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On the Existence and Exponential Attractivity of a Unique Positive Almost Periodic Solution to an Impulsive Hematopoiesis Model with Delays

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Abstract In this paper, a generalized model of hematopoiesis with delays and impulses is considered. By employing the contraction mapping principle and a novel type of impulsive delay inequality, we prove the existence of a unique positive almost periodic solution of the model. It is also proved that, under the proposed conditions in this paper, the unique positive almost periodic solution is globally exponentially attractive. A numerical example is given to illustrate the effectiveness of the obtained results.

Keywords Hematopoiesis model · Almost periodic solution · Impulsive systems

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1 Introduction

The nonlinear delay differential equation

$$\dot{x}(t) = -ax(t) + \frac{b}{1 + x^n(t-\tau)}, \quad n > 0, \tag{1.1}$$

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where a, b, τ are positive constants, proposed by Mackey and Glass [17], has been used as an appropriate model for the dynamics of hematopoiesis (blood cells production) [5, 7, 17]. In medical terms, $x(t)$ denotes the density of mature cells in blood circulation at time t and τ is the time delay between the production of immature cells in the bone marrow and their maturation for release in circulating bloodstream.

As we may know, the periodic or almost periodic phenomena are popular in various natural problems of real-world applications [2, 5, 6, 9, 10, 16, 19, 22, 23]. In comparing with periodicity, almost periodicity is more frequent in nature and much more complicated in studying for such model [20, 21]. On the other hand, many dynamical systems which describe the real phenomena depend on the history as well as undergo abrupt changes in their states. This kind of models are best described by impulsive delay differential equations [3, 18, 20]. A great deal of effort from researchers has been devoted to the study of existence and asymptotic behavior of almost periodic solutions of (1.1) and its generalizations due to their extensively realistic significance. We refer the reader to [8, 13, 15, 21, 24, 25] and the references therein. Particularly, in [21], Wang and Zhang investigated the existence, nonexistence, and uniqueness of positive almost periodic solution of the following model

$$\dot{x}(t) = -a(t)x(t) + \frac{b(t)x(t-\tau(t))}{1+x^n(t-\tau(t))}, \quad n > 1, \tag{1.2}$$

by using a new fixed point theorem in the cone. Very recently, using a fixed point theorem for contraction mapping combining with the Lyapunov functional method, Zhang et al. [25] obtained sufficient conditions for the existence and exponential stability of a positive almost periodic solution to a generalized model of (1.1)

$$\dot{x}(t) = -a(t)x(t) + \sum_{i=1}^{m} \frac{b_i(t)}{1+x^n(t-\tau_i(t))}, \quad n > 0. \tag{1.3}$$

By employing a novel argument, a delay-independent criterion was established in [13] ensuring the existence, uniqueness and global exponential stability of positive almost periodic solutions of a non-autonomous delayed model of hematopoiesis with almost periodic coefficients and delays. In [1], Alzabut et al. considered the following model of hematopoiesis with impulses

$$\begin{aligned} &\dot{x}(t) = -a(t)x(t) + \tfrac{b(t)}{1+x^n(t-\tau)}, \ t \ge 0, \ t \ne t_k, \\ &\Delta x(t_k) = \gamma_k x\left(t_k^-\right) + \delta_k, \ k \in \mathbb{N}, \end{aligned} \tag{1.4}$$

where t_k represents the instant at which the density suffers an increment of δ_k unit and $\Delta x(t_k) = x\left(t_k^+\right) - x\left(t_k^-\right)$. The density of mature cells in blood circulation decreases at prescribed instant t_k by some medication and it is proportional to the density at that time t_k^-. By employing the contraction mapping principle and applying Gronwall-Bellman's inequality, sufficient conditions which guarantee the existence and exponential stability of a positive almost periodic solution of system (1.4) were given in [1] as follows.

Theorem 1.1 ([1]) *Assume that*

(C1) *The function $a \in C\left(\mathbb{R}^+, \mathbb{R}^+\right)$ is almost periodic in the sense of Bohr and there exists a positive constant μ such that $a(t) \ge \mu$.*

(C2) *The sequence $\{\gamma_k\}$ is almost periodic and $-1 \le \gamma_k \le 0, \ k \in \mathbb{N}$.*

(C3) *The sequences $\left\{t_k^p\right\}$ are uniformly almost periodic and there exists a positive constant η such that $\inf_{k\in\mathbb{N}} t_k^1 = \eta$, where $0 < \sigma \le t_k < t_{k+1}$ $\forall k \in \mathbb{N}$, $\lim_{k\to\infty} t_k = \infty$ and $t_k^p = t_{k+p} - t_k, k, p \in \mathbb{N}$.*

(C4) *The function* $b \in C(\mathbb{R}^+, \mathbb{R}^+)$ *is almost periodic in the sense of Bohr,* $b(0) = 0$*, and there exists a positive constant* ν *such that* $\sup_{t \in \mathbb{R}^+} |b(t)| < \nu$.

(C5) *The sequence* $\{\delta_k\}$ *is almost periodic and there exists a constant* $\delta > 0$ *such that* $\sup_{k \in \mathbb{N}} |\delta_k| < \delta$.

If $\nu < \mu$*, then* (1.4) *has a unique positive almost periodic solution.*

Unfortunately, the above theorem is incorrect. For this, let us consider the following example.

Example 1.1 Consider the following equation

$$\dot{x}(t) = -x(t), \quad t \geq 0, t \neq k \in \mathbb{N}, \quad \Delta x(k) = -1, \ k \in \mathbb{N}. \tag{1.5}$$

Note that (1.5) is a special case of (1.4). Moreover, we can easily see that (1.5) satisfies conditions (C1)–(C5), where $t_k = k$, $\gamma_k = 0$ and $\delta_k = -1$.

Suppose that system (1.5) has a positive almost periodic solution $x^*(t)$. It is obvious that

$$x^*(t) = e^{-t}x^*(0) - \sum_{k \in \mathbb{N},\, k \leq t} e^{-(t-k)}, \quad t > 0.$$

For any positive integer n, we have

$$0 < x^*(n) = e^{-n}x^*(0) - e^{-1}\left(\frac{1 - e^{-n}}{1 - e^{-1}}\right) \longrightarrow \frac{-e^{-1}}{1 - e^{-1}} < 0 \text{ as } n \to \infty$$

which yields a contradiction. This shows that (1.5) has no positive almost periodic solution. Thus, Theorem 1.1 is incorrect, and Theorem 3.2 in [1] is also incorrect.

Motivated by the aforementioned discussions, in this paper, we consider a generalized model of hematopoiesis with delays, harvesting terms [4, 12, 14] and impulses of the form

$$\begin{aligned}
\dot{x}(t) = -a(t)x(t) \;+\; &\sum_{i=1}^{m}\Bigg[\frac{b_i(t)}{1 + x^{\alpha_i}(t - \tau_i(t))} \\
&+\; c_i(t)\int_0^T \frac{v_i(s)}{1 + x^{\beta_i}(t-s)}ds - H_i(t, x(t - \sigma_i(t)))\Bigg], \quad t \neq t_k, \\
\Delta x(t_k) = x\left(t_k^+\right) \;-\; &x\left(t_k^-\right) = \gamma_k x\left(t_k^-\right) + \delta_k, \quad k \in \mathbb{Z},
\end{aligned} \tag{1.6}$$

where m is a given positive integer, $a, b_i, c_i : \mathbb{R} \to \mathbb{R}$, $i \in \underline{m} := \{1, 2, \ldots, m\}$, are nonnegative functions; $H_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$, $i \in \underline{m}$, are nonnegative functions represent harvesting terms; $\tau_i(t), \sigma_i(t) \geq 0, i \in \underline{m}$, are time delays; $\alpha_i, \beta_i, i \in \underline{m}$, are positive numbers and $T > 0$ is a constant; $\gamma_k, \delta_k, k \in \mathbb{Z}$, are constants; $v_i(t), i \in \underline{m}$, are nonnegative integrable functions on $[0, T]$ with $\int_0^T v_i(s)ds = 1$; $\{t_k\}, k \in \mathbb{Z}$, is an increasing sequence involving the fixed impulsive points with $\lim_{k \to \pm\infty} t_k = \pm\infty$.

The main goal of the present paper is to establish conditions for the existence of a unique positive almost periodic solution of model (1.6). It is also proved that, under the proposed conditions, the unique positive almost periodic solution of (1.6) is globally exponentially attractive.

The rest of this paper is organized as follows. Section 2 introduces some notations, basic definitions and technical lemmas. Main results on the existence and the exponential attractivity of a unique positive almost periodic solution of (1.6) are presented in Section 3. An illustrative example is given in Section 4. The paper ends with the conclusion and cited references.

2 Preliminaries

Let $\{t_k\}_{k\in\mathbb{Z}}$ be a fixed sequence of real numbers satisfying $t_k < t_{k+1}$ for all $k \in \mathbb{Z}$, $\lim_{k\to\pm\infty} t_k = \pm\infty$. Let X be an interval of $\mathbb{R}$, denote by $PLC(X, \mathbb{R})$ the space of all piecewise left continuous functions $\phi : X \to \mathbb{R}$ with points of discontinuity of the first kind at $t = t_k$, $k \in \mathbb{Z}$.

The following notations will be used in this paper. For bounded functions $f : \mathbb{R} \to \mathbb{R}$, $F : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ and a bounded sequence $\{z_k\}$, we set

$$f_L = \inf_{t\in\mathbb{R}} f(t),\ \ f_M = \sup_{t\in\mathbb{R}} f(t),$$
$$F_L = \inf_{(t,x)\in\mathbb{R}\times\mathbb{R}_+} F(t,x),\ \ F_M = \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}_+} F(t,x),$$
$$z_L = \inf_{k\in\mathbb{Z}} z_k,\ \ z_M = \sup_{k\in\mathbb{Z}} z_k.$$

The following definitions are borrowed from [18].

Definition 2.1 ([18, 20]) The set of sequences $\left\{t_k^p\right\}$, where $t_k^p = t_{k+p} - t_k$, $p, k \in \mathbb{Z}$, is said to be uniformly almost periodic if for any positive number ϵ, there exists a relatively dense set of ϵ-almost periods common for all sequences.

Definition 2.2 ([10, 18]) A function $\phi \in PLC(\mathbb{R}, \mathbb{R})$ is said to be almost periodic if the following conditions hold

(i) The set of sequences $\left\{t_k^p\right\}$ is uniformly almost periodic.
(ii) For any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that, if $t, \bar{t}$ belong to the same interval of continuity of $\phi(t)$, $|t - \bar{t}| < \delta$, then $|\phi(t) - \phi(\bar{t})| < \epsilon$.
(iii) For any $\epsilon > 0$, there exists a relatively dense set Ω of ϵ-almost periods such that, if $\omega \in \Omega$ then $|\phi(t + \omega) - \phi(t)| < \epsilon$ for all $t \in \mathbb{R}$, $k \in \mathbb{Z}$ satisfying $|t - t_k| > \epsilon$.

For (1.6), we introduce the following assumptions

(A1) The function $a(t)$ is almost periodic in the sense of Bohr and $a_L > 0$.
(A2) The functions $b_i(t), c_i(t), i \in \underline{m}$, are nonnegative and almost periodic in the sense of Bohr.
(A3) The function $H_i(t, x), i \in \underline{m}$, are bounded nonnegative and almost periodic in the sense of Bohr in $t \in \mathbb{R}$ uniformly in $x \in \mathbb{R}_+$ and there exist positive constants L_i such that

$$|H_i(t, x) - H_i(t, y)| \le L_i|x - y|\ \forall(t, x), (t, y) \in \mathbb{R} \times \mathbb{R}_+.$$

(A4) The functions $\tau_i(t), \sigma_i(t), i \in \underline{m}$, are almost periodic in the sense of Bohr, $\dot{\tau}_i(t), \dot{\sigma}_i(t)$ are bounded, $\inf_{t\in\mathbb{R}} (1 - \dot{\tau}_i(t)) > 0$, $\inf_{t\in\mathbb{R}} (1 - \dot{\sigma}_i(t)) > 0$.
(A5) The sequence $\{\delta_k\}$ is almost periodic.
(A6) The sequence $\{\gamma_k\}$ is almost periodic satisfying

$$\gamma_L > -1, \quad \Gamma_M = \sup_{p,q\in\mathbb{Z},\, p\ge q} \Gamma(q, p) < \infty, \quad \Gamma_L = \inf_{p,q\in\mathbb{Z},\, p\ge q} \Gamma(q, p) > 0,$$

where $\Gamma(q, p) = \prod_{i=q}^{p} (1 + \gamma_i)$, $p \ge q$.
(A7) The set of sequences $\left\{t_k^p\right\}$ is uniformly almost periodic, $\eta = \inf_{k\in\mathbb{Z}} t_k^1 > 0$.

Remark 2.1 It should be noted that model (1.6) includes (1.4) as a special case. For that model, assumptions (A3), (A4) obviously are removed. Furthermore, we make assumption (A6) in order to correct condition (C2) in [1].

The following lemmas will be used in the proof of our main results.

Lemma 2.1 ([18]) *Let assumption* (A7) *hold. Assume that functions $g_i(t), i \in \underline{m}$, are almost periodic in the sense of Bohr, a function $\phi(t)$ and sequences $\{\delta_k\}$, $\{\gamma_k\}$ are almost periodic. Then for any $\epsilon > 0$, there exist $\epsilon_1 \in (0, \epsilon)$, relatively dense sets $\Omega \subset \mathbb{R}$, $\mathcal{P} \subset \mathbb{Z}$ such that*

(a1) $|\phi(t+\omega) - \phi(t)| < \epsilon, t \in \mathbb{R}, |t - t_k| > \epsilon, k \in \mathbb{Z}$;
(a2) $|g_i(t+\omega) - g_i(t)| < \epsilon, t \in \mathbb{R}, i \in \underline{m}$;
(a3) $|\gamma_{k+p} - \gamma_k| < \epsilon, |\delta_{k+p} - \delta_k| < \epsilon, |t_k^p - \omega| < \epsilon_1, \omega \in \Omega, p \in \mathcal{P}, k \in \mathbb{Z}$.

Lemma 2.2 *Let assumption* (A7) *hold. Assume that functions $f_i(t, x), i \in \underline{m}$, are almost periodic in $t \in \mathbb{R}$ uniformly in $x \in \mathbb{R}$ in the sense of Bohr, a function $\phi(t)$ and sequences $\{\delta_k\}$, $\{\gamma_k\}$ are almost periodic. Then for any compact set $\mathcal{M} \subset \mathbb{R}$ and positive number ϵ, there exist $\epsilon_1 \in (0, \epsilon)$, relatively dense sets $\Omega \subset \mathbb{R}$, $\mathcal{P} \subset \mathbb{Z}$ such that*

(b1) $|\phi(t+\omega) - \phi(t)| < \epsilon, t \in \mathbb{R}, |t - t_k| > \epsilon, \omega \in \Omega$;
(b2) $|f_i(t+\omega, x) - f_i(t, x)| < \epsilon, t \in \mathbb{R}, x \in \mathcal{M}, \omega \in \Omega, i \in \underline{m}$;
(b3) $|\gamma_{k+p} - \gamma_k| < \epsilon, |\delta_{k+p} - \delta_k| < \epsilon, |t_k^p - \omega| < \epsilon_1, \omega \in \Omega, p \in \mathcal{P}, k \in \mathbb{Z}$.

Proof The proof of this lemma is similar to the proof of Lemma 2.1 in [20] so let us omit it here. □

Lemma 2.3 *For given $\epsilon > \epsilon_1 > 0$, real number ω and integers k, p such that $|t_k^p - \omega| < \epsilon_1$, if $|t - t_i| > \epsilon$ for all $i \in \mathbb{Z}$ and $t_{k-1} < t < t_k$ then $t_{k+p-1} < t + \omega < t_{k+p}$.*

Proof The proof is straightforward, so let us omit it here. □

Lemma 2.4 *Let assumptions* (A6) *and* (A7) *hold. If $p \in \mathbb{Z}$ satisfies $|\gamma_{i+p} - \gamma_i| \le \epsilon$ for all $i \in \mathbb{Z}$, then*

$$|\Gamma(n+p, k+p) - \Gamma(n, k)| \le \frac{\Gamma_M}{1+\gamma_L}(k-n+1)\epsilon \quad \forall k, n \in \mathbb{Z}, k \ge n.$$

Proof Using the facts that $|e^u - e^v| \le |u - v| \max\{e^u, e^v\}$ for all $u, v \in \mathbb{R}$ and $|\ln(1+u) - \ln(1+v)| \le \frac{1}{1+\min\{u,v\}}|u - v|$ for all $u, v > -1$, from (A6) we have

$$\begin{aligned}|\Gamma(n+p, k+p) - \Gamma(n, k)| &\le \left|\exp\left(\sum_{i=n+p}^{k+p} \ln(1+\gamma_i)\right) - \exp\left(\sum_{i=n}^{k} \ln(1+\gamma_i)\right)\right| \\ &\le \frac{\Gamma_M}{1+\gamma_L} \sum_{i=n}^{k} |\gamma_{i+p} - \gamma_i| \le \frac{\Gamma_M}{1+\gamma_L}(k-n+1)\epsilon, \ k \ge n.\end{aligned}$$

The proof is completed. □

Lemma 2.5 *Let assumption* (A7) *hold. For any* $\alpha > 0$, $0 < \epsilon < \eta/2$, *we have*

$$\sum_{t_k<t} e^{-\alpha(t-t_k)} \le \frac{1}{1-e^{-\alpha\eta}}, \quad \sum_{t_k<t} \int_{t_k-\epsilon}^{t_k+\epsilon} e^{-\alpha(t-s)} ds \le 2e^{\frac{1}{2}\alpha\eta} \frac{\epsilon}{1-e^{-\alpha\eta}}.$$

Proof The proof follows by some direct estimates and, thus, is omitted here. □

Now, let $\sigma = \max_{i\in \underline{m}}\{T, \tau_{iM}, \sigma_{iM}\}$, so $0 < \sigma < +\infty$. From biomedical significance, it is only necessary to consider the initial condition

$$x(s) = \xi(s) \ge 0,\ s \in [\alpha - \sigma, \alpha),\ \xi(\alpha) > 0,\ \xi \in PLC([\alpha - \sigma, \alpha], \mathbb{R}). \tag{2.1}$$

It should be noted that problem (1.6) and (2.1) has a unique solution $x(t) = x(t; \alpha, \xi)$ defined on $[\alpha - \sigma, \infty)$ which is piecewise continuous with points of discontinuity of the first kind, namely $t_k, k \in \mathbb{Z}$, at which it is left continuous and the following relations are satisfied [18]

$$x\left(t_k^-\right) = x(t_k),\ \Delta x(t_k) := x\left(t_k^+\right) - x\left(t_k^-\right) = \gamma_k x\left(t_k^-\right) + \delta_k.$$

Related to (1.6), we consider the following linear equation

$$\dot{y}(t) = -a(t)y(t),\ t \ne t_k, \quad \Delta y(t_k) = \gamma_k y\left(t_k^-\right),\ k \in \mathbb{Z}. \tag{2.2}$$

Lemma 2.6 *Let assumptions* (A1), (A6) *and* (A7) *hold. Then*

$$Be^{-a_M(t-s)} \le H(t,s) \le Ae^{-a_L(t-s)},\ s \le t,$$

where

$$H(t,s) = \begin{cases} \exp\left(-\int_s^t a(r)dr\right) & \text{if } t_{k-1} < s \le t \le t_k, \\ \Gamma(n,k)\exp\left(-\int_s^t a(r)dr\right) & \text{if } t_{n-1} < s \le t_n \le t_k < t \le t_{k+1} \end{cases} \tag{2.3}$$

is the Cauchy matrix of (2.2), $A = \max\{\Gamma_M, 1\}$ *and* $B = \min\{\Gamma_L, 1\}$.

Proof The proof is straightforward from (2.3), so let us omit it here. □

Similar to Lemma 36 in [18] and Lemma 2.6 in [20] we have the following lemma.

Lemma 2.7 *Let assumptions* (A1), (A6) *and* (A7) *hold. Then, for given* $0 < \epsilon_1 < \epsilon$, *relatively dense sets* $\Omega \subset \mathbb{R}$, $\mathcal{P} \subset \mathbb{Z}$, *satisfying*

(c1) $|a(t+\omega) - a(t)| < \epsilon,\ t \in \mathbb{R}, \omega \in \Omega;$
(c2) $|\gamma_{k+p} - \gamma_k| < \epsilon,\ |t_k^p - \omega| < \epsilon_1, \omega \in \Omega,\ p \in \mathcal{P},\ k \in \mathbb{Z},$

the following estimate holds

$$|H(t+\omega, s+\omega) - H(t,s)| \le \epsilon M e^{-\frac{1}{2}a_L(t-s)}$$

for any $\omega \in \Omega$, $t, s \in \mathbb{R}$ *satisfying* $t \ge s$, $|t - t_k| > \epsilon$, $|s - t_k| > \epsilon$, $k \in \mathbb{Z}$, *where*

$$M = \max\left\{\frac{2}{a_L}, \Gamma_M\left[\frac{2}{a_L} + \frac{1}{1+\gamma_L}\left(1 + \frac{2}{a_L\eta}\right)\right]\right\}. \tag{2.4}$$

Proof We divide the proof into two possible cases as follows.

Case 1: $t_{k-1} < s \le t \le t_k$. By Lemma 2.3, $t_{k+p-1} < s+\omega \le t+\omega < t_{k+p}$. Since $|a(t+\omega)-a(t)| \le \epsilon$ for all $t \in \mathbb{R}$, $\epsilon < \eta/2$ and $\frac{1}{2}a_L(t-s)e^{-\frac{1}{2}a_L(t-s)} < 1$, it follows from (2.3), (2.4) that

$$\begin{aligned} |H(t+\omega,s+\omega)-H(t,s)| &= \left|\exp\left(-\int_s^t a(r+\omega)dr\right)-\exp\left(-\int_s^t a(r)dr\right)\right| \\ &\le e^{-a_L(t-s)}\int_s^t |a(r+\omega)-a(r)|dr \\ &\le \frac{2}{a_L}\epsilon e^{-\frac{1}{2}a_L(t-s)} \le \epsilon M e^{-\frac{1}{2}a_L(t-s)}. \end{aligned} \tag{2.5}$$

Case 2: $t_{n-1} < s \le t_n \le t_k < t \le t_{k+1}$. Similarly, we have

$$t_{n+p-1} < s+\omega < t_{n+p} \le t+\omega < t_{k+p+1}.$$

By Lemma 2.4, from (2.3)–(2.5) we obtain

$$\begin{aligned} &|H(t+\omega,s+\omega)-H(t,s)| \\ &= \Gamma(n+p,k+p)\left|\exp\left(-\int_{s+\omega}^{t+\omega} a(r)dr\right)-\exp\left(-\int_s^t a(r)dr\right)\right| \\ &\quad +|\Gamma(n+p,k+p)-\Gamma(n,k)|\exp\left(-\int_s^t a(r)dr\right) \\ &\le \frac{2\Gamma_M\epsilon}{a_L}e^{-\frac{1}{2}a_L(t-s)}+\frac{\Gamma_M\epsilon}{1+\gamma_L}(k-n+1)e^{-a_L(t-s)} \\ &\le \frac{2\Gamma_M\epsilon}{a_L}e^{-\frac{1}{2}a_L(t-s)}+\frac{\Gamma_M\epsilon}{1+\gamma_L}\left(\frac{t-s}{\eta}+1\right)e^{-a_L(t-s)} \\ &\le \frac{2\Gamma_M\epsilon}{a_L}e^{-\frac{1}{2}a_L(t-s)}+\frac{\Gamma_M\epsilon}{1+\gamma_L}\left(1+\frac{2}{a_L\eta}\right)e^{-\frac{1}{2}a_L(t-s)} \\ &\le \epsilon M e^{-\frac{1}{2}a_L(t-s)}. \end{aligned}$$

The proof is completed. □

It is worth noting that, the proof of Lemma 2.7 is different from those in [1, 18, 20]. By employing Lemma 2.4, we obtain a new bound for constant M given in (1.4).

Lemma 2.8 ([11]) *Assume that there exist constants $R, S > 0, \tau \ge 0$, $T_0 \in \mathbb{R}$ and a function $y \in PLC([T_0-\tau,\infty),\mathbb{R}^+)$ satisfying*

(d1) $\Delta y(t_k) \le \gamma_k y\left(t_k^-\right)$ *for* $t_k \ge T_0$, *where* $\gamma_k > -1$ *and* $\max_{t_k\ge T_0}\left\{(\gamma_k+1)^{-1},1\right\} < \dfrac{R}{S}$;

(d2) $D^+y(t) \le -Ry(t)+S\overline{y}(t)$ *for* $t \ge T_0, t \ne t_k$, *where* $\overline{y}(t) = \sup_{t-\tau\le s\le t} y(s)$ *and* D^+ *denotes the upper-right Dini derivative;*

(d3) $\tau \le t_k - t_{k-1}$ *for all* $k \in \mathbb{Z}$ *satisfying* $t_k \ge T_0$.

Then

$$y(t) \le \overline{y}(T_0)\left(\prod_{T_0<t_k\le t}(\gamma_k+1)\right)e^{-\lambda(t-T_0)}\ \forall t \ge T_0,$$

where $0 < \lambda \le R - S\max_{t_k\ge T_0}\left\{(\gamma_k+1)^{-1},1\right\}e^{\lambda\tau}$.

3 Main Results

Let us set $\mathcal{D}_1 = \{\phi \in PLC(\mathbb{R}, \mathbb{R})$: ϕ is almost periodic, $\phi(t) \geq 0$ for all $t \in \mathbb{R}\}$ and $\|\phi\| = \sup_{t\in\mathbb{R}} |\phi(t)|$. We define an operator $F : \mathcal{D}_1 \to PLC(\mathbb{R}, \mathbb{R})$ as follows

$$F\phi(t) = \int_{-\infty}^{t} H(t,s) \sum_{i=1}^{m} \left\{ \frac{b_i(s)}{1+\phi^{\alpha_i}(s-\tau_i(s))} + c_i(s) \int_0^T \frac{v_i(r)}{1+\phi^{\beta_i}(s-r)} dr \right.$$
$$\left. - H_i(s, \phi(s-\sigma_i(s))) \right\} ds + \sum_{t_k<t} H(t,t_k)\delta_k. \tag{3.1}$$

It can be verified that $x^*(t) = \phi(t)$ is an almost periodic solution on $\mathcal{D}_1$ of (1.6) if and only if $F\phi = \phi$.

We define the following constants

$$\underline{\delta} = \inf_{k\in\mathbb{Z}} |\delta_k|, \ \overline{\delta} = \sup_{k\in\mathbb{Z}} |\delta_k|, \ \overline{\eta} = \sup_{k\in\mathbb{Z}} t_k^1,$$
$$M_1 = \frac{A}{a_L} \sum_{i=1}^{m} (b_{iM} + c_{iM} - H_{iL}) + \frac{A\delta_M}{1-e^{-a_L\eta}},$$
$$M_2 = \begin{cases} \dfrac{B}{a_M} \displaystyle\sum_{i=1}^{m} \left(\frac{b_{iL}}{1+M_1^{\alpha_i}} + \frac{c_{iL}}{1+M_1^{\beta_i}} - H_{iM} \right) + \dfrac{B\delta_L e^{-a_M\overline{\eta}}}{1-e^{-a_M\bar{\eta}}} & \text{if } \delta_L \geq 0, \\ \dfrac{B}{a_M} \sum_{i=1}^{m} \left(\dfrac{b_{iL}}{1+M_1^{\alpha_i}} + \dfrac{c_{iL}}{1+M_1^{\beta_i}} - H_{iM} \right) + \dfrac{A\delta_L}{1-e^{-a_L\eta}} & \text{if } \delta_L < 0. \end{cases} \tag{3.2}$$

Lemma 3.1 *Let assumptions* (A1)–(A7) *hold. If* $\phi \in \mathcal{D}_1$ *then*

$$M_2 \leq F\phi(t) \leq M_1 \ \forall t \in \mathbb{R}.$$

Proof Let $\phi \in \mathcal{D}_1$. By Lemma 2.5 and Lemma 2.6, from (3.1), we have

$$F\phi(t) \leq \int_{-\infty}^{t} Ae^{-a_L(t-s)} \sum_{i=1}^{m} (b_{iM} + c_{iM} - H_{iL})\, ds + A\delta_M \sum_{t_k<t} e^{-a_L(t-t_k)}$$
$$\leq \frac{A}{a_L} \sum_{i=1}^{m} (b_{iM} + c_{iM} - H_{iL}) + \frac{A\delta_M}{1-e^{-a_L\eta}} = M_1 \ \forall t \in \mathbb{R}. \tag{3.3}$$

For each $t \in \mathbb{R}$, let n_0 be an integer such that $t_{n_0} < t \leq t_{n_0+1}$. If $\delta_L \geq 0$ then, by Lemma 2.6, it follows from the fact $(n-k)\eta \leq t_n - t_k \leq (n-k)\bar{\eta}$ for all $k \leq n$ that

$$\sum_{t_k<t} H(t,t_k)\delta_k \geq \sum_{t_k<t} B\delta_L e^{-a_M(t-t_k)} \geq \sum_{t_k\leq t_{n_0}} B\delta_L e^{-a_M(t_{n_0+1}-t_k)}$$
$$= \sum_{k\leq n_0} B\delta_L e^{-a_M(t_{n_0+1}-t_k)} \geq \sum_{q=1}^{\infty} B\delta_L e^{-a_M\bar{\eta}q} = \frac{B\delta_L e^{-a_M\bar{\eta}}}{1-e^{-a_M\bar{\eta}}}. \tag{3.4}$$

If $\delta_L < 0$ then from (A7) and Lemma 2.6, we have

$$\sum_{t_k<t} H(t,t_k)\delta_k \geq \sum_{t_k<t} A\delta_L e^{-a_L(t-t_k)} \geq \sum_{t_k\leq t_{n_0}} A\delta_L e^{-a_L(t_{n_0}-t_k)} \geq \sum_{q=0}^{\infty} A\delta_L e^{-a_L q\eta} = \frac{A\delta}{1-e^{-a_L\eta}}. \tag{3.5}$$

From (3.4) and (3.5), we obtain

$$F\phi(t) \geq \frac{B}{a_M}\sum_{i=1}^{m}\left(\frac{b_{iL}}{1+M_1^{\alpha_i}} + \frac{c_{iL}}{1+M_1^{\beta_i}} - H_{iM}\right) + \sum_{t_k<t} H(t,t_k)\delta_k \geq M_2. \tag{3.6}$$

The proof is completed. □

Now we are in position to introduce our main results as follows.

Theorem 3.1 *Under the assumptions* (A1)–(A7), *if* $\phi \in \mathcal{D}_1$ *then* $F\phi(t)$ *is almost periodic.*

Proof Let $\phi \in \mathcal{D}_1$. For given $\epsilon \in (0, \eta/2)$, there exists $0 < \delta < \epsilon/2$ such that, if $t, \bar{t}$ belong to the same interval of continuity of $\phi(t)$ then

$$|\phi(t) - \phi(\bar{t})| < \epsilon, \ |t - \bar{t}| < \delta. \tag{3.7}$$

By Lemma 2.2 and Lemma 2.7, there exist $0 < \epsilon_1 < \delta$, relatively dense sets $\Omega \subset \mathbb{R}$, $\mathcal{P} \subset \mathbb{Z}$ such that, for all $\omega \in \Omega$, we have

$$\begin{aligned}
&|H(t+\omega, s+\omega) - H(t,s)| \leq \delta M e^{-\frac{1}{2}a_L(t-s)},\ t \geq s, |t-t_k| > \delta, |s-t_k| > \delta;\\
&|\phi(t+\omega) - \phi(t)| < \delta,\ t \in \mathbb{R}, |t-t_k| > \delta, k \in \mathbb{Z};\\
&|H_i(t+\omega, x) - H_i(t,x)| < \delta,\ t \in \mathbb{R}, x \in [\phi_L, \|\phi\|], i \in \underline{m};\\
&|a(t+\omega) - a(t)| < \delta,\ t \in \mathbb{R};\\
&|b_i(t+\omega) - b_i(t)| < \delta,\ |c_i(t+\omega) - c_i(t)| < \delta,\ t \in \mathbb{R}, i \in \underline{m};\\
&|\tau_i(t+\omega) - \tau_i(t)| < \delta,\ |\sigma_i(t+\omega) - \sigma_i(t)| < \delta,\ t \in \mathbb{R}, i \in \underline{m};\\
&|\gamma_{k+p} - \gamma_k| < \delta,\ |\delta_{k+p} - \delta_k| < \delta,\ |t_k^p - \omega| < \epsilon_1, p \in \mathcal{P}, k \in \mathbb{Z}.
\end{aligned} \tag{3.8}$$

Let $\omega \in \Omega$, $p \in \mathcal{P}$. One can easily see that

$$\begin{aligned}
F\phi(t+\omega) = &\int_{-\infty}^{t} H(t+\omega, s+\omega)\sum_{i=1}^{m}\left\{\frac{b_i(s+\omega)}{1+\phi^{\alpha_i}(s+\omega-\tau_i(s+\omega))}\right.\\
&\left. + \int_0^T \frac{c_i(s+\omega)v_i(r)}{1+\phi^{\beta_i}(s+\omega-r)}dr - H_i(s+\omega, \phi(s+\omega-\sigma_i(s+\omega)))\right\} ds\\
&+ \sum_{t_k<t} H(t+\omega, t_{k+p})\delta_{k+p}.
\end{aligned} \tag{3.9}$$

We define $E_\epsilon(\{t_k\}) = \{t \in \mathbb{R} : \ |t - t_k| > \epsilon \ \forall k \in \mathbb{Z}\}$. For $t \in E_\epsilon(\{t_k\})$, $i \in \underline{m}$, let us set

$$
\begin{aligned}
C_i &= \int_{-\infty}^{t} \left| \frac{H(t+\omega, s+\omega) b_i(s+\omega)}{1 + \phi^{\alpha_i}(s+\omega-\tau_i(s+\omega))} - \frac{H(t,s) b_i(s)}{1+\phi^{\alpha_i}(s-\tau_i(s))} \right| ds, \\
D_i &= \int_{-\infty}^{t} \left| H(t+\omega, s+\omega) \int_0^T \frac{c_i(s+\omega) v_i(r)}{1+\phi^{\beta_i}(s+\omega-r)} dr \right. \\
&\quad \left. - H(t,s) \int_0^T \frac{c_i(s) v_i(r)}{1+\phi^{\beta_i}(s-r)} dr \right| ds, \\
E_i &= \int_{-\infty}^{t} |H(t+\omega, s+\omega) H_i(s+\omega, \phi(s+\omega-\sigma_i(s+\omega))) \\
&\quad - H(t,s) H_i(s, \phi(s-\sigma_i(s)))| \, ds, \\
G &= \sum_{t_k < t} \left| H(t+\omega, t_{k+p}) \delta_{k+p} - H(t, t_k) \delta_k \right|,
\end{aligned}
\tag{3.10}
$$

then we have

$$
|F\phi(t+\omega) - F\phi(t)| \le \sum_{i=1}^{m} (C_i + D_i + E_i) + G, \ \ t \in E_\epsilon(\{t_k\}). \tag{3.11}
$$

We also define

$$
\begin{aligned}
C_{i1} &= \int_{-\infty}^{t} |H(t+\omega, s+\omega) - H(t,s)| \frac{|b_i(s+\omega)|}{1+\phi^{\alpha_i}(s+\omega-\tau_i(s+\omega))} ds, \\
C_{i2} &= \int_{-\infty}^{t} H(t,s) \frac{|b_i(s+\omega) - b_i(s)|}{1+\phi^{\alpha_i}(s+\omega-\tau_i(s+\omega))} ds, \\
C_{i3} &= \int_{-\infty}^{t} H(t,s) |\phi(s+\omega-\tau_i(s+\omega)) - \phi(s-\tau_i(s+\omega))| ds, \\
C_{i4} &= \int_{-\infty}^{t} H(t,s) |\phi(s-\tau_i(s+\omega)) - \phi(s-\tau_i(s))| ds
\end{aligned}
\tag{3.12}
$$

and $K_i = \sup_{\phi_L \le x \le \|\phi\|} \alpha_i x^{\alpha_i - 1}$. It can be seen from (3.10) and (3.12) that

$$
C_i \le C_{i1} + C_{i2} + b_{iM} K_i (C_{i3} + C_{i4}), \ \ i \in \underline{m}. \tag{3.13}
$$

By Lemma 2.5 and Lemma 2.6, from (3.8), (3.12) and the fact that $\int_{-\infty}^{t} e^{-\frac{1}{2} a_L (t-s)} ds = 2/a_L$, we have

$$
C_{i1} \le \frac{2 b_{iM} M}{a_L} + \sum_{t_k < t} 2 A b_{iM} \int_{t_k - \epsilon}^{t_k + \epsilon} e^{-a_L(t-s)} ds \le b_{iM} \left(\frac{2M}{a_L} + \frac{4 A e^{\frac{1}{2} a_L \eta}}{1 - e^{-a_L \eta}} \right) \epsilon, \tag{3.14}
$$

and

$$
\begin{aligned}
C_{i2} &\le \int_{-\infty}^{t} A \epsilon e^{-a_L(t-s)} ds = \frac{A}{a_L} \epsilon, \\
C_{i3} &\le \int_{-\infty}^{t} A \epsilon e^{-a_L(t-s)} ds + 2A \|\phi\| \sum_{t_k < t} \int_{\{s : |s - \tau_i(s+\omega) - t_k| < \epsilon, s \le t\}} e^{-a_L(t-s)} ds.
\end{aligned}
\tag{3.15}
$$

It should be noted that, by (A4), $t-\tau_i(t)$, $i\in \underline{m}$, are strictly increasing functions, and thus, there exist the inverse functions $\tau_i^*(t)$ of $t-\tau_i(t)$. For each $t\in\mathbb{R}$, denote $\bar{t}=t-\epsilon-\tau_i(t+\omega)$ then

$$t+\omega=\tau_i^*(\bar{t}+\omega+\epsilon). \tag{3.16}$$

Let $\underline{\lambda}_i=\inf_{s\in\mathbb{R}}\dot{\tau}_i^*(s)$, $\overline{\lambda}_i=\sup_{s\in\mathbb{R}}\dot{\tau}_i^*(s)$, $i\in\underline{m}$, then, by (A4), $0<\underline{\lambda}_i,\overline{\lambda}_i<\infty$. Therefore,

$$\tau_i^*(\bar{t}+\omega+\epsilon)-\tau_i^*(t_k+\omega+\epsilon)\ge\underline{\lambda}_i(\bar{t}-t_k),\ t_k<\bar{t}, \tag{3.17}$$

and hence, from (3.15)–(3.17), we have

$$\begin{aligned}
C_{i3}\ &\le\ \frac{A\epsilon}{a_L}+2A\|\phi\|\sum_{t_k<\bar{t}}\int_{\tau_i^*(t_k+\omega-\epsilon)-\omega}^{\tau_i^*(t_k+\omega+\epsilon)-\omega}e^{-a_L(t-s)}ds\\
&\le\ \frac{A\epsilon}{a_L}+2A\|\phi\|\sum_{t_k<\bar{t}}e^{-a_L\left[\tau_i^*(\bar{t}+\omega+\epsilon)-\tau_i^*(t_k+\omega+\epsilon)\right]}\left[\tau_i^*(t_k+\omega+\epsilon)-\tau_i^*(t_k+\omega-\epsilon)\right]\\
&\le\ \frac{A\epsilon}{a_L}+4A\|\phi\|\overline{\lambda}_i\epsilon\sum_{t_k<\bar{t}}e^{-a_L\underline{\lambda}_i(\bar{t}-t_k)}\le\left(\frac{1}{a_L}+\frac{4\overline{\lambda}_i\|\phi\|}{1-e^{-a_L\underline{\lambda}_i\eta}}\right)A\epsilon.
\end{aligned}\tag{3.18}$$

By the same arguments used in deriving (3.18), we obtain

$$\begin{aligned}
C_{i4}\ &\le\ A\epsilon\int_{-\infty}^{t}e^{-a_L(t-s)}ds+2A\|\phi\|\sum_{t_k<t}\int_{\tau_i^*(t_k-\epsilon)}^{\tau_i^*(t_k+\epsilon)}e^{-a_L(t-s)}ds\\
&\le\left(\frac{1}{a_L}+\frac{4\overline{\lambda}_i\|\phi\|}{1-e^{-a_L\underline{\lambda}_i\eta}}\right)A\epsilon.
\end{aligned}\tag{3.19}$$

Combining (3.13)–(3.15), (3.18), and (3.19), we readily obtain

$$C_i\le\left[\frac{A}{a_L}+\frac{2b_{iM}(M+AK_i)}{a_L}+Ab_{iM}\left(\frac{4e^{\frac{1}{2}a_L\eta}}{1-e^{-a_L\eta}}+\frac{8K_i\overline{\lambda}_i\|\phi\|}{1-e^{-a_L\underline{\lambda}_i\eta}}\right)\right]\epsilon. \tag{3.20}$$

Next, let us set

$$\begin{aligned}
D_{i1}\ &=\ \int_{-\infty}^{t}|H(t+\omega,s+\omega)-H(t,s)|\int_0^T\frac{v_i(r)}{1+\phi^{\beta_i}(s+\omega-r)}drds,\\
D_{i2}\ &=\ \int_{-\infty}^{t}H(t,s)|c_i(s+\omega)-c_i(s)|\int_0^T\frac{v_i(r)}{1+\phi^{\beta_i}(s+\omega-r)}drds,\\
D_{i3}\ &=\ \int_{-\infty}^{t}H(t,s)\int_0^T v_i(r)\left|\frac{1}{1+\phi^{\beta_i}(s+\omega-r)}-\frac{1}{1+\phi^{\beta_i}(s-r)}\right|drds.
\end{aligned}\tag{3.21}$$

It follows from (3.10) and (3.21) that

$$D_i\le c_{iM}D_{i1}+D_{i2}+c_{iM}D_{i3},\ i\in\underline{m}. \tag{3.22}$$

By Lemmas 2.5 and 2.6, from (3.8), we have

$$\begin{aligned}
D_{i1}\ &\le\ \int_{-\infty}^{t}M\epsilon e^{-\frac{1}{2}a_L(t-s)}ds+\sum_{t_k<t}2A\int_{t_k-\epsilon}^{t_k+\epsilon}e^{-a_L(t-s)}\le\left(\frac{2M}{a_L}+\frac{4Ae^{\frac{1}{2}a_L\eta}}{1-e^{-a_L\eta}}\right)\epsilon,\\
D_{i2}\ &\le\ \int_{-\infty}^{t}A\epsilon e^{-a_L(t-s)}ds=\frac{A}{a_L}\epsilon,
\end{aligned}\tag{3.23}$$

and

$$
\begin{aligned}
D_{i3} &\le \int_{-\infty}^{t} Ae^{-a_L(t-s)}\int_0^T G_i v_i(r)|\phi(s+\omega-r)-\phi(s-r)|drds\\
&\le \int_{-\infty}^{t} AG_i\epsilon e^{-a_L(t-s)}ds + 2AG_i\|\phi\|\int_0^T v_i(r)\left(\sum_{t_k+r<t}\int_{t_k-r-\epsilon}^{t_k+r+\epsilon} e^{-a_L(t-s)}ds\right)dr\\
&\le \frac{AG_i\epsilon}{a_L} + 4AG_i\|\phi\|\epsilon\int_0^T v_i(r)\left(\sum_{t_k+r<t} e^{-a_L(t-t_k-r-\epsilon)}\right)dr\\
&\le \left(\frac{1}{a_L}+\frac{4e^{\frac{1}{2}a_L\eta}\|\phi\|}{1-e^{-a_L\eta}}\right)AG_i\epsilon, \qquad (3.24)
\end{aligned}
$$

where $G_i = \sup_{\phi_L\le x\le\|\phi\|}\beta_i x^{\beta_i-1}$. From (3.22)–(3.24), we readily obtain

$$
D_i \le \left[\frac{A+2Mc_{iM}+Ac_{iM}G_i}{a_L}+\frac{4Ac_{iM}e^{\frac{1}{2}a_L\eta}(1+G_i\|\phi\|)}{1-e^{-a_L\eta}}\right]\epsilon. \qquad (3.25)
$$

Now, we define

$$
\begin{aligned}
E_{i1} &= \int_{-\infty}^{t} |H(t+\omega,s+\omega)-H(t,s)|H_i(s+\omega,\phi(s+\omega-\sigma_i(s+\omega)))ds,\\
E_{i2} &= \int_{-\infty}^{t} H(t,s)|H_i(s+\omega,\phi(s+\omega-\sigma_i(s+\omega)))-H_i(s+\omega,\phi(s-\sigma_i(s+\omega)))|ds,\\
E_{i3} &= \int_{-\infty}^{t} H(t,s)|H_i(s+\omega,\phi(s-\sigma_i(s+\omega)))-H_i(s+\omega,\phi(s-\sigma_i(s)))|ds,\\
E_{i4} &= \int_{-\infty}^{t} H(t,s)|H_i(s+\omega,\phi(s-\sigma_i(s)))-H_i(s,\phi(s-\sigma_i(s)))|ds, \qquad (3.26)
\end{aligned}
$$

then, from (3.10) and (3.26), we have

$$
E_i \le E_{i1}+E_{i2}+E_{i3}+E_{i4},\ i\in\underline{m}. \qquad (3.27)
$$

Also using Lemma 2.5 and Lemma 2.6, from (3.8) and the fact that $\int_{-\infty}^{t} e^{-\frac{1}{2}a_L(t-s)}ds = 2/a_L$, we obtain

$$
\begin{aligned}
E_{i1} &\le \frac{2H_{iM}M\epsilon}{a_L}+\sum_{t_k<t}2AH_{iM}\int_{t_k-\epsilon}^{t_k+\epsilon} e^{-a_L(t-s)}ds \le 2\left(\frac{M}{a_L}+\frac{2Ae^{\frac{1}{2}a_L\eta}}{1-e^{-a_L\eta}}\right)H_{iM}\epsilon,\\
E_{i4} &\le \int_{-\infty}^{t} A\epsilon e^{-a_L(t-s)}ds \le \frac{A}{a_L}\epsilon. \qquad (3.28)
\end{aligned}
$$

Let $\underline{\xi}_i = \inf_{t\in\mathbb{R}} \dot{\sigma}_i^*(t)$, $\overline{\xi}_i = \sup_{t\in\mathbb{R}} \dot{\sigma}_i^*(t)$, $i \in \underline{m}$. Similarly to (3.18) and (3.19), we readily obtain

$$\begin{aligned} E_{i2} &\le L_i \int_{-\infty}^{t} H(t,s)|\phi(s+\omega-\sigma_i(s+\omega)) - \phi(s-\sigma_i(s+\omega))|ds \\ &\le \left(\frac{1}{a_L} + \frac{4\overline{\xi}_i\|\phi\|}{1-e^{-a_L\underline{\xi}_i\eta}}\right) AL_i\epsilon, \\ E_{i3} &\le L_i \int_{-\infty}^{t} H(t,s)|\phi(s-\sigma_i(s+\omega)) - \phi(s-\sigma_i(s))|ds \\ &\le \left(\frac{1}{a_L} + \frac{4\overline{\xi}_i\|\phi\|}{1-e^{-a_L\underline{\xi}_i\eta}}\right) AL_i\epsilon. \end{aligned} \tag{3.29}$$

Inequalities (3.27)–(3.29) yield

$$E_i \le \left(\frac{A(2L_i+1)+2H_{iM}M}{a_L} + \frac{4Ae^{\frac{1}{2}a_L\eta}H_{iM}}{1-e^{-a_L\eta}} + \frac{8AL_i\overline{\xi}_i\|\phi\|}{1-e^{-a_L\underline{\xi}_i\eta}}\right)\epsilon. \tag{3.30}$$

Let us set

$$G_1 = \sum_{t_k<t} |H(t+\omega, t_{k+p}) - H(t,t_k)|, \quad G_2 = \sum_{t_k<t} |H(t,t_k)| \tag{3.31}$$

then

$$G \le \sum_{t_k<t} |H(t+\omega, t_{k+p}) - H(t,t_k)||\delta_{k+p}| + \sum_{t_k<t} |H(t,t_k)||\delta_{k+p}-\delta_k| \le \bar{\delta}G_1 + \epsilon G_2. \tag{3.32}$$

For each $t \ge t_k$, there exists a unique integer $l = l(t)$ such that $t_l < t \le t_{l+1}$. By Lemma 2.3 we, have $t_{l+p} < t+\omega < t_{l+p+1}$. Thus

$$\begin{aligned} G_1 &\le \sum_{t_k<t} \left(\Gamma_M e^{-a_L(t-t_k-\epsilon)} \left|\int_{t_{k+p}}^{t+\omega} a(s)ds - \int_{t_k}^{t} a(s)ds\right| \right. \\ &\quad \left. + e^{-a_L(t-t_k)} |\Gamma(k+p, l+p) - \Gamma(k,l)| \right) \\ &\le \Gamma_M I_1 + I_2, \end{aligned} \tag{3.33}$$

where

$$\begin{aligned} I_1 &= \sum_{t_k<t} e^{-a_L(t-t_k-\epsilon)} \left|\int_{t_{k+p}}^{t+\omega} a(s)ds - \int_{t_k}^{t} a(s)ds\right|, \\ I_2 &= \sum_{t_k<t} e^{-a_L(t-t_k)} |\Gamma(k+p,l+p) - \Gamma(k,l)|. \end{aligned} \tag{3.34}$$

Note that $\epsilon < \eta/2$, $\frac{1}{2}a_L(t-t_k)e^{-\frac{1}{2}a_L(t-t_k)} < 1$ and $|t_k^p - \omega| < \epsilon_1 < \epsilon$, from (3.8) and Lemma 2.5, we have

$$
\begin{aligned}
I_1 &\le \sum_{t_k<t} e^{-a_L(t-t_k-\epsilon)} \left(\int_{t_k}^{t} |a(s+\omega) - a(s)| ds + \left| \int_{t_{k+p}-\omega}^{t_k} a(s) ds \right| \right) \\
&\le \epsilon \sum_{t_k<t} e^{-a_L(t-t_k-\epsilon)} (t - t_k + a_M) \\
&\le \left(a_M \epsilon e^{\frac{1}{2}a_L\eta} + \frac{2\epsilon}{a_L} e^{\frac{1}{2}a_L\eta} \right) \sum_{t_k<t} e^{-\frac{1}{2}a_L(t-t_k)} \\
&\le \frac{e^{\frac{1}{2}a_L\eta}}{1 - e^{-\frac{1}{2}a_L\eta}} \left(a_M + \frac{2}{a_L} \right) \epsilon. \qquad (3.35)
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
I_2 &\le \frac{\Gamma_M}{1+\gamma_L} \sum_{t_k<t} e^{-a_L(t-t_k)} (l - k + 1)\epsilon \le \frac{\Gamma_M}{1+\gamma_L} \sum_{t_k<t} e^{-a_L(t-t_k)} \left(\frac{t-t_k}{\eta} + 1 \right) \epsilon \\
&\le \frac{\Gamma_M}{(1+\gamma_L)\left(1 - e^{-\frac{1}{2}a_L\eta}\right)} \left(\frac{2}{a_L\eta} + 1 \right) \epsilon. \qquad (3.36)
\end{aligned}
$$

It follows from (3.33)–(3.36) that

$$
G_1 \le \frac{\Gamma_M}{1 - e^{-\frac{1}{2}a_L\eta}} \left[e^{\frac{1}{2}a_L\eta} \left(a_M + \frac{2}{a_L} \right) + \frac{1}{1+\gamma_L} \left(\frac{2}{a_L\eta} + 1 \right) \right] \epsilon. \qquad (3.37)
$$

We also have

$$
G_2 \le A \sum_{t_k<t} e^{-a_L(t-t_k)} \le \frac{A}{1 - e^{-a_L\eta}}. \qquad (3.38)
$$

Therefore

$$
G \le \frac{\epsilon \overline{\delta} \Gamma_M}{1 - e^{-\frac{1}{2}a_L\eta}} \left[e^{\frac{1}{2}a_L\eta} \left(a_M + \frac{2}{a_L} \right) + \frac{1}{1+\gamma_L} \left(\frac{2}{a_L\eta} + 1 \right) \right] + \frac{A\epsilon}{1 - e^{-\frac{1}{2}a_L\eta}}. \qquad (3.39)
$$

We can see clearly from (3.11), (3.20), (3.25), (3.30), and (3.39) that there exists a positive constant Λ such that $|F(\phi(t+\omega) - F\phi(t)| \le \Lambda\epsilon$ for all $t \in \mathbb{R}$, $|t - t_k| > \epsilon$, $k \in \mathbb{Z}$. This shows that $F\phi(t)$ is almost periodic. The proof is completed. □

Theorem 3.2 *Let assumptions* (A1)–(A7) *hold. If M_2, defined in* (3.2)*, is positive and*

$$
\frac{A}{a_L} \sum_{i=1}^{m} \left(b_{iM} K_i^* + c_{iM} G_i^* + L_i \right) < 1, \qquad (3.40)
$$

where $K_i^* = \sup_{M_2 \le x \le M_1} \alpha_i x^{\alpha_i - 1}$, $G_i^* = \sup_{M_2 \le x \le M_1} \beta_i x^{\beta_i - 1}$, then (1.6) *has a unique positive almost periodic solution.*

Proof We define $\mathcal{D}_2 = \{\phi \in \mathcal{D}_1 : M_2 \le \phi(t) \le M_1,\ t \in \mathbb{R}\}$. It is worth noting that, from Lemma 3.1, Theorem 3.1 and the assumption $M_2 > 0$, we have $F(\mathcal{D}_2) \subset \mathcal{D}_2$. For any $\phi, \psi \in \mathcal{D}_2$, applying Lemma 2.6 we obtain

$$\begin{aligned} |F\phi(t) - F\psi(t)| &\le \int_{-\infty}^{t} H(t,s) \sum_{i=1}^{m} \Big\{ L_i |\phi(s-\sigma_i(s)) - \psi(s-\sigma_i(s))| \\ &\quad + c_{iM} \int_0^T v_i(r) \left| \frac{1}{1+\phi^{\beta_i}(s-r)} - \frac{1}{1+\psi^{\beta_i}(s-r)} \right| dr \\ &\quad + b_{iM} \left| \frac{1}{1+\phi^{\alpha_i}(s-\tau_i(s))} - \frac{1}{1+\psi^{\alpha_i}(s-\tau_i(s))} \right| \Big\} ds \\ &\le \frac{A}{a_L} \sum_{i=1}^{m} \left(b_{iM} K_i^* + c_{iM} G_i^* + L_i \right) \|\phi - \psi\|. \end{aligned}$$

Therefore

$$\|F\phi - F\psi\| \le \frac{A}{a_L} \sum_{i=1}^{m} \left(b_{iM} K_i^* + c_{iM} G_i^* + L_i \right) \|\phi - \psi\|$$

which shows that F is a contraction mapping on $\mathcal{D}_2$ by condition (3.40). Then F has a unique fixed point in $\mathcal{D}_2$, namely ϕ_0. It should be noted that $F(\mathcal{D}_1) \subset \mathcal{D}_2$, and hence, F also has a unique fixed point ϕ_0 in $\mathcal{D}_1$. This shows that (1.6) has a unique positive almost periodic solution $x^*(t) = \phi_0(t)$. The proof is complete. □

Theorem 3.3 *Let assumptions* (A1)–(A7) *hold. If* $M_2 > 0, \sigma \le \eta$ *and*

$$\frac{1}{a_L} \max\left\{ A, (\gamma_L + 1)^{-1} \right\} \sum_{i=1}^{m} \left(b_{iM} K_i^* + c_{iM} G_i^* + L_i \right) < 1, \tag{3.41}$$

then (1.6) *has a unique positive almost periodic solution* $x^*(t)$. *Moreover, every solution* $x(t) = x(t, \alpha, \xi)$ *of* (1.6) *converges exponentially to* $x^*(t)$ *as* $t \to \infty$.

Proof By Theorem 3.2, (1.6) has a unique positive almost periodic solution $x^*(t)$. Let $x(t) = x(t, \alpha, \xi)$ be a solution of (1.6) and (2.1). We define $V(t) = |x(t) - x^*(t)|$ then

$$\begin{aligned} D^+V(t) &\le -a_L |x(t) - x^*(t)| + \sum_{i=1}^{m} \Big[b_{iM} K_i^* |x(t-\tau_i(t)) - x^*(t-\tau_i(t))| \\ &\quad + c_{iM} G_i^* \int_0^T v_i(s) |x(t-s) - x^*(t-s)| ds + L_i |x(t-\sigma_i(t)) - x^*(t-\sigma_i(t))| \Big] \\ &\le -a_L V(t) + \sum_{i=1}^{m} \left(b_{iM} K_i^* + c_{iM} G_i^* + L_i \right) \overline{V}(t),\ t \ne t_k,\ t \ge \alpha, \\ \varDelta V(t_k) &= \gamma_k V\left(t_k^-\right),\ t_k \ge \alpha, k \in \mathbb{Z}, \end{aligned}$$

where $\overline{V}(t) = \sup_{t-\sigma \le s \le t} V(s)$. By Lemma 2.8, there exists a positive constant λ such that

$$V(t) \le \overline{V}(\alpha) \prod_{\alpha < t_k \le t} (\gamma_k + 1) e^{-\lambda(t-\alpha)} \le \varGamma_M \overline{V}(\alpha) e^{-\lambda(t-\alpha)},\ t \ge \alpha.$$

This shows that $x(t)$ converges exponentially to $x^*(t)$ as $t \to \infty$. The proof is complete. □

The existence and exponential stability of positive almost periodic solution of (1.4) is presented in the following corollary as an application of our obtained results with $m = 1$.

Corollary 3.1 *Under assumptions* (A1), (A2) *(with* $c(t) = 0$*) and* (A5)-(A7)*, if* $M_2 > 0$, $\tau \leq \eta$ *and*

$$\frac{1}{a_L} \max \left\{ A, (\gamma_L + 1)^{-1} \right\} b_M K^* < 1, \tag{3.42}$$

where $K^* = \sup_{M_2 \leq x \leq M_1} \alpha x^{\alpha - 1}$*, then* (1.4) *has a unique positive almost periodic solution which is exponentially stable.*

4 An Illustrative Example

In this section, we give a numerical example to illustrate the effectiveness of our conditions. For illustrating purpose, let us consider the following equation

$$\begin{aligned} \dot{x}(t) &= -a(t)x(t) + \frac{b(t)}{1 + x^2(t - \tau(t))} + \int_0^1 \frac{c(t)ds}{1 + x^2(t-s)} \\ &\quad -d(t)\frac{|x(t - \sigma(t))|}{10 + |x(t - \sigma(t))|}, \quad t \neq k, \\ \Delta x(k) &= \gamma_k x(k - 0) + \delta_k, \quad k \in \mathbb{Z}, \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} a(t) &= 5 + |\sin\left(t\sqrt{2}\right)|, \ b(t) = \frac{1}{10}\left(1 + |\sin\left(t\sqrt{3}\right)|\right), \ c(t) = \frac{1}{10}\left(1 + |\cos\left(t\sqrt{3}\right)|\right), \\ d(t) &= \frac{1}{20}\sin^2\left(t\sqrt{3}\right), \ \tau(t) = \sin^2\left(\frac{\sqrt{3}}{2}t\right), \ \sigma(t) = \cos^2\left(\frac{\sqrt{3}}{2}t\right), \\ \gamma_{2m} &= -\frac{1}{2}, \ \gamma_{2m+1} = 1, \ \delta_{2m} = 1, \ \delta_{2m+1} = \frac{1}{2}, \ m \in \mathbb{Z}. \end{aligned}$$

It should be noted that the functions $a(t)$, $b(t)$, $c(t)$, $\tau(t)$, and $\sigma(t)$ are almost periodic in the sense of Bohr, $H(t, x) = d(t)\frac{|x|}{10+|x|}$ is almost periodic in $t \in \mathbb{R}$ uniformly in $x \in \mathbb{R}_+$, $|H(t, x) - H(t, y)| \leq \frac{1}{2}|x - y|$. Therefore, assumptions (A1)–(A5) and (A7) are satisfied. On the other hand, $\Gamma(q, p) = \prod_{i=q}^{p}(1 + \gamma_i) \in \left\{\frac{1}{2}, 1, 2\right\}$ for any $p, q \in \mathbb{Z}$, $p \geq q$. Thus, $\Gamma_M = 2$, $\Gamma_L = \dfrac{1}{2}$ and assumption (A6) is satisfied. Taking some computations, we obtain

$$\begin{aligned} &a_L = 5, \ a_M = 6, \ b_L = c_L = 0.1, \ b_M = c_M = 0.2, L \leq 0.5, H_M = \frac{1}{20}, H_L = 0, \\ &A = 2, \ B = 0.5, \ M_1 = 2.1736, \ M_2 = 0.0027, \ K^* = G^* = 2M_1 \end{aligned}$$

and $\frac{A}{a_L}(b_M K^* + c_M G^* + L) \leq 0.8956$. By Theorem 3.2, (4.1) has a unique positive almost periodic solution $x^*(t)$. Furthermore, it can be seen that $\gamma_L = -\frac{1}{2}$, and hence, $\frac{1}{a_L}\max\left\{A, (\gamma_L + 1)^{-1}\right\}(b_M K^* + c_M G^* + L) \leq 0.8956$. By Theorem 3.3, every solution $x(t, \alpha, \xi)$ of (4.1) converges exponentially to $x^*(t)$ as t tends to infinity. As presented in Fig. 1, state trajectories of (4.1) with different initial conditions converge to the unique positive almost periodic solution of (4.1).

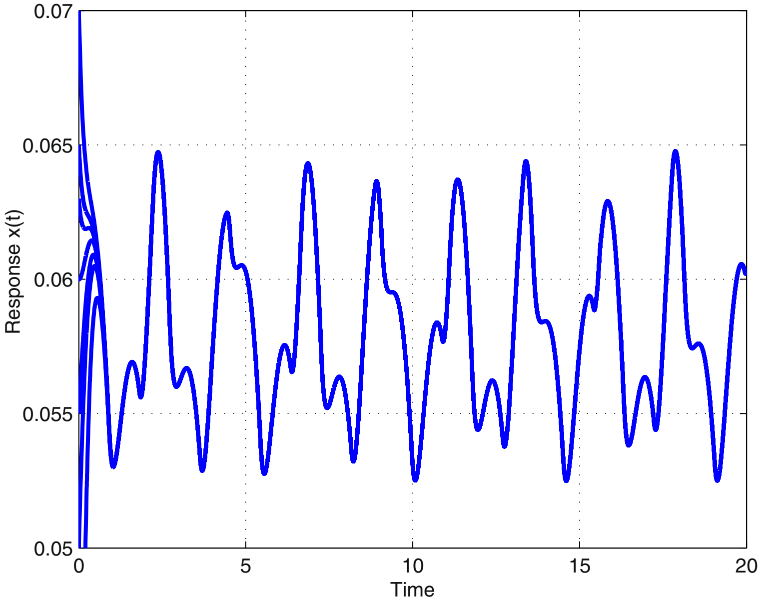

Fig. 1 State trajectories of (4.1) converge to the unique positive almost periodic solution

5 Conclusion

This paper has dealt with the existence and exponential attractivity of a unique positive almost periodic solution for a generalized model of hematopoiesis with delays and impulses. Using the contraction mapping principle and a novel type of impulsive delay inequality, new sufficient conditions have been derived ensuring that all solutions of the model converge exponentially to the unique positive almost periodic solution.

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