

$$K_1 = \begin{bmatrix} 1.1080 \\ 0.8292 \end{bmatrix}, \quad K_2 = -0.1840, \; L_1 = \begin{bmatrix} 122.8090 & -114.7042 \end{bmatrix}, \; L_2 = -2.209 \times 10^{-7},$$

$$L_3 = \begin{bmatrix} 0.01092 & -0.02124 \end{bmatrix}.$$

By choosing $\mu_1 = 1$, $\mu_2 = 1$, $\theta_1 = 1$, $\theta_2 = 1$, $\epsilon = 35$, $\alpha = 0.1$, the optimal value of the disturbance attenuation level obtained with theorem 2 is $\lambda = 5.7922$. Now we simulate the system including the same input f with the same sample time T_e, qualitatively we obtain a response similar to the one shown in Figure 8.1.

7. Conclusion.

In this paper, a simple and useful state observer-based controller for a class of nonlinear systems subjected to unknown inputs has been established. Owing to an artificial unknown input model, a simultaneous $\mathcal{H}_\infty$ state estimator has been developed and incorporated into a relevant nonlinear controller. After several linearization techniques, sufficient conditions for asymptotic convergence are expressed in terms of LMIs easily tractable by convex optimization techniques. High performances and efficiency of the proposed approach are shown through an example under severe conditions.

8. Appendix.

In this Appendix. Firstly, we present the proof of the theorem 2. Secondly, we show that the inequality (4.6) implies the two conditions (4.4) and (4.5).

8.1. Proof of Theorem 2. Let us consider the quadratic Lyapunov function

$$V_k = \begin{bmatrix} e(k) \\ \Delta h_k \\ x(k) \end{bmatrix}^T \begin{bmatrix} P & S & 0 \\ S^T & Q & 0 \\ 0 & 0 & R \end{bmatrix} \begin{bmatrix} e(k) \\ \Delta h_k \\ x(k) \end{bmatrix} \tag{8.1}$$

where $\begin{bmatrix} P & S \\ S^T & Q \end{bmatrix} > 0$ and $R > 0$. The variation $\Delta V = V_{k+1} - V_k$ of this Lyapunov function is given by

$$\Delta V = e^T(k+1)Pe(k+1) - e^T(k)Pe(k) + x^T(k+1)Rx(k+1) - x^T(k)Rx(k)$$

$$+\Delta h_{k+1}^T Q \Delta h_{k+1} - \Delta h_k^T Q \Delta h_k + 2e^T(k+1) S \Delta h_{k+1} - 2e^T(k) S \Delta h_k \quad (8.2)$$

Following similar steps (3.16)-(3.20) as in the previous proof and by replacing Δh_k and Δh_{k+1} by $F_x \Delta g_k$ and $F_x \Delta g_{k+1}$, respectively, we obtain

$$\begin{aligned}\Delta V \leqslant{}& \eta e^T(k+1) P e(k+1) + e^T(k)\left(-P + \delta_1 I\right) e(k) + \Delta g_{k+1}^T \mathbb{M}_{44} \Delta g_{k+1} + \Delta g_k^T \mathbb{M}_{33} \Delta g_k \\ &+ 2e^T(k) \mathbb{S} \Delta g_k + 2e^T(k+1) \mathbb{M}_{23} \Delta g_{k+1} + x^T(k+1) R x(k+1) - x^T(k) R x(k) \quad (8.3)\end{aligned}$$

where $\mathbb{S}$, $\mathbb{M}_{23}$, $\mathbb{M}_{33}$ and $\mathbb{M}_{44}$ are given by (4.10).

Using (4.3), it follows that

$$x^T(k+1) R x(k+1) = \xi^T(k) \mathcal{H}^T R \mathcal{H} \xi(k) \quad (8.4)$$

where

$$\xi^T(k) = \begin{bmatrix} e^T(k) & x^T(k) & \Delta g_K^T & \Delta g_{K+1}^T & g_K^T & f^T(k) & w_f^T(k) & w^T(k) \end{bmatrix} \quad (8.5)$$

and

$$\mathcal{H} = \begin{bmatrix} B_x L & \mathcal{H}_1^T R & B_x L_3 & 0 & \mathcal{H}_3^T & \mathcal{H}_2^T & 0 & D_x \end{bmatrix} \quad (8.6)$$

with $\mathcal{H}_1 = R^{-1}(A_x - B_x L_1)^T$, $\mathcal{H}_2 = (E_x - B_x L_2)^T$ and $\mathcal{H}_3 = (F_x - B_x L_3)^T$.

On the other hand, from the one-sided Lipschitz and the quadratically inner-bounded conditions (4.1), we deduce the following inequality

$$\begin{cases} \theta_1 \rho \left(\|x\|^2 + \|f\|^2\right) - \theta_1 \langle h(x,f), x\rangle \geqslant 0 \\ \theta_2 \beta \left(\|x\|^2 + \|f\|^2\right) + \theta_2 \gamma \langle x, h(x,f)\rangle - \theta_2 \|h(x,f)\|^2 \geqslant 0. \end{cases} \quad (8.7)$$

Based on the Lyapunov stability theory, the convergence of the estimation error is guaranteed, as long as $\mathcal{V}$ is negative definite, which is equivalent to

$$\xi^T \overline{\mathbb{M}} \xi < 0 \quad (8.8)$$

where $\overline{\mathbb{M}}$ is obtained by adding (8.3), (8.4) and (8.7) and given by

$$\overline{\mathbb{M}} = \overline{\mathbb{M}}_1 + \mathcal{H}^T R \mathcal{H} \quad (8.9)$$

where

$$\overline{\mathbb{M}}_1 = \begin{bmatrix} (1.1) & 0 & \eta\mathcal{M}_{12}F + \mathbb{S} & \mathcal{M}_{14} & 0 & 0 & \eta\mathcal{M}_{12}W_1 & \eta\mathcal{M}_{12}P^{-1}\mathcal{M}_{15}^T \\ \star & (1+\delta_2)I_n - R & 0 & 0 & \delta_3 F_x & 0 & 0 & 0 \\ \star & \star & (3.3) & F^T\mathbb{M}_{23} & 0 & 0 & \eta F^T P W_1 & \eta F^T \mathcal{M}_{15}^T \\ \star & \star & \star & \mathbb{M}_{44} & 0 & 0 & \mathbb{M}_{23}^T W_1 & \mathcal{M}_{48} \\ \star & \star & \star & \star & \mathbb{M}_{55} & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \mathbb{M}_{66} & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \mathbb{M}_{77} & \eta W_1^T \mathcal{M}_{15}^T \\ \star & \star & \star & \star & \star & \star & \star & (8.8) \end{bmatrix} \quad (8.10)$$

with

$$\begin{cases} (1.1) = \mathbb{M}_{11} + \eta\mathcal{M}_{12}P^{-1}\mathcal{M}_{12}^T, \ (3.3) = \mathbb{M}_{33} + \eta F^T P F, \ \mathcal{M}_{12} = (A - KC)^T P, \\ (8.8) = \eta\mathcal{M}_{15}P^{-1}\mathcal{M}_{15}^T - \lambda I_t, \ \mathcal{M}_{15} = (W_2 - KD_y)^T P, \ \mathcal{M}_{14} = \mathcal{M}_{12}P^{-1}\mathbb{M}_{23}, \\ \mathcal{M}_{48} = \mathbb{M}_{23}^T P^{-1}\mathcal{M}_{15}^T, \ \mathbb{M}_{77} = \eta W_1^T P W_1 - \lambda I_r. \end{cases}$$

The BMI problem of (8.8)-(8.10) is not convex and the existing LMI computational procedures can not be applied. To linearize the BMI (8.8)-(8.10), we proceed in three steps.

In the first step, using the Schur lemma [38], the inequality (8.8) is equivalent to the following LMI

$$\widetilde{\mathbb{M}}=\begin{bmatrix} (1.1) & 0 & \eta\mathcal{M}_{12}F+\mathbb{S} & \mathcal{M}_{14} & 0 & 0 & \eta\mathcal{M}_{12}W_1 & \eta\mathcal{M}_{12}P^{-1}\mathcal{M}_{15}^T & L^TB_x^T \\ \star & (1+\delta_2)I_n-R & 0 & 0 & \delta_3F_x & 0 & 0 & 0 & R\mathcal{H}_1 \\ \star & \star & (3.3) & F^T\mathbb{M}_{23} & 0 & 0 & \eta F^TPW_1 & \eta F^T\mathcal{M}_{15}^T & L_3^TB_x^T \\ \star & \star & \star & \mathbb{M}_{44} & 0 & 0 & \mathbb{M}_{23}^TW_1 & \mathcal{M}_{48} & 0 \\ \star & \star & \star & \star & \mathbb{M}_{55} & 0 & 0 & 0 & \mathcal{H}_3 \\ \star & \star & \star & \star & \star & \mathbb{M}_{66} & 0 & 0 & \mathcal{H}_2 \\ \star & \star & \star & \star & \star & \star & \mathbb{M}_{77} & \eta W_1^T\mathcal{M}_{15}^T & 0 \\ \star & \star & \star & \star & \star & \star & \star & (8.8) & D_x^T \\ \star & \star & \star & \star & \star & \star & \star & \star & -R^{-1} \end{bmatrix}<0. \tag{8.11}$$

In the second step, we multiply $\widetilde{\mathbb{M}}$ from both sides by Ξ given by

$$\Xi = \text{bdiag}(I_{n+r}, R^{-1}, I_q, I_q, I_q, I_r, I_r, I_t, I_n) \tag{8.12}$$

where bdiag(.) stands for a block-diagonal matrix, and we obtain

$$\mathcal{M}=\begin{bmatrix} (1.1) & 0 & \eta\mathcal{M}_{12}F+\mathbb{S} & \mathcal{M}_{14} & 0 & 0 & \eta\mathcal{M}_{12}W_1 & \eta\mathcal{M}_{12}P^{-1}\mathcal{M}_{15}^T & L^TB_x^T \\ \star & (1+\delta_2)R^{-2}-R^{-1} & 0 & 0 & \delta_3R^{-1}F_x & 0 & 0 & 0 & \mathcal{H}_1 \\ \star & \star & (3.3) & F^T\mathbb{M}_{23} & 0 & 0 & \eta F^TPW_1 & \eta F^T\mathcal{M}_{15}^T & L_3^TB_x^T \\ \star & \star & \star & \mathbb{M}_{44} & 0 & 0 & \mathbb{M}_{23}^TW_1 & \mathcal{M}_{48} & 0 \\ \star & \star & \star & \star & \mathbb{M}_{55} & 0 & 0 & 0 & \mathcal{H}_3 \\ \star & \star & \star & \star & \star & \mathbb{M}_{66} & 0 & 0 & \mathcal{H}_2 \\ \star & \star & \star & \star & \star & \star & \mathbb{M}_{77} & \eta W_1^T\mathcal{M}_{15}^T & 0 \\ \star & \star & \star & \star & \star & \star & \star & (8.8) & D_x^T \\ \star & \star & \star & \star & \star & \star & \star & \star & -R^{-1} \end{bmatrix} \tag{8.13}$$

The third step concerns matrices $\mathcal{M}_{14}$, $\mathcal{M}_{48}$ and $L^TB_x^T$. Since we have P, K, S, L_1 and L_2 acting separately in (8.13), and $\mathcal{M}_{14}$, $\mathcal{M}_{48}$ and $L^TB_x^T$ are a function of P, K, S, L_1 and L_2, it is necessary to rewrite $\mathcal{M}_{14}$, $\mathcal{M}_{48}$ and $L^TB_x^T$ under a form so that P, K, S, L_1 and L_2 appear separately in the equation as follows. The matrix $L^TB_x^T$ can be rewritten as

$$\begin{aligned} L^TB_x^T &= \mathcal{E}_L^TRR^{-1}L_1^TB_x^T + \mathcal{E}_r^TL_2^TB_x^T \\ &= \mathcal{E}_L^TRR^{-1}L_1^TB_x^T + \mathbb{M}_{19} \end{aligned}$$

where $\mathbb{M}_{19}$ is given by (4.9) and $\mathcal{E}_L = [\, I_n \;\; 0_{\mathbb{R}^{n\times r}} \,]$, $\mathcal{E}_r = [\, 0_{\mathbb{R}^{r\times n}} \;\; I_r \,]$.

Then the matrix $\mathcal{M}$ is rewritten as follows

$$\mathcal{M} = \mathcal{M}_1 + \Psi\Phi^T + \Phi\Psi^T \tag{8.14}$$

where $\Phi = \tilde{\Phi}\Omega$, and

$$\mathcal{M}_1 = \begin{bmatrix} (1.1) & 0 & \eta\mathcal{M}_{12}F + \mathbb{S} & \mathbf{0} & 0 & 0 & \eta\mathcal{M}_{12}W_1 & \eta\mathcal{M}_{12}P^{-1}\mathcal{M}_{15}^T & \mathbb{M}_{19} \\ \star & (1+\delta_2)R^{-2} - R^{-1} & 0 & 0 & \delta_3 R^{-1}F_x & 0 & 0 & 0 & \mathcal{H}_1 \\ \star & \star & (3.3) & F^T\mathbb{M}_{23} & 0 & 0 & \eta F^T P W_1 & \eta F^T\mathcal{M}_{15}^T & L_3^T B_x^T \\ \star & \star & \star & \mathbb{M}_{44} & 0 & 0 & \mathbb{M}_{23}^T W_1 & \mathbf{0} & 0 \\ \star & \star & \star & \star & \mathbb{M}_{55} & 0 & 0 & 0 & \mathcal{H}_3 \\ \star & \star & \star & \star & \star & \mathbb{M}_{66} & 0 & 0 & \mathcal{H}_2 \\ \star & \star & \star & \star & \star & \star & \mathbb{M}_{77} & \eta W_1^T\mathcal{M}_{15}^T & 0 \\ \star & \star & \star & \star & \star & \star & \star & (8.8) & D_x^T \\ \star & \star & \star & \star & \star & \star & \star & \star & -R^{-1} \end{bmatrix} \tag{8.15}$$

and

$$\begin{cases} \Psi^T = \begin{bmatrix} \mathcal{M}_{12}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathcal{E}_L & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{M}_{23} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ \tilde{\Phi}^T = \begin{bmatrix} 0 & 0 & 0 & \mathbb{M}_{23} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & R^{-1}L_1^T B_x^T \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{M}_{15}^T & 0 \end{bmatrix} \\ \Omega = \mathrm{bdiag}(P^{-1}, R, P^{-1}). \end{cases}$$

In the third step, we use lemma 1 with $\Sigma = \Omega^{-1}$ to obtain an upper bound of the term $\Psi\Phi^T + \Phi\Psi^T$ in (8.14) and the following inequality holds for any positive scalar ϵ

$$\Psi\Phi^T + \Phi\Psi^T \leqslant \epsilon \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathbb{M}_{23}^T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathcal{M}_{15} \\ 0 & B_x L_1 R^{-1} & 0 \end{bmatrix} \Omega \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mathbb{M}_{23}^T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathcal{M}_{15} \\ 0 & B_x L_1 R^{-1} & 0 \end{bmatrix}^T + \frac{1}{\epsilon} \begin{bmatrix} \mathcal{M}_{12} & \mathcal{E}_L^T & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbb{M}_{23}^T \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Omega \begin{bmatrix} \mathcal{M}_{12} & \mathcal{E}_L^T & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbb{M}_{23}^T \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^T$$

This implies that if the following linear matrix inequality

$$\mathbf{M} = \begin{bmatrix} \mathbb{M}_{11} + \Upsilon_1^T\Pi\Upsilon_1 & 0 & \Upsilon_1^T\Pi\Upsilon_3 + \mathbb{S} & 0 & 0 & 0 & \Upsilon_1^T\Pi\Upsilon_7 & \Upsilon_1^T\Pi\Upsilon_8 & \mathbb{M}_{19} \\ \star & \Upsilon_2^T\Pi\Upsilon_2 - \Gamma & 0 & 0 & \mathbb{M}_{25} & 0 & 0 & 0 & \mathcal{H}_1 \\ \star & \star & \mathbb{M}_{33} + \Upsilon_3^T\Pi\Upsilon_3 & F^T\mathbb{M}_{23} & 0 & 0 & \Upsilon_3^T\Pi\Upsilon_7 & \Upsilon_3^T\Pi\Upsilon_8 & L_3^T B_x^T \\ \star & \star & \star & \mathbb{M}_{44} + \Upsilon_4^T\Pi\Upsilon_4 & 0 & 0 & \mathbb{M}_{23}^T W_1 & 0 & 0 \\ \star & \star & \star & \star & \mathbb{M}_{55} & 0 & 0 & 0 & \mathcal{H}_3 \\ \star & \star & \star & \star & \star & \mathbb{M}_{66} & 0 & 0 & \mathcal{H}_2 \\ \star & \star & \star & \star & \star & \star & \Upsilon_7^T\Pi\Upsilon_7 - \lambda I_r & \Upsilon_7^T\Pi\Upsilon_8 & 0 \\ \star & \star & \star & \star & \star & \star & \star & \Upsilon_8^T\Pi\Upsilon_8 - \lambda I_t & D_x^T \\ \star & \star & \star & \star & \star & \star & \star & \star & \Upsilon_9^T\Pi\Upsilon_9 - \Gamma \end{bmatrix} < 0 \tag{8.16}$$

is verified then $\mathcal{V} < 0$, where $\Gamma = R^{-1}$, $\mathcal{Y} = L_1\Gamma$ and

$$\begin{cases} \Upsilon_1^T = \begin{bmatrix}\mathcal{M}_{12} & \mathcal{M}_{12} & \mathcal{E}_L^T & 0 & 0 & 0 & 0\end{bmatrix}, \\ \Upsilon_2^T = \begin{bmatrix}0 & 0 & 0 & \Gamma & 0 & 0 & 0\end{bmatrix}, \\ \Upsilon_3^T = \begin{bmatrix}F^TP & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}, \\ \Upsilon_4^T = \begin{bmatrix}0 & 0 & 0 & 0 & \mathbb{M}_{23}^T & 0 & 0\end{bmatrix}, \\ \Upsilon_7^T = \begin{bmatrix}W_1^TP & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}, \\ \Upsilon_8^T = \begin{bmatrix}\mathcal{M}_{15} & 0 & 0 & 0 & 0 & \mathcal{M}_{15} & 0\end{bmatrix}, \\ \Upsilon_9^T = \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0 & B_x\mathcal{Y}\end{bmatrix}, \\ \Pi = \mathrm{bdiag}(\eta P^{-1}, \epsilon^{-1}P^{-1}, \epsilon^{-1}R, \tau_3^{-1}I_n, \tau_4^{-1}P^{-1}, \epsilon P^{-1}, \epsilon R). \end{cases}$$

We also let,

$$\mathbf{M} = \mathbf{M}_1 + \Upsilon\Pi\Upsilon^T \tag{8.17}$$

where

$$\Upsilon^T = \begin{bmatrix}\Upsilon_1 & \Upsilon_2 & \Upsilon_3 & \Upsilon_4 & 0 & 0 & \Upsilon_7 & \Upsilon_8 & \Upsilon_9\end{bmatrix}$$

and

$$\mathbf{M}_1 = \begin{bmatrix} \mathbb{M}_{11} & 0 & \mathbb{S} & 0 & 0 & 0 & 0 & 0 & \mathbb{M}_{19} \\ \star & -\Gamma & 0 & 0 & \mathbb{M}_{25} & 0 & 0 & 0 & \mathcal{H}_1 \\ \star & \star & \mathbb{M}_{33} & F^T\mathbb{M}_{23} & 0 & 0 & 0 & 0 & L_3^TB_x^T \\ \star & \star & \star & \mathbb{M}_{44} & 0 & 0 & \mathbb{M}_{23}^TW_1 & 0 & 0 \\ \star & \star & \star & \star & \mathbb{M}_{55} & 0 & 0 & 0 & \mathcal{H}_3 \\ \star & \star & \star & \star & \star & \mathbb{M}_{66} & 0 & 0 & \mathcal{H}_2 \\ \star & \star & \star & \star & \star & \star & -\lambda I_r & 0 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\lambda I_t & D_x^T \\ \star & \star & \star & \star & \star & \star & \star & \star & -\Gamma \end{bmatrix} \tag{8.18}$$

Now using the Schur lemma and the notations $\mathcal{X} = K^TP$, $\mathcal{Y} = L_1\Gamma$, the inequality (4.7) and (8.17) are equivalent, which completes the proof.

8.2. The inequality (4.6) **implies the conditions** (4.4)**-**(4.5)**.** By using a quite similar argument as the one given in [23], it is easy to show that the inequality (4.6) implies the two conditions (4.4) and (4.5). Indeed, If $\begin{bmatrix} f^T & w_f^T & w^T \end{bmatrix}^T \neq 0$, then by taking the sum of the inequality (4.6) from $k = 0$ to $k \to N$, we get

$$\sum_{k=0}^{k=N} \left\| \begin{bmatrix} e(k) \\ x(k) \end{bmatrix} \right\|^2 - \lambda \sum_{k=0}^{k=N} \left(\|f\|^2 + \|w_f\|^2 + \|w\|^2 \right) + \sum_{k=0}^{k=N} (V_{K+1} - V_K) < 0. \tag{8.19}$$

which is equivalent to

$$\sum_{k=0}^{k=N} \left\| \begin{bmatrix} e(k) \\ x(k) \end{bmatrix} \right\|^2 < \lambda \sum_{k=0}^{k=N} \left(\|f(k)\|^2 + \|w_f(k)\|^2 + \|w(k)\|^2 \right) - V_{N+1} + V_0$$

When $N \to \infty$, we have: $\lim\limits_{N\to\infty} \sum\limits_{k=0}^{k=N} \left\| \begin{bmatrix} e(k) \\ x(k) \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} e(k) \\ x(k) \end{bmatrix} \right\|_{\ell_2}^2$, $\lim\limits_{N\to\infty} \sum\limits_{k=0}^{k=N} \|f(k)\|^2 = \|f(k)\|_{\ell_2}^2$, $\lim\limits_{N\to\infty} \sum\limits_{k=0}^{k=N} \|w_f(k)\|^2 = \|w_f(k)\|_{\ell_2}^2$, and $\lim\limits_{N\to\infty} \sum\limits_{k=0}^{k=N} \|w(k)\|^2 = \|w(k)\|_{\ell_2}^2$. Since

$V_{N+1} \geqslant 0$, then $\lim_{N\to\infty} V_{N+1} \geqslant 0$. Therefore, knowing that $V_0 = V([e_0; \Delta h_0; x_0])$, we obtain

$$\left\| \begin{bmatrix} e(k) \\ x(k) \end{bmatrix} \right\|_{\ell_2}^2 < \lambda \left(\|f(k)\|_{\ell_2}^2 + \|w_f(k)\|_{\ell_2}^2 + \|w(k)\|_{\ell_2}^2 \right) + V \left(\begin{bmatrix} e_0 \\ \Delta h_0 \\ x_0 \end{bmatrix} \right) \tag{8.20}$$

Besides, using the Lyapunov function (8.1), we get

$$V \left(\begin{bmatrix} e_0 \\ \Delta h_0 \\ x_0 \end{bmatrix} \right) = \begin{bmatrix} e(0) \\ \Delta h_0 \\ x(0) \end{bmatrix}^T \Xi \begin{bmatrix} e(0) \\ \Delta h_0 \\ x(0) \end{bmatrix} \leqslant \lambda_{max}(\Xi) \left\| \begin{bmatrix} e(0) \\ \Delta h_0 \\ x(0) \end{bmatrix} \right\|^2 \tag{8.21}$$

where

$$\Xi = \begin{bmatrix} P & S & 0 \\ S^T & Q & 0 \\ 0 & 0 & R \end{bmatrix}$$

On the other hand, using the one-sided Lipschitz and the inner-bounded conditions (2.4) and (2.5), it follows that

$$\|\Delta h_0\|^2 \leqslant (\gamma\rho + \beta) \|e(0)\|^2 \tag{8.22}$$

From (8.21) and (8.22), we obtain the following inequality

$$V \left(\begin{bmatrix} e_0 \\ \Delta h_0 \\ x_0 \end{bmatrix} \right) \leqslant \kappa_0 \left\| \begin{bmatrix} e(0) \\ x(0) \end{bmatrix} \right\|^2 \tag{8.23}$$

where $\kappa_0 = \max\left((1 + \gamma\rho + \beta)\lambda_{max}(\Xi), \lambda_{max}(\Xi)\right)$.

Finally, by substituting the left hand-side of the inequality (8.23) in (8.20) we get

$$\left\| \begin{bmatrix} e(k) \\ x(k) \end{bmatrix} \right\|_{\ell_2}^2 < \lambda \left\| \begin{bmatrix} f(k) \\ w_f(k) \\ w(k) \end{bmatrix} \right\|_{\ell_2}^2 + \kappa_0 \left\| \begin{bmatrix} e(0) \\ x(0) \end{bmatrix} \right\|^2$$

that leads to (4.5).

Whereas, if $\begin{bmatrix} f^T & w_f^T & w^T \end{bmatrix}^T = 0$, then the inequality (4.6) implies that $\Delta V + \xi^T \xi < 0$. Thus, from the Lyapunov theory, we deduce that the augmented vector $\xi(k) = \begin{bmatrix} e(k) \\ x(k) \end{bmatrix}$ converges exponentially toward zero, and then the existence of $\alpha, \beta > 0$ so that

$$\|\xi(k)\| \leqslant \beta \|\xi(0)\| e^{-\alpha k}.$$

which means to

$$0 \leq \sum_{k=0}^{k=\infty} \|\xi(k)\|^2 \leq \frac{\beta^2}{1 - e^{-\alpha}} \|\xi(0)\|^2$$

we deduce that

$$\lim_{k \to +\infty} (\|\xi(k)\|) = 0.$$

In this appendix, we only demonstrate the equivalence between (4.6) and (4.4)-(4.5), the equivalence between (3.9) and (3.7)-(3.8) can be obtained in the same manner and by assuming that $x(k) = 0$, $f(k) = 0$ in (8.19). This ends the proof.

REFERENCES

[1] M. Blanke, M. Kinnaert, J. Lunze, and M. Staroswiecki, *Diagnosis and fault-tolerant control*. Berlin: Springer, 2003.

[2] Z. Gao and S. X. Ding, "Actuator fault robust estimation and fault-tolerant control for a class of nonlinear descriptor systems," *Automatica*, vol. 43, pp. 912–920, 2007.

[3] M. Arcak and P. Kokotović, "Nonlinear observers: A circle criterion design and robustness analysis," *Automatica*, vol. 37, pp. 1923–1930, 2001.

[4] X. Fan and M. Arcak, "Observer design for systems with multivariable monotone nonlinearities," *Syst. & Contr. Letters*, vol. 50, pp. 319–330, 2003.

[5] H. Trinh, T. Tran, and T. Fernando, "Disturbance decoupled observers for systems with unknown inputs," *IEEE Trans. Aut. Control*, vol. 53, pp. 2397–2402, 2008.

[6] S. Ibrir, W. F. Xie, and C. Y. Su, "Observer-based control of discrete-time lipschitzian nonlinear systems: application to one-link flexible joint robot," *Int. J. Contr.*, vol. 78, pp. 385–395, 2005.

[7] S. Ibrir, "Circle-criterion approach to discrete-time nonlinear observer design," *Automatica*, vol. 43, pp. 1432–1441, 2007.

[8] M. Abbaszadeh and H. Marquez, "Robust $\mathcal{H}_\infty$ observer design for sampled-data Lipschitz nonlinear systems with exact and Euler approximate models," *Automatica*, vol. 44, pp. 799–806, 2008.

[9] F. E. Thau, "Observing the state of non-linear dynamic systems," *Int. J. Contr.*, vol. 17, pp. 471–479, 1973.

[10] G. Phanomchoeng and R. Rajamani, "The bounded Jacobian approach to nonlinear observer design," in *Proc. IEEE American Control Conf.*, Baltimore, USA, 2010, pp. 6083–6088.

[11] A. Zemouche, M. Boutayeb, and G. I. Bara, "Observers for a class of Lipschitz systems with extension to $\mathcal{H}_\infty$ performance analysis," *Syst. & Contr. Letters*, vol. 57, pp. 18–27, 2008.

[12] G. D. Hu, "Observers for one-sided Lipschitz non-linear systems," *IMA J. Mathematical Control & Information*, vol. 23, pp. 395–401, 2006.

[13] M. Benallouch, M. Boutayeb, and M. Zasadzinski, "Observers design for one-sided Lipschitz discrete-time systems," *Syst. & Contr. Letters*, vol. 61, pp. 879–886, 2012.

[14] G. D. Hu, "A note on observer for one-sided Lipschitz non-linear systems," *IMA J. Mathematical Control & Information*, vol. 25, pp. 297–303, 2008.

[15] M. Xu, G. D. Hu, and Y. Zhao, "Reduced-order observer design for one-sided Lipschitz nonlinear systems," *IMA J. Mathematical Control & Information*, vol. 26, pp. 299–317, 2009.

[16] Y. Zhao, J. Tao, and N. Z. Shi, "A note on observer design for one-sided Lipschitz nonlinear systems," *Syst. & Contr. Letters*, vol. 59, pp. 66–71, 2010.

[17] M. Abbaszadeh and H. J. Marquez, "Nonlinear observer design for one-sided Lipschitz systems," in *Proc. IEEE American Control Conf.*, Baltimore, USA, 2010, pp. 5284–5289.

[18] W. Zhang, H. Su, F. Zhu, and D. Yue, "A note on observers for discrete-time lipschitz nonlinear systems," *IEEE Trans. Circ. Syst. II: Express Briefs*, vol. 59, pp. 123–127, 2012.

[19] D. H. S. Maithripala, W. P. Dayawansa, and J. M. Berg, "Intrinsic observer-based stabilization for simple mechanical systems on lie groups," *SIAM J. Contr. Opt.*, vol. 44, pp. 1691–1711, 2005.

[20] B. Song and J. Hedrick, "Observer-based dynamic surface control for a class of nonlinear systems: An LMI approach," *IEEE Trans. Aut. Control*, vol. 49, pp. 1995–2001, 2004.

[21] C. Lien, "Robust observer-based control of systems with state perturbations via LMI approach," *IEEE Trans. Aut. Control*, vol. 49, pp. 1365–1370, 2004.

[22] M. Abbaszadeh and H. J. Marquez, "LMI optimization approach to robust $\mathcal{H}_\infty$ observer design and static output feedback stabilization for discrete-time nonlinear uncertain systems," *Int. J. Robust & Nonlinear Contr.*, vol. 19, pp. 313–340, 2009.

[23] B. Grandvallet, A. Zemouche, H. Souley-Ali, and M. Boutayeb, "New LMI condition for observer-based $\mathcal{H}_\infty$ stabilization of a class of nonlinear discrete-time systems," *SIAM J. Contr. Opt.*, vol. 51, pp. 784–800, 2013.

[24] M. Wu, Y. He, and J. H. She, *Stability analysis and robust control of time-delay systems.* Berlin Heidelberg: Springer, 2010.

[25] S. Ibrir and S. Diop, "Novel LMI conditions for observer-based stabilization of Lipschitzian nonlinear systems and uncertain linear systems in discrete-time," *Applied Mathematics and Computation*, vol. 206, pp. 579–588, 2008.

[26] W. P. M. H. Heemels, J. Daafouz, and G. Millerioux, "Observer-based control of discrete-time LPV systems with uncertain parameters," *IEEE Trans. Aut. Control*, vol. 55, pp. 2130–2135, 2010.

[27] S. Wang, E. J. Davison, and P. Dorato, "Observing the states of systems with unmeasurable disturbances," *IEEE Trans. Aut. Control*, vol. 20, pp. 716–717, 1975.

[28] S. P. Bhattacharyya, "Observer design for linear systems with unknown inputs," *IEEE Trans. Aut. Control*, vol. 23, pp. 483–484, 1978.

[29] N. Kobayashi and T. Nakamizo, "An observer design for linear systems with unkown inputs," *Int. J. Contr.*, vol. 35, pp. 605–619, 1982.

[30] F. W. Fairman, S. Mahil, and L. Luk, "Disturbance decoupled observer design via singular value decomposition," *IEEE Trans. Aut. Control*, vol. 29, pp. 84–86, 1984.

[31] M. Hou and P. C. Müller, "Design of observers for linear systems with unknown inputs," *IEEE Trans. Aut. Control*, vol. 37, pp. 871–875, 1992.

[32] M. Darouach, "On the novel approach to the design of unknown input observers," *IEEE Trans. Aut. Control*, vol. 39, pp. 698–699, 1994.

[33] P. Kabore and H. Wang, "Design of fault diagnosis filters and fault-tolerant control for a class of nonlinear systems," *IEEE Trans. Aut. Control*, vol. 46, pp. 1805–1810, 2001.

[34] M. Benosman and K. Lum, "Passive actuators fault-tolerant control for affine nonlinear systems," *IEEE Trans. Control Syst. Techn.*, vol. 18, pp. 152–163, 2010.

[35] B. Jiang, M. Staroswiecki, and V. Cocquempot, "Fault accommodation for nonlinear dynamic systems," *IEEE Trans. Aut. Control*, vol. 51, pp. 1578–1583, 2006.

[36] L. Xie and C. de Souza, "Robust $\mathcal{H}_\infty$ control for linear systems with norm-bounded time-varying uncertainties," *IEEE Trans. Aut. Control*, vol. 37, pp. 1188–1191, 1992.

[37] A. van der Schaft, "$\mathcal{L}_2$-gain analysis of nonlinear systems and nonlinear state-feedback $\mathcal{H}_\infty$ control," *IEEE Trans. Aut. Control*, vol. 37, pp. 770–784, 1992.

[38] S. Boyd, L. El Ghaoui, E. Féron, and V. Balakrishnan, *Linear Matrix Inequalities in Systems and Control Theory.* Philadelphia: SIAM, 1994.

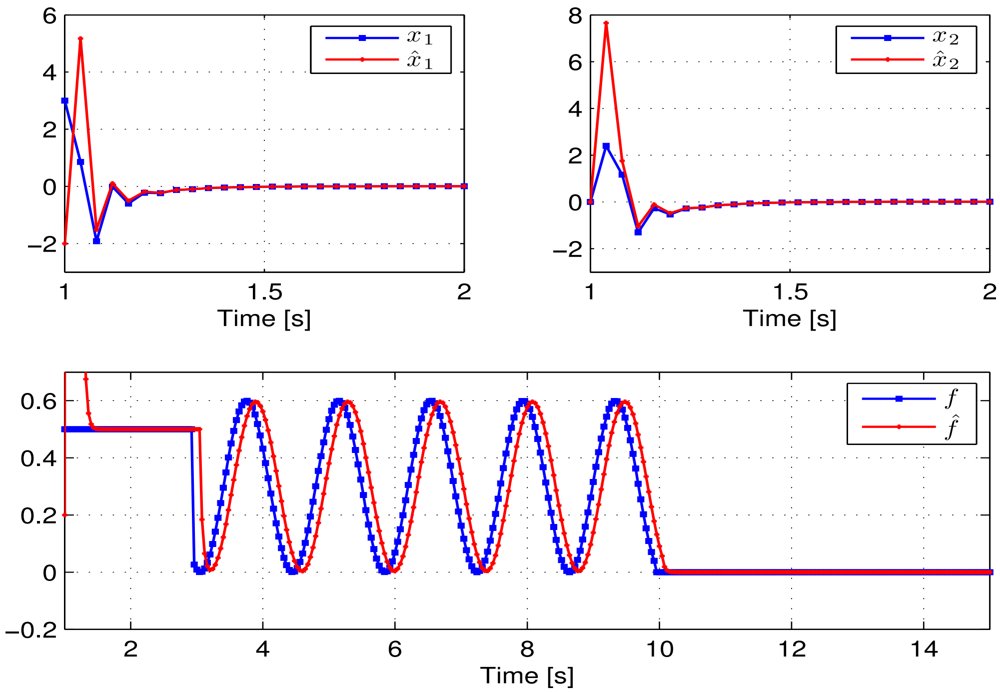

FIG. 8.1. *State vector and its estimate in section 6.0.1.*