On large bipartite graphs of diameter 3

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Abstract

We consider the bipartite version of the *degree/diameter problem*, namely, given natural numbers $d \geq 2$ and $D \geq 2$, find the maximum number $\mathrm{N}^b(d,D)$ of vertices in a bipartite graph of maximum degree d and diameter D. In this context, the bipartite Moore bound $\mathrm{M}^b(d,D)$ represents a general upper bound for $\mathrm{N}^b(d,D)$. Bipartite graphs of order $\mathrm{M}^b(d,D)$ are very rare, and determining $\mathrm{N}^b(d,D)$ still remains an open problem for most (d,D) pairs.

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This paper is a follow-up to our earlier paper [4], where a study on bipartite $(d, D, -4)$-graphs (that is, bipartite graphs of order $\mathrm{M}^b(d, D) - 4$) was carried out. Here we first present some structural properties of bipartite $(d, 3, -4)$-graphs, and later prove there are no bipartite $(7, 3, -4)$-graphs. This result implies that the known bipartite $(7, 3, -6)$-graph is optimal, and therefore $\mathrm{N}^b(7, 3) = 80$. Our approach also bears a proof of the uniqueness of the known bipartite $(5, 3, -4)$-graph, and the non-existence of bipartite $(6, 3, -4)$-graphs.
In addition, we discover three new largest known bipartite (and also vertex-transitive) graphs of degree 11, diameter 3 and order 190, result which improves by 4 vertices the previous lower bound for $\mathrm{N}^b(11, 3)$.

Keywords: Degree/diameter problem for bipartite graphs, bipartite Moore bound, large bipartite graphs, defect.
AMS Subject Classification: 05C35, 05C75.

1 Introduction

Consider the *degree/diameter problem for bipartite graphs*, stated as follows:

> Given natural numbers $d \geq 2$ and $D \geq 2$, find the largest possible number $\mathrm{N}^b(d, D)$ of vertices in a bipartite graph of maximum degree d and diameter D.

It is well known that an upper bound for $\mathrm{N}^b(d, D)$ is given by the *bipartite Moore bound* $\mathrm{M}^b(d, D)$, defined below:

$$\mathrm{M}^b(d, D) = 2\left(1 + (d-1) + \cdots + (d-1)^{D-1}\right).$$

Bipartite graphs of degree d, diameter D and order $\mathrm{M}^b(d, D)$ are called *bipartite Moore graphs*. Bipartite Moore graphs are very scarce; when $d \geq 3$ and $D \geq 3$ they may only exist for $D = 3, 4$ or 6 (see [3]). It has also turned out to be very difficult to determine $\mathrm{N}^b(d, D)$ even for particular instances; in fact, with the exception of $\mathrm{N}^b(3, 5) = \mathrm{M}^b(3, 5) - 6$ settled in [5], the other known values of $\mathrm{N}^b(d,D)$ are those for which a bipartite Moore graph is known to exist.

Research in this area falls then into two main directions. On one hand, the efforts to improve the upper bounds for $\mathrm{N}^b(d,D)$ by studying the existence or otherwise of bipartite graphs of maximum degree d, diameter D and order $\mathrm{M}^b(d,D)-\epsilon$ for small $\epsilon>0$ (that is, bipartite $(d,D,-\epsilon)$-graphs, where the parameter ϵ is called the *defect*). On the other hand, the studies to improve the lower bounds for $\mathrm{N}^b(d,D)$ by constructing ever larger bipartite graphs with given maximum degree and diameter. In spite of these efforts and the wide range of techniques and approaches used to tackle these problems (see [8]), in most cases there is still a significant gap between the current lower and upper bound for $\mathrm{N}^b(d,D)$.

In this paper we restrict ourselves to the case of bipartite graphs of diameter 3, and present some modest contributions in both directions. When $D=3$ there is a bipartite Moore graph whenever $d-1$ is a prime power (namely, the incidence graphs of projective planes); however, there is no Moore bipartite graph of diameter 3 for $d=7$ ([9]) or $d=11$ ([6]). The existence of Moore bipartite graphs of diameter 3 for other degrees remains an open problem. In [1] the authors proved that bipartite $(d,3,-2)$-graphs may only exist for certain values of d; in particular, they do not exist for $d=7$.

The results and ideas exposed here are, in a great extent, a continuation of the precursory work initiated in [4]. We provide structural properties for bipartite $(d,3,-4)$-graphs and, most important, prove the non-existence of bipartite $(7,3,-4)$-graphs. Such outcome implies that the only known bipartite $(7,3,-6)$-graph – found by Paul Hafner and independently by Eyal Loz ([7]) – is optimal, and therefore $\mathrm{N}^b(7,3)=80$. This is just the second value settled for $\mathrm{N}^b(d,D)$ other than a bipartite Moore bound. Our approach can also be used to show the uniqueness of the known bipartite $(5,3,-4)$-graph, as well as the non-existence of bipartite $(6,3,-4)$-graphs.

Finally, we also find three largest known bipartite (and vertex-transitive) graphs of degree 11 and diameter 3. This settles $190 \leq \mathrm{N}^b(11,3)$, which improves by 4 vertices the previous lower bound for $\mathrm{N}^b(11,3)$. Adjacency lists of these graphs are available at `http://guillermo.com.au/wiki/List_of_Publications` under the name of this paper.

We conclude this introduction by depicting all the known bipartite $(d,3,-4)$ graphs. Figure 1 shows all the bipartite $(3,3,-4)$-graphs, Figure 2 all the bipartite $(3,3,-4)$-graphs, and Figure 3 the – after this paper unique – bipartite $(5,3,-4)$-graph.

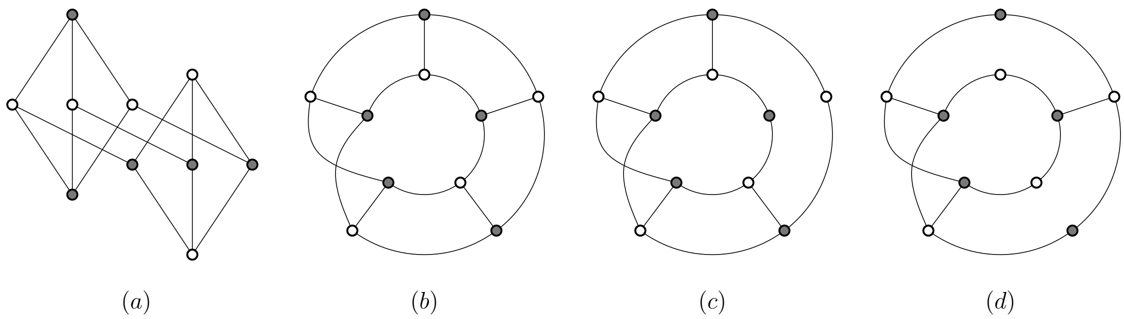

Figure 1: All the bipartite $(3,3,-4)$-graphs.

2 Notation and Terminology

Our notation and terminology follows from [4], which is standard and consistent with that used in [2].

All graphs considered are simple. The vertex set of a graph Γ is denoted by $V(\Gamma)$, and its edge set by $E(\Gamma)$. For an edge $e = \{x, y\}$ we write $x \sim y$. The set of edges in a graph Γ joining a vertex x in $X \subseteq V(\Gamma)$ to a vertex y in $Y \subseteq V(\Gamma)$ is denoted by $E(X,Y)$. A vertex of degree at least 3 is called a *branch vertex* of Γ.

A cycle of length k is called a *k-cycle.* In a bipartite $(d,D,-4)$-graph we call a cycle of length at most $2D-2$ a *short cycle.* If two short cycles C^1 and C^2 are non-disjoint we say that C^1 and C^2 are *neighbors.*

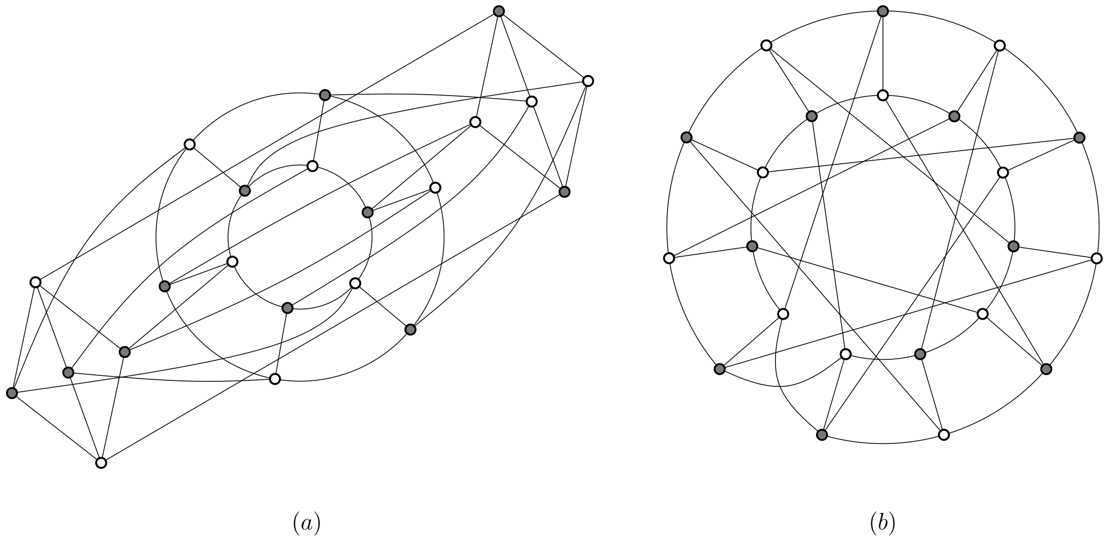

Figure 2: All the bipartite $(4, 3, -4)$-graphs.

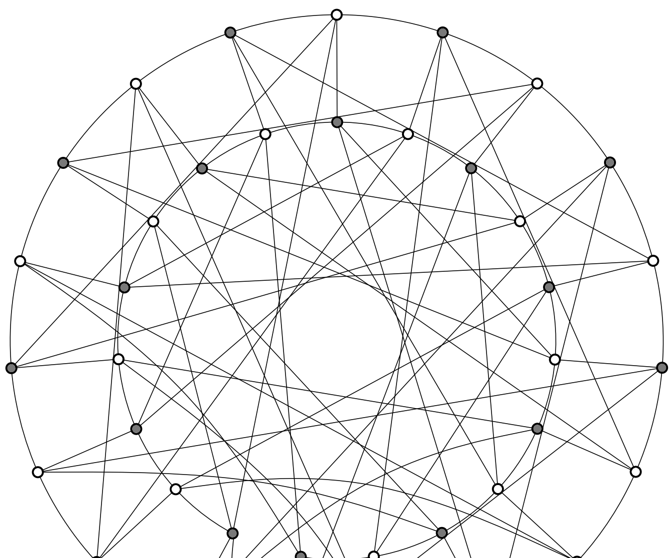

Figure 3: The unique bipartite $(5, 3, -4)$-graph.

For a vertex x lying on a short cycle C, we denote by $\mathrm{rep}^C(x)$ the vertex x' in C such that $d(x, x') = D - 1$, where $d(x, x')$ denotes the distance between x and x'. In this case, we say x' is a *repeat* of x in C and vice versa, or simply that x and x' are *repeats* in C. A *closed set of repeats* in a bipartite $(d, D, -4)$-graph Γ is a subset of $V(\Gamma)$ which is closed under the repeat relation. A closed set of repeats is *minimal* if it does not have a proper closed subset of repeats.

Finally, we introduce some special graphs. The union of three independent paths of length t with common endvertices is denoted by Θ_t. For an integer $m \geq 5$, Φ_m denotes the bipartite graph with vertex set $V = \{x_i | 0 \leq i \leq m-1\} \cup \{y_i | 0 \leq i \leq m-1\}$ and edge set $E = \{x_i \sim y_i, x_i \sim y_{i+1}, x_i \sim y_{i-1} | 0 \leq i \leq m-1\}$. Note that Φ_m is vertex-transitive. Throughout this paper we do addition modulo m on the vertex subscripts of a Φ_m.

3 Preliminaries

We begin with the regularity condition for bipartite graphs with small defect.

Proposition 3.1 ([1]) *For $\epsilon < 1 + (d-1) + (d-1)^2 + \ldots + (d-1)^{D-2}$, $d \geq 3$ and $D \geq 3$, a bipartite $(d, D, -\epsilon)$-graph is regular.*

Proposition 3.2 ([1]) *For $\epsilon < 2\left((d-1) + (d-1)^3 + \ldots + (d-1)^{D-2}\right)$,$d \geq 3$ and odd $D \geq 3$, a bipartite $(d, D, -\epsilon)$-graph is regular.*

In particular, we will implicitly use the fact that a bipartite $(d, 3, -4)$-graph with $d \geq 4$ must be regular, and therefore its partite sets must have the same cardinality. Also note that, since bipartite $(d, 3, -\epsilon)$ graphs with $d \geq 4$ and $\epsilon = 3, 5$ are not regular, the above propositions imply their non-existence.

From the paper [4] we borrow the following results:

Proposition 3.3 ([4]) *The girth of a regular bipartite $(d, D, -4)$-graph Γ with $d \geq 3$ and $D \geq 3$ is $2D - 2$. Furthermore, any vertex x of Γ lies on* *the short cycles specified below and no other short cycle, and we have the following cases:*

x **is contained in exactly three** $(2D-2)$**-cycles**. *Then*

(i) *x is a branch vertex of one Θ_{D-1}, or*

x **is contained in two** $(2D-2)$**-cycles**. *Then*

(ii) *x lies on* **exactly two** $(2D-2)$**-cycles**, *whose intersection is a ℓ-path with $\ell \in \{0, \ldots, D-1\}$.*

As in [4], often our arguments revolve around the identification of the elements in the set S_x of short cycles containing a given vertex x; we call this process *saturating* the vertex x. A vertex x is called *saturated* if the elements in S_x have been completely identified.

Lemma 3.1 ([4], Saturating Lemma) *Let $\mathcal{C}$ be a $(2D-2)$-cycle in a bipartite $(d, D, -4)$-graph Γ with $d \geq 4$ and $D \geq 3$, and α, α' two vertices in $\mathcal{C}$ such that $\alpha' = \mathrm{rep}^{\mathcal{C}}(\alpha)$. Let γ be a neighbor of α not contained in $\mathcal{C}$, and $\mu_1, \mu_2, \ldots, \mu_{d-2}$ the neighbors of α' not contained in $\mathcal{C}$. Suppose there is no short cycle in Γ containing the edge $\alpha \sim \gamma$ and intersecting $\mathcal{C}$ at a path of length greater than $D-3$.*

Then, in Γ there exist a vertex $\mu \in \{\mu_1, \mu_2, \ldots, \mu_{d-2}\}$ and a short cycle $\mathcal{C}^1$ such that γ and μ are repeats in $\mathcal{C}^1$, and $\mathcal{C} \cap \mathcal{C}^1 = \emptyset$.

Lemma 3.2 ([4], Repeat Cycle Lemma) *Let C be a short cycle in a bipartite $(d, D, -4)$-graph Γ with $d \geq 4$ and $D \geq 3$, $\{C^1, C^2, \ldots, C^k\}$ the set of neighbors of C, and $I_i = C^i \cap C$ for $1 \leq i \leq k$. Suppose at least one I_j, for $j \in \{1, \ldots, k\}$, is a path of length smaller than $D-2$. Then there is an additional short cycle C' in Γ intersecting C^i at $I'_i = \mathrm{rep}^{C^i}(I_i)$, where $1 \leq i \leq k$.*

Proposition 3.4 ([4]) *The set $S(\Gamma)$ of short cycles in a bipartite $(d, D, -4)$-graph Γ with $d \geq 3$ and $D \geq 3$ can be partitioned into sets $S_{D-1}(\Gamma)$, $S_{D-2}(\Gamma)$ and $S_{D-3}(\Gamma)$, where*

$S_{D-1}(\Gamma)$ is the set of short cycles in Γ whose intersections with neighbor cycles are $(D-1)$-paths,

$S_{D-2}(\Gamma)$ is the set of short cycles in Γ whose intersections with neighbor cycles are $(D-2)$-paths, and

$S_{D-3}(\Gamma)$ is the set of short cycles in Γ whose intersections with neighbor cycles are paths of length at most $D-3$.

Proposition 3.5 ([4]) *The set $V(\Gamma)$ of vertices in a bipartite $(d, D, -4)$-graph Γ with $d \geq 4$ and $D \geq 3$ can be partitioned into sets $V_{D-1}(\Gamma)$, $V_{D-2}(\Gamma)$ and $V_{D-3}(\Gamma)$, where*

$V_{D-1}(\Gamma)$ is the set of vertices contained in cycles of $S_{D-1}(\Gamma)$,

$V_{D-2}(\Gamma)$ is the set of vertices contained in cycles of $S_{D-2}(\Gamma)$,

$V_{D-3}(\Gamma)$ is the set of vertices contained in cycles of $S_{D-3}(\Gamma)$,

and $S_{D-1}(\Gamma)$, $S_{D-2}(\Gamma)$, $S_{D-3}(\Gamma)$ are defined as in Proposition 3.4.

3.1 On bipartite graphs of diameter 3 and defect 4

In this section we present additional structural properties for bipartite graphs of diameter 3 and defect 4.

Let Γ be a bipartite $(d, 3, -4)$-graphs with $d \geq 4$. We set $\Gamma_i = \bigcup_{C \in S_i(\Gamma)} C$ for $i = 0, 1, 2$. Note that Γ_2 is the union of all graphs in Γ isomorphic to Θ_2; these graphs are pairwise disjoint, so they are the connected components of Γ_2. In addition, Γ_1 is the union of all graphs in Γ isomorphic to some Φ_m for $m \geq 5$; similarly, these Φ_m are the connected components of Γ_1.

If G is a connected component in $\Gamma_2 \cup \Gamma_1 \cup \Gamma_0$ then $V(G)$ is a closed set of repeats. The branch vertices of a $\Theta_2 \subset \Gamma_2$ constitute a minimal closed set of repeats, as well as its non-branch vertices. In the case of a $\Phi_m \subset \Gamma_1$, the vertices x_i's form a minimal closed set of repeats, the same as the vertices y_i's. According to the Repeat Cycle Lemma, every minimal closed set of repeats in Γ_0 contains exactly 4 vertices. Observe that all vertices in a minimal closed set of repeats in Γ belong to the same partite set.

Some further observations about Γ follow from the systematic application of the Saturating Lemma:

Observation 3.1 *Let Γ be a bipartite $(d, 3, -4)$-graph with $d \geq 4$. There is no edge in Γ joining a branch vertex in Γ_2 to a non-branch vertex of a different connected component of Γ_2.*

Proof.

Let G, G' be two connected components in Γ_0 such that a branch vertex x'_0 in G' is adjacent to a non-branch vertex y_0 in G. Let x'_1, x_0, x_1, y_1, y_2 be as in Figure 4 (a). We apply the Saturating Lemma (by mapping the cycle $x_0y_0x_1y_1x_0$ to $\mathcal{C}$, y_0 to α, y_1 to α' and x'_0 to γ), and obtain that y_1 is adjacent to x'_1. Similarly, y_2 is also adjacent to x'_1 (see Figure 4 (b)), but then there is a fourth short cycle $x_0y_1x'_1y_2x_0$ in Γ containing x_0, a contradiction. $\square$

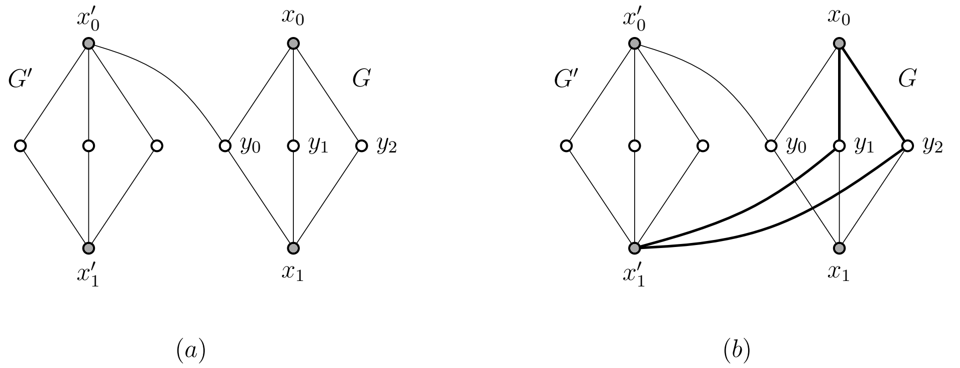

Figure 4: Auxiliary figure for Observation 3.1

Observation 3.2 *Let Γ be a bipartite $(d, 3, -4)$-graph with $d \geq 4$. There is no edge in Γ joining a branch vertex in Γ_2 to a vertex in Γ_1.*

Proof.

Let G, G' be two connected components of Γ_1 and Γ_2 respectively, such that a branch vertex x'_0 in G' is adjacent to a vertex y_i in G. Let $x'_1, y_{i+1}, y_{i-1}, x_i, x_{i+1}, x_{i-1}$

be as in Figure 5 (a). We apply the Saturating Lemma (by mapping cycle $y_i x_{i-1} y_{i-1} x_i y_i$ to $\mathcal{C}$, y_i to α, y_{i-1} to α' and x'_0 to γ), and obtain that y_{i-1} is adjacent to x'_1. Similarly, y_{i+1} is also adjacent to x'_1 (see Figure 5 (b)). But then, there is a third short cycle $y_{i+1} x_i y_{i-1} x'_1 y_{i+1}$ in Γ containing x_i, a contradiction. □

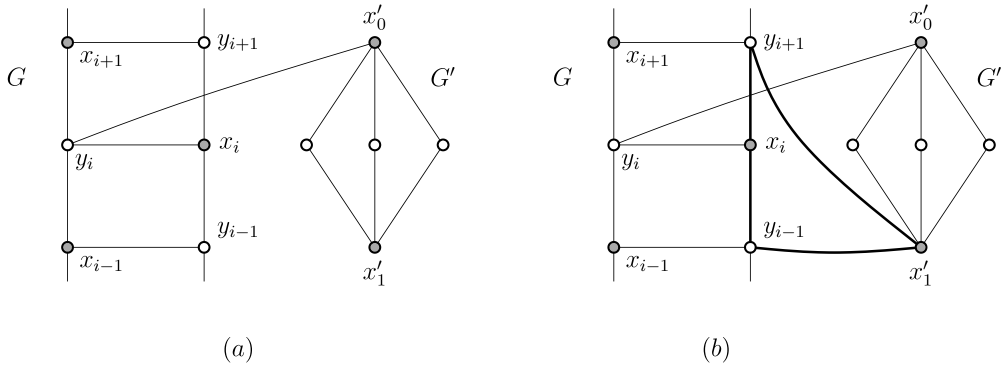

Figure 5: Auxiliary figure for Observation 3.2

Observation 3.3 *Let Γ be a bipartite $(d, 3, -4)$-graph with $d \geq 4$. There is no edge in Γ joining a non-branch vertex in Γ_2 to a vertex in Γ_0.*

Proof.

Let G' be a connected component in Γ_2 with a non-branch vertex y'_0 adjacent to a vertex x_0 in Γ_0. Let $\{x_0, x_1, x_2, x_3\}$ be the minimal closed set of repeats containing x_0 (x_2 not being a repeat of x_0), and let the vertices x'_0, x'_1, y'_1, y'_2 be as in Figure 6 (a). We first apply the Saturating Lemma (by mapping the cycle $x'_0 y'_0 x'_1 y'_1 x'_0$ to $\mathcal{C}$, y'_0 to α, y'_1 to α' and x_0 to γ), and obtain that y'_1 is adjacent to a repeat of x_0 (say x_1). Similarly, mapping the cycle $x'_0 y'_0 x'_1 y'_1 x'_0$ to $\mathcal{C}$, y'_1 to α, y'_2 to α' and x_1 to γ, we obtain that y'_2 is adjacent to x_2 (as it cannot be adjacent to x_0). Analogously, y'_0 is adjacent to x_3 (see Figure 6 (b)), but then there is a third short cycle in Γ containing x_0, a contradiction. □

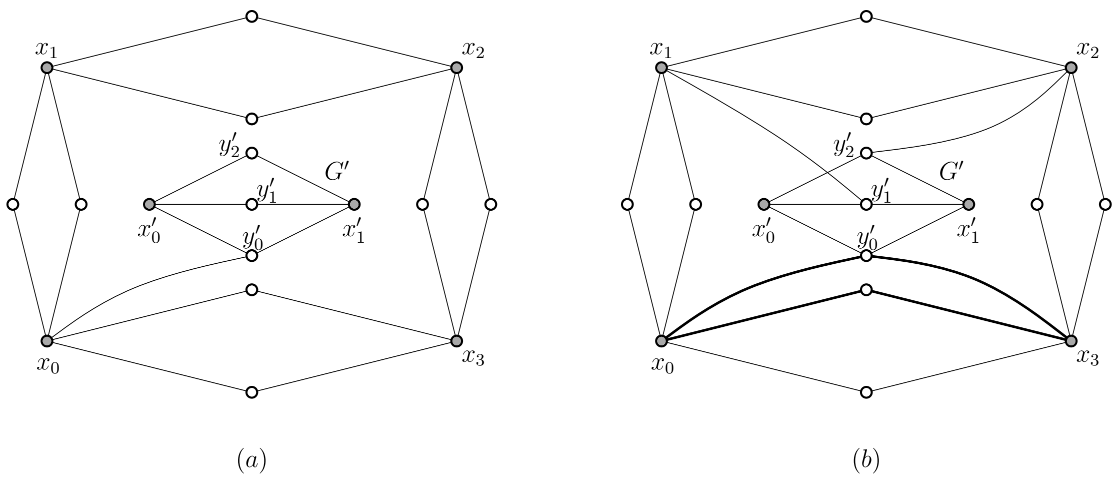

Figure 6: Auxiliary figure for Observation 3.3

Observation 3.4 *Let* Γ *be a bipartite* $(d, 3, -4)$*-graph with* $d \geq 4$*, and* $G = \Phi_m$ *a connected component in* Γ_1*. Given* $x_i \in G$*, if* $x_i \sim y_j \in E(\Gamma)$ *for some* j *then* $x_{i+k} \sim y_{j+k} \in E(\Gamma)$ *for every* k*.*

Proof.

This clearly holds when $j \in \{i, i+1, i-1\}$; see the description of Φ_m.

Suppose $j \notin \{i, i+1, i-1\}$. Since all the vertices in G are saturated, we have $|i-j| \geq 4$. According to the Saturating Lemma (by mapping the cycle $x_i y_i x_{i+1} y_{i+1} x_i$ to $\mathcal{C}$, x_i to α, x_{i+1} to α' and y_j to γ) we have either $x_{i+1} \sim y_{j+1} \in E(\Gamma)$ or $x_{i+1} \sim y_{j-1} \in E(\Gamma)$. But in case $x_{i+1} \sim y_{j-1} \in E(\Gamma)$, it is easy to see that, by repeatedly applying the Saturating Lemma (to the cycles $x_{i+p} y_{i+p} x_{i+p+1} y_{i+p+1} x_{i+p}$ for $p = 1, 2, \ldots$) we obtain there is an edge $x_r \sim y_s$ in Γ such that $2 \leq |r-s| \leq 3$, which is not possible. Thus $x_{i+1} \sim y_{j+1} \in E(\Gamma)$ and, by induction, $x_{i+k} \sim y_{j+k} \in E(\Gamma)$ for every k. $\square$

Observation 3.5 *Let* Γ *be a bipartite* $(d, 3, -4)$*-graph with* $d \geq 3$*, and* G, G' *two connected components in* Γ_1 *of order* $2m$ *and* $2m'$ *respectively (*$m \leq m'$*). Suppose there is at least one edge in* Γ *joining a vertex in* G *to a vertex in* G'*. Then* $m' = km$*, with* $1 \leq k \leq d-3$*.*

Proof.

Denote the vertices of $G = \Phi_m$ by $x_0, \ldots, x_{m-1}, y_0, \ldots, y_{m-1}$, and the vertices of $G' = \Phi_{m'}$ by $x'_0, \ldots, x'_{m'-1}, y'_0, \ldots, y'_{m'-1}$. With an appropriate labelling we may assume there is an edge $x_0 \sim y'_0$ in Γ and, by the Saturating Lemma (on the cycle $x_0y_0x_1y_1x_0$), also an edge $x_1 \sim y'_1$ in Γ.

Suppose $m' = km + r$, with $1 \leq r \leq m - 1$ and $k \geq 1$. Then, by repeatedly applying the Saturating Lemma on the cycles $x_iy_ix_{i+1}y_{i+1}x_i$ with $i = 1, \ldots, m-1$, we find the edges $x_i \sim y'_i$ for $i = 2, \ldots, m$ are all present in Γ. In particular, y'_m is a neighbor of x_0 and, inductively, the vertices $y'_{2m}, \ldots, y'_{km}, y'_{m-r}, y'_{2m-r}, \ldots$ also are. But similarly, x_{m-r} has also neighbors y'_{m-r} and y'_{2m-r}; this way, we obtain there is in Γ a third short cycle $x_0y'_{m-r}x_{m-r}y'_{2m-r}x_0$ containing x_0, a contradiction.

Since a vertex in G has at least 3 neighbors in G, it follows that $k \leq d-3$.
□

Observation 3.6 *Let Γ be a bipartite $(7, 3, -4)$-graph. If $\Gamma_0 \neq \emptyset$ then $|\Gamma_0| = 8k$, with $k \geq 3$.*

Proof.

If t is the number of short cycles in Γ_0 then, by a simple counting argument, Γ_0 has $2t$ vertices, half of them in each partite set. Recall that $V(\Gamma_0)$ is a closed set of repeats. Since a minimal closed set of repeats in Γ_0 contains exactly 4 vertices belonging to the same partite set, we have $t = 4k$ and then $|\Gamma_0| = 8k$.

Also, the Repeat Cycle Lemma ensures that the graph G depicted in Figure 7 is a subgraph of Γ_0. Since any vertex in Γ_0 must have at least 4 neighbors in Γ_0, we have $|\Gamma_0| > 16$ and $k \geq 3$. □

Observation 3.7 *Let Γ be a bipartite $(d, 3, -4)$-graph with $d \geq 4$, G a connected component in Γ_2, and G' a connected component in Γ_1 of order $2m'$. Suppose there is in Γ at least one edge joining a vertex in G to a vertex in G'. Then $m' = 3k$ with $2 \leq k \leq d-2$.*

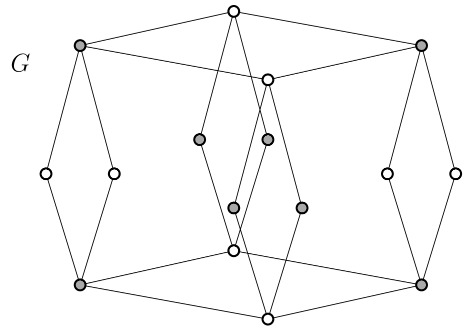

Figure 7: Auxiliary figure for Observation 3.6

Proof.

Let x_0, x_1, x_2 be the non-branch vertices of $G = \Theta_2$, and denote by $x'_0, \ldots, x'_{m'-1}, y'_0, \ldots, y'_{m'-1}$ the vertices of $G' = \Phi_{m'}$.

By Observation 3.2 any edge between Γ_2 and Γ_1 involves only non-branch vertices of Γ_2. We may assume there are edges $x_0 \sim y'_0$ and $x_1 \sim y'_1$ in Γ. Suppose $m' = 3k + r$, with $1 \leq r \leq 2$ and $k \geq 1$. Then, by repeatedly applying the Saturating Lemma on the three short cycles of G, we obtain that x_0 has neighbors $y'_0, y'_3, y'_6, \ldots, y'_{3k}, y'_{3-r}, y'_{6-r}, \ldots$ But similarly, x_{3-r} has also neighbors y'_{3-r} and y'_{6-r}; hence, we obtain there is in Γ a third short cycle $x_0 y'_{3-r} x_{3-r} y'_{6-r} x_0$ containing x_0, a contradiction.

Since each x_i has 2 neighbors in G and $m' \geq 5$, it follows that $2 \leq k \leq d - 2$. □

Observation 3.8 *Let Γ be a bipartite $(d, 3, -4)$-graph with $d \geq 4$ and G' a connected component in Γ_1 of order $2m'$. Suppose there is in Γ an edge joining a vertex in Γ_0 to a vertex in G'. Then $m' = 4k$ with $2 \leq k \leq d - 4$.*

Proof.

Let $x_0 \in V(\Gamma_0)$ and let $\{x_0, x_1, x_2, x_3\}$ be the minimal closed set of repeats containing x_0 (x_2 not being a repeat of x_0). Denote by $x'_0, \ldots, x'_{m'-1}, y'_0, \ldots, y'_{m'-1}$ the vertices of $G' = \Phi_{m'}$. We may assume there are edges $x_0 \sim y'_0$ and $x_1 \sim y'_1$ in Γ. Suppose $m' = 4k + r$, with $1 \leq r \leq 3$ and $k \geq 1$. Then, by repeatedly applying the Saturating Lemma on the cycles $x'_i y'_i x'_{i+1} y'_{i+1} x'_i$ $(i = 1, 2, \ldots)$ of G', we obtain that x_0 has neighbors $y'_0, y'_4, y'_8, \ldots, y'_{4k}, y'_{4-r}, y'_{8-r}, \ldots$ But analogously, x_{4-r} has also neighbors y'_{4-r} and y'_{8-r}; hence, we obtain there is in Γ a third short cycle $x_0 y'_{4-r} x_{4-r} y'_{8-r} x_0$ containing x_0, a contradiction.

Since x_0 has at least 4 neighbors in Γ_0 and $m' \geq 5$, it follows that $2 \leq k \leq d - 4$. □

The statements in Observations 3.1, 3.3, 3.5, 3.7 and 3.8 are better summarized in the following, more compact assertion.

Proposition 3.6 *Let Γ be a bipartite $(d, 3, -4)$-graph with $d \geq 4$, and M, M' two minimal closed set of repeats in Γ such that $E(M, M') \neq \emptyset$. Then $|M|$ divides $|M'|$ or $|M'|$ divides $|M|$, except when $M \cup M'$ is the set of five the vertices in a Θ_2 .*

□

4 Non-existence of bipartite $(7, 3, -4)$-graphs

In this section we prove that there are no bipartite $(7, 3, -4)$-graphs, and consequently that $\mathrm{N}^b(7, 3) = 80$.

Proposition 4.1 *Let Γ be a bipartite $(7, 3, -4)$-graph. Then Γ_2 cannot be a spanning subgraph of Γ.*

Proof.

Since the connected components of Γ_2 are graphs isomorphic to Θ_2, we have that 5 must divide $|\Gamma_2| = 82$, a contradiction. □

Proposition 4.2 *Let Γ be a bipartite $(7, 3, -4)$-graph. Then Γ_1 cannot be a spanning subgraph of Γ.*

Proof.

This is a computer-assisted proof.

Suppose that Γ_1 contains exactly one connected component G, isomorphic to Φ_{41}. Denote by $x_0, \ldots, x_{40}, y_0, \ldots, y_{40}$ the vertices of G. By virtue of Observation 3.4, if the vertex x_0 has neighbors $y_0, y_1, y_{-1}, y_{i_1}, y_{i_2}, y_{i_3}, y_{i_4}$ in G then x_k has neighbors $y_k, y_{k+1}, y_{k-1}, y_{k+i_1}, y_{k+i_2}, y_{k+i_3}, y_{k+i_4}$ for every k. Exhaustive computer search through the feasible choices for the vertices $y_{i_1}, y_{i_2}, y_{i_3}, y_{i_4}$ yields no graph of diameter 3, and so there is more than one conected component in Γ_1.

Now suppose then that Γ_1 has exactly n connected components $G_1, G_2, \ldots, G_n$, isomorphic to $\Phi_{m_1}, \Phi_{m_2}, \ldots, \Phi_{m_n}$, respectively. Note that $5 \leq m_i \leq 36$, $2 \leq n \leq 8$ and $m_1 + \ldots + m_n = 41$. We define the graph $H(G_1, G_1, \ldots, G_n)$ as follows: every G_i contracts to a vertex v_i in H, and there is an edge $v_i - v_j$ in H if and only if – according to Observation 3.5 – there could be an edge from G_i to G_j in Γ (that is, if m_i divides m_j or vice versa). Clearly, if Γ has diameter 3 then H has diameter at most 2. However, we could verify that none of the feasible values for n and the m_i's yields a graph H of diameter at most 2.

Consequently, $V(\Gamma_1)$ cannot span Γ. □

Proposition 4.3 *Let* Γ *be a bipartite* $(7, 3, -4)$*-graph. Then* Γ_0 *cannot be a spanning subgraph of* Γ*.*

Proof.

From Observation 3.6 we have $82 = |\Gamma_0| = 8k$, a contradiction. □

Proposition 4.4 *Let* Γ *be a bipartite* $(7, 3, -4)$*-graph. Then* $\Gamma_2 \cup \Gamma_1$ *cannot be a spanning subgraph of* Γ*.*

Proof.

Suppose $\Gamma_2 \neq \emptyset$ and $\Gamma_1 \neq \emptyset$. On one hand, from a branch vertex in Γ_2 it is possible to reach in exactly two steps at most 15 vertices of Γ_1 (see Observations 3.1 and 3.2). Therefore, $|\Gamma_1| \leq 30$. On the other hand, from a vertex in Γ_1 it is possible to reach in exactly two steps at most 8 branch vertices of Γ_2, and $|\Gamma_2| \leq 40$. This means $|\Gamma| \leq 70$, a contradiction. □

Proposition 4.5 *Let* Γ *be a bipartite* $(7,3,-4)$*-graph. Then* $\Gamma_2 \cup \Gamma_0$ *cannot be a spanning subgraph of* Γ*.*

Proof.

Suppose $\Gamma_2 \neq \emptyset$ and $\Gamma_0 \neq \emptyset$. From a non-branch vertex in Γ_2 it is possible to reach in two steps at most 8 vertices of Γ_0 (see Observations 3.1 and 3.3). Therefore, $|\Gamma_0| \leq 16$, which contradicts Observation 3.6. $\square$

Proposition 4.6 *Let* Γ *be a bipartite* $(7,3,-4)$*-graph. Then* $\Gamma_1 \cup \Gamma_0$ *cannot be a spanning subgraph of* Γ*.*

Proof.

Let $G = \Phi_m$ be a connected component in Γ_1. We prove that m is even. If G has a neighbor in Γ_0 then, by Observation 3.8, we have $m \in \{8, 12\}$. If instead G has no neighbor in Γ_0 and m is odd, then there must be a connected component G' in Γ_1 isomorphic to some $\Phi_{m'}$ such that G has a neighbor in G' and G' has a neighbor in Γ_0. But again we have $m' \in \{8, 12\}$ and, according to Observation 3.5, $m \geq 5$ must be an odd divisor of m', which is not possible.

From the above and Observation 3.6 it follows that $|\Gamma| \equiv 0 \pmod 4$, which contradicts $|\Gamma| = 82$. $\square$

Proposition 4.7 *Let* Γ *be a bipartite* $(7,3,-4)$*-graph. Then* $\Gamma_2 \cup \Gamma_1 \cup \Gamma_0$ *cannot be a spanning subgraph of* Γ*.*

Proof.

Let $\Gamma_i \neq \emptyset$ for $i = 0, 1, 2$.

Claim 1. Every connected component of Γ_1 has a neighbor in Γ_0.

Proof of Claim 1.

Suppose there is a connected component G of Γ_1 with no neighbors in Γ_0, and take a vertex x in G. According to Observation 3.2, x must have at least one non-branch neighbor in Γ_2 for it can reach in 2 steps the branch vertices of Γ_2 belonging to its partite set. But then from x it is possible to

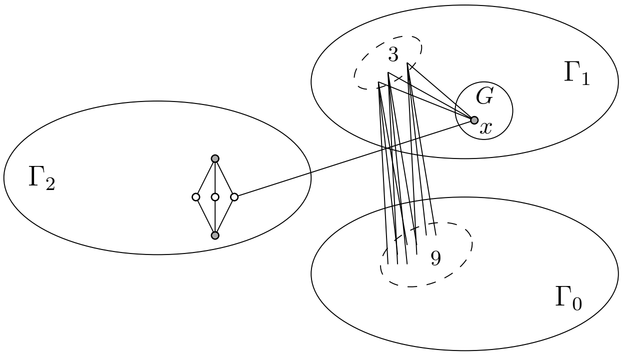

Figure 8: Auxiliary figure for Claim1

reach at most 9 vertices of Γ_0 in exactly 2 steps (see Figure 8). This implies $|\Gamma_0| \leq 18$, which contradicts Observation 3.6. □

Claim 2. Every connected component of Γ_1 has a neighbor in Γ_2.

Proof of Claim 2.

Suppose there is a connected component G of Γ_1 with no neighbors in Γ_2. First note that Γ_2 must have the same number of vertices in each partite set of Γ, so $|\Gamma_2| \geq 10$. From a vertex x in G we must reach in exactly two steps at least three non-branch vertices in a connected component of Γ_2, and other two branch vertices in a different connected component of Γ_2. However, it is only possible to reach from x at most 4 of such 5 vertices (see Figure 9). □

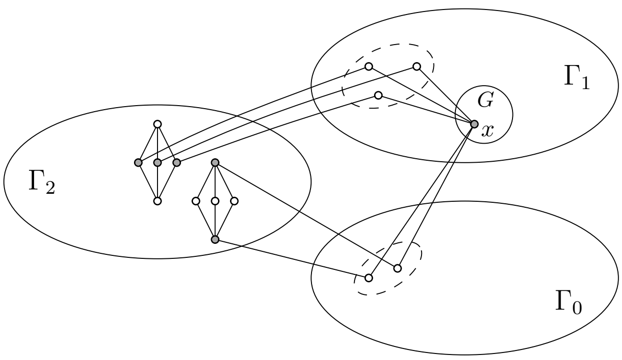

Figure 9: Auxiliary figure for Claim 2

From Claim 1 and Observation 3.8 we can deduce that if $G = \Phi_m$ is a connected component of Γ_1 then $m \in \{8, 12\}$. But from Claim 2 and Observation 3.7 it follows that $m \equiv 0 \pmod 3$, and therefore $m = 12$. In other words, every connected component of Γ_1 has 24 vertices.

In addition, since $|\Gamma_0| \geq 24$ and $|\Gamma_1| \geq 24$ we have that $|\Gamma_2| \leq 34$. But 5 (and hence 10) must divide $|\Gamma_2|$, and $|\Gamma_0| \equiv |\Gamma_1| \equiv 0 \pmod 8$; consequently, $|\Gamma_2| = 10$.

To complete the proof we only need to consider two possibilities left. If $|\Gamma_2| = 10$, $|\Gamma_1| = 48$ and $|\Gamma_0| = 24$ then from a branch vertex x in Γ_2 it is possible to reach in exactly two steps at most 23 vertices of Γ_1 in the same partite set as x, a contradiction (see Figure 10 (a)). Similarly, if $|\Gamma_2| = 10$, $|\Gamma_1| = 24$ and $|\Gamma_0| = 48$ then from a non-branch vertex y in Γ_2 it is possible to reach in exactly two steps at most 23 vertices of Γ_0 in the same partite set as y, a contradiction as well (see Figure 10 (b)). □

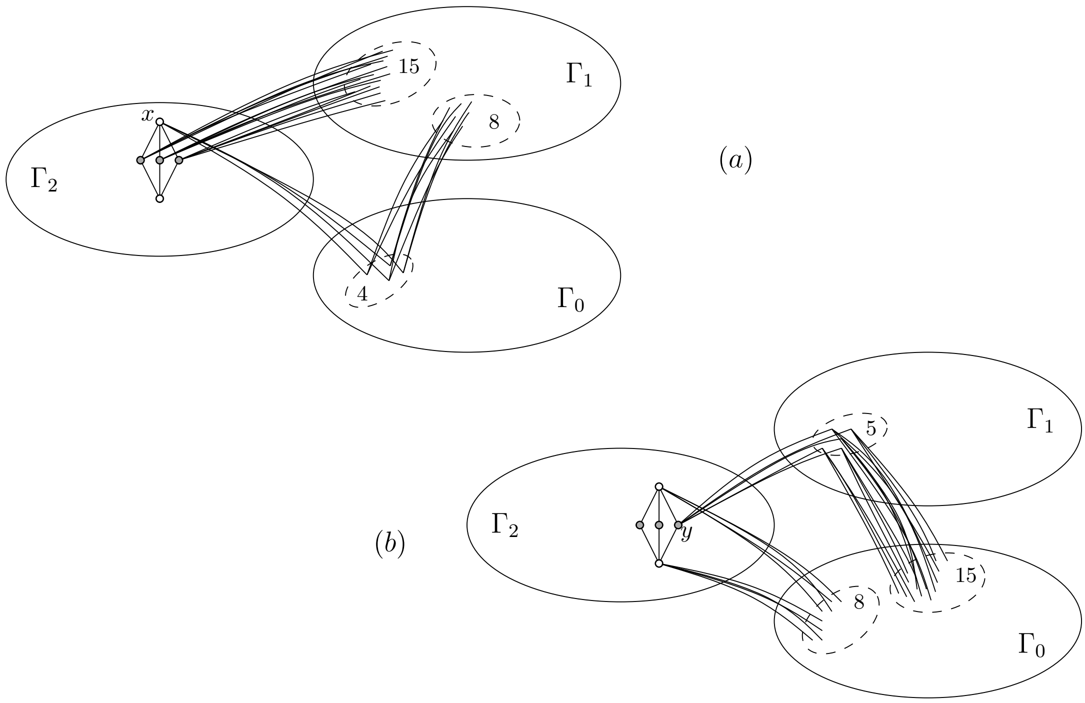

Figure 10: Auxiliary figure for Proposition 4.7

From Proposition 4.7 it immediately follows the main result of this section.

Theorem 4.1 *There is no bipartite* $(7, 3, -4)$*-graph.*

Theorem 4.1 settles the optimality of the known bipartite $(7, 3, -6)$-graph, and therefore $\mathrm{N}^b(7, 3) = 80$.

5 Three largest known bipartite graphs of diameter 3

In this section we present three new largest known bipartite graphs of degree 11, diameter 3 and order 190. This improves by 4 vertices the former lower bound for $\mathrm{N}^b(11, 3)$.

To obtain such graphs we were inspired by Observation 3.4, which tells us about the overall structure of a –hypothetical – bipartite $(d, 3, -4)$-graph Γ in the particular case of Γ_1 being a spanning subgraph of Γ with exactly one connected component Φ_m.

Corollary 5.1 *Let* Γ *be a bipartite* $(d, 3, -4)$*-graph such that* Γ_1 *has exactly one connected component* $G = \Phi_{d^2-d-1}$ *and* $V(G)$ *spans* Γ*. If the vertex* x_0 *in* G *has neighbors* $y_0, y_1, y_{-1}, y_{i_1}, y_{i_2}, \ldots, y_{i_{d-3}}$ *in* G *then* x_k *has neighbors* $y_k, y_{k+1}, y_{k-1}, y_{k+i_1}, y_{k+i_2}, \ldots, y_{k+i_{d-3}}$ *for every* k*.*

When $d = 4$ or $d = 5$ we have as examples the existing graphs depicted in Figures 2 (b) and 3. It is then natural to ask if similar graphs exist for greater values of d.

Problem 1 *Is there a a bipartite* $(d, 3, -4)$*-graph with* $d \geq 5$ *such that* Γ_1 *has exactly one connected component* $G = \Phi_{d^2-d-1}$ *and* $V(G)$ *spans* Γ*?*

By computer search we obtained that for small degrees ($d = 6, 7, 8, 9$) such graphs do not exist. This is a strong indication that for all $d \geq 6$ the answer to the above problem is no. Thus, we shift our interest to a more general problem.

We first introduce an extension to the construction of a Φ_m. Let $d \geq 4$ and $a_1, a_2, \ldots, a_{d-3}$ be such that $2 \leq a_j \leq m-2$ and $a_j \neq a_k$ when $j \neq k$. Then $\Phi_m(a_1, a_2, \ldots, a_{d-3})$ denotes the graph with vertex set $V = \{x_0, x_1, \ldots, x_{m-1}\} \cup \{y_0, y_1, \ldots, y_{m-1}\}$ and edge set $E = \{x_i \sim y_i, x_i \sim y_{i+1}, x_i \sim y_{i-1}, x_i \sim y_{i+a_j} | 0 \leq i \leq m-1,\ 1 \leq j \leq d-3\}$. As before, we do addition modulo m on the vertex subscripts. Note that $\Phi_m(a_1, a_2, \ldots, a_{d-3})$ is a bipartite vertex-transitive graph.

Problem 2 *Given a natural number $d \geq 6$, find the largest natural number $m(d)$ for which there exist natural numbers $a_1, a_2, \ldots, a_{d-3}$ $(2 \leq a_j \leq m-2)$ such that the graph $\Phi_{m(d)}(a_1, a_2, \ldots, a_{d-3})$ has diameter* 3.

If we take a vertex x_0 of a $\Phi_{m(d)}(a_1, a_2, \ldots, a_{d-3})$ and assume that x_0 has neighbors $y_0, y_1, y_{-1}, y_{a_1}, y_{a_2}, \ldots, y_{a_{d-3}}$ then x_0 can reach in at exactly two steps the – not necessarily distinct – vertices $x_0, x_1, x_{-1}, x_2, x_{-2}, x_{a_i}, x_{-a_i}, x_{a_i+1}, x_{-a_i-1}, x_{a_i-1}, x_{-a_i+1}$ and $x_{a_i-a_j}$, and no other vertex. Since $\Phi_{m(d)}(a_1, a_2, \ldots, a_{d-3})$ is vertex-transitive, Problem 2 amounts to the following congruence-related problem:

Problem 3 *Given a natural number $d \geq 6$, find the largest natural number $m(d)$ for which there exist natural numbers $a_1, a_2, \ldots, a_{d-3}$ such that the collection $0, 1, -1, 2, -2, a_i, -a_i, a_i+1, -a_i-1, a_i-1, -a_i+1, a_i-a_j$ of (not necessarily distinct) numbers contains a full set of residues modulo $m(d)$.*

It is not difficult to verify that $m(d) \leq d^2 - d - 1 = (\mathrm{M}^b(d,3) - 4)/2$.

With the aid of computer search we found the non-isomorphic bipartite $(11, 3, -32)$-graphs $\Phi_{95}(4, 7, 16, 27, 38, 52, 62, 81)$, $\Phi_{95}(4, 16, 30, 43, 51, 62, 71, 89)$ and $\Phi_{95}(11, 15, 21, 28, 37, 40, 45, 63)$. This discovery implies that $m(11) \geq 95$ and $\mathrm{N}^b(11, 3) \geq 190$. Adjacency lists of these graphs are available at `http://guillermo.com.au/wiki/List_of_Publications` under the name of this paper.

6 Conclusions

In this paper we offered several structural properties for bipartite graphs of diameter 3 and defect 4. Using these properties we showed the non-existence of bipartite $(7, 3, -4)$-graphs, which proves the optimality of the known bipartite $(7, 3, -6)$-graph on 80 vertices. This is just the second non-Moore bipartite graph known to be optimal.

We would also like to emphasize that, using the results of Section 3 and reasoning as in Section 4, it is possible to prove as well the uniqueness of the only known bipartite $(5, 3, -4)$-graph depicted in Figure 3, and the non-existence of bipartite $(6, 3, -4)$-graphs.

In addition, some of the results in Section 4 could have been stated for any bipartite $(d, 3, -4)$-graph by providing a more elaborate proof. However, we decided to omit this extension as it does not lead to any conclusive outcome on the existence or otherwise of bipartite graphs of diameter 3 and defect 4 in general. We nevertheless feel that the following conjecture is valid.

Conjecture 6.1 *There is no bipartite $(d, 3, -4)$-graph with $d \geq 6$.*

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