THE LINKEDNESS OF CUBICAL POLYTOPES: BEYOND THE CUBE

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Abstract. A cubical polytope is a polytope with all its facets being combinatorially equivalent to cubes. The paper is concerned with the linkedness of the graphs of cubical polytopes.

A graph with at least k vertices is $\lfloor k/2\rfloor$*-linked* if, for every set of $2\lfloor k/2\rfloor$ distinct vertices organised in arbitrary $\lfloor k/2\rfloor$ unordered pairs of vertices, there are $\lfloor k/2\rfloor$ vertex-disjoint paths joining the vertices in the pairs. In a previous paper [2] we proved that every cubical d-polytope is $\lfloor d/2\rfloor$-linked. Here we strengthen this result by establishing the $\lfloor (d+1)/2\rfloor$-linkedness of cubical d-polytopes, for every $d\neq 3$.

A graph is *strongly* $\lfloor k/2\rfloor$*-linked* if it has at least k vertices and, for every set X of exactly k vertices organised in arbitrary $\lfloor k/2\rfloor$ unordered pairs of vertices, there are $\lfloor k/2\rfloor$ vertex-disjoint paths joining the vertices in the pairs and avoiding the vertices in X not being paired. We say that a polytope is (strongly) $\lfloor k/2\rfloor$*-linked* if its graph is (strongly) $\lfloor k/2\rfloor$-linked. In this paper, we also prove that every cubical d-polytope is strongly $\lfloor (d+1)/2\rfloor$-linked, for every $d\neq 3$.

These results are best possible for such a class of polytopes.

1. Introduction

The *graph* $G(P)$ of a polytope P is the undirected graph formed by the vertices and edges of the polytope. This paper studies the linkedness of *cubical d-polytopes*,

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d-dimensional polytopes with all their facets being cubes. A *d-dimensional cube* is the convex hull in $\mathbb{R}^d$ of the 2^d vectors $(\pm 1, \ldots, \pm 1)$. By a cube we mean any polytope whose face lattice is isomorphic to the face lattice of a cube.

Denote by $V(X)$ the vertex set of a graph or a polytope X. Given sets A, B of vertices in a graph, a path from A to B, called an *$A-B$ path*, is a (vertex-edge) path $L := u_0 \ldots u_n$ in the graph such that $V(L) \cap A = \{u_0\}$ and $V(L) \cap B = \{u_n\}$. We write $a-B$ path instead of $\{a\}-B$ path, and likewise, write $A-b$ path instead of $A-\{b\}$.

Let G be a graph and X a subset of $2k$ distinct vertices of G. The elements of X are called *terminals*. Let $Y := \{\{s_1, t_1\}, \ldots, \{s_k, t_k\}\}$ be an arbitrary labelling and (unordered) pairing of all the vertices in X. We say that Y is *linked* in G if we can find disjoint $s_i - t_i$ paths for $i \in [1, k]$, where $[1, k]$ denotes the interval $1, \ldots, k$. The set X is *linked* in G if every such pairing of its vertices is linked in G. Throughout this paper, by a set of disjoint paths, we mean a set of vertex-disjoint paths. If G has at least $2k$ vertices and every set of exactly $2k$ vertices is linked in G, we say that G is *k-linked*. If the graph of a polytope is k-linked, we say that the polytope is also *k-linked*.

Larman and Mani [4, Thm. 2] proved that every d-polytope is $\lfloor (d+1)/3 \rfloor$-linked, a result that was slightly improved to $\lfloor (d+2)/3 \rfloor$ in [7, Thm. 2.2]. Concerning cubical polytopes, in a previous paper [2], we answered a question of Wotzlaw [8, Question 5.4.12] proving the following theorem:

Theorem 1. *For every $d \geq 1$, a cubical d-polytope is $\lfloor d/2 \rfloor$-linked.*

In this paper, we extend this result:

Theorem 2 (Linkedness of cubical polytopes)**.** *For every $d \neq 3$, a cubical d-polytope is $\lfloor (d+1)/2 \rfloor$-linked.*

Our methodology relies on results on the connectivity of strongly connected subcomplexes of cubical polytopes, whose proof ideas were first developed in [1], and a number of new insights into the structure of d-cube exposed in [2]. One obstacle that forces some tedious analysis is the fact that the 3-cube is not 2-linked.

In our paper [2], we introduce the notion of strong linkedness. We say that a d-polytope P is *strongly $\lfloor (d+1)/2 \rfloor$-linked* if, for every set X of exactly $d+1$ vertices and every pairing Y with $\lfloor (d+1)/2 \rfloor$ pairs from X, the set Y is linked in $G(P)$ and each path joining a pair in Y avoids the vertices in X not being paired in Y. For odd $d = 2k-1$ the properties of strongly k-linkedness and k-linkedness coincide, since every vertex in X is paired in Y; but they differ for even $d = 2k$–namely strong k-linkedness implies k-linkedness, but the converse is not necessarily true. In this paper, we show that cubical d-polytopes are strongly $\lfloor (d+1)/2 \rfloor$-linked, for $d \neq 3$.

Theorem 3 (Strong linkedness of cubical polytopes)**.** *For every $d \neq 3$, a cubical d-polytope is strongly $\lfloor (d+1)/2 \rfloor$-linked.*

Unless otherwise stated, the graph theoretical notation and terminology follow from [3] and the polytope theoretical notation and terminology from [9]. Moreover, when referring to graph-theoretical properties of a polytope such as minimum degree, linkedness and connectivity, we mean properties of its graph.

2. Preliminary results

This section groups a number of results that will be used in the paper.

Let X be a set of vertices in a graph G. Denote by $G[X]$ the subgraph of G *induced* by X, the subgraph of G that contains all the edges of G with vertices in X. Write $G-X$ for $G[V(G)\setminus X]$. A path in the graph is called *X-valid* if no inner vertex of the path is in X. The *distance* between two vertices s and t in a graph G, denoted $\operatorname{dist}_G(s,t)$, is the length of a shortest path between the vertices.

We start with some results on the linkedness of polytopes in dimensions 3 and 4, which will be useful to establish the base cases in our proofs:

Proposition 4 ([2, Prop. 4 and Cor. 5])**.** *Let G be the graph of a 3-polytope and let X be a set of four vertices of G. The set X is linked in G if and only if there is no facet of the polytope containing all the vertices of X. In particular, no nonsimplicial 3-polytope is 2-linked.*

Proposition 5 ([2, Prop. 6])**.** *Every 4-polytope is 2-linked.*

The next set of results concerns polytopal complexes. A *polytopal complex* $\mathcal{C}$ is a finite nonempty collection of polytopes in $\mathbb{R}^d$ where the faces of each polytope in $\mathcal{C}$ all belong to $\mathcal{C}$ and where polytopes intersect only at faces (if $P_1\in\mathcal{C}$ and $P_2\in\mathcal{C}$ then $P_1\cap P_2$ is a face of both P_1 and P_2). The empty polytope is always in $\mathcal{C}$. The *dimension* of a complex $\mathcal{C}$ is the largest dimension of a polytope in $\mathcal{C}$; if $\mathcal{C}$ has dimension d we say that C is a *d-complex*. Faces of a complex $\mathcal{C}$ of largest and second largest dimension are called *facets* and *ridges*, respectively. If each of the faces of a complex $\mathcal{C}$ is contained in some facet we say that $\mathcal{C}$ is *pure*. Given a polytopal complex $\mathcal{C}$ with vertex set V and a subset X of V, the subcomplex of $\mathcal{C}$ formed by all the faces of $\mathcal{C}$ containing only vertices from X is called *induced* and is denoted by $\mathcal{C}[X]$. Removing from $\mathcal{C}$ all the vertices in a subset $X\subset V(\mathcal{C})$ results in the subcomplex $\mathcal{C}[V(\mathcal{C})\setminus X]$, which we write as $\mathcal{C}-X$. If $X=\{x\}$ we write $\mathcal{C}-x$ rather than $\mathcal{C}-\{x\}$. We say that a subcomplex $\mathcal{C}'$ of a complex $\mathcal{C}$ is a *spanning* subcomplex of $\mathcal{C}$ if $V(\mathcal{C}')=V(\mathcal{C})$. The *graph* of a complex is the undirected graph formed by the vertices and edges of the complex; as in the case of polytopes, we denote the graph of a complex $\mathcal{C}$ by $G(\mathcal{C})$. A pure polytopal complex $\mathcal{C}$ is *strongly connected* if every pair of facets F and F' is connected by a path $F_1\ldots F_n$ of facets in $\mathcal{C}$ such that $F_i\cap F_{i+1}$ is a ridge of $\mathcal{C}$ for $i\in[1,n-1]$, $F_1=F$ and $F_n=F'$; we say that such a path is a *$(d-1,d-2)$-path* or a *facet-ridge path* if the dimensions of the faces can be deduced from the context.

The relevance of strongly connected complexes stems from a result of Sallee that is described below.

Proposition 6 ([6, Sec. 2]). *For every $d \geq 1$, the graph of a strongly connected d-complex is d-connected.*

Strongly connected complexes can be defined from a d-polytope P. Two basic examples are given by the complex of all faces of P, called the *complex* of P and denoted by $\mathcal{C}(P)$, and the complex of all proper faces of P, called the *boundary complex* of P and denoted by $\mathcal{B}(P)$. For a polytopal complex $\mathcal{C}$, the *star* of a face F of $\mathcal{C}$, denoted $\operatorname{star}(F, \mathcal{C})$, is the subcomplex of $\mathcal{C}$ formed by all the faces containing F, and their faces; the *antistar* of a face F of $\mathcal{C}$, denoted $\operatorname{astar}(F, \mathcal{C})$, is the subcomplex of $\mathcal{C}$ formed by all the faces disjoint from F; and the *link* of a face F, denoted $\operatorname{link}(F, \mathcal{C})$, is the subcomplex of $\mathcal{C}$ formed by all the faces of $\operatorname{star}(F, \mathcal{C})$ that are disjoint from F. That is, $\operatorname{astar}(F, \mathcal{C}) = \mathcal{C} - V(F)$ and $\operatorname{link}(F, \mathcal{C}) = \operatorname{star}(F, \mathcal{C}) - V(F)$. Unless otherwise stated, when defining stars, antistars and links in a polytope, we always assume that the underlying complex is the boundary complex of the polytope.

Proposition 7 ([2, Prop. 22]). *For every $d \geq 2$ such that $d \neq 3$, the link of a vertex in a $(d+1)$-cube Q_{d+1} is $\lfloor (d+1)/2 \rfloor$-linked.*

2.1. **d-cube.** In this section we briefly recall results from [2] that we will use in our proofs. One of the main results of that paper is that the d-cube is strongly $\lfloor (d+1)/2 \rfloor$-linked:

Theorem 8 (Strong linkedness of the cube). *For every $d \neq 3$, a d-cube is strongly $\lfloor (d+1)/2 \rfloor$-linked.*

Let v be a vertex in a d-cube Q_d and let v^o denote the vertex at distance d from v, called the vertex *opposite* to v. In the d-cube Q_d, the facet disjoint from a facet F is denoted by F^o, and we say that F and F^o is a pair of *opposite* facets.

Definition 9 (Projection π). For a pair of opposite facets $\{F, F^o\}$ of Q_d, define a projection $\pi_{F^o}^{Q_d}$ from Q_d to F^o by sending a vertex $x \in F$ to the unique neighbour $x_{F^o}^p$ of x in F^o, and a vertex $x \in F^o$ to itself (that is, $\pi_{F^o}^{Q_d}(x) = x$); write $\pi_{F^o}^{Q_d}(x) = x_{F^o}^p$ to be precise, or write $\pi(x)$ or x^p if the cube Q_d and the facet F^o are understood from the context.

We extend this projection to sets of vertices: given a pair $\{F, F^o\}$ of opposite facets and a set $X \subseteq V(F)$, the projection $X_{F^o}^p$ or $\pi_{F^o}^{Q_d}(X)$ of X onto F^o is the set of the projections of the vertices in X onto F^o. For an i-face $J \subseteq F$, the projection $J_{F^o}^p$ or $\pi_{F^o}^{Q_d}(J)$ of J onto F^o is the i-face consisting of the projections of all the vertices of J onto F^o. For a pair $\{F, F^o\}$ of opposite facets in Q^d, the restrictions of the projection π_{F^o} to F and the projection π_F to F^o are bijections.

Let Z be a set of vertices in the graph of a d-cube Q_d. If, for some pair of opposite facets $\{F, F^o\}$, the set Z contains both a vertex $z \in V(F) \cap Z$ and its projection $z_{F^o}^p \in V(F^o) \cap Z$, we say that the pair $\{F, F^o\}$ is *associated* with the set Z in Q_d and that $\{z, z^p\}$ is an *associating pair*. Note that an associating pair can associate only one pair of opposite facets.

The next lemma lies at the core of our methodology.

Lemma 10 ([2, Lemma 6]). *Let Z be a nonempty subset of $V(Q_d)$. Then the number of pairs $\{F, F^o\}$ of opposite facets associated with Z is at most $|Z|-1$.*

The relevance of the lemma stems from the fact that a pair of opposite facets $\{F, F^o\}$ not associated with a given set of vertices Z allows each vertex z in Z to have "free projection"; that is, for every $z \in Z \cap V(F)$ the projection $\pi_{F^o}(z)$ is not in Z, and for $z \in Z \cap V(F^o)$ the projection $\pi_F(z)$ is not in Z.

Given sets A, B, X of vertices in a graph G, the set X *separates* A from B if every $A-B$ path in the graph contains a vertex from X. A set X separates two vertices a, b not in X if it separates $\{a\}$ from $\{b\}$. We call the set X a *separator* of the graph.

A set of vertices in a graph is *independent* if no two of its elements are adjacent.

Corollary 11 ([2, Corollary 9]). *A separator of cardinality d in a d-cube is an independent set.*

Remark 12. If x and y are vertices of a cube, then they share at most two neighbours. In other words, the complete bipartite graph $K_{2,3}$ is not a subgraph of the cube; in fact, it is not an induced subgraph of any simple polytope [5, Cor. 1.12(iii)].

The following theorem is a consequence of Menger's theorem:

Theorem 13 ([2, Theorem 9]). *Let G be a k-connected graph, and let A and B be two subsets of its vertices, each of cardinality at least k. Then there are k disjoint $A-B$ paths in G.*

We need the following three technical, but useful, lemmas in our proof. For a set $Y := \{\{s_1, t_1\}, \dots, \{s_k, t_k\}\}$ of pairs of vertices in a graph, a *Y-linkage* $\{L_1, \dots, L_k\}$ is a set of disjoint paths with the path L_i joining the pair $\{s_i, t_i\}$ for $i \in [1, k]$. For a path $L := u_0 \dots u_n$ we often write $u_i L u_j$ for $0 \le i \le j \le n$ to denote the subpath $u_i \dots u_j$.

Lemma 14 ([2, Lemma 14]). *Let P be a cubical d-polytope with $d \ge 4$. Let X be a set of $d+1$ vertices in P, all contained in a facet F. Let $k := \lfloor (d+1)/2 \rfloor$. Arbitrarily label and pair $2k$ vertices in X to obtain $Y := \{\{s_1, t_1\}, \dots, \{s_k, t_k\}\}$. Then, for at least $k-1$ of these pairs $\{s_i, t_i\}$, there is an X-valid s_i-t_i path in F.*

Lemma 15. *Let $\ell \le k$. Let X be a set of 2ℓ distinct vertices of a k-linked graph G, let Y be a labelling and pairing of the vertices in X, and let Z be a set of $2k-2\ell$ vertices in G such that $X \cap Z = \emptyset$. Then there exists a Y-linkage in G that avoids every vertex in Z.*

Lemma 16 ([7, Sec. 3]). *Let G be a $2k$-connected graph and let G' be a k-linked subgraph of G. Then G is k-linked.*

3. Connectivity of cubical polytopes

The aim of this section is to present a couple of results related to the connectivity of strongly connected complexes in cubical polytopes. The first results are from [1].

Lemma 17 ([1, Lem. 8])**.** *Let F be a proper face in the d-cube Q_d. Then the antistar of F is a strongly connected $(d-1)$-complex.*

Proposition 18 ([1, Prop. 13])**.** *Let F be a facet in the star $\mathcal{S}$ of a vertex in a cubical d-polytope. Then the antistar of F in $\mathcal{S}$ is a strongly connected $(d-2)$-subcomplex of $\mathcal{S}$.*

We proceed with two simple but useful remarks.

Remark 19. Let P be a cubical d-polytope. Let v be a vertex of P and let F be a face of P containing v. In addition, let v^o be the vertex of F opposite to v. The smallest face in the polytope containing both v and v^o is precisely F.

Remark 20. For any two faces F, J of a polytope, with F not contained in J, there is a facet containing J but not F. In particular, for any two distinct vertices of a polytope, there is a facet containing one but not the other.

The proof idea in Proposition 18 can be pushed a bit further to obtain a rather technical result that we prove next. Two vertex-edge paths are *independent* if they share no inner vertex.

Lemma 21. *Let P be a cubical d-polytope with $d \ge 4$. Let s_1 be any vertex in P and let $\mathcal{S}_1$ be the star of s_1 in the boundary complex of P. Let s_2 be any vertex in $\mathcal{S}_1$, other than s_1. Define the following sets:*

- *F_1 in $\mathcal{S}_1$, a facet containing s_1 but not s_2;*
- *F_{12} in $\mathcal{S}_1$, a facet containing s_1 and s_2;*
- *$\mathcal{S}_{12}$, the star of s_2 in $\mathcal{S}_1$ (that is, the subcomplex of $\mathcal{S}_1$ formed by the facets of P in $\mathcal{S}_1$ containing s_2);*
- *$\mathcal{A}_1$, the antistar of F_1 in $\mathcal{S}_1$; and*
- *$\mathcal{A}_{12}$, the subcomplex of $\mathcal{S}_{12}$ induced by $V(\mathcal{S}_{12}) \setminus (V(F_1) \cup V(F_{12}))$.*

Then the following assertions hold.

(i) The complex $\mathcal{S}_{12}$ is a strongly connected $(d-1)$-subcomplex of $\mathcal{S}_1$.

(ii) If there are more than two facets in $\mathcal{S}_{12}$, then, between any two facets of $\mathcal{S}_{12}$ that are different from F_{12}, there exists a $(d-1, d-2)$-path in $\mathcal{S}_{12}$ that does not contain the facet F_{12}.

(iii) If $\mathcal{S}_{12}$ contains more than one facet, then the subcomplex $\mathcal{A}_{12}$ of $\mathcal{S}_{12}$ contains a spanning strongly connected $(d-3)$-subcomplex.

Proof. Let us prove (i). Let ψ define the natural anti-isomorphism from the face lattice of P to the face lattice of its dual P^*. The facets in $\mathcal{S}_1$ correspond to the vertices in the facet $\psi(s_1)$ in P^* corresponding to s_1; likewise for the facets in

$\mathrm{star}(s_2, \mathcal{B}(P))$ and the vertices in $\psi(s_2)$. The facets in $\mathcal{S}_{12}$ correspond to the vertices in the nonempty face $\psi(s_1) \cap \psi(s_2)$ of P^*. The existence a facet-ridge path in $\mathcal{S}_{12}$ between any two facets J_1 and J_2 of $\mathcal{S}_{12}$ amounts to the existence of a vertex-edge path in $\psi(s_1) \cap \psi(s_2)$ between $\psi(J_1)$ and $\psi(J_2)$. That $\mathcal{S}_{12}$ is a strongly connected $(d-1)$-complex now follows from the connectivity of the graph of $\psi(s_1) \cap \psi(s_2)$ (Balinski's theorem), as desired.

We proceed with the proof of (ii). Let J_1 and J_2 be two facets of $\mathcal{S}_{12}$, other than F_{12}. If there are more than two facets in $\mathcal{S}_{12}$, then the face $\psi(s_1) \cap \psi(s_2)$ is at least bidimensional. As a result, the graph of $\psi(s_1) \cap \psi(s_2)$ is at least 2-connected by Balinski's theorem. By Menger's theorem, there are at least two independent vertex-edge paths in $\psi(s_1) \cap \psi(s_2)$ between $\psi(J_1)$ and $\psi(J_2)$. Pick one such path L^* that avoids the vertex $\psi(F_{12})$ of $\psi(s_1) \cap \psi(s_2)$. Dualising this path L^* gives a $(d-1, d-2)$-path between J_1 and J_2 in $\mathcal{S}_{12}$ that does not contain the facet F_{12}.

We finally prove (iii). Assume that $\mathcal{S}_{12}$ contains more than one facet. We need some additional notation.

- Let F be a facet in $\mathcal{S}_{12}$ other than F_{12}; it exists by our assumption on $\mathcal{S}_{12}$.
- Let $\mathcal{A}_1^F$ denote the subcomplex $F - V(F_1)$; that is, $\mathcal{A}_1^F$ is the antistar of $F \cap F_1$ in F.
- Let $\mathcal{A}_{12}^F$ denote the subcomplex $F - (V(F_1) \cup V(F_{12}))$, the subcomplex of F induced by $V(F) \setminus (V(F_1) \cup V(F_{12}))$.

We require the following claim.

Claim 1. $\mathcal{A}_{12}^F$ contains a spanning strongly connected $(d-3)$-subcomplex $\mathcal{C}^F$.

Proof. We first show that $\mathcal{A}_{12}^F \neq \emptyset$. Denoting by s_1^o the vertex in F opposite to s_1, we have that s_1^o is not in F_1 or in F_{12} by Remark 19. So s_1^o is in $\mathcal{A}_{12}^F$.

Notice that $s_1 \notin \mathcal{A}_1^F$. From Lemma 17 it follows that $\mathcal{A}_1^F$ is a strongly connected $(d-2)$-subcomplex of F. Write

$$\mathcal{A}_1^F = \mathcal{C}(R_1) \cup \cdots \cup \mathcal{C}(R_m),$$

where R_i is a $(d-2)$-face of F for $i \in [1, m]$. No ridge R_i is contained in F_{12}; otherwise $R_i = F \cap F_{12}$, which implies that $s_1 \in R_i$, and therefore that $s_1 \in \mathcal{A}_1^F$, a contradiction. Moreover, $s_1^o \in R_i$ for every $i \in [1, m]$, since every ridge of F contains either s_1 or s_1^o, and $s_1 \notin R_i$.

Let $\mathcal{C}_i := \mathcal{B}(R_i) - V(F_{12})$. As $R_i \not\subset F_{12}$, we have $\dim R_i \cap F_{12} \leq d-3$. Hence $\mathcal{C}_i$ is nonempty. If $R_i \cap F_{12} \neq \emptyset$, then $\mathcal{C}_i$ denotes the antistar of $R_i \cap F_{12}$ in R_i, a spanning strongly connected $(d-3)$-subcomplex of R_i by Lemma 17. If $R_i \cap F_{12} = \emptyset$, then $\mathcal{C}_i$ denotes the boundary complex of R_i, again a spanning strongly connected $(d-3)$-subcomplex of R_i.

Let

$$\mathcal{C}^F := \bigcup \mathcal{C}_i.$$

Then the complex $\mathcal{C}^F$ is a spanning $(d-3)$-subcomplex of $\mathcal{A}_{12}^F$; we show it is strongly connected.

Take any two $(d-3)$-faces W and W' in $\mathcal{C}^F$. We find a $(d-3,d-4)$-path L in $\mathcal{C}^F$ between W and W'. There exist ridges R and R' in $\mathcal{A}_1^F$ with $W \subset R$ and $W' \subset R'$. Since $\mathcal{A}_1^F$ is a strongly connected $(d-2)$-complex, there is a $(d-2,d-3)$-path $R_{i_1} \ldots R_{i_p}$ in $\mathcal{A}_1^F$ between $R_{i_1} = R$ and $R_{i_p} = R'$, with $R_{i_j} \in \mathcal{A}_1^F$ for $j \in [1,p]$. We will show by induction on the length p of the $(d-2,d-3)$-path $R_{i_1} \ldots R_{i_p}$ that there is a $(d-3,d-4)$-path in $\mathcal{C}^F$ between W and W'.

If $p = 1$, then $R_{i_1} = R_{i_p} = R = R'$. The existence of the path follows from the strong connectivity of $\mathcal{C}_{i_1}$.

Suppose that the claim is true when the length of the path is $p-1$. We already established that $s_1^o \in R_{i_j}$ for every $j \in [1,p]$ and that $s_1^o \notin F_{12}$. Consequently, we get that $R_{i_{p-1}} \cap R_{i_p} \not\subset F_{12}$, and therefore, that $\dim R_{i_{p-1}} \cap R_{i_p} \cap F_{12} \le d-4$. Hence the subcomplex $\mathcal{B}_{i_{p-1}} := \mathcal{B}(R_{i_{p-1}} \cap R_{i_p}) - V(F_{12})$ of $\mathcal{B}(R_{i_{p-1}} \cap R_{i_p})$ is a nonempty, strongly connected $(d-4)$-complex by Lemma 17; in particular, it contains a $(d-4)$-face U_{i_p}. Furthermore, $\mathcal{B}_{i_{p-1}} \subset \mathcal{C}_{i_{p-1}} \cap \mathcal{C}_{i_p}$.

Let $W_{i_{p-1}}$ and W_{i_p} be $(d-3)$-faces in $\mathcal{C}_{i_{p-1}}$ and $\mathcal{C}_{i_p}$ containing U_{i_p} respectively. By the induction hypothesis, the existence of the $(d-2,d-3)$-path $R_{i_1} \ldots R_{i_{p-1}}$ implies the existence of a $(d-3,d-4)$-path L_{p-1} in $\mathcal{C}^F$ from W to $W_{i_{p-1}}$. The strong connectivity of $\mathcal{C}_{i_p}$ gives the existence of a path L_p from W_{i_p} to W'. Finally, the desired path L is the concatenation of these two paths: $L = L_{p-1}L_p$. The existence of the path L between W and W' completes the proof of Claim 1. $\square$

We are now ready to complete the proof of (iii). The proof goes along the lines of the proof of Claim 1. We let

$$\mathcal{S}_{12} = \bigcup_{i=1}^{m} \mathcal{C}(J_i),$$

where the facets $J_1, \ldots, J_m$ are all the facets in P containing s_1 and s_2.

For every $i \in [1,m]$ we let $\mathcal{C}^{J_i}$ be the spanning strongly connected $(d-3)$-subcomplex in $\mathcal{A}_{12}^{J_i}$ given by Claim 1. And we let

$$\mathcal{C} := \bigcup \mathcal{C}^{J_i}.$$

Then $\mathcal{C}$ is a spanning $(d-3)$-subcomplex of $\mathcal{A}_{12}$; we show it is strongly connected.

If there are exactly two facets in $\mathcal{S}_{12}$, namely F_{12} and some other facet F, then the complex $\mathcal{A}_{12}$ coincides with the complex $\mathcal{A}_{12}^F$. The strong $(d-3)$-connectivity of $\mathcal{A}_{12}^F$ is then settled by Claim 1. Hence assume that there are more than two facets in $\mathcal{S}_{12}$; this implies that the smallest face containing s_1 and s_2 in $\mathcal{S}_{12}$ is at most $(d-3)$-dimensional.

Take any two $(d-3)$-faces W and W' in $\mathcal{C}$. Let $J \ne F_{12}$ and $J' \ne F_{12}$ be facets of $\mathcal{S}_{12}$ such that $W \subset J$ and $W' \subset J'$. By (ii), we can find a $(d-1,d-2)$-path $J_{i_1} \ldots J_{i_q}$ in $\mathcal{S}_{12}$ between $J_{i_1} = J$ and $J_{i_q} = J'$ such that $J_{i_j} \ne F_{12}$ for $j \in [1,q]$. We will show that a $(d-3,d-4)$-path L exists between W and W' in $\mathcal{C}$, using an induction on the length q of the path $J_{i_1} \ldots J_{i_q}$.

If $q = 1$, then W and W' belong to the same facet F in $\mathcal{S}_{12}$, which is different from F_{12}. In this case, W and W' are both in $\mathcal{A}_{12}^F$, and consequently, Claim 1 gives the desired $(d-3, d-4)$-path between W and W' in $\mathcal{A}_{12}^F \subseteq \mathcal{C}$.

Suppose that the induction hypothesis holds when the length of the path is $q-1$. First, we show that there exists a $(d-4)$-face U_q in $C^{J_{i_{q-1}}} \cap C^{J_{i_q}}$. As $J_{i_{q-1}}, J_{i_q} \neq F_{12}$, we obtain that $\mathcal{B}(J_{i_{q-1}} \cap J_{i_q}) - V(F_{12})$ is a nonempty, strongly connected $(d-3)$-subcomplex (Lemma 17); in particular, it contains a $(d-3)$-face K_q. We pick U_q in $\mathcal{B}(K_q) - V(F_1)$ as follows. It holds that $K_q \not\subset F_1$; otherwise $K_q = J_{i_{q-1}} \cap J_{i_q} \cap F_1$, a contradiction because $s_1 \notin K_q$ but $s_1 \in J_{i_{q-1}} \cap J_{i_q} \cap F_1$. As a consequence, $\mathcal{B}(K_q) - V(F_1)$ is a nonempty, strongly connected $(d-4)$-subcomplex (Lemma 17 again); in particular, it contains a desired $(d-4)$-face U_q.

Pick $(d-3)$-faces $W_{q-1} \in \mathcal{C}^{J_{i_{q-1}}}$ and $W_q \in \mathcal{C}^{J_{i_q}}$ such that both contain the $(d-4)$ face U_q. The induction hypothesis tells us that there exists a $(d-3, d-4)$-path L_{q-1} from W to W_{q-1} in $\mathcal{C}$. And the strong $(d-3)$-connectivity of $\mathcal{C}^{J_{i_q}}$ ensures that there exists a $(d-3, d-4)$-path L_q from W_q to W'. By concatenating these two paths, we can obtain the path $L = WL_{q-1}W_{q-1}W_qL_qW'$. This completes the proof of the lemma. $\square$

4. Linkedness of cubical polytopes

The aim of this section is to prove that, for every $d \neq 3$, a cubical d-polytope is $\lfloor (d+1)/2 \rfloor$-linked (Theorem 2). It suffices to prove Theorem 2 for odd $d \geq 5$; since $\lfloor d/2 \rfloor = \lfloor (d+1)/2 \rfloor$ for even d, Theorem 1 trivially establishes Theorem 2 in this case.

The proof of Theorem 2 heavily relies on Lemma 23. To state the lemma we require the following definitions.

Definition 22 (Configuration dF)**.** Let $d \geq 3$ be odd and let X be a set of at least $d+1$ terminals in a cubical d-polytope P. In addition, let Y be a labelling and pairing of the vertices in X. A terminal of X, say s_1, is in *Configuration dF* if the following conditions are satisfied:

(i) at least $d+1$ vertices of X appear in a facet F of P;
(ii) the terminals in the pair $\{s_1, t_1\} \in Y$ are at distance $d-1$ in F (that is, $\operatorname{dist}_F(s_1, t_1) = d-1$); and
(iii) the neighbours of t_1 in F are all vertices of X.

Lemma 23. *Let $d \geq 5$ be odd and let $k := (d+1)/2$. Let s_1 be a vertex in a cubical d-polytope and let $\mathcal{S}_1$ be the star of s_1 in the polytope. Moreover, let $Y := \{\{s_1, t_1\}, \ldots, \{s_k, t_k\}\}$ be a labelling and pairing of $2k$ distinct vertices of $\mathcal{S}_1$. Then the set Y is linked in $\mathcal{S}_1$ if and only if the vertex s_1 is not in Configuration dF.*

We defer the proof of Lemma 23 to Subsection 4.1. We are now ready to prove our main result, assuming Lemma 23.

Proof of Theorem 2 (Linkedness of cubical polytopes). Theorem 1 settled the case of even d, so we assume d is odd.

Let d be odd and $d \geq 5$ and let $k := (d+1)/2$. Let X be any set of $2k$ vertices in the graph G of a cubical d-polytope P. Recall the vertices in X are called terminals. Also let $Y := \{\{s_1, t_1\}, \ldots, \{s_k, t_k\}\}$ be a labelling and pairing of the vertices of X. We aim to find a Y-linkage $\{L_1, \ldots, L_k\}$ in G where L_i joins the pair $\{s_i, t_i\}$ for $i = 1, \ldots, k$. Recall that a path is X-valid if it contains no inner vertex from X.

The first step of the proof is to reduce the analysis space from the whole polytope to a more manageable space, the star $\mathcal{S}_1$ of a terminal vertex in the boundary complex of P, say that of s_1. We do so by considering $d = 2k - 1$ disjoint paths $S_i := s_i - \mathcal{S}_1$ $(i \in [2, k])$ and $T_j := t_j - \mathcal{S}_1$ $(j \in [1, k])$ from the terminals into $\mathcal{S}_1$. Here we resort to the d-connectivity of G. In addition, let $S_1 := s_1$. We then denote by $\bar{s}_i$ and $\bar{t}_j$ the intersection of the paths S_i and T_j with $\mathcal{S}_1$. Using the vertices $\bar{s}_i$ and $\bar{t}_i$ for $i \in [1, k]$, define sets $\bar{X}$ and $\bar{Y}$ in $\mathcal{S}_1$, counterparts to the sets X and Y of G. In an abuse of terminology, we also say that the vertices $\bar{s}_i$ and $\bar{t}_i$ are terminals. In this way, the existence of a $\bar{Y}$-linkage $\{\bar{L}_1, \ldots, \bar{L}_k\}$ with $\bar{L}_i := \bar{s}_i - \bar{t}_i$ in $G(\mathcal{S}_1)$ implies the existence of a Y-linkage $\{L_1, \ldots, L_k\}$ in $G(P)$, since each path $\bar{L}_i$ $(i \in [1, k])$ can be extended with the paths S_i and T_i to obtain the corresponding path $L_i = s_i S_i \bar{s}_i \bar{L}_i \bar{t}_i T_i t_i$.

The second step of the proof is to find a $\bar{Y}$-linkage $\{\bar{L}_1, \ldots, \bar{L}_k\}$ in $G(\mathcal{S}_1)$, whenever possible. According to Lemma 23, there is a $\bar{Y}$-linkage in $G(\mathcal{S}_1)$ provided that the vertex s_1 is not in Configuration dF. The existence of a $\bar{Y}$-linkage in turn gives the existence of a Y-linkage, and completes the proof of the theorem in this case.

The third and final step is to deal with Configuration dF for s_1. Hence assume that the vertex s_1 is in Configuration dF. This is implies that

(i) there exists a unique facet F_1 of $\mathcal{S}_1$ containing $\bar{t}_1$; that
(ii) $|\bar{X} \cap V(F_1)| = d + 1$; and that
(iii) $\text{dist}_{F_1}(\bar{s}_1, \bar{t}_1) = d - 1$ and all the $d - 1$ neighbours of $\bar{t}_1$ in F_1, and thus in $\mathcal{S}_1$, belong to $\bar{X}$.

Let R be a $(d-2)$-face of F_1 containing $s_1^o = \bar{t}_1$, then $s_1 \notin R$. Denote by R_{F_1} the $(d-2)$-face of F_1 disjoint from R. Let J be the other facet of P containing R and let R_J denote the $(d-2)$-face of J disjoint from R. Then R_J is disjoint from F_1. Partition the vertex set $V(R_J)$ of R_J into the vertex sets of two induced subgraphs G_{bad} and G_{good} such that G_{bad} contains the neighbours of the terminals in R, namely $V(G_{\text{bad}}) = \pi^J_{R_J}(\bar{X} \cap V(R))$ and $V(G_{\text{good}}) = V(R_J) \setminus V(G_{\text{bad}})$. Then $\pi^J_R(V(G_{\text{bad}})) \subseteq \bar{X}$ and $\pi^J_R(V(G_{\text{good}})) \cap \bar{X} = \emptyset$. See Fig. 1(a).

Consider again the paths S_i and T_j that bring the vertices s_i $(i \in [2, k])$ and t_j $(j \in [1, k])$ into $\mathcal{S}_1$. Also recall that the paths S_i and T_j intersect $\mathcal{S}_1$ at $\bar{s}_i$ and $\bar{t}_j$, respectively. We distinguish two cases: either at least one path S_i or T_j touches R_J or no path S_i or T_j touches R_J. In the former case we redirect one aforementioned path S_i or T_j to break Configuration dF for s_1 and use Lemma 23, while in the latter case we find the $\bar{Y}$-linkage using the antistar of s_1.

Case 1. Suppose at least one path S_i or T_j touches R_J.

If possible, pick one such path, say S_ℓ, for which it holds that $V(S_\ell)\cap V(G_{\text{good}})\neq\emptyset$. Otherwise, pick one such path, say S_ℓ, that does not contain $\pi^J_{R_J}(t_1)$, if it is possible. If none of these two selections are possible, then there is exactly one path S_i or T_j touching R_J, say S_ℓ, in which case $\pi^J_{R_J}(t_1)\in V(S_\ell)$.

We replace the path S_ℓ by a new path s_ℓ–$\mathcal{S}_1$ that is disjoint from the other paths S_i and T_j and we replace the old terminal $\bar{s}$ by a new terminal that causes s_1 not to be in Configuration dF. First suppose that there exists s'_ℓ in $V(S_\ell)\cap V(G_{\text{good}})$. Then the old path S_ℓ is replaced by the path $s_\ell S_\ell s'_\ell\pi^J_R(s'_\ell)$, and the old terminal $\bar{s}_\ell$ is replaced by $\pi^J_R(s'_\ell)$. Now suppose that $V(S_\ell)\cap V(G_{\text{good}})=\emptyset$. Then every path S_i and T_j that touches R_J is disjoint from G_{good}. Denote by s'_ℓ the first intersection of S_ℓ with R_J. Let M_ℓ be a shortest path in R_J from $s'_\ell\in V(G_{\text{bad}})$ to a vertex $s''_\ell\in V(G_{\text{good}})$. By our selection of S_ℓ this path M_ℓ always exists. If $s''_\ell\in V(G_{\text{good}})\setminus V(\mathcal{S}_1)$ then the old path S_ℓ is replaced by the path $s_\ell S_\ell s'_\ell M_\ell s''_\ell\pi^J_R(s''_\ell)$, and the old terminal $\bar{s}_\ell$ is replaced by $\pi^J_R(s''_\ell)$. If instead $s''_\ell\in V(G_{\text{good}})\cap V(\mathcal{S}_1)$ then the old path S_ℓ is replaced by the path $s_\ell S_\ell s'_\ell M_\ell s''_\ell$, and the old terminal $\bar{s}_\ell$ is replaced by s''_ℓ. Refer to Fig. 1(b) for a depiction of this case.

In any case, the replacement of the old vertex $\bar{s}_\ell$ with the new $\bar{s}_\ell$ forces s_1 out of Configuration dF, and we can apply Lemma 23 to find a $\bar{Y}$-linkage. The case of S_ℓ being equal to T_1 requires a bit more explanation in order to make sure that the vertex s_1 does not end up in a new configuration dF. Let $\mathcal{A}_1$ be the antistar of F_1 in $\mathcal{S}_1$. The new vertex $\bar{t}_1$ is either in F_1 or in $\mathcal{A}_1$. If the new $\bar{t}_1$ is in F_1 then it is plain that s_1 is not in Configuration dF. If the new vertex $\bar{t}_1$ is in $\mathcal{A}_1$, then a new facet F_1 containing s_1 and the new $\bar{t}_1$ cannot contain all the $d-1$ neighbours of the old $\bar{t}_1$ in the old F_1, since the intersection between the new and the old F_1 is at most $(d-2)$-dimensional and no $(d-2)$-dimensional face of the old F_1 contains all the $d-1$ neighbours of the old $\bar{t}_1$. This completes the proof of the case.

Case 2. For any ridge R of F_1 that contains $\bar{t}_1$, the aforementioned ridge R_J in the facet J is disjoint from all the paths S_i and T_j.

Consider the vertex $\bar{t}_1$ in F_1, an aforementioned ridge R, and the corresponding facet J and ridge R_J. There is a unique neighbour of $\bar{t}_1$ in R_{F_1}, say $\bar{s}_k$, while every other neighbour of $\bar{t}_1$ in F_1 is in R. Let $\bar{X}^p:=\pi^J_{R_J}(\bar{X}\setminus\{s_1,\bar{s}_k,\bar{t}_k\})$ and let $s_1^{pp}:=\pi^J_{R_J}(\pi^{F_1}_R(s_1))$. See Fig. 1(c). The $d-1$ vertices in $\bar{X}^p\cup\{s_1^{pp}\}$ can be linked in R_J (Theorem 8) by a linkage $\{\bar{L}'_1,\ldots,\bar{L}'_{k-1}\}$. Observe that, for the special case of $d=5$ where R_J is a 3-cube, the sequence $s_1^{pp},\pi^J_{R_J}(\bar{s}_2),\pi^J_{R_J}(\bar{t}_1),\pi^J_{R_J}(\bar{t}_2)$ cannot be in a 2-face in cyclic order, since $\operatorname{dist}_{R_J}(s_1^{pp},\pi^J_{R_J}(\bar{t}_1))=3$. The linkage $\{\bar{L}'_1,\ldots,\bar{L}'_{k-1}\}$ together with the two-path $\bar{L}_k:=\bar{s}_k\pi^{F_1}_{R_{F_1}}(\bar{t}_k)\bar{t}_k$ can be extended to a linkage $\{\bar{L}_1,\ldots,\bar{L}_k\}$ given by

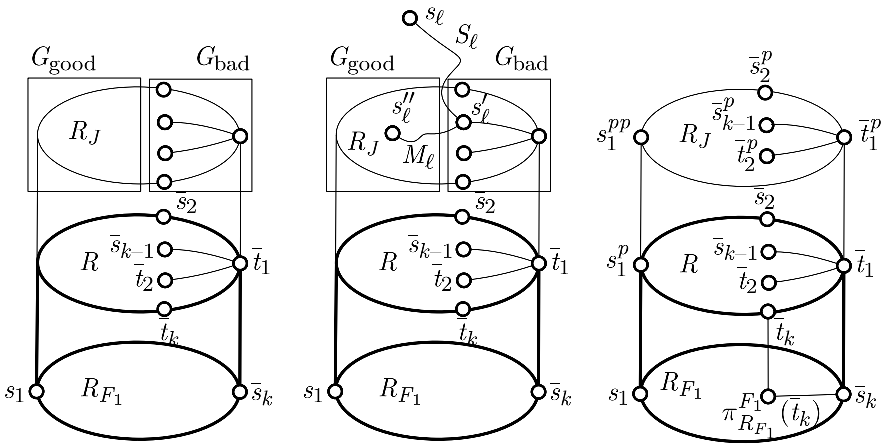

FIGURE 1. Auxiliary figure for Theorem 2, where the facet F_1 is highlighted in bold. **(a)** A depiction of the subgraphs G_{good} and G_{bad} of R_J. **(b)** A configuration where a path S_i or T_j touches R_J. **(c)** A configuration where no path S_i or T_j touches R_J.

$$\bar{L}_i := \begin{cases} s_1\pi^{F_1}_{R}(s_1)s_1^{pp}\bar{L}'_1\pi^{J}_{R_J}(\bar{t}_1)\bar{t}_1, & \text{for } i=1;\\ \bar{s}_i\pi^{J}_{R_J}(\bar{s}_i)\bar{L}'_i\pi^{J}_{R_J}(\bar{t}_i)\bar{t}_i, & \text{for } i\in[2,k-1];\\ \bar{s}_k\pi^{F_1}_{R_{F_1}}(\bar{t}_k)\bar{t}_k, & \text{for } i=k. \end{cases}$$

Concatenating the paths S_i ($i \in [2,k]$) and T_j ($j \in [1,k]$) with the linkage $\{\bar{L}_1,\ldots,\bar{L}_k\}$ gives the desired Y-linkage. This completes the proof of the case, and with it the proof of the theorem. □

4.1. **Proof of Lemma 23.** This section is devoted to proving Lemma 23. Before starting the proof, we require a couple of results.

Proposition 24. *Let F be a facet in the star $\mathcal{S}$ of a vertex in a cubical d-polytope. Then, for every $d \ge 2$, the antistar of F in $\mathcal{S}$ is $\lfloor (d-2)/2 \rfloor$-linked.*

Proof. Let $\mathcal{S}$ be the star of a vertex s in a cubical d-polytope and let F be a facet in the star $\mathcal{S}$. Let $\mathcal{A}$ denote the antistar of F in $\mathcal{S}$.

The case of $d = 2, 3$ imposes no demand on $\mathcal{A}$, while the case $d = 4, 5$ amounts to establishing that the graph of $\mathcal{A}$ is connected. The graph of $\mathcal{A}$ is in fact $(d-2)$-connected, since $\mathcal{A}$ is a strongly connected $(d-2)$-complex (Proposition 18). See also Proposition 6. So assume $d \ge 6$.

There is a $(d-2)$-face R in $\mathcal{A}$. Indeed, take a $(d-2)$-face R' in F containing s and consider the other facet F' in $\mathcal{S}$ containing R'; the $(d-2)$-face of F' disjoint from R' is the desired R. By Theorem 8 the ridge R is $\lfloor (d-1)/2 \rfloor$-linked but we only require it to be $\lfloor (d-2)/2 \rfloor$-linked. By Propositions 6 and 18 the graph of $\mathcal{A}$ is $(d-2)$-connected. Combining the linkedness of R and the connectivity of the graph of $\mathcal{A}$ settles the proposition by virtue of Lemma 16. □

For a pair of opposite facets $\{F, F^o\}$ in a cube, the restriction of the projection $\pi_{F^o} : Q_d \to F^o$ (Definition 9) to F is a bijection from $V(F)$ to $V(F^o)$. With the help of π, given the star $\mathcal{S}$ of a vertex s in a cubical polytope and a facet F in $\mathcal{S}$, we can define an injection from the vertices in F, except the vertex opposite to s, to the antistar of F in $\mathcal{S}$. Defining this injection is the purpose of Lemma 25.

Lemma 25. *Let F be a facet in the star $\mathcal{S}$ of a vertex s in a cubical d-polytope. Then there is an injective function, defined on the vertices of F except the vertex s^o opposite to s, that maps each such vertex in F to a neighbour in $V(\mathcal{S}) \setminus V(F)$.*

Proof. We construct the aforementioned injection f between $V(F)\setminus\{s^o\}$ and $V(\mathcal{S})\setminus V(F)$ as follows. Let $R_1, \ldots, R_{d-1}$ be the $(d-2)$-faces of F containing s, and let $J_1, \ldots, J_{d-1}$ be the other facets of $\mathcal{S}$ containing $R_1, \ldots, R_{d-1}$, respectively. Every vertex in F other than s^o lies in $R_1 \cup \cdots \cup R_{d-1}$. Let R_i^o be the $(d-2)$-face in J_i that is opposite to R_i for $i \in [1, d-1]$. For every vertex v in $V(R_j)\setminus(V(R_1)\cup\cdots\cup V(R_{j-1}))$ define $f(v)$ as the projection π in J_j of v onto $V(R_j^o)$, namely $f(v) := \pi_{R_j^o}(v)$; observe that $\pi_{R_j^o}(v) \in V(R_j^o) \setminus (V(R_1^o) \cup \cdots \cup V(R_{j-1}^o))$. Here R_{-1} and R_{-1}^o are empty sets. The function f is well defined as R_i and R_i^o are opposite $(d-2)$-cubes in the $(d-1)$-cube J_i.

To see that f is an injection, take distinct vertices $v_1, v_2 \in V(F) \setminus \{s^o\}$, where $v_1 \in V(R_i) \setminus (V(R_1) \cup \cdots \cup V(R_{i-1}))$ and $v_2 \in V(R_j) \setminus (V(R_1) \cup \cdots \cup V(R_{j-1}))$ for $i \le j$. If $i = j$ then $f(v_1) = \pi_{R_i^o}(v_1) \neq \pi_{R_i^o}(v_2) = f(v_2)$. If instead $i < j$ then $f(v_1) \in V(R_i^o) \subseteq V(R_1^o) \cup \cdots \cup V(R_{j-1}^o)$, while $f(v_2) \notin V(R_1^o) \cup \cdots \cup V(R_{j-1}^o)$. $\square$

Proof of Lemma 23. Let $d \ge 5$ be odd and let $k := (d+1)/2$. Let s_1 be a vertex in a cubical d-polytope P and let $\mathcal{S}_1$ denote the star of s_1 in $\mathcal{B}(P)$. Let X be any set of $2k$ vertices in the graph $G(\mathcal{S}_1)$ of $\mathcal{S}_1$. The vertices in X are our terminals. Also let $Y := \{\{s_1, t_1\}, \ldots, \{s_k, t_k\}\}$ be a labelling and pairing of the vertices of X. We aim to find a Y-linkage $\{L_1, \ldots, L_k\}$ in G where L_i joins the pair $\{s_i, t_i\}$ for $i = 1, \ldots, k$. Recall that a path is X-valid if it contains no inner vertex from X.

We consider a facet F_1 of $\mathcal{S}_1$ containing t_1 and having the largest possible number of terminals.

The necessary condition of Y being linked in $\mathcal{S}_1$ is easy to prove. Suppose that the vertex s_1 is in Configuration dF. Since $\mathrm{dist}_{F_1}(s_1, t_1) = d-1$, it follows that F_1 is the only facet of $\mathcal{S}_1$ that contains t_1. Then all the neighbours of t_1 in F_1, and thus, in $\mathcal{S}_1$ are in X. As a consequence, every $s_1 - t_1$ path in $\mathcal{S}_1$ must touch X. Hence Y is not linked.

We decompose the sufficiency proof into four cases based on the number of terminals in F_1, proceeding from the more manageable case to the more involved one.

Case 1. $|X \cap V(F_1)| = d$.
Case 2. $3 \le |X \cap V(F_1)| \le d-1$.
Case 3. $|X \cap V(F_1)| = 2$.
Case 4. $|X \cap V(F_1)| = d+1$ and the vertex s_1 is not in Configuration dF.

The sufficiency proof of Lemma 23 is long, so we outline the main ideas. We let $\mathcal{A}_1$ be the antistar of F_1 in $\mathcal{S}_1$ and let $\mathcal{L}_1$ be the link of s_1 in F_1. Using the $(k-1)$-linkedness of F_1 (Theorem 8), we link as many pairs of terminals in F_1 as possible through disjoint X-valid paths $L_i := s_i{-}t_i$. For those terminals that cannot be linked in F_1, if possible we use the injection from $V(F_1)$ to $V(\mathcal{A}_1)$ granted by Lemma 25 to find a set $N_{\mathcal{A}_1}$ of pairwise distinct neighbours in $\mathcal{A}_1$ not in X. Then, using the $(k-2)$-linkedness of $\mathcal{A}_1$ (Proposition 24), we link the corresponding pairs of terminals in $\mathcal{A}_1$ and vertices in $N_{\mathcal{A}_1}$ accordingly. This general scheme does not always work, as the vertex s_1^o opposite to s_1 in F_1 does not have an image in $\mathcal{A}_1$ under the aforementioned injection or the image of a vertex in F_1 under the injection may be a terminal. In those scenarios we resort to ad hoc methods, including linking corresponding pairs in the link of s_1 in F_1, which is $(k-1)$-linked by Proposition 7 and does not contain s_1 or s_1^o, or linking corresponding pairs in ridges disjoint from F_1, which are $(k-1)$-linked by Theorem 8.

To aid the reader, each case is broken down into subcases highlighted in bold.

Recall that, given a pair $\{F, F^o\}$ of opposite facets in a cube Q, for every vertex $z \in V(F)$ we denote by $z^p_{F^o}$ or $\pi^Q_{F^o}(z)$ the unique neighbour of z in F^o.

Case 1. $|X \cap V(F_1)| = d$.

Without loss of generality, assume that $t_2 \notin V(F_1)$.

Suppose first that $\operatorname{dist}_{F_1}(s_2, s_1) < d-1$. There exists a neighbour s_2' of s_2 in $\mathcal{A}_1$. With the use of the strong $(k-1)$-linkedness of F_1 (Theorem 8), find disjoint paths $L_1 := s_1 - t_1$ and $L_i := s_i - t_i$ ($i \in [3, k]$) in F_1, each avoiding s_2. Find a path L_2 in $\mathcal{S}_1$ between s_2 and t_2 that consists of the edge $s_2 s_2'$ and a subpath in $\mathcal{A}_1$ between s_2' and t_2, using the connectivity of $\mathcal{A}_1$ (see Proposition 18). The paths L_i ($i \in [1, k]$) give the desired Y-linkage.

Now assume $\operatorname{dist}_{F_1}(s_2, s_1) = d-1$. Since $2k-1 = d$ and there are $d-1$ pairs of opposite $(d-2)$-faces in F_1, by Lemma 10 there exists a pair $\{R, R^o\}$ of opposite ridges of F_1 that is not associated with the set $X_{s_2} := (X \cap V(F_1)) \setminus \{s_2\}$, whose cardinality is $d-1$. Assume $s_2 \in R$. Then $s_1 \in R^o$.

Suppose all the neighbours of s_2 in R are in X; that is, $N_R(s_2) = X \setminus \{s_1, s_2, t_2\}$. The projection $\pi^{F_1}_{R^o}(s_2)$ of s_2 onto R^o is not in X since s_1 is the only terminal in R^o and $\operatorname{dist}_{F_1}(s_2, s_1) = d-1 \geq 2$. Next find disjoint paths $L_i := s_i - t_i$ for $i \in [3, k]$ in R that do not touch s_2 or t_1, using the $(k-1)$-linkedness of R if $d \geq 7$ (Lemma 15) or the 3-connectivity of R if $d = 5$. With the help of Lemma 25, find a neighbour s_2' of $\pi^{F_1}_{R^o}(s_2)$ in $\mathcal{A}_1$, and with the connectivity of $\mathcal{A}_1$, a path L_2 between s_2 and t_2 that consists of the length-two path $s_2\pi^{F_1}_{R^o}(s_2)s_2'$ and a subpath in $\mathcal{A}_1$ between s_2' and t_2. Finally, find a path L_1 in F_1 between s_1 and t_1 that consists of the edge $t_1\pi^{F_1}_{R^o}(t_1)$ and a subpath in R^o disjoint from $\pi^{F_1}_{R^o}(s_2)$ (here use the 2-connectivity of R^o). The paths L_i ($i \in [1, k]$) give the desired Y-linkage.

Thus assume there exists a neighbour $\bar{s}_2$ of s_2 in $V(R)\setminus X$. Let $X_{R^o} := \pi^{F_1}_{R^o}(X\setminus\{s_2,t_2\})$. Find a path L_2 between t_2 and s_2 that consists of the edge $s_2\bar{s}_2$ and a subpath in $\mathcal{A}_1$ between t_2 and a neighbour s_2' of $\bar{s}_2$ in $\mathcal{A}_1$.

Let $d\ge 7$. Find disjoint paths $L_i := \pi^{F_1}_{R^o}(s_i) - \pi^{F_1}_{R^o}(t_i)$ ($i\in[1,k]$ and $i\ne 2$) in R^o linking the $d-1$ vertices in X_{R^o} using the $(k-1)$-linkedness of R^o; add the edge $\pi^{F_1}_{R^o}(t_i)t_i$ to L_i if $t_i\in R$ or the edge $\pi^{F_1}_{R^o}(s_i)s_i$ to L_i if $s_i\in R$. The disjoint paths L_i ($i\in[1,k]$) gives the desired Y-linkage.

Let $d=5$. If the sequence $s_1,\pi^{F_1}_{R^o}(s_3),\pi^{F_1}_{R^o}(t_1),\pi^{F_1}_{R^o}(t_3)$ in X_{R^o} is not in a 2-face of R^o in cyclic order, then the same reasoning as in the case of $d\ge 7$ applies. Thus assume otherwise. This in turn implies that $\pi^{F_1}_{R}(s_3)\notin\{s_2,s_2'\}$ and $\pi^{F_1}_{R}(t_3)\notin\{s_2,s_2'\}$, since $\mathrm{dist}_{F_1}(s_1,s_2)=4$.

Find a path L_3' in R between $\pi^{F_1}_{R}(s_3)$ and $\pi^{F_1}_{R}(t_3)$ such that L_3' is disjoint from both s_2 and s_2' and disjoint from t_1 if $t_1\in R$; here use Corollary 11, which ensures that the vertices s_2, s_2' and t_1, if they are all in R, cannot separate $\pi^{F_1}_{R}(s_3)$ from $\pi^{F_1}_{R}(t_3)$ in R, since a separator of size three in R must be an independent set. Extend the path L_3' in R to a path $L_3 := s_3\pi^{F_1}_{R}(s_3)L_3'\pi^{F_1}_{R}(t_3)t_3$ in F_1, if necessary. Find a path $L_1' := s_1 - \pi^{F_1}_{R^o}(t_1)$ in R^o disjoint from $\pi^{F_1}_{R^o}(s_3)$ and $\pi^{F_1}_{R^o}(t_3)$, using the 3-connectivity of R^o. Extend L_1' to a path $L_1 := s_1L_1'\pi^{F_1}_{R^o}(t_1)t_1$ in F_1, if necessary. The linkage $\{L_1,L_2,L_3\}$ is a Y-linkage. This completes the proof of Case 1.

Case 2. $3\le |X\cap V(F_1)|\le d-1$.

Since $2k-1=d$ and there are $d-1$ pairs of opposite facets in F_1, by Lemma 10 there exists a pair $\{R,R^o\}$ of opposite ridges of F_1 that is not associated with $X\cap V(F_1)$. Assume $s_1\in R$. We consider two subcases according to whether $t_1\in R$ or $t_1\in R^o$.

Suppose first that $t_1\in R$. The $(d-2)$-connectivity of R ensures the existence of an X-valid path $L_1 := s_1 - t_1$ in R. Let

$$X_{R^o} := \pi^{F_1}_{R^o}((X\setminus\{s_1,t_1\})\cap V(F_1)).$$

Then $1\le |X_{R^o}|\le d-3$. Let s_1^o be the vertex opposite to s_1 in F_1; the vertex s_1^o has no neighbour in $\mathcal{A}_1$.

Let $\bar{Z}$ be a set of $|V(\mathcal{A}_1)\cap X|$ distinct vertices in $V(R^o)\setminus(X_{R^o}\cup\{s_1^o\})$. Use Lemma 25 to obtain a set Z in $\mathcal{A}_1$ of $|\bar{Z}|$ distinct vertices adjacent to vertices in $\bar{Z}$. Then $|Z|=|V(\mathcal{A}_1)\cap X|\le d-2$. To see that $|\bar{Z}|\le |V(R^o)\setminus(X_{R^o}\cup\{s_1^o\})|$, observe that, for $d\ge 5$ and $|X_{R^o}|\le d-3$, we get

$$|V(R^o)\setminus(X_{R^o}\cup\{s_1^o\})|\ge 2^{d-2}-(d-3)-1\ge d-2\ge|\bar{Z}|=|Z|.$$

Using the $(d-2)$-connectivity of $\mathcal{A}_1$ (Proposition 18) and Menger's theorem, find disjoint paths $\bar{S}_i$ and $\bar{T}_j$ ($i,j\ne 1$) in $\mathcal{A}_1$ between $V(\mathcal{A}_1)\cap X$ and Z. Then produce disjoint paths S_i and T_j ($i,j\ne 1$) from terminals s_i and t_j in $\mathcal{A}_1$, respectively, to R^o by adding edges $z_\ell\bar{z}_\ell$ with $z_\ell\in Z$ and $\bar{z}_\ell\in\bar{Z}$ to the corresponding paths $\bar{S}_i$ and $\bar{T}_j$. If s_i or t_j is already in R^o, let $S_i := s_i$ or $T_j := t_j$, accordingly. If instead s_i or t_j is in R, let S_i be the edge $s_i\pi^{F_1}_{R^o}(s_i)$ or let T_j be the edge $t_j\pi^{F_1}_{R^o}(t_j)$. It follows

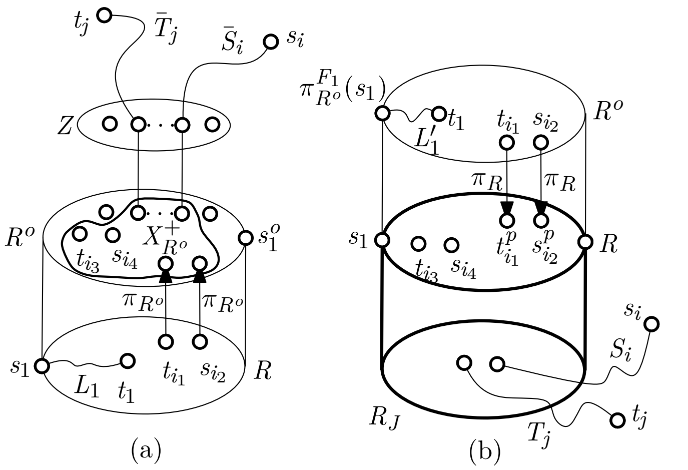

FIGURE 2. Auxiliary figure for Case 2 of Lemma 23. **(a)** A configuration where $t_1 \in R$ and the subset $X^+_{R^o}$ of R^o is highlighted in bold. **(b)** A configuration where $t_1 \in R^o$ and the facet J is highlighted in bold.

that the paths S_i and T_i for $i \in [2,k]$ are all pairwise disjoint. Let $X^+_{R^o}$ be the intersections of R^o and the paths S_i and T_j $(i, j \neq 1)$. Then $|X^+_{R^o}| = d-1$. Suppose that $X^+_{R^o} = \{\bar{s}_2, \bar{t}_2, \ldots, \bar{s}_k, \bar{t}_k\}$. The corresponding pairing $Y^+_{R^o}$ of the vertices in $X^+_{R^o}$ can be linked through paths $\bar{L}_i := \bar{s}_i - \bar{t}_i$ $(i \in [2,k])$ in R^o using the $(k-1)$-linkedness of R^o (Theorem 8). See Fig. 2(a) for a depiction of this configuration. In this case, the desired Y-linkage is given by the following paths.

$$L_i := \begin{cases} s_1 L_1 t_1, & \text{for } i = 1; \\ s_i S_i \bar{s}_i \bar{L}_i \bar{t}_i T_i t_i, & \text{otherwise.} \end{cases}$$

Some comments for $d = 5$ are in order. By virtue of Proposition 4, we need to make sure that the sequence $\bar{s}_2, \bar{s}_3, \bar{t}_2, \bar{t}_3$ in $X^+_{R^o}$ is not in a 2-face of R^o in cyclic order. To ensure this, we need to be a bit more careful when selecting the vertices in $\bar{Z}$. Indeed, if there are already two vertices in X_{R^o} at distance three in R^o, no care is needed when selecting $\bar{Z}$, so proceed as in the case of $d \geq 7$. Otherwise, pick a vertex $\bar{z} \in \bar{Z} \subseteq V(R^o) \setminus (X_{R^o} \cup \{s^o_1\})$ such that $\bar{z}$ is the unique vertex in R^o with $\operatorname{dist}_{R^o}(\bar{z}, x) = 3$ for some vertex $x \in X_{R^o}$; this vertex x exists because $|X \cap V(F_1)| \geq 3$. Selecting such a $\bar{z} \neq s^o_1$ is always possible because s^o_1 is not at distance three in R^o from *any* vertex in X_{R^o}: the unique vertex in R^o at distance three from s^o_1 is $\pi^{F_1}_{R^o}(s_1)$, and $\pi^{F_1}_{R^o}(s_1) \notin X$ because the pair $\{R, R^o\}$ is not associated with $X \cap V(F_1)$. Once $\bar{z}$ is selected, the set Z will contain a neighbour z of $\bar{z}$. In this way, some path S_i or T_j bringing terminals s_i or t_j in $\mathcal{A}_1$ into R^o through Z would use the vertex z, thereby ensuring that x and $\bar{z}$ would be both in $X^+_{R^o}$. This will cause the the sequence $\bar{s}_2, \bar{s}_3, \bar{t}_2, \bar{t}_3$ not to be in a 2-face, and thus, not in cyclic order.

Suppose now that $t_1 \in R^o$. Let

$$X_R := \pi_R^{F_1}((X \setminus \{t_1\}) \cap V(F_1)).$$

There are at most $d-2$ terminal vertices in R^o. Therefore, the $(d-2)$-connectivity of R^o ensures the existence of an X-valid $\pi_{R^o}^{F_1}(s_1) - t_1$ path $\bar{L}_1$ in R^o. Then let $L_1 := s_1\pi_{R^o}^{F_1}(s_1)\bar{L}_1 t_1$. Let J be the other facet in $\mathcal{S}_1$ containing R and let R_J be the $(d-2)$-face of J disjoint from R. Then $R_J \subset \mathcal{A}_1$. Since there are at most $d-2$ terminals in $\mathcal{A}_1$ and since $\mathcal{A}_1$ is $(d-2)$-connected (Proposition 18), we can find corresponding disjoint paths S_i and T_j bringing the terminals in $\mathcal{A}_1$ to R_J (Theorem 13). For terminals s_i and t_j in $X \cap V(R)$, let $S_i := s_i$ and $T_j := t_j$ for $i, j \neq 1$, while for terminals s_i and t_j in $X \cap V(R^o)$, let $S_i := s_i\pi_R^{F_1}(s_i)$ and $T_j := t_j\pi_R^{F_1}(t_j)$ for $i, j \neq 1$. Let X_J be the set of the intersections of the paths S_i and T_j with J plus the vertex s_1. Then $X_J \subset V(J)$ and $|X_J| = d$ (since $t_1 \in R^o$). Suppose that $X_J = \{s_1, \bar{s}_2, \bar{t}_2, \ldots, \bar{s}_k, \bar{t}_k\}$ and let $Y_J = \{\{\bar{s}_2, \bar{t}_2\}, \ldots, \{\bar{s}_k, \bar{t}_k\}\}$ be a pairing of $X_J \setminus \{s_1\}$.

Resorting to the strong $(k-1)$-linkedness of the facet J (Theorem 8), we obtain $k-1$ disjoint paths $\bar{L}_i := \bar{s}_i - \bar{t}_i$ for $i \neq 1$ that correspondingly link Y_J in J, with all the paths avoiding s_1. See Fig. 2(b) for a depiction of this configuration. In this case, the desired Y-linkage is given by the following paths.

$$L_i := \begin{cases} s_1 L_1 t_1, & \text{for } i = 1; \\ s_i S_i \bar{L}_i T_i t_i, & \text{otherwise.} \end{cases}$$

Case 3. $|X \cap V(F_1)| = 2$.

In this case, we have that $|V(\mathcal{A}_1) \cap X| = d-1$. The proof of this case requires the definition of several sets. For quick reference and ease of readability, we place most of these definitions in itemised lists. We begin with the following sets:

- $\mathcal{S}_{12}$, the star of s_2 in $\mathcal{S}_1$ (that is, the complex formed by the facets of P containing s_1 and s_2);
- $G(\mathcal{S}_{12})$, the graph of $\mathcal{S}_{12}$; and
- Γ_{12}, the subgraph of $G(\mathcal{S}_{12})$ and $G(\mathcal{A}_1)$ that is induced by $V(\mathcal{S}_{12}) \setminus V(F_1)$.

It follows that every neighbour in $G(\mathcal{A}_1)$ of s_2 is in Γ_{12}; in other words, the set of neighbours of s_2 in each subgraph is the same:

$$N_{\Gamma_{12}}(s_2) = N_{G(\mathcal{A}_1)}(s_2). \tag{1}$$

The first step for this case is to bring the terminals in $\mathcal{A}_1$ into Γ_{12}. Denote by S_i an X-valid path in $\mathcal{A}_1$ from the terminal $s_i \in \mathcal{A}_1$ to Γ_{12}. Let $V(S_i) \cap V(\Gamma_{12}) = \{\hat{s}_i\}$. Similarly, define T_j and $\hat{t}_j$. The existence of these $d-2$ pairwise disjoint X-valid paths S_i and T_j is ensured by the $(d-2)$-connectivity of the graph $G(\mathcal{A}_1)$ of $\mathcal{A}_1$, which in turn is guaranteed by Proposition 18. By (1) each path S_i or T_j touches Γ_{12} at a vertex other than s_2; this is so because each such path will need to reach the neighbourhood of s_2 in Γ_{12} before reaching s_2. Every terminal vertex x already in Γ_{12} is also denoted by $\hat{x}$, and the corresponding path

S_i or T_j consists only of the vertex $\hat{x}$. We also let $\hat{s}_2$ denote s_2. The set of vertices $\hat{x}$ is accordingly denoted by $\hat{X}$. Then $|\hat{X}| = d-1$. Abusing terminology, since there is no potential for confusion, we call the vertices in $\hat{X}$ terminals as well. Figure 3(a) depicts this configuration.

Pick a facet

- F_{12} in $\mathcal{S}_{12}$ that contains $\hat{t}_2$.

An important point is that t_1 is not in F_{12}; otherwise F_{12} would contain s_1,s_2 and t_1, and it should have been chosen instead of F_1.

The second step is to find a path L_1 in F_1 between s_1 and t_1 such that $V(L_1) \cap V(F_{12}) = \{s_1\}$.

To see the existence of such a path, note that the intersection of F_{12} and F_1 is at most a $(d-2)$-face containing s_1 (but not t_1), which is contained in a $(d-2)$-face R of F_1 containing s_1 but not t_1 (Remark 20). Find a path L'_1 in R^o, the ridge of F_1 disjoint from R and containing t_1, between $\pi^{F_1}_{R^o}(s_1)$ and t_1 and let $L_1 := s_1\pi^{F_1}_{R^o}(s_1)L'_1t_1$.

The third step is to bring the $d-1$ terminal vertices $\hat{x} \in \mathit{\Gamma}_{12}$ into the facet F_{12} so that they can be linked there, avoiding s_1. We consider two cases depending on the number of facets in $\mathcal{S}_{12}$.

Suppose $\mathcal{S}_{12}$ only consists of F_{12}. Then

$$\hat{X} = \{\hat{s}_2, \ldots, \hat{s}_k, \hat{t}_2, \ldots, \hat{t}_k\} \subset V(\mathit{\Gamma}_{12}) \subset V(F_{12}).$$

With the help of the strong $(k-1)$-linkedness of F_{12} (Theorem 8), we can link the pairs $\{\hat{s}_i, \hat{t}_i\}$ for $i \in [2, k]$ in F_{12} through disjoint paths $\hat{L}_i$, all avoiding s_1. The paths $\hat{L}_i$ concatenated with the paths S_i and T_i for $i \in [2, k]$ give a $(Y \setminus \{s_1, t_1\})$-linkage $\{L_2, \ldots, L_k\}$. Hence the desired Y-linkage is as follows.

$$L_i := \begin{cases} s_1\pi^{F_1}_{R^o}(s_1)L'_1t_1, & \text{for } i = 1; \\ s_iS_i\hat{s}_i\hat{L}_i\hat{t}_iT_it_i, & \text{otherwise.} \end{cases}$$

Assume $\mathcal{S}_{12}$ has more than one facet. We have that

$$\hat{X} = \{\hat{s}_2, \ldots, \hat{s}_k, \hat{t}_2, \ldots, \hat{t}_k\} \subset V(\mathit{\Gamma}_{12}).$$

Define

- $\mathcal{A}_{12}$ as the complex of $\mathcal{S}_{12}$ induced by $V(\mathcal{S}_{12}) \setminus (V(F_1) \cup V(F_{12}))$.

Then the graph $G(\mathcal{A}_{12})$ of $\mathcal{A}_{12}$ coincides with the subgraph of $\mathit{\Gamma}_{12}$ induced by $V(\mathit{\Gamma}_{12}) \setminus V(F_{12})$. Figure 3(b) depicts this configuration.

Our strategy is first to bring the $d-3$ terminal vertices $\hat{x}$ in $\mathit{\Gamma}_{12}$ other than $\hat{s}_2$ and $\hat{t}_2$ into $F_{12} \setminus F_1$ through disjoint paths $\hat{S}_i$ and $\hat{T}_j$, without touching $\hat{s}_2$ and $\hat{t}_2$. Second, denoting by $\tilde{s}_i$ and $\tilde{t}_j$ the intersection of $\hat{S}_i$ and $\hat{T}_j$ with $V(F_{12}) \setminus V(F_1)$, respectively, we link the pairs $\{\tilde{s}_i, \tilde{t}_i\}$ for $i = [2, k]$ in F_{12} through disjoint paths $\tilde{L}_i$, without touching s_1; here we resort to the strong $(k-1)$-linkedness of F_{12}. We develop these ideas below.

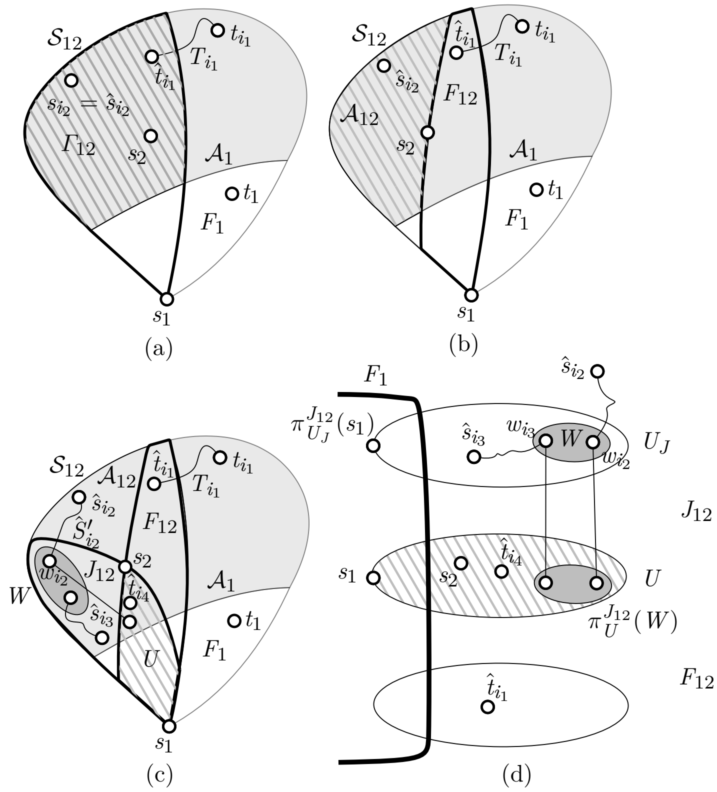

FIGURE 3. Auxiliary figure for Case 3 of Lemma 23. A representation of $\mathcal{S}_1$. (a) A configuration where the subgraph Γ_{12} is tiled in falling pattern and the complex $\mathcal{A}_1$ is coloured in grey. (b) A depiction of $\mathcal{S}_{12}$ with more than one facet; the facet F_{12} is highlighted in bold, the complex $\mathcal{A}_1$ is coloured in grey and the complex $\mathcal{A}_{12}$ is highlighted in falling pattern. (c) A depiction of $\mathcal{S}_{12}$ with more than one facet; the facets F_{12} and J_{12} are highlighted in bold and their intersection U is highlighted in falling pattern; the set W in J_{12} is coloured in dark grey. (d) A depiction of a portion of $\mathcal{S}_{12}$, zooming in on the facets F_{12} and J_{12}; each facet is represented as the convex hull of two disjoint $(d-2)$-faces, and their intersection U is highlighted in falling pattern. The sets W and $\pi^{J_{12}}_{U}(W)$ in J_{12} are coloured in dark grey.

From Lemma 21(iii), it follows that $\mathcal{A}_{12}$ is nonempty and contains a spanning strongly connected $(d-3)$-subcomplex, thereby implying, by Proposition 6, that

$$G(\mathcal{A}_{12}) \text{ is } (d-3)\text{-connected.}$$

Since $\mathcal{S}_{12}$ contains more than one facet, the following sets exist:

- U, a $(d-2)$-face in F_{12} that contains s_1 and $\hat{s}_2$ $(= s_2)$ (Remark 19);
- J_{12}, the other facet in $\mathcal{S}_{12}$ containing U;
- U_J, the $(d-2)$-face in J_{12} disjoint from U, and as a consequence, disjoint from F_{12};

- $\mathcal{C}_U$, the subcomplex of $\mathcal{B}(U)$ induced by $V(U)\setminus V(F_1)$, namely the antistar of $U\cap F_1$ in U; and
- $\mathcal{C}_{U_J}$, the subcomplex of $\mathcal{B}(U_J)$ induced by $V(U_J)\setminus V(F_1)$.

The subcomplex $\mathcal{C}_U$ is nonempty, since $\hat{s}_2 \in V(U)\setminus V(F_1)$, and so, thanks to Lemma 17, it is a strongly connected $(d-3)$-complex. Then, from C_U containing a $(d-3)$-face it follows that

$$|V(\mathcal{C}_U)| = |V(U)\setminus V(F_1))| \geq 2^{d-3} \geq d-1 \text{ for } d\geq 5. \tag{2}$$

The subcomplex $\mathcal{C}_{U_J}$ is nonempty: if $U_J\cap F_1=\emptyset$ then $\mathcal{C}_{U_J}=\mathcal{B}(U_J)$; otherwise $\mathcal{C}_{U_J}$ is the antistar of $U_J\cap F_1$ in U_J, and since $U\cap F_1\neq\emptyset$ (s_1 is in both), it follows that $U_J\not\subseteq F_1$. Put differently, the vertex in J_{12} opposite to s_1 is not in U, since $s_1\in U$, nor is it in F_1, and so it must be in $\mathcal{C}_{U_J}$. Therefore, according to Lemma 17, $\mathcal{C}_{U_J}$ is a strongly connected $(d-3)$-complex. Hence, in both instances,

$$|V(\mathcal{C}_{U_J})| = |V(U_J)\setminus V(F_1))| \geq 2^{d-3} \geq d-1 \text{ for } d\geq 5. \tag{3}$$

Recall that we want to bring every vertex in the set $\hat{X}$, which is contained in Γ_{12}, into $F_{12}\setminus F_1$. We construct $|\hat{X}\cap V(\mathcal{A}_{12})|$ pairwise disjoint paths $\hat{S}_i$ and $\hat{T}_j$ from $\hat{s}_i\in\mathcal{A}_{12}$ and $\hat{t}_j\in\mathcal{A}_{12}$, respectively, to $V(F_{12})\setminus V(F_1)$ as follows. Pick a set

$$W\subset V(\mathcal{C}_{U_J})\setminus \pi_{U_J}^{J_{12}}\left((\hat{X}\cup\{s_1\})\cap U\right)$$

of $|\hat{X}\cap V(\mathcal{A}_{12})|$ vertices in $\mathcal{C}_{U_J}$. Then $\pi_U^{J_{12}}(W)$ is disjoint from $(\hat{X}\cup\{s_1\})\cap U$. In other words, the vertices in W are in $\mathcal{C}_{U_J}$ and are not projections of the vertices in $(\hat{X}\cup\{s_1\})\cap U$ onto U_J. We show that the set W exists, which amounts to showing that $\mathcal{C}_{U_J}$ has enough vertices to accommodate W.

First note that

$$\begin{aligned}|\hat{X}\cap V(\mathcal{A}_{12})| + |(\hat{X}\cup\{s_1\})\cap V(F_{12})| &= |\hat{X}\cup\{s_1\}| = d,\\ (\hat{X}\cup\{s_1\})\cap V(U) &\subseteq (\hat{X}\cup\{s_1\})\cap V(F_{12}).\end{aligned} \tag{4}$$

If $U_J\cap F_1=\emptyset$ then $\mathcal{C}_{U_J}=\mathcal{B}(U_J)$. And (4) together with $|V(U_J)|=2^{d-2}\geq d$ for $d\geq 5$ gives the following chain of inequalities

$$\begin{aligned}\left|V(C_{U_J})\setminus \pi_{U_J}^{J_{12}}\left((\hat{X}\cup\{s_1\})\cap V(U)\right)\right| &\geq d-\left|(\hat{X}\cup\{s_1\})\cap V(U)\right|\\ &\geq \left|\hat{X}\cup\{s_1\}\right| - \left|(\hat{X}\cup\{s_1\})\cap V(F_{12})\right| = \left|\hat{X}\cap V(\mathcal{A}_{12})\right| = |W|,\end{aligned}$$

as desired.

Suppose now $U_J\cap F_1\neq\emptyset$. Since $s_1\in U\cap F_1$ and $J_{12}=\operatorname{conv}\{U\cup U_J\}$, the cube $J_{12}\cap F_1$ has opposite facets $U_J\cap F_1$ and $U\cap F_1$. From $s_1\in U\cap F_1$ it follows that $\pi_{U_J}^{J_{12}}(s_1)\in U_J\cap F_1$, and thus, that $\pi_{U_J}^{J_{12}}(s_1)\notin\mathcal{C}_{U_J}$; here we use the following remark.

Remark 26. Let (K,K^o) be opposite facets in a cube Q and let B be a proper face of Q such that $B\cap K\neq\emptyset$ and $B\cap K^o\neq\emptyset$. Then $\pi_{K^o}^{Q}(B\cap K)=B\cap K^o$.

Since $\pi_{U_J}^{J_{12}}(s_1) \notin \mathcal{C}_{U_J}$, using (3) and (4) we get

$$\left|V(C_{U_J}) \setminus \pi_{U_J}^{J_{12}}\left((\hat{X} \cup \{s_1\}) \cap V(U)\right)\right| = \left|V(C_{U_J}) \setminus \pi_{U_J}^{J_{12}}\left(\hat{X} \cap V(U)\right)\right|$$
$$\geq d-1-\left|\hat{X} \cap V(U)\right| \geq \left|\hat{X}\right| - \left|\hat{X} \cap V(F_{12})\right| = \left|\hat{X} \cap V(\mathcal{A}_{12})\right| = |W|.$$

In this way, we have shown that $\mathcal{C}_{U_J}$ can accommodate the set W. We now finalise the case.

There are at most $d-3$ vertices $\hat{x}$ in $\hat{X} \cap V(\mathcal{A}_{12})$ because $\hat{s}_2$ and $\hat{t}_2$ are already in $V(F_{12}) \setminus V(F_1)$. Since $G(\mathcal{A}_{12})$ is $(d-3)$-connected, we can find $|W| = |\hat{X} \cap V(\mathcal{A}_{12})|$ pairwise disjoint paths $\hat{S}_i'$ and $\hat{T}_j'$ in $\mathcal{A}_{12}$ from the terminals $\hat{s}_i$ and $\hat{t}_j$ in $\hat{X} \cap V(\mathcal{A}_{12})$ to W. The $\hat{X}$-valid path $\hat{S}_i$ from $\hat{s}_i \in \mathcal{A}_{12}$ to $V(F_{12}) \setminus V(F_1)$ then consists of the subpath $\hat{S}_i' := \hat{s}_i - w_i$ with $w_i \in W$ plus the edge $w_i \pi_U^{J_{12}}(w_i)$; from the choice of W it follows that $\pi_U^{J_{12}}(w_i) \notin \hat{X} \cup \{s_1\}$. The paths $\hat{T}_j'$ and $\hat{T}_j$ are defined analogously. Figure 3(c)-(d) depicts this configuration.

Denote by $\tilde{s}_i$ the intersection of $\hat{S}_i$ and $V(F_{12}) \setminus V(F_1)$; similarly, define $\tilde{t}_j$. Every terminal vertex $\hat{x}$ already in F_{12} is also denoted by $\tilde{x}$, and in this case we let $\hat{S}_i$ or $\hat{T}_j$ be the vertex $\tilde{x}$.

Now F_{12} contains the pairs $\{\tilde{s}_i, \tilde{t}_i\}$ for $i \in [2, k]$ and the terminal s_1, as desired. Link these pairs in F_{12} through disjoint paths $\tilde{L}_i$, each avoiding s_1, with the use of the strong $(k-1)$-linkedness of F_{12} (Theorem 8). The paths $\tilde{L}_i$ concatenated with the paths S_i, $\hat{S}_i$, T_i and $\hat{T}_i$ for $i \in [2, k]$ give a $(Y \setminus \{s_1, t_1\})$-linkage $\{L_2, \ldots, L_k\}$. Hence the desired Y-linkage is as follows.

$$L_i := \begin{cases} s_1 \pi_{R^o}^{F_1}(s_1) L_1' t_1, & \text{for } i = 1; \\ s_i S_i \hat{s}_i \hat{S}_i \tilde{s}_i \tilde{L}_i \tilde{t}_i \hat{T}_i \hat{t}_i T_i t_i, & \text{otherwise.} \end{cases}$$

Case 4. $|X \cap V(F_1)| = d + 1$ and the vertex s_1 is not in Configuration dF.

Here we have that $V(\mathcal{A}_1) \cap X = \emptyset$. This case is decomposed into three main subcases A, B and C, based on the nature of the vertex s_1^o opposite to s_1 in F_1, which is the only vertex in F_1 that does not have an image under the injection from F_1 to $\mathcal{A}_1$ defined in Lemma 25. And each subcase is then analysed for $d \geq 7$ and $d = 5$ separately. The difficulty with $d = 5$ stems from the $(d-2)$-faces of the polytope not being 2-linked (Proposition 4).

SUBCASE A for $d \geq 7$. The vertex s_1^o opposite to s_1 in F_1 does not belong to X. Let $X' := X \setminus \{t_1\}$ and let $Y' := Y \setminus \{\{s_1, t_1\}\}$. Since $|X'| = d$, the strong $(k-1)$-linkedness of F_1 (Theorem 8) gives a Y'-linkage $\{L_2, \ldots, L_k\}$ in the facet F_1 with each path $L_i := s_i - t_i$ $(i \in [2, k])$ avoiding s_1. We find pairwise distinct neighbours s_1' and t_1' in $\mathcal{A}_1$ of s_1 and t_1, respectively. If none of the paths L_i touches t_1, we find a path $L_1 := s_1 - t_1$ in $\mathcal{S}_1$ that contains a subpath in $\mathcal{A}_1$ between s_1' and t_1' (here use the connectivity of $\mathcal{A}_1$, Proposition 18), and we are home. Otherwise, assume that the path L_j contains t_1. With the help of Lemma 25, find pairwise distinct neighbours s_j' and t_j' in $\mathcal{A}_1$ of s_j and t_j, respectively, such that the vertices

s_1', t_1', s_j' and t_j' are pairwise distinct. According to Proposition 24, the complex $\mathcal{A}_1$ is 2-linked for $d \geq 7$. Hence, we can find disjoint paths $L_1' := s_1' - t_1'$ and $L_j' := s_j' - t_j'$ in $\mathcal{A}_1$, respectively; these paths naturally give rise to paths $L_1 := s_1 s_1' L_1' t_1' t_1$ in $\mathcal{S}_1$ and $L_j := s_j s_j' L_j' t_j' t_j$ in $\mathcal{S}_1$. The paths $\{L_1, \ldots, L_k\}$ give the desired Y-linkage.

SUBCASE B for $d \geq 7$. The vertex s_1^o opposite to s_1 in F_1 belongs to X but is different from t_1, say $s_1^o = s_2$. First find a neighbour s_1' of s_1 and a neighbour t_1' of t_1 in $\mathcal{A}_1$. There is a neighbour $s_2^{F_1}$ of s_2 in F_1 that is either t_2 or a vertex not in X: $\{s_1, s_2\} \cap N_{F_1}(s_2) = \emptyset$ and $|N_{F_1}(s_2)| = d - 1$. The link $\mathcal{L}_1$ of s_1 in F_1 contains all the vertices in F_1 except s_1 and s_2.

Suppose $s_2^{F_1} = t_2$. Let $L_2 := s_2 t_2$, and using the $(k-1)$-linkedness of $\mathcal{L}_1$ (Proposition 7), find disjoint paths $t_1 - t_2$ and $L_i := s_i - t_i$ for $i \in [3, k]$ in $\mathcal{L}_1$. Then define a path $L_1 := s_1 - t_1$ in $\mathcal{S}_1$ that contains a subpath in $\mathcal{A}_1$ between s_1' and t_1'; here we use the connectivity of $\mathcal{A}_1$ (Proposition 18). The paths $\{L_1, \ldots, L_k\}$ give the desired Y-linkage.

Assume $s_2^{F_1}$ is not in X. Observe that $|(X \setminus \{s_1, s_2\}) \cup \{s_2^{F_1}\}| = d$. Using the $(k-1)$-linkedness of $\mathcal{L}_1$ for $d \geq 7$ (Proposition 7), find in $\mathcal{L}_1$ disjoint paths $L_2' := s_2^{F_1} - t_2$ and $L_i' := s_i - t_i$ for $i \in [3, k]$. Since t_1 is also in $\mathcal{L}_1$ it may happen that it lies in one of the paths L_i'. If t_1 does not belong to any of the paths L_i' for $i \in [2, k]$, then find a path $L_1 := s_1 s_1' L_1' t_1' t_1$ in $\mathcal{S}_1$ where L_1' is a subpath in $\mathcal{A}_1$ between s_1' and t_1', using the connectivity of $\mathcal{A}_1$ (Proposition 18). In this scenario, let $L_2 := s_2 s_2^{F_1} L_2' t_2$ and $L_i := L_i'$ for $i \in [3, k]$; the desired Y-linkage is given by the paths $\{L_1, \ldots, L_k\}$.

If t_1 belongs to one of the paths L_i' with $i \in [2, k]$, say L_j', then consider in $\mathcal{A}_1$ a neighbour t_j' of t_j and, either a neighbour s_j' of s_j if $j \neq 2$ or a neighbour s_2' of $s_2^{F_1}$. From Lemma 25 it follows that the vertices s_1', t_1', s_j' and t_j' can be taken pairwise distinct. Since $\mathcal{A}_1$ is 2-linked for $d \geq 7$ (see Proposition 24), find in $\mathcal{A}_1$ a path L_1' between s_1' and t_1' and a path L_j'' between s_j' and t_j'. As a consequence, we obtain in $\mathcal{S}_1$ a path $L_1 := s_1 s_1' L_1' t_1' t_1$ and, either a path $L_j := s_j s_j' L_j'' t_j' t_j$ if $j \neq 2$ or a path $L_2 := s_2 s_2^{F_1} s_2' L_2'' t_2' t_2$. In addition, let $L_i := L_i'$ for $i \in [3, k]$ and $i \neq j$. The paths $\{L_1, \ldots, L_k\}$ give the desired Y-linkage.

SUBCASES A AND B for $d = 5$. The vertex s_1^o opposite to s_1 in F_1 either does not belong to X or belongs to X but is different from t_1. Let $X := \{s_1, s_2, s_3, t_1, t_2, t_3\}$ be any set of six vertices in the graph G of a cubical 5-polytope P. Also let $Y := \{\{s_1, t_1\}, \{s_2, t_2\}, \{s_3, t_3\}\}$. We aim to find a Y-linkage $\{L_1, L_2, L_3\}$ in G where L_i joins the pair $\{s_i, t_i\}$ for $i = 1, 2, 3$.

In both subcases there is a 3-face R of F_1 containing both s_1 and t_1. Let J_1 be the other facet in $\mathcal{S}_1$ containing R. Denote by R_J and R_F the ridges in J_1 and F_1, respectively, that are disjoint from R. Then $s_1^o \in R_F$. We need the following claim.

Claim 1. If a 3-cube contains three pairs of terminals, there must exist two pairs of terminals in the 3-cube, say $\{s_1, t_1\}$ and $\{s_2, t_2\}$, that are not arranged in the cyclic order s_1, s_2, t_1, t_2 in a 2-face of the cube.

Proof. If no terminal in the cube is in Configuration 3F, we are done. So suppose that one is, say s_1, and that the sequence s_1, x_1, t_1, x_2 of vertices of X is present in cyclic order in a 2-face. Without loss of generality, assume that $s_2 \notin \{x_1, x_2\}$. Then s_2 cannot be adjacent to both s_1 and t_1, since the bipartite graph $K_{2,3}$ is not a subgraph of $G(Q_3)$ (Remark 12). Thus the sequence s_1, s_2, t_1, t_2 cannot be in a 2-face in cyclic order. □

Suppose all the six terminals are in the 3-face R. By virtue of Claim 1, we may assume that the pairs $\{s_1, t_1\}$ and $\{s_2, t_2\}$ are not arranged in the cyclic order s_1, s_2, t_1, t_2 in a 2-face of R. Proposition 4 ensures that the pairs $\{\pi^{J_1}_{R_J}(s_1), \pi^{J_1}_{R_J}(t_1)\}$ and $\{\pi^{J_1}_{R_J}(s_2), \pi^{J_1}_{R_J}(t_2)\}$ in R_J can be linked in R_J through disjoint paths L'_1 and L'_2, since the sequence $\pi^{J_1}_{R_J}(s_1), \pi^{J_1}_{R_J}(s_2), \pi^{J_1}_{R_J}(t_1), \pi^{J_1}_{R_J}(t_2)$ cannot be in a 2-face of R_J in cyclic order. Moreover, by the connectivity of R_F, there is a path L'_3 in R_F linking the pair $\{\pi^{F_1}_{R_F}(s_3), \pi^{F_1}_{R_F}(t_3)\}$. The linkage $\{L'_1, L'_2, L'_3\}$ can naturally be extended to a Y-linkage $\{L_1, L_2, L_3\}$ as follows.

$$L_i := \begin{cases} s_i\pi^{J_1}_{R_J}(s_i)L'_i\pi^{J_1}_{R_J}(t_i)t_i, & \text{for } i = 1, 2; \\ s_3\pi^{F_1}_{R_F}(s_3)L'_3\pi^{F_1}_{R_F}(t_3)t_3, & \text{otherwise.} \end{cases}$$

Suppose that R contains a pair $\{s_i, t_i\}$ for $i = 2, 3$, say $\{s_2, t_2\}$. There are at most five terminals in R, and consequently, applying Lemma 14 to the polytope F_1 and its facet R, we obtain an X-valid path $L_1 := s_1 - t_1$ in R or an X-valid path $L_2 := s_2 - t_2$ in R. For the sake of concreteness, say an X-valid path L_2 exists in R. From the connectivity of R_F and R_J follows the existence of a path L'_3 in R_F between $\pi^{F_1}_{R_F}(s_3)$ and $\pi^{F_1}_{R_F}(t_3)$, and of a path L'_1 in R_J between $\pi^{J_1}_{R_J}(s_1)$ and $\pi^{J_1}_{R_J}(t_1)$. The linkage $\{L'_1, L'_2, L'_3\}$ can be extended to a linkage $\{s_1 - t_1, s_2 - t_2, s_3 - t_3\}$ in $\mathcal{S}_1$.

Suppose that the ridge R contains no other pair from Y and that the ridge R_F contains a pair (s_i, t_i) $(i = 2, 3)$. Without loss of generality, assume s_2 and t_2 are in R_F.

First suppose that $s_3 \in R$, which implies that $t_3 \in R_F$. Further suppose that there is a path T_3 of length at most two from t_3 to R that is disjoint from $X \setminus \{s_3, t_3\}$. Let $\{t'_3\} := V(T_3) \cap V(R)$. Use the 2-linkedness of J_1 (Proposition 5) to find disjoint paths $L_1 := s_1 - t_1$ and $L'_3 := s_3 - t'_3$ in J_1. Let $L_3 := s_3L'_3t'_3T_3t_3$. Use the 3-connectivity of R_F to find an X-valid path $L_2 := s_2 - t_2$ in R_F that is disjoint from $V(T_3)$; note that $|V(T_3) \cap V(R_F)| \leq 2$. The paths $\{L_1, L_2, L_3\}$ give the desired Y-linkage. Now suppose there is no such path T_3 from t_3 to R. Then, the projection $\pi^{F_1}_R(t_3)$ is in $\{s_1, t_1\}$, say $\pi^{F_1}_R(t_3) = t_1$; the projection $\pi^{F_1}_{R_F}(s_1)$ is a neighbour of t_3 in R_F; and both s_2 and t_2 are neighbours of t_3 in R_F. This configuration implies that s_1 and t_1 are adjacent in R. Let $L_1 := s_1t_1$. Find a path $L_2 := s_2 - t_2$ in R_F that is disjoint from t_3, using the 3-connectivity of R_F. Then find a neighbour s'_3 in $\mathcal{A}_1$ of s_3 and a neighbour t'_3 in $\mathcal{A}_1$ of t_3; note that, since $\operatorname{dist}_{F_1}(s_1, t_3) \leq 2$, we have that $t_3 \neq s^o_1$. Find a path L_3 in $\mathcal{S}_1$ between s_3 and t_3 that contains a subpath L'_3 in $\mathcal{A}_1$ between s'_3 and t'_3; here use the connectivity of $\mathcal{A}_1$ (Proposition 18): $L_3 := s_3 s_3' L_3' t_3' t_3$. The linkage $\{L_1, L_2, L_3\}$ is the desired Y-linkage.

Assume that $s_3 \in R_F$; by symmetry we can further assume that $t_3 \in R_F$. The connectivity of R ensures the existence of a path $L_1 := s_1 - t_1$ therein. In the case of $s_1^o \in X$, without loss of generality, assume $s_1^o = s_2$. The 3-connectivity of R_F ensures the existence of an X-valid path $L_2 := s_2 - t_2$ therein. Use Lemma 25 to find pairwise distinct neighbours s_3' of s_3 and t_3' of t_3 in $\mathcal{A}_1$; these exist since $s_3 \neq s_1^o$ and $t_3 \neq s_1^o$. Using the connectivity of $\mathcal{A}_1$ (Proposition 18), find a path $L_3 := s_3 - t_3$ in $\mathcal{S}_1$ that contains a subpath $s_3' - t_3'$ in $\mathcal{A}_1$. The linkage $\{L_1, L_2, L_3\}$ is the desired Y-linkage.

Assume neither R nor R_F contains a pair $\{s_i, t_i\}$ $(i = 2, 3)$. Without loss of generality, assume that $s_2, s_3 \in R$, that $t_2, t_3 \in R_F$ and that $t_2 \neq s_1^o$.

First suppose that there exists a path S_3 in F_1 from s_3 to R_F that is of length at most two and is disjoint from $X \setminus \{s_3, t_3\}$. Let $\{\hat{s}_3\} := V(S_3) \cap V(R_F)$. Find pairwise distinct neighbours s_2' and t_2' of s_2 and t_2, respectively, in $\mathcal{A}_1$. And find a path $L_2 := s_2 - t_2$ in $\mathcal{S}_1$ that contains a subpath $s_2' - t_2'$ in $\mathcal{A}_1$ (using the connectivity of $\mathcal{A}_1$). Using the 3-connectivity of R_F link the pair $\{\hat{s}_3, t_3\}$ in R_F through a path L_3' that is disjoint from t_2. Let $L_3 := s_3 S_3 \hat{s}_3 L_3' t_3$. Since Corollary 11 ensures that any separator of size three in a 3-cube must be independent, we can find a path $L_1 := s_1 - t_1$ in R that is disjoint from s_2 and $V(S_3) \cap V(R)$; the set $V(S_3) \cap V(R)$ has either cardinality one or contains an edge. The paths $\{L_1, L_2, L_3\}$ form the desired Y-linkage.

Assume that there is no such path S_3. In this case, the neighbours of s_3 in F_1 are s_1, t_1, s_2 from R and t_2 from R_F. Use Lemma 25 to find a neighbour s_3' of s_3 in $\mathcal{A}_1$. Again use Lemma 25 either to find a neighbour t_3' of t_3 if $t_3 \neq s_1^o$ or to find a neighbour t_3' of a neighbour u of t_3 in R_F (with $u \neq t_2$) if $t_3 = s_1^o$. Let T_3 be the path of length at most two from t_3 to $\mathcal{A}_1$ defined as $T_3 = t_3 t_3'$ if $t_3 \neq s_1^o$ and $T_3 = t_3 u t_3'$ if $t_3 = s_1^o$. Find a path L_3 in $\mathcal{S}_1$ between s_3 and t_3 that contains a subpath in $\mathcal{A}_1$ between s_3' and t_3'; here use the connectivity of $\mathcal{A}_1$ (Proposition 18). We next find a path S_2 in F_1 from s_2 to R_F that is of length at most two and is disjoint from $V(T_3) \cup \{s_1, t_1, s_3\}$. There are exactly four disjoint such $s_2 - R_F$ paths of length at most two, one through each of the neighbours of s_2 in F_1. One such path is $s_2 s_3 t_2$. Among the remaining three $s_2 - R_F$ paths, since none of them contains s_1 or t_1 and since $|V(T_3) \cap V(R_F)| \leq 2$, we find the path S_2. Let $\hat{s}_2 := V(S_2) \cap V(R_F)$. Find a path $L_2' := \hat{s}_2 - t_2$ in R_F that is disjoint from $V(T_3)$, using the 3-connectivity of R_F. Let $L_2 := s_2 S_2 \hat{s}_2 L_2' t_2$. Since the vertices in $(V(S_2) \cap V(R)) \cup \{s_3\}$ cannot separate s_1 from t_1 in R (Corollary 11), find a path $L_1 := s_1 - t_1$ in R disjoint from $V(S_2) \cap V(R) \cup \{s_3\}$; the set $V(S_2)$ has cardinality one or contains one edge. The paths $\{L_1, L_2, L_3\}$ form the desired Y-linkage.

SUBCASE C for $d \geq 7$. The vertex opposite to s_1 in F_1 coincides with t_1. And the vertex s_1 is not in Configuration dF.. Then t_1 has no neighbour in $\mathcal{A}_1$. In fact, F_1 is the only facet in $\mathcal{S}_1$ containing t_1.

Because the vertex s_1 is not in Configuration dF, t_1 has a neighbour $t_1^{F_1}$ in F_1 that is not in X. Here we reason as in the scenario in which $s_2 = s_1^o$ and s_2 has a neighbour not in X.

First, using the $(k-1)$-linkedness of $\mathcal{L}_1$ (Proposition 7) find disjoint paths $L_i := s_i - t_i$ in $\mathcal{L}_1$ for $i \in [2,k]$. It may happen that $t_1^{F_1}$ is in one of the paths L_i for $i \in [2,k]$. Second, consider neighbours s_1' and t_1' in $\mathcal{A}_1$ of s_1 and $t_1^{F_1}$, respectively.

If $t_1^{F_1}$ doesn't belong to any path L_i, then find a path $L_1 := s_1 - t_1$ that contains the edge $t_1 t_1^{F_1}$ and a subpath L_1' in $\mathcal{A}_1$ between s_1' and t_1'; that is, $L_1 = s_1 s_1' L_1' t_1' t_1^{F_1} t_1$. The desired Y-linkage is given by $\{L_1, \ldots, L_k\}$.

If $t_1^{F_1}$ belongs to one of the paths L_i with $i \in [2,k]$, say L_j, then disregard this path L_j and consider in $\mathcal{A}_1$ a neighbour s_j' of s_j and a neighbour t_j' of t_j. From Lemma 25, it follows that the vertices s_1', t_1', s_j' and t_j' can be taken pairwise distinct. Using the 2-linkedness of $\mathcal{A}_1$ for $d \geq 7$, find a path L_1' in $\mathcal{A}_1$ between s_1' and t_1' and a path L_j' in $\mathcal{A}_1$ between s_j' and t_j'. Let $L_1 := s_1 s_1' L_1' t_1' t_1^{F_1} t_1$ and let $L_j := s_j s_j' L_j' t_j' t_j$ be the new $s_j - t_j$ path. The paths $\{L_1, \ldots, L_k\}$ form the desired Y-linkage.

SUBCASE C for $d = 5$. The vertex opposite to s_1 in F_1 coincides with t_1. And the vertex s_1 is not in Configuration dF.. Hence we may suppose that t_1 has a neighbour t_1' not in X. We reason as in Subcases A and B for $d = 5$. We give the details for the sake of completeness.

Let R denote the 3-face in F_1 containing both s_1 and t_1'; $\operatorname{dist}_R(s_1, t_1') = 3$. Let R_F be the 3-face of F_1 disjoint from R. Let J_1 be the other facet in $\mathcal{S}_1$ containing R and let R_J be the 3-face of J_1 disjoint from R.

Suppose R contains a pair $\{s_i, t_i\}$ $(i = 2, 3)$, say (s_2, t_2). There are at most five terminals in R. Since the smallest face in R containing s_1 and t_1' is 3-dimensional, the sequence $\pi_{R_J}^{J_1}(s_1), \pi_{R_J}^{J_1}(s_2), \pi_{R_J}^{J_1}(t_1'), \pi_{R_J}^{J_1}(t_2)$ cannot appear in a 2-face of R_J in cyclic order. As a consequence, the pairs $\{\pi_{R_J}^{J_1}(s_1), \pi_{R_J}^{J_1}(t_1')\}$ and $\{\pi_{R_J}^{J_1}(s_2), \pi_{R_J}^{J_1}(t_2)\}$ can be linked in R_J through disjoint paths L_1' and L_2', thanks to Proposition 4. Let $L_1 := s_1 \pi_{R_J}^{J_1}(s_1) L_1' \pi_{R_J}^{J_1}(t_1') t_1' t_1$ and $L_2 := s_2 \pi_{R_J}^{J_1}(s_2) L_2' \pi_{R_J}^{J_1}(t_2) t_2$. From the 3-connectivity of R_F follows the existence of a path L_3' in R_F between $\pi_{R_F}^{F_1}(s_3)$ and $\pi_{R_F}^{F_1}(t_3)$ that avoids t_1. Let $L_3 := s_3 \pi_{R_F}^{F_1}(s_3) L_3' \pi_{R_F}^{F_1}(t_3) t_3$. The paths $\{L_1, L_2, L_3\}$ form the desired Y-linkage.

Suppose that the ridge R contains no pair $\{s_i, t_i\}$ $(i = 2, 3)$ and that the ridge R_F contains a pair $\{s_i, t_i\}$ $(i = 2, 3)$, say $\{s_2, t_2\}$. Then, there are at most five terminals in R_F. If there are at most four terminals in R_F, the 3-connectivity of R_F ensures the existence of an X-valid path $L_2 := s_2 - t_2$ in R_F; if there are exactly five terminals in R_F, applying Lemma 14 to the polytope F_1 and its facet R_F gives either an X-valid path $L_2 := s_2 - t_2$ or an X-valid path $L_3 := s_3 - t_3$ in R_F. As a result, regardless of the number of terminals in R_F, we can assume there is an X-valid path $L_2 := s_2 - t_2$ in R_F. Find pairwise distinct neighbours s_3' and t_3' in $\mathcal{A}_1$ of s_3 and t_3, respectively, and a path L_3 in $\mathcal{S}_1$ between s_3 and t_3 that contains a subpath in $\mathcal{A}_1$ between s_3' and t_3'; here use the connectivity of $\mathcal{A}_1$ (Proposition 18).

In addition, let L_1' be a path in R between s_1 and t_1'; here use the 3-connectivity of R to avoid any terminal in R. Let $L_1 := s_1L_1't_1't_1$. The Y-linkage is given by the paths $\{L_1, L_2, L_3\}$.

Assume neither R nor R_{F_1} contains a pair $\{s_i, t_i\}$ $(i = 2, 3)$. Without loss of generality, we can assume $s_2, s_3 \in R$ and $t_2, t_3 \in R_F$.

For *some* $i = 2, 3$, there exists a path S_i in F_1 from s_i to R_F that is of length at most two and is disjoint from t_1' and $X \setminus \{s_i, t_i\}$. Suppose there is no such path $S_3 = s_3 - R_F$. Then the neighbours of s_3 in F_1 would be s_1, t_1', s_2 from R and t_2 from R_F. But, since there are exactly four $s_2 - R_F$ paths of length at most two in F_1 and since the vertex s_2 could not be adjacent to $\{s_1, t_1'\}$, the existence of such a path $S_2 = s_2 - R_F$ would be guaranteed. Hence assume the existence of such a path $S_3 = s_3 - R_F$. Let $\{\hat{s}_3\} := V(S_3) \cap V(R_F)$. Find an X-valid path $L_3' := \hat{s}_3 - t_3$ in R_F using its 3-connectivity. Let $L_3 := s_3S_3\hat{s}_3L_3't_3$. Then find neighbours s_2' and t_2' of s_2 and t_2, respectively, in $\mathcal{A}_1$, and a path $L_2 := s_2 - t_2$ in $\mathcal{S}_1$ that contains a subpath $s_2' - t_2'$ in $\mathcal{A}_1$ (using the connectivity of $\mathcal{A}_1$). Since Corollary 11 ensures that any separator of size three in a 3-cube must be independent, we can find an $L_1' := s_1 - t_1'$ in R that is disjoint from s_2 and $V(S_3) \cap V(R)$; the set $V(S_3) \cap V(R)$ has either cardinality one or contains an edge. Let $L_1 := s_1L_1't_1't_1$. The paths $\{L_1, L_2, L_3\}$ form the desired Y-linkage.

And finally, the proof of Lemma 23 is complete. □

5. Strong linkedness of cubical polytopes

Proof of Theorem 3 (Strong linkedness of cubical polytopes). Let P be a cubical d-polytope. For odd d Theorems 2 and 3 are equivalent. So assume $d = 2k \geq 4$. Let X be a set of $d + 1$ vertices in P. Arbitrarily pair $2k$ vertices in X to obtain $Y := \{\{s_1, t_1\}, \ldots, \{s_k, t_k\}\}$. Let x be the vertex of X not paired in Y. We find a Y-linkage $\{L_1, \ldots, L_k\}$ where each path L_i joins the pair $\{s_i, t_i\}$ and avoids the vertex x.

Using the d-connectivity of $G(P)$ and Menger's theorem, bring the $d = 2k$ terminals in $X \setminus \{x\}$ to the link of x in the boundary complex of P through $2k$ disjoint paths L_{s_i} and L_{t_i} for $i \in [1, k]$. Let $s_i' := V(L_{s_i}) \cap \operatorname{link}(x)$ and $t_i' := V(L_{t_i}) \cap \operatorname{link}(x)$ for $i \in [1, k]$. Thanks to Proposition 7, the link of x is k-linked. Using the k-linkedness of $\operatorname{link}(x)$, find disjoint paths $L_i' := s_i' - t_i'$ in $\operatorname{link}(x)$. Observe that all these k paths $\{L_1', \ldots, L_k'\}$ avoid x. Extend each path L_i' with L_{s_i} and L_{t_i} to form a path $L_i := s_i - t_i$ for $i \in [1, k]$. The paths $\{L_1, \ldots, L_k\}$ form the desired Y-linkage. □

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