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On large bipartite graphs of diameter 3

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ABSTRACT

We consider the bipartite version of the *degree/diameter problem*, namely, given natural numbers $d \geq 2$ and $D \geq 2$, find the maximum number $\mathrm{N}^b(d, D)$ of vertices in a bipartite graph of maximum degree d and diameter D. In this context, the bipartite Moore bound $\mathrm{M}^b(d, D)$ represents a general upper bound for $\mathrm{N}^b(d, D)$. Bipartite graphs of order $\mathrm{M}^b(d, D)$ are very rare, and determining $\mathrm{N}^b(d, D)$ still remains an open problem for most (d, D) pairs.

This paper is a follow-up of our earlier paper (Feria-Purón and Pineda-Villavicencio, 2012 [5]), where a study on bipartite $(d, D, -4)$-graphs (that is, bipartite graphs of order $\mathrm{M}^b(d, D) - 4$) was carried out. Here we first present some structural properties of bipartite $(d, 3, -4)$-graphs, and later prove that there are no bipartite $(7, 3, -4)$-graphs. This result implies that the known bipartite $(7, 3, -6)$-graph is optimal, and therefore $\mathrm{N}^b(7, 3) = 80$. We dub this graph the Hafner–Loz graph after its first discoverers Paul Hafner and Eyal Loz.

The approach here presented also provides a proof of the uniqueness of the known bipartite $(5, 3, -4)$-graph, and the non-existence of bipartite $(6, 3, -4)$-graphs.

In addition, we discover at least one new largest known bipartite – and also vertex-transitive – graph of degree 11, diameter 3 and order 190, a result which improves by four vertices the previous lower bound for $\mathrm{N}^b(11, 3)$.

1. Introduction

Consider the *degree/diameter problem for bipartite graphs*, stated as follows:

Given natural numbers $d \geq 2$ and $D \geq 2$, find the largest possible number $\mathrm{N}^b(d, D)$ of vertices in a bipartite graph of maximum degree d and diameter D.

It is well known that an upper bound for $\mathrm{N}^b(d, D)$ is given by the *bipartite Moore bound* $\mathrm{M}^b(d, D)$, defined below:

$$\mathrm{M}^b(d, D) = 2\left(1 + (d-1) + \cdots + (d-1)^{D-1}\right).$$

Bipartite graphs of degree d, diameter D and order $\mathrm{M}^b(d, D)$ are called *bipartite Moore graphs*. Bipartite Moore graphs are very scarce; when $d \geq 3$ and $D \geq 3$ they may only exist for $D = 3$, 4 or 6 (see [4]). It has also turned out to be very difficult to determine $\mathrm{N}^b(d, D)$ even for particular instances; in fact, with the exception of $\mathrm{N}^b(3, 5) = \mathrm{M}^b(3, 5) - 6$ settled in [6], the known values of $\mathrm{N}^b(d, D)$ are those for which a bipartite Moore graph is known to exist.

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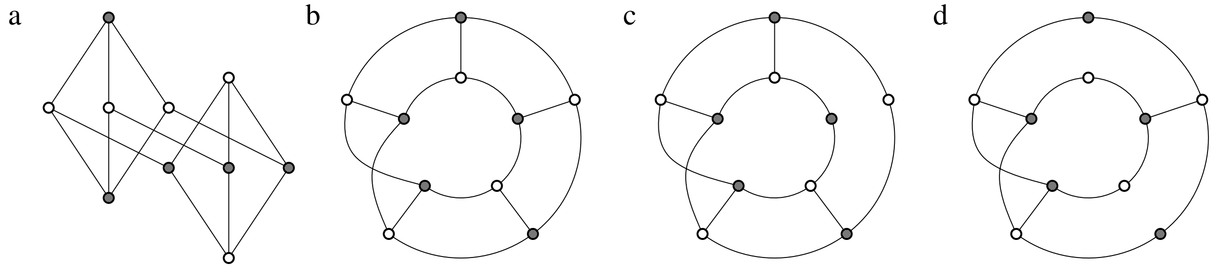

Fig. 1. All the bipartite $(3, 3, -4)$-graphs.

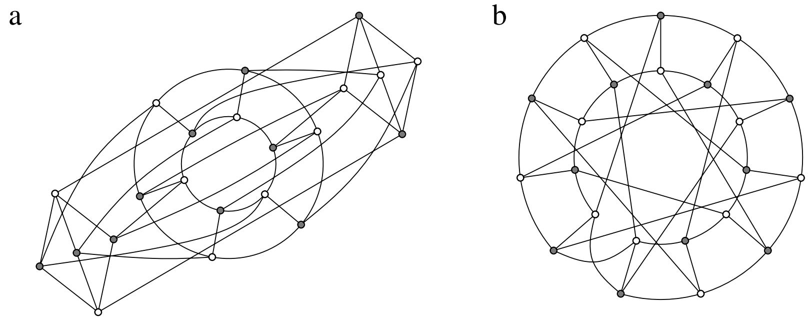

Fig. 2. All the bipartite $(4, 3, -4)$-graphs.

Research in this area falls into two main directions. On one hand, there are efforts to improve the upper bounds for $\mathrm{N}^b(d, D)$ by studying the existence or otherwise of bipartite graphs of maximum degree d, diameter D and order $\mathrm{M}^b(d, D) - \epsilon$ for small $\epsilon > 0$, that is, bipartite $(d, D, -\epsilon)$-graphs. The parameter ϵ is called the *defect*. On the other hand, there are studies aiming to improve the lower bounds for $\mathrm{N}^b(d, D)$ by constructing ever larger bipartite graphs with given maximum degree and diameter. In spite of these efforts and the wide range of techniques and approaches used to tackle these problems (see [10]), in most cases there is still a significant gap between the current lower and upper bound for $\mathrm{N}^b(d, D)$.

In this paper we restrict ourselves to the case of bipartite graphs of diameter 3, and present some modest contributions in both directions. When $D = 3$ there is a bipartite Moore graph whenever $d - 1$ is a prime power (namely, the incidence graphs of projective planes); however, there is no bipartite Moore graph of diameter 3 for $d = 7$ [11] or $d = 11$ [7]. The existence of bipartite Moore graphs of diameter 3 for other degrees remains an open problem. In [2] the authors proved that bipartite $(d, 3, -2)$-graphs may only exist for certain values of d; in particular, they do not exist for $d = 7$.

The results and ideas exposed here are, to a great extent, a continuation of the precursory work initiated in [5]. We provide structural properties for bipartite $(d, 3, -4)$-graphs and, most importantly, prove the non-existence of bipartite $(7, 3, -4)$-graphs. Such an outcome implies that the only known bipartite $(7, 3, -6)$-graph – found by Paul Hafner and independently by Eyal Loz [9] – is optimal, and therefore $\mathrm{N}^b(7, 3) = 80$. This is just the second value settled for $\mathrm{N}^b(d, D)$ other than a bipartite Moore bound. We call this graph the Hafner–Loz graph in honor of its discoverers. Our approach can also be used to show the uniqueness of the known bipartite $(5, 3, -4)$-graph, as well as the non-existence of bipartite $(6, 3, -4)$-graphs.

Finally, we also find at least one largest known bipartite (also vertex-transitive) graph of degree 11 and diameter 3. This gives $190 \leq \mathrm{N}^b(11, 3)$, which improves by four vertices the previous lower bound for $\mathrm{N}^b(11, 3)$. Adjacency lists of these graphs are available at [1] under the name of this paper.

We conclude this introduction by depicting all the known bipartite $(d, 3, -4)$ graphs. Fig. 1 shows all the bipartite $(3, 3, -4)$-graphs, Fig. 2 all the bipartite $(3, 3, -4)$-graphs, and Fig. 3 the unique bipartite $(5, 3, -4)$-graph.

2. Notation and terminology

Our notation and terminology follow from [5], and are standard and consistent with those used in [3].

All graphs considered are simple. The vertex set of a graph Γ is denoted by $V(\Gamma)$, and its edge set by $E(\Gamma)$. For an edge $e = \{x, y\}$ we write $x \sim y$. The set of edges in a graph Γ joining a vertex x in $X \subseteq V(\Gamma)$ to a vertex y in $Y \subseteq V(\Gamma)$ is denoted by $E(X, Y)$. A vertex of degree at least 3 is called a *branch vertex* of Γ.

A cycle of length k is called a *k-cycle*. In a bipartite $(d, D, -4)$-graph we call a cycle of length at most $2D - 2$ a *short cycle*. If two short cycles C^1 and C^2 are non-disjoint we say that C^1 and C^2 are *neighbors*.

For a vertex x lying on a short cycle C, we denote by $\mathrm{rep}^C(x)$ the vertex x' in C such that $d(x, x') = D - 1$, where $d(x, x')$ denotes the distance between x and x'. In this case, we say x' is a *repeat* of x in C and vice versa, or simply that x and x' are

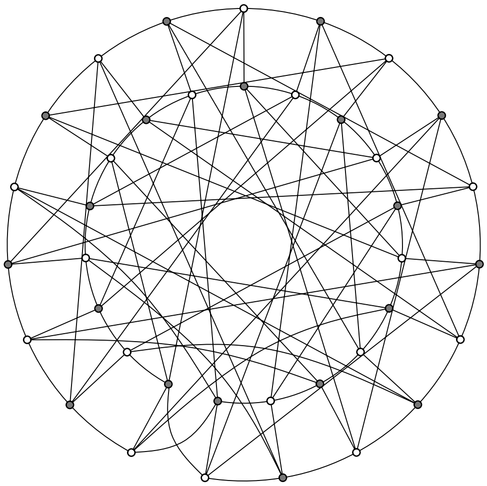

Fig. 3. The unique bipartite $(5, 3, -4)$-graph.

repeats in C. A *closed set of repeats* in a bipartite $(d, D, -4)$-graph Γ is a subset of $V(\Gamma)$ which is closed under the repeat relation. A closed set of repeats is *minimal* if it does not have a proper closed subset of repeats.

Finally, we introduce notation for some particular graphs. The union of three independent paths of length t with common endvertices is denoted by Θ_t. For an integer $m \geq 5$, Φ_m denotes the bipartite graph with vertex set $V = \{x_i | 0 \leq i \leq m-1\} \cup \{y_i | 0 \leq i \leq m-1\}$ and edge set $E = \{x_i \sim y_i, x_i \sim y_{i+1}, x_i \sim y_{i-1} | 0 \leq i \leq m-1\}$. Throughout this paper we do addition modulo m on the vertex subscripts of a Φ_m.

3. Preliminaries

We begin with the regularity condition for bipartite graphs with small defect.

Proposition 3.1 ([2]). *For $\epsilon < 1 + (d-1) + (d-1)^2 + \cdots + (d-1)^{D-2}$, $d \geq 3$ and $D \geq 3$, a bipartite $(d, D, -\epsilon)$-graph is regular.*

Proposition 3.2 ([2]). *For $\epsilon < 2\left((d-1) + (d-1)^3 + \cdots + (d-1)^{D-2}\right)$, $d \geq 3$ and odd $D \geq 3$, a bipartite $(d, D, -\epsilon)$-graph is regular.*

In particular, we will implicitly use the fact that a bipartite $(d, 3, -4)$-graph with $d \geq 4$ must be regular, and therefore its partite sets must have the same cardinality. Also note that, since bipartite $(d, 3, -\epsilon)$ graphs with $d \geq 4$ and $\epsilon = 3, 5$ are not regular, the above propositions imply their non-existence.

From the paper [5] we borrow the following results:

Proposition 3.3 ([5]). *The girth of a regular bipartite $(d, D, -4)$-graph Γ with $d \geq 3$ and $D \geq 3$ is $2D-2$. Furthermore, any vertex x of Γ lies on the short cycles specified below and no other short cycle. We have the following cases:*

x is contained in exactly three $(2D-2)$-cycles. *Then:*

(i) *x is a branch vertex of one Θ_{D-1}, or*

x is contained in two $(2D-2)$-cycles. *Then:*

(ii) *x lies on* exactly two $(2D-2)$-cycles, *whose intersection is a ℓ-path with $\ell \in \{0, \ldots, D-1\}$.*

As in [5], often our arguments revolve around the identification of the elements in the set S_x of short cycles containing a given vertex x; we call this process *saturating* the vertex x. A vertex x is called *saturated* if the elements in S_x have been completely identified.

Lemma 3.1 ([5], *Saturating Lemma*). *Let $\mathcal{C}$ be a $(2D-2)$-cycle in a bipartite $(d, D, -4)$-graph Γ with $d \geq 4$ and $D \geq 3$, and α, α' two vertices in $\mathcal{C}$ such that $\alpha' = \mathrm{rep}^{\mathcal{C}}(\alpha)$. Let γ be a neighbor of α not contained in $\mathcal{C}$, and $\mu_1, \mu_2, \ldots, \mu_{d-2}$ the neighbors of α' not contained in $\mathcal{C}$. Suppose there is no short cycle in Γ containing the edge $\alpha \sim \gamma$ and intersecting $\mathcal{C}$ at a path of length greater than $D-3$.*

Then, in Γ there exist a vertex $\mu \in \{\mu_1, \mu_2, \ldots, \mu_{d-2}\}$ and a short cycle $\mathcal{C}^1$ such that γ and μ are repeats in $\mathcal{C}^1$, and $\mathcal{C} \cap \mathcal{C}^1 = \emptyset$.

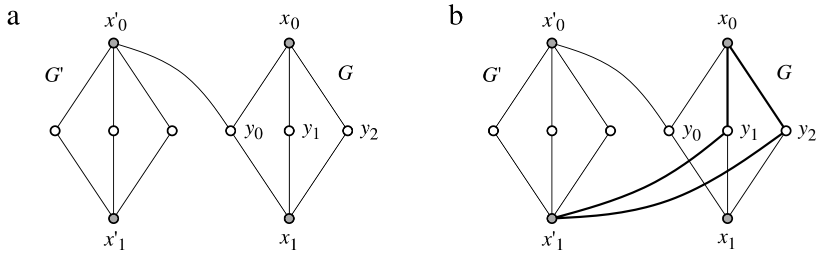

Fig. 4. Auxiliary figure for Observation 3.1.

Lemma 3.2 ([5], *Repeat Cycle Lemma*). *Let C be a short cycle in a bipartite $(d, D, -4)$-graph Γ with $d \geq 4$ and $D \geq 3$, $\{C^1, C^2, \ldots, C^k\}$ the set of neighbors of C, and $I_i = C^i \cap C$ for $1 \leq i \leq k$. Suppose at least one I_j, for $j \in \{1, \ldots, k\}$, is a path of length smaller than $D - 2$. Then there is an additional short cycle C' in Γ intersecting C^i at $I'_i = \mathrm{rep}^{C^i}(I_i)$, where $1 \leq i \leq k$.*

Proposition 3.4 ([5]). *The set $S(\Gamma)$ of short cycles in a bipartite $(d, D, -4)$-graph Γ with $d \geq 3$ and $D \geq 3$ can be partitioned into sets $S_{D-1}(\Gamma)$, $S_{D-2}(\Gamma)$ and $S_{D-3}(\Gamma)$, where*

$S_{D-1}(\Gamma)$ is the set of short cycles in Γ whose intersections with neighbor cycles are $(D-1)$-paths,
$S_{D-2}(\Gamma)$ is the set of short cycles in Γ whose intersections with neighbor cycles are $(D-2)$-paths, and
$S_{D-3}(\Gamma)$ is the set of short cycles in Γ whose intersections with neighbor cycles are paths of length at most $D - 3$.

Proposition 3.5 ([5]). *The set $V(\Gamma)$ of vertices in a bipartite $(d, D, -4)$-graph Γ with $d \geq 4$ and $D \geq 3$ can be partitioned into sets $V_{D-1}(\Gamma)$, $V_{D-2}(\Gamma)$ and $V_{D-3}(\Gamma)$, where*

$V_{D-1}(\Gamma)$ is the set of vertices contained in cycles of $S_{D-1}(\Gamma)$,
$V_{D-2}(\Gamma)$ is the set of vertices contained in cycles of $S_{D-2}(\Gamma)$,
$V_{D-3}(\Gamma)$ is the set of vertices contained in cycles of $S_{D-3}(\Gamma)$,

and $S_{D-1}(\Gamma)$, $S_{D-2}(\Gamma)$, $S_{D-3}(\Gamma)$ are defined as in Proposition 3.4.

3.1. On bipartite graphs of diameter 3 and defect 4

In this section we present additional structural properties for bipartite graphs of diameter 3 and defect 4.

Let Γ be a bipartite $(d, 3, -4)$-graph with $d \geq 4$. We set $\Gamma_i = \bigcup_{C \in S_i(\Gamma)} C$ for $i = 0, 1, 2$. Note that Γ_2 is the union of all graphs in Γ isomorphic to Θ_2; these graphs are pairwise disjoint, so they are the connected components of Γ_2. In addition, Γ_1 is the union of all graphs in Γ isomorphic to some Φ_m for $m \geq 5$; similarly, these Φ_m are the connected components of Γ_1.

If G is a connected component in $\Gamma_2 \cup \Gamma_1 \cup \Gamma_0$ then $V(G)$ is a closed set of repeats. The branch vertices of a $\Theta_2 \subset \Gamma_2$ constitute a minimal closed set of repeats, as well as its non-branch vertices. In the case of a $\Phi_m \subset \Gamma_1$, the vertices x_i form a minimal closed set of repeats, and so do the vertices y_i. According to the Repeat Cycle Lemma, every minimal closed set of repeats in Γ_0 contains exactly four vertices. Observe that all vertices in a minimal closed set of repeats in Γ belong to the same partite set.

Some further observations about Γ follow from the systematic application of the Saturating Lemma.

Observation 3.1. *Let Γ be a bipartite $(d, 3, -4)$-graph with $d \geq 4$. There is no edge in Γ joining a branch vertex in Γ_2 to a non-branch vertex of a different connected component of Γ_2.*

Proof. Let G, G' be two connected components in Γ_0 such that a branch vertex x'_0 in G' is adjacent to a non-branch vertex y_0 in G. Let x'_1, x_0, x_1, y_1, y_2 be as in Fig. 4(a). We apply the Saturating Lemma (by mapping the cycle $x_0y_0x_1y_1x_0$ to $\mathcal{C}$, y_0 to α, y_1 to α' and x'_0 to γ), and obtain that y_1 is adjacent to x'_1. Similarly, y_2 is also adjacent to x'_1 (see Fig. 4(b)), but then there is a fourth short cycle $x_0y_1x'_1y_2x_0$ in Γ containing x_0, a contradiction. $\square$

Observation 3.2. *Let Γ be a bipartite $(d, 3, -4)$-graph with $d \geq 4$. There is no edge in Γ joining a branch vertex in Γ_2 to a vertex in Γ_1.*

Proof. Let G, G' be two connected components of Γ_1 and Γ_2 respectively, such that a branch vertex x'_0 in G' is adjacent to a vertex y_i in G. Let $x'_1, y_{i+1}, y_{i-1}, x_i, x_{i+1}$ and x_{i-1} be as in Fig. 5(a). We apply the Saturating Lemma (by mapping cycle $y_ix_{i-1}y_{i-1}x_iy_i$ to $\mathcal{C}$, y_i to α, y_{i-1} to α' and x'_0 to γ), and obtain that y_{i-1} is adjacent to x'_1. Similarly, y_{i+1} is also adjacent to x'_1 (see Fig. 5(b)). But then, there is a third short cycle $y_{i+1}x_iy_{i-1}x'_1y_{i+1}$ in Γ containing x_i, a contradiction. $\square$

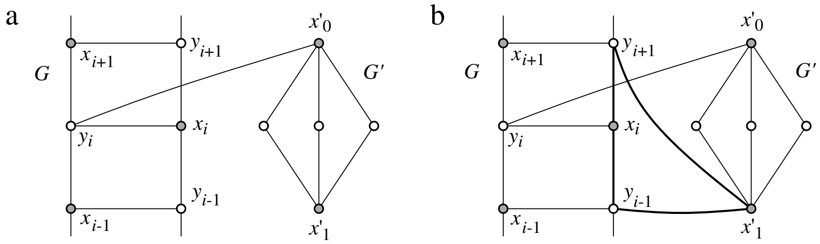

Fig. 5. Auxiliary figure for Observation 3.2.

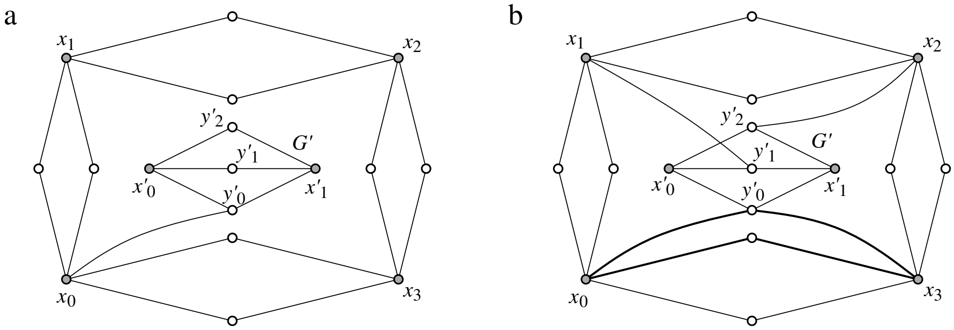

Fig. 6. Auxiliary figure for Observation 3.3.

Observation 3.3. *Let Γ be a bipartite $(d, 3, -4)$-graph with $d \geq 4$. There is no edge in Γ joining a non-branch vertex in Γ_2 to a vertex in Γ_0.*

Proof. Let G' be a connected component in Γ_2 with a non-branch vertex y'_0 adjacent to a vertex x_0 in Γ_0. Let $\{x_0, x_1, x_2, x_3\}$ be the minimal closed set of repeats containing x_0 (x_2 not being a repeat of x_0), and let the vertices x'_0, x'_1, y'_1, y'_2 be as in Fig. 6(a). We first apply the Saturating Lemma (by mapping the cycle $x'_0y'_0x'_1y'_1x'_0$ to $\mathcal{C}$, y'_0 to α, y'_1 to α' and x_0 to γ), and obtain that y'_1 is adjacent to a repeat of x_0 (say x_1). Similarly, mapping the cycle $x'_0y'_0x'_1y'_1x'_0$ to $\mathcal{C}$, y'_1 to α, y'_2 to α' and x_1 to γ, we obtain that y'_2 is adjacent to x_2 (as it cannot be adjacent to x_0). Analogously, y'_0 is adjacent to x_3 (see Fig. 6(b)), but then there is a third short cycle in Γ containing x_0, a contradiction. □

Observation 3.4. *Let Γ be a bipartite $(d, 3, -4)$-graph with $d \geq 4$, and $G = \Phi_m$ a connected component in Γ_1. Given $x_i \in G$, if $x_i \sim y_j \in E(\Gamma)$ for some j then $x_{i+k} \sim y_{j+k} \in E(\Gamma)$ for every k.*

Proof. This clearly holds when $j \in \{i, i+1, i-1\}$; see the description of Φ_m.

Suppose $j \notin \{i, i+1, i-1\}$. Since all the vertices in G are saturated, we have $|i-j| \geq 4$. According to the Saturating Lemma (by mapping the cycle $x_iy_ix_{i+1}y_{i+1}x_i$ to $\mathcal{C}$, x_i to α, x_{i+1} to α' and y_j to γ) we have either $x_{i+1} \sim y_{j+1} \in E(\Gamma)$ or $x_{i+1} \sim y_{j-1} \in E(\Gamma)$. But in the case where $x_{i+1} \sim y_{j-1} \in E(\Gamma)$, it is easy to see that, by repeatedly applying the Saturating Lemma (to the cycles $x_{i+p}y_{i+p}x_{i+p+1}y_{i+p+1}x_{i+p}$ for $p = 1, 2, \ldots$) we obtain that there is an edge $x_r \sim y_s$ in Γ such that $2 \leq |r-s| \leq 3$, which is not possible. Thus $x_{i+1} \sim y_{j+1} \in E(\Gamma)$ and, by induction, $x_{i+k} \sim y_{j+k} \in E(\Gamma)$ for every k. □

Observation 3.5. *Let Γ be a bipartite $(d, 3, -4)$-graph with $d \geq 3$, and G, G' two connected components in Γ_1 of order $2m$ and $2m'$ respectively ($m \leq m'$). Suppose there is at least one edge in Γ joining a vertex in G to a vertex in G'. Then $m' = km$, with $1 \leq k \leq d-3$.*

Proof. Denote the vertices of $G = \Phi_m$ by $x_0, \ldots, x_{m-1}, y_0, \ldots, y_{m-1}$, and the vertices of $G' = \Phi_{m'}$ by $x'_0, \ldots, x'_{m'-1}$, $y'_0, \ldots, y'_{m'-1}$. With an appropriate labeling we may assume there is an edge $x_0 \sim y'_0$ in Γ and, by the Saturating Lemma (on the cycle $x_0y_0x_1y_1x_0$), also an edge $x_1 \sim y'_1$ in Γ.

Suppose $m' = km + r$, with $1 \leq r \leq m-1$ and $k \geq 1$. Then, by repeatedly applying the Saturating Lemma on the cycles $x_iy_ix_{i+1}y_{i+1}x_i$ with $i = 1, \ldots, m-1$, we find that the edges $x_i \sim y'_i$ for $i = 2, \ldots, m$ are all present in Γ. In particular, y'_m is a neighbor of x_0 and, inductively, the vertices $y'_{2m}, \ldots, y'_{km}, y'_{m-r}, y'_{2m-r}, \ldots$ also are. But similarly, x_{m-r} has also neighbors y'_{m-r} and y'_{2m-r}; this way, we obtain that there is in Γ a third short cycle $x_0y'_{m-r}x_{m-r}y'_{2m-r}x_0$ containing x_0, a contradiction.

Since a vertex in G has at least three neighbors in G, it follows that $k \leq d-3$. □

Observation 3.6. *Let Γ be a bipartite $(7, 3, -4)$-graph. If $\Gamma_0 \neq \emptyset$ then $|\Gamma_0| = 8k$, with $k \geq 3$.*

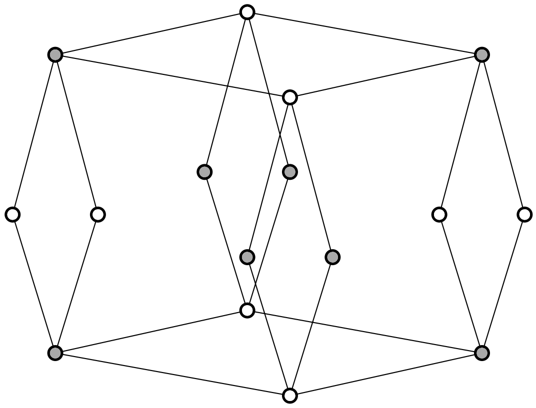

Fig. 7. Auxiliary figure for Observation 3.6.

Proof. If t is the number of short cycles in Γ_0 then, by a simple counting argument, Γ_0 has $2t$ vertices, half of them in each partite set. Recall that $V(\Gamma_0)$ is a closed set of repeats. Since a minimal closed set of repeats in Γ_0 contains exactly four vertices belonging to the same partite set, we have $t = 4k$ and then $|\Gamma_0| = 8k$.

Also, the Repeat Cycle Lemma ensures that the graph G depicted in Fig. 7 is a subgraph of Γ_0. Since any vertex in Γ_0 must have at least four neighbors in Γ_0, we have $|\Gamma_0| > 16$ and $k \geq 3$. $\square$

Observation 3.7. *Let Γ be a bipartite $(d, 3, -4)$-graph with $d \geq 4$, G a connected component in Γ_2, and G' a connected component in Γ_1 of order $2m'$. Suppose there is in Γ at least one edge joining a vertex in G to a vertex in G'. Then $m' = 3k$ with $2 \leq k \leq d - 2$.*

Proof. Let x_0, x_1, x_2 be the non-branch vertices of $G = \Theta_2$, and denote by $x'_0, \ldots, x'_{m'-1}, y'_0, \ldots, y'_{m'-1}$ the vertices of $G' = \Phi_{m'}$.

By Observation 3.2 any edge between Γ_2 and Γ_1 involves only non-branch vertices of Γ_2. We may assume there are edges $x_0 \sim y'_0$ and $x_1 \sim y'_1$ in Γ. Suppose $m' = 3k + r$, with $1 \leq r \leq 2$ and $k \geq 1$. Then, by repeatedly applying the Saturating Lemma on the three short cycles of G, we obtain that x_0 has neighbors $y'_0, y'_3, y'_6, \ldots, y'_{3k}, y'_{3-r}, y'_{6-r}, \ldots$. But similarly, x_{3-r} has also neighbors y'_{3-r} and y'_{6-r}; hence, we obtain that there is in Γ a third short cycle $x_0y'_{3-r}x_{3-r}y'_{6-r}x_0$ containing x_0, a contradiction.

Since each x_i has two neighbors in G and $m' \geq 5$, it follows that $2 \leq k \leq d - 2$. $\square$

Observation 3.8. *Let Γ be a bipartite $(d, 3, -4)$-graph with $d \geq 4$ and G' a connected component in Γ_1 of order $2m'$. Suppose there is in Γ an edge joining a vertex in Γ_0 to a vertex in G'. Then $m' = 4k$ with $2 \leq k \leq d - 4$.*

Proof. Let $x_0 \in V(\Gamma_0)$ and let $\{x_0, x_1, x_2, x_3\}$ be the minimal closed set of repeats containing x_0 (x_2 not being a repeat of x_0). Denote by $x'_0, \ldots, x'_{m'-1}, y'_0, \ldots, y'_{m'-1}$ the vertices of $G' = \Phi_{m'}$. We may assume there are edges $x_0 \sim y'_0$ and $x_1 \sim y'_1$ in Γ. Suppose $m' = 4k + r$, with $1 \leq r \leq 3$ and $k \geq 1$. Then, by repeatedly applying the Saturating Lemma on the cycles $x'_iy'_ix'_{i+1}y'_{i+1}x'_i$ ($i = 1, 2, \ldots$) of G', we obtain that x_0 has neighbors $y'_0, y'_4, y'_8, \ldots, y'_{4k}, y'_{4-r}, y'_{8-r}, \ldots$. But analogously, x_{4-r} has also neighbors y'_{4-r} and y'_{8-r}; hence, we obtain that there is in Γ a third short cycle $x_0y'_{4-r}x_{4-r}y'_{8-r}x_0$ containing x_0, a contradiction.

Since x_0 has at least four neighbors in Γ_0 and $m' \geq 5$, it follows that $2 \leq k \leq d - 4$. $\square$

The statements in Observations 3.1, 3.3, 3.5, 3.7 and 3.8 are better summarized in the following, more compact assertion.

Proposition 3.6. *Let Γ be a bipartite $(d, 3, -4)$-graph with $d \geq 4$, and M, M' two minimal closed set of repeats in Γ such that $E(M, M') \neq \emptyset$. Then $|M|$ divides $|M'|$ or $|M'|$ divides $|M|$, except when $M \cup M'$ is the set of the five vertices in a Θ_2.* $\square$

4. Non-existence of bipartite (7, 3, −4)-graphs

In this section we prove that there are no bipartite $(7, 3, -4)$-graphs, and consequently that $\mathrm{N}^b(7, 3) = 80$.

Proposition 4.1. *Let Γ be a bipartite $(7, 3, -4)$-graph. Then Γ_2 cannot be a spanning subgraph of Γ.*

Proof. Since the connected components of Γ_2 are graphs isomorphic to Θ_2, we have that 5 must divide $|\Gamma_2| = 82$, a contradiction. $\square$

Proposition 4.2. *Let Γ be a bipartite $(7, 3, -4)$-graph. Then Γ_1 cannot be a spanning subgraph of Γ.*

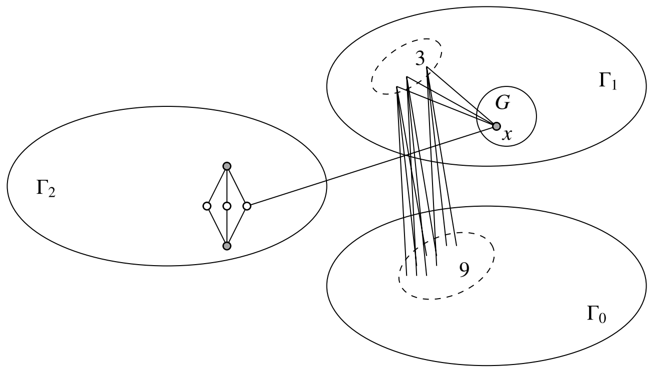

Fig. 8. Auxiliary figure for Claim 1.

Proof. This is a computer-assisted proof.

Suppose that Γ_1 contains exactly one connected component G, isomorphic to Φ_{41}. Denote by $x_0, \ldots, x_{40}, y_0, \ldots, y_{40}$ the vertices of G. By virtue of Observation 3.4, if the vertex x_0 has neighbors $y_0, y_1, y_{-1}, y_{i_1}, y_{i_2}, y_{i_3}, y_{i_4}$ in G then x_k has neighbors $y_k, y_{k+1}, y_{k-1}, y_{k+i_1}, y_{k+i_2}, y_{k+i_3}, y_{k+i_4}$ for every k. Exhaustive computer search through the feasible choices for the vertices $y_{i_1}, y_{i_2}, y_{i_3}, y_{i_4}$ yields no graph of diameter 3, and so there is more than one connected component in Γ_1.

Now suppose that Γ_1 has exactly n connected components $G_1, G_2, \ldots, G_n$, isomorphic to $\Phi_{m_1}, \Phi_{m_2}, \ldots, \Phi_{m_n}$, respectively. Note that $5 \leq m_i \leq 36, 2 \leq n \leq 8$ and $m_1 + \cdots + m_n = 41$. We define the graph $H(G_1, G_1, \ldots, G_n)$ as follows: every G_i contracts to a vertex v_i in H, and there is an edge $v_i - v_j$ in H if and only if – according to Observation 3.5 – there could be an edge from G_i to G_j in Γ (that is, if m_i divides m_j or vice versa). Clearly, if Γ has diameter 3 then H has diameter at most 2. However, we could verify that none of the feasible values for n and the m_i's yields a graph H of diameter at most 2.

Consequently, $V(\Gamma_1)$ cannot span Γ. $\square$

Proposition 4.3. *Let Γ be a bipartite $(7, 3, -4)$-graph. Then Γ_0 cannot be a spanning subgraph of Γ.*

Proof. From Observation 3.6 we have $82 = |\Gamma_0| = 8k$, a contradiction. $\square$

Proposition 4.4. *Let Γ be a bipartite $(7, 3, -4)$-graph. Then $\Gamma_2 \cup \Gamma_1$ cannot be a spanning subgraph of Γ.*

Proof. Suppose $\Gamma_2 \neq \emptyset$ and $\Gamma_1 \neq \emptyset$. On one hand, from a branch vertex in Γ_2 it is possible to reach in exactly two steps at most fifteen vertices of Γ_1 (see Observations 3.1 and 3.2). Therefore, $|\Gamma_1| \leq 30$. On the other hand, from a vertex in Γ_1 it is possible to reach in exactly two steps at most eight branch vertices of Γ_2, and $|\Gamma_2| \leq 40$. This means $|\Gamma| \leq 70$, a contradiction. $\square$

Proposition 4.5. *Let Γ be a bipartite $(7, 3, -4)$-graph. Then $\Gamma_2 \cup \Gamma_0$ cannot be a spanning subgraph of Γ.*

Proof. Suppose $\Gamma_2 \neq \emptyset$ and $\Gamma_0 \neq \emptyset$. From a non-branch vertex in Γ_2 it is possible to reach in two steps at most eight vertices of Γ_0 (see Observations 3.1 and 3.3). Therefore, $|\Gamma_0| \leq 16$, which contradicts Observation 3.6. $\square$

Proposition 4.6. *Let Γ be a bipartite $(7, 3, -4)$-graph. Then $\Gamma_1 \cup \Gamma_0$ cannot be a spanning subgraph of Γ.*

Proof. Let $G = \Phi_m$ be a connected component in Γ_1. We prove that m is even. If G has a neighbor in Γ_0 then, by Observation 3.8, we have $m \in \{8, 12\}$. If instead G has no neighbor in Γ_0 and m is odd, then there must be a connected component G' in Γ_1 isomorphic to some $\Phi_{m'}$ such that G has a neighbor in G' and G' has a neighbor in Γ_0. But again we have $m' \in \{8, 12\}$ and, according to Observation 3.5, $m \geq 5$ must be an odd divisor of m', which is not possible.

From the above and Observation 3.6 it follows that $|\Gamma| \equiv 0 \pmod 4$, which contradicts $|\Gamma| = 82$. $\square$

Proposition 4.7. *Let Γ be a bipartite $(7, 3, -4)$-graph. Then $\Gamma_2 \cup \Gamma_1 \cup \Gamma_0$ cannot be a spanning subgraph of Γ.*

Proof. Let $\Gamma_i \neq \emptyset$ for $i = 0, 1, 2$.

Claim 1. *Every connected component of Γ_1 has a neighbor in Γ_0.*

Proof of Claim 1. Suppose there is a connected component G of Γ_1 with no neighbors in Γ_0, and take a vertex x in G. According to Observation 3.2, x must have at least one non-branch neighbor in Γ_2, in order to reach in two steps the branch vertices of Γ_2 belonging to the same partite set as x. But then from x it is possible to reach at most nine vertices of Γ_0 in exactly two steps (see Fig. 8). This implies $|\Gamma_0| \leq 18$, which contradicts Observation 3.6. This completes the proof of Claim 1. $\square$

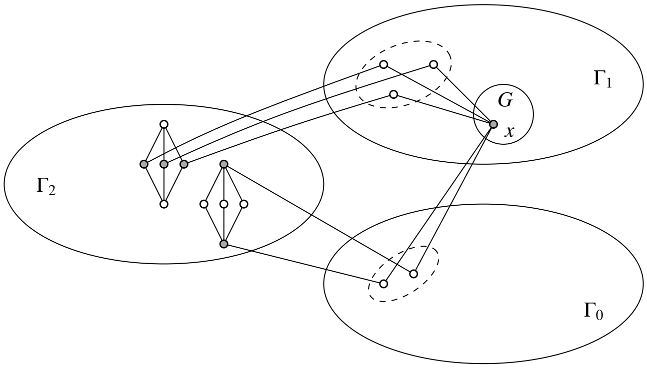

Fig. 9. Auxiliary figure for Claim 2.

Claim 2. *Every connected component of Γ_1 has a neighbor in Γ_2.*

Proof of Claim 2. Suppose there is a connected component G of Γ_1 with no neighbors in Γ_2. First note that Γ_2 must have the same number of vertices in each partite set of Γ, so $|\Gamma_2| \geq 10$. From a vertex x in G we must reach in exactly two steps at least three non-branch vertices in a connected component of Γ_2, and two other branch vertices in a different connected component of Γ_2. However, it is only possible to reach from x at most four of five such vertices (see Fig. 9). This completes the proof of Claim 2. $\square$

From Claim 1 and Observation 3.8 we can deduce that if $G = \Phi_m$ is a connected component of Γ_1 then $m \in \{8, 12\}$. But from Claim 2 and Observation 3.7 it follows that $m \equiv 0 \pmod 3$, and therefore $m = 12$. In other words, every connected component of Γ_1 has 24 vertices.

In addition, since $|\Gamma_0| \geq 24$ and $|\Gamma_1| \geq 24$ we have that $|\Gamma_2| \leq 34$. But 5 (and hence 10) must divide $|\Gamma_2|$, and $|\Gamma_0| \equiv |\Gamma_1| \equiv 0 \pmod 8$; consequently, $|\Gamma_2| = 10$.

To complete the proof we only need to consider the two remaining possibilities. If $|\Gamma_2| = 10$, $|\Gamma_1| = 48$ and $|\Gamma_0| = 24$ then from a branch vertex x in Γ_2 it is possible to reach in exactly two steps at most 23 vertices of Γ_1 in the same partite set as x, a contradiction (see Fig. 10(a)). Similarly, if $|\Gamma_2| = 10$, $|\Gamma_1| = 24$ and $|\Gamma_0| = 48$ then from a non-branch vertex y in Γ_2 it is possible to reach in exactly two steps at most 23 vertices of Γ_0 in the same partite set as y, a contradiction (see Fig. 10(b)). $\square$

The main result of this section immediately follows from Proposition 4.7:

Theorem 4.1. *There is no bipartite* $(7, 3, -4)$*-graph.*

Theorem 4.1 settles the optimality of the known bipartite $(7, 3, -6)$-graph, and therefore $\mathrm{N}^b(7, 3) = 80$.

The significance of such a graph certainly increases after this outcome, as it becomes just the second bipartite non-Moore graph known to be optimal. Therefore, we consider that it deserves to be named the *Hafner–Loz* graph henceforth. For a construction of the graph, the reader familiar with voltage graphs can find a description in [8, p. 72], as part of a much more extensive work developed by Loz. Loz obtained the graph from a *dipole* (i.e., a two-vertex graph) with seven parallel edges and no loops as the quotient graph, and the semidirect product $\mathbb{Z}_5 \rtimes_2 \mathbb{Z}_8$ as the voltage group. The specific values for the voltages on the parallel edges of the quotient graph were $(0, 0)$, $(3, 4)$, $(4, 4)$, $(3, 7)$, $(3, 3)$, $(1, 6)$ and $(4, 6)$.

5. Largest known bipartite graphs of diameter 3

In this section we present at least one new largest known bipartite graph of degree 11, diameter 3 and order 190. This improves the former lower bound for $\mathrm{N}^b(11, 3)$ by four vertices.

To obtain such graphs we were inspired by Observation 3.4, which tells us about the overall structure of a – hypothetical – bipartite $(d, 3, -4)$-graph Γ in the particular case of Γ_1 being a spanning subgraph of Γ with exactly one connected component Φ_m.

Corollary 5.1. *Let Γ be a bipartite $(d, 3, -4)$-graph such that Γ_1 has exactly one connected component $G = \Phi_{d^2-d-1}$ and $V(G)$ spans Γ. If the vertex x_0 in G has neighbors $y_0, y_1, y_{-1}, y_{i_1}, y_{i_2}, \ldots, y_{i_{d-3}}$ in G then x_k has neighbors $y_k, y_{k+1}, y_{k-1}, y_{k+i_1}, y_{k+i_2}, \ldots, y_{k+i_{d-3}}$ for every k.*

When $d = 4$ or $d = 5$ we have as examples the existing graphs depicted in Figs. 2(b) and 3. It is then natural to ask whether similar graphs exist for greater values of d.

Problem 1. Is there a bipartite $(d, 3, -4)$-graph with $d \geq 5$ such that Γ_1 has exactly one connected component $G = \Phi_{d^2-d-1}$ and $V(G)$ spans Γ?

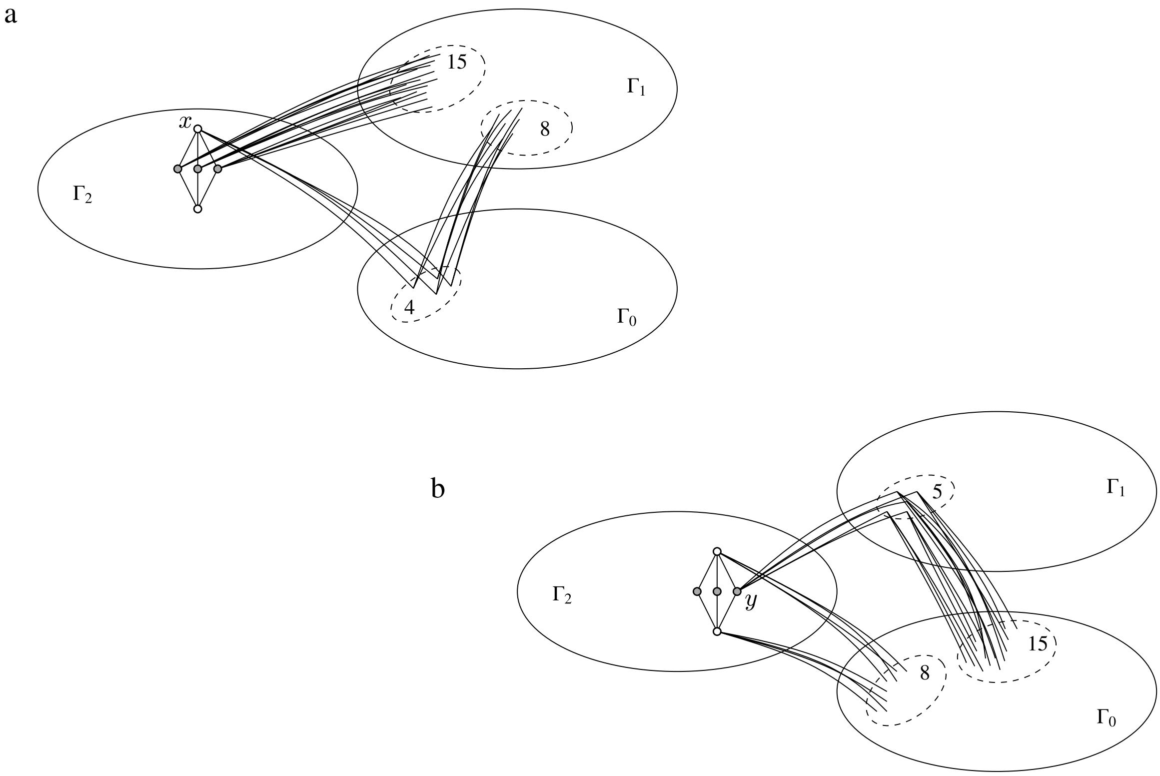

Fig. 10. Auxiliary figure for Proposition 4.7.

By computer search we obtained that for small degrees ($d = 6, 7, 8, 9$) such graphs do not exist.

Before shifting our interest to a more general problem, we introduce an extension to the construction of a Φ_m. Let $d \geq 4$ and $a_1, a_2, \ldots, a_{d-3}$ be such that $2 \leq a_j \leq m-2$ and $a_j \neq a_k$ when $j \neq k$. Then $\Phi_m(a_1, a_2, \ldots, a_{d-3})$ denotes the graph with vertex set $V = \{x_0, x_1, \ldots, x_{m-1}\} \cup \{y_0, y_1, \ldots, y_{m-1}\}$ and edge set $E = \{x_i \sim y_i, x_i \sim y_{i+1}, x_i \sim y_{i-1}, x_i \sim y_{i+a_j} | 0 \leq i \leq m-1, 1 \leq j \leq d-3\}$. Note that $\Phi_m(a_1, a_2, \ldots, a_{d-3})$ can be defined equivalently as the (bipartite) circulant graph $G(2m; \{m, m+1, 2m-1, m+a_1, m+a_2, \ldots, m+a_{d-3}\})$. As before, we do addition modulo m on the vertex subscripts.

Problem 2. Given a natural number $d \geq 6$, find the largest natural number $m(d)$ for which there exist natural numbers $a_1, a_2, \ldots, a_{d-3} (2 \leq a_j \leq m-2)$ such that the graph $\Phi_{m(d)}(a_1, a_2, \ldots, a_{d-3})$ has diameter 3.

If we take a vertex x_0 of a $\Phi_{m(d)}(a_1, a_2, \ldots, a_{d-3})$ and assume that x_0 has neighbors $y_0, y_1, y_{-1}, y_{a_1}, y_{a_2}, \ldots, y_{a_{d-3}}$ then x_0 can reach in exactly two steps the – not necessarily distinct – vertices $x_0, x_1, x_{-1}, x_2, x_{-2}, x_{a_i}, x_{-a_i}, x_{a_i+1}, x_{-a_i-1}, x_{a_i-1}, x_{-a_i+1}$ and $x_{a_i-a_j}$, and no other vertex. Problem 2 amounts to the following congruence-related problem since $\Phi_{m(d)}(a_1, a_2, \ldots, a_{d-3})$ is vertex-transitive.

Problem 3. Given a natural number $d \geq 6$, find the largest natural number $m(d)$ for which there exist natural numbers $a_1, a_2, \ldots, a_{d-3}$ such that the collection $0, 1, -1, 2, -2, a_i, -a_i, a_i+1, -a_i-1, a_i-1, -a_i+1, a_i-a_j$ of (not necessarily distinct) numbers contains a full set of residues modulo $m(d)$.

It is not difficult to verify that $m(d) \leq d^2 - d - 1 = (\mathrm{M}^b(d, 3) - 4)/2$.

With the aid of computer search and a simple pruning algorithm we found the bipartite $(11, 3, -32)$-graphs $\Phi_{95}(4, 7, 16, 27, 38, 52, 62, 81)$, $\Phi_{95}(4, 16, 30, 43, 51, 62, 71, 89)$ and $\Phi_{95}(11, 15, 21, 28, 37, 40, 45, 63)$. This discovery implies that $m(11) \geq 95$ and $\mathrm{N}^b(11, 3) \geq 190$. Adjacency lists of these graphs are available at [1] under the name of this paper. The order 190 is already large when considering isomorphism check between graphs. Unfortunately, we have not foreseen an approach for performing isomorphism testing efficiently on such graphs.

6. Conclusions

In this paper we offered several structural properties for bipartite graphs of diameter 3 and defect 4. Using these properties we showed the non-existence of bipartite $(7, 3, -4)$-graphs, which proves the optimality of the known bipartite

$(7, 3, -6)$-graph on 80 vertices. This is just the second bipartite graph known to be optimal, other than the bipartite Moore graphs.

We would also like to emphasize that, using the results of Section 3 and reasoning as in Section 4, it is possible to prove also the uniqueness of the only known bipartite $(5, 3, -4)$-graph depicted in Fig. 3, and the non-existence of bipartite $(6, 3, -4)$-graphs.

In addition, some of the results in Section 4 could have been stated for any bipartite $(d, 3, -4)$-graph by providing a more elaborate proof. However, we decided to omit this extension as it does not lead to any conclusive outcome on the existence or otherwise of bipartite graphs of diameter 3 and defect 4 in general. We nevertheless feel that the following conjecture is valid.

Conjecture 6.1. *There is no bipartite $(d, 3, -4)$-graph with $d \geq 6$.*

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