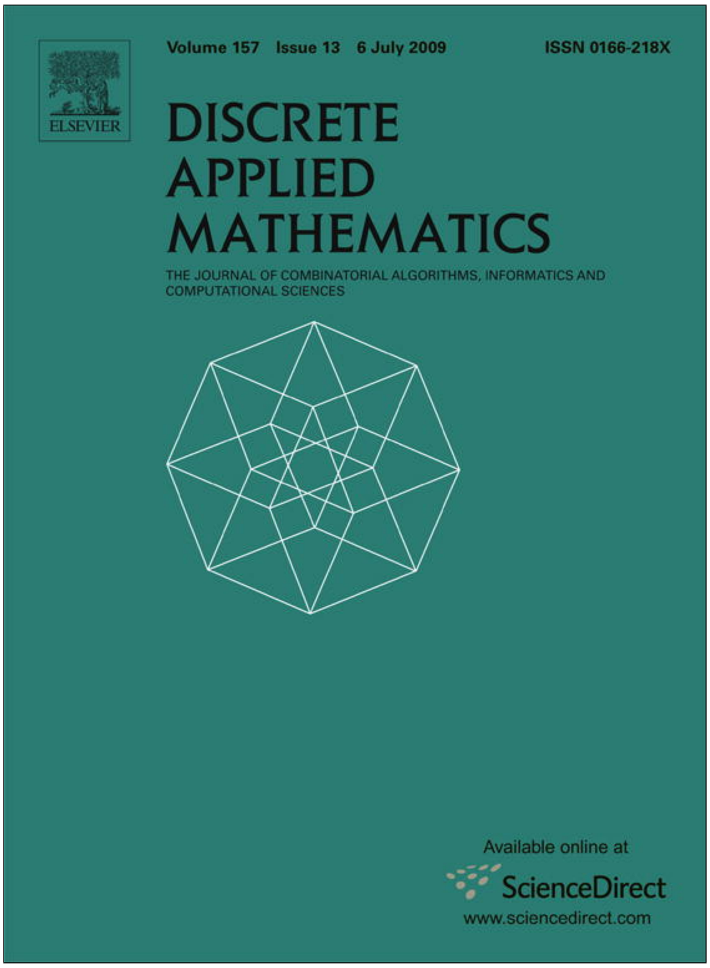
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Complete catalogue of graphs of maximum degree 3 and defect at most 4

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ABSTRACT

We consider graphs of maximum degree 3, diameter $D \geq 2$ and at most 4 vertices less than the Moore bound $M_{3,D}$, that is, $(3, D, -\epsilon)$-graphs for $\epsilon \leq 4$.

We prove the non-existence of $(3, D, -4)$-graphs for $D \geq 5$, completing in this way the catalogue of $(3, D, -\epsilon)$-graphs with $D \geq 2$ and $\epsilon \leq 4$. Our results also give an improvement to the upper bound on the largest possible number $N_{3,D}$ of vertices in a graph of maximum degree 3 and diameter D, so that $N_{3,D} \leq M_{3,D} - 6$ for $D \geq 5$.

1. Introduction

The optimality of a network has been interpreted in various ways, see, for instance, [16]. One possible interpretation can be stated as follows:

An *optimal network* contains the maximum possible number of nodes, given a limitation on the number of connections attached to a node and a limitation on the number of traversed links between any two farthest nodes.

In graph-theoretical terms, the preceding interpretation leads to the

Degree/diameter problem: Given natural numbers $\Delta \geq 2$ and $D \geq 1$, find the largest possible number of vertices $N_{\Delta,D}$ in a graph of maximum degree Δ and diameter D.

It is straightforward to verify that $N_{\Delta,D}$ is defined for $\Delta \geq 2$ and $D \geq 1$. An upper bound on $N_{\Delta,D}$ is given by the following expression [3,13].

$$\begin{aligned} N_{\Delta,D} &\leq 1 + \Delta + \Delta(\Delta-1) + \cdots + \Delta(\Delta-1)^{D-1} \\ &= 1 + \Delta[1 + (\Delta-1) + \cdots + (\Delta-1)^{D-1}] \\ &= \begin{cases} 1 + \Delta \dfrac{(\Delta-1)^D - 1}{\Delta - 2} & \text{if } \Delta > 2 \\ 2D+1 & \text{if } \Delta = 2. \end{cases} \end{aligned} \tag{1}$$

This expression is known as the *Moore bound*, and is denoted by $M_{\Delta,D}$. A graph whose order is equal to the Moore bound is called a *Moore graph*.

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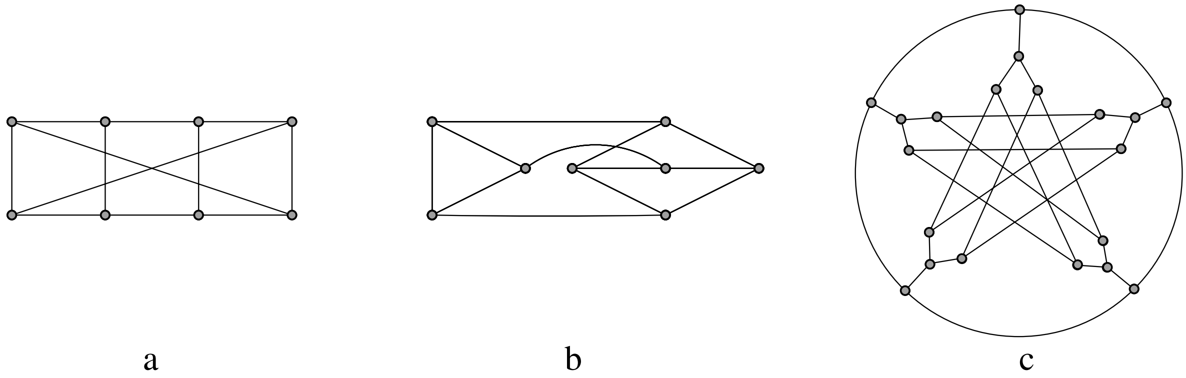

Fig. 1. All the $(3, D, -2)$-graphs for $D \geq 2$.

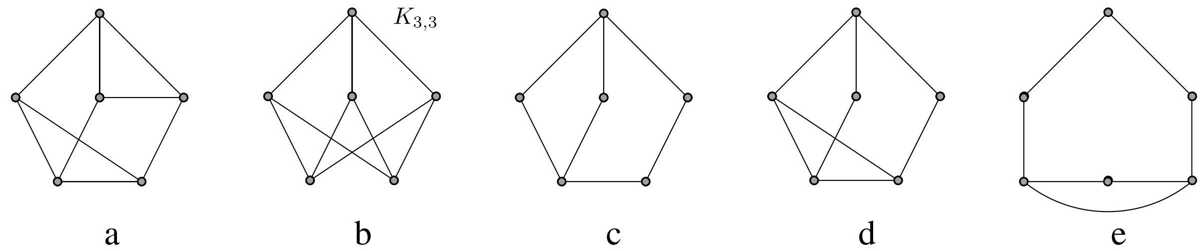

Fig. 2. All the $(3, 2, -4)$-graphs.

Moore graphs exist only for certain special values of maximum degree and diameter. To be more precise, for diameter $D = 1$ and degree $\Delta \geq 1$, Moore graphs are the complete graphs of order $\Delta + 1$. For diameter $D = 2$ Hoffman and Singleton [9] proved that Moore graphs exist for $\Delta = 2, 3, 7$ and possibly 57, but not for any other degree. Finally, for $D \geq 3$ and $\Delta = 2$, Moore graphs are the cycles on $2D + 1$ vertices. The fact that Moore graphs do not exist for $D \geq 3$ and $\Delta \geq 3$ was shown by Damerell [5] and, independently, also by Bannai and Ito [1].

Therefore, we are interested in studying the existence of large graphs of given maximum degree Δ, diameter D and order $M_{\Delta,D} - \epsilon$, for $\epsilon > 0$, that is, $(\Delta, D, -\epsilon)$-graphs, where ϵ is called the *defect*.

Since the case $\Delta = 2$ is completely settled ($N_{2,D} = 2D + 1$, for $D \geq 3$), in this paper, we consider the next case, $\Delta = 3$.

For $D \geq 2$, if a $(3, D, -\epsilon)$-graph had a vertex of degree at most 2 then the order of such a graph would be at most $\frac{2}{3}M_{3,D} + \frac{1}{3}$; see [10]. Therefore, we can state the following proposition.

Proposition 1.1 ([10]). *If* $\epsilon < \frac{M_{3,D}}{3} - \frac{1}{3}$ *then a* $(3, D, -\epsilon)$*-graph is regular.*

By Proposition 1.1, for $\epsilon < \frac{M_{3,D}}{3} - \frac{1}{3}$, odd ϵ, and $D \geq 2$, a $(3, D, -\epsilon)$-graph is *cubic*, and must have an even number of vertices. Therefore, these graphs do not exist when $\epsilon = 1, 3$. Thus, the next interesting cases occur when $\epsilon = 2$ and 4.

The case of $\epsilon = 2$ was analyzed by Jørgensen [10]. Jørgensen proved that for $D \geq 4$ there are no $(3, D, -2)$-graphs and showed the uniqueness of the two known $(3, 2, -2)$-graphs (graphs (a) and (b) in Fig. 1) and of the $(3, 3, -2)$-graph (graph (c) in Fig. 1).

The case $\epsilon = 4$ and $D = 2$ or 3 was considered in [14], where we presented all the $(3, 2, -4)$-graphs. The unique $(3, 3, -4)$-graph was constructed initially by Faradžev [8], and later rediscovered by McKay and Royle [12], who proved its uniqueness; see Figs. 2 and 3.

For diameter 4 the non-existence of $(3, 4, -4)$-graphs was proved by Jørgensen [11].

A simple counting argument shows that a $(3, D, -4)$-graph, $D \geq 3$, has girth at least $2D - 2$. In [14] we proved that the girth must be at least $2D - 1$, and conjectured that its real value is $2D$.

In this paper we prove that if a $(3, D, -4)$-graph with $D \geq 5$ exists then its girth must be $2D$. Moreover, using this result about the girth of such graphs, we show that there are no $(3, D, -4)$-graphs for $D \geq 5$, thus completing the census of $(3, D, -\epsilon)$-graphs with $D \geq 2$ and $\epsilon \leq 4$.

Note that some parts of our proof are inspired by the reasoning used by Jørgensen in [11].

Values of $N_{3,D}$ are known only for $D = 2$, 3 and 4. For $D = 2$, $N_{3,2} = M_{3,2}$, and the unique graph is the Petersen graph; see [9]. For $D = 3$, $N_{3,3} = M_{3,3} - 2$, and the unique graph, depicted in Fig. 1(c), was found by Bermond, Delorme and Farhi [2,10]. For $D = 4$, by proving the non-existence of $(3, 4, -6)$-graphs, Buset[4] showed that $N_{3,4} = M_{3,4} - 8$, and the two known (non-isomorphic) graphs, constructed by Doty [7] and by von Conta [15], therefore became the largest graphs when $\Delta = 3$ and $D = 4$.

Our results give an improvement on the upper bound of $N_{3,D}$, so that $N_{3,D} \leq M_{3,D} - 6$ for $D \geq 5$.

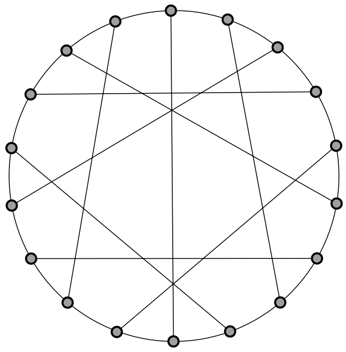

Fig. 3. The unique $(3, 3, -4)$-graph.

The rest of this paper is structured as follows: in Section 2, we settle the notation and terminology used throughout this paper and we give some preliminary results. Section 3 is devoted to proving that if a $(3, D, -4)$-graph with $D \geq 5$ exists then it must have girth $2D$. In Section 4 we prove the non-existence of $(3, D, -4)$-graphs with $D \geq 5$; and in Section 5 we give a summary of our results.

It is perhaps worth noting that the case of $(3, D, -4)$-graphs is particularly interesting, because it is the first result concerning $(\Delta, D, -\epsilon)$-graphs of defect greater than the maximum degree of the graph.

2. Terminology and preliminary results

All graphs considered in this paper are simple, that is, they have neither loops nor multiple edges.

Throughout this paper, it is assumed that the reader is already familiar with basic graph theory, and therefore with its main concepts and results. Thus, the only objective of this section is to settle those notations that could vary among texts.

The terminology and notation used in this paper is standard and consistent with that used in [6].

The vertex set of a graph Γ is denoted by $V(\Gamma)$, and its edge set by $E(\Gamma)$. In Γ a vertex of degree at least 3 is called a *branch vertex* of Γ. For an edge $e = \{x, y\}$, we write $e = xy$, or simply xy, or alternatively, $x \sim y$. If two vertices u and u are not adjacent then we write $x \nsim y$. The *length* of a path P is the number of edges in P. A path of length k is called a *k-path*. A path from a vertex x to a vertex y is denoted by $x - y$. Whenever we refer to paths, we mean shortest paths. A cycle of length k is called a *k-cycle*.

We will also use the following notations for subpaths of a path $P = x_0x_1 \ldots x_k$: $x_iPx_j = x_i \ldots x_j$, where $0 \leq i \leq j \leq k$.

The set of vertices at distance k from a vertex x is denoted by $N_k(x)$. The set of neighbors of a vertex x in Γ is simply denoted by $N(x)$. The set of edges in the graph Γ joining a vertex x in $X \subseteq V(\Gamma)$ to a vertex y in $Y \subseteq V(\Gamma)$ is denoted by $E(X, Y)$; for simplicity, instead of $E(X, X)$, we write $E(X)$.

The *difference* between the graphs Γ and Γ', denoted by $\Gamma - \Gamma'$, is the graph with vertex set $V(\Gamma) - V(\Gamma')$ and edge set formed by all the edges with both endvertices in $V(\Gamma) - V(\Gamma')$.

The union of three independent paths of length D with common endvertices is denoted by Θ_D.

Finally, we call a cycle of length at most $2D$ a *short cycle*, and we call a vertex x a *saturated* vertex if x cannot belong to any further short cycle.

From now on, let Γ be a $(3, D, -4)$-graph for $D \geq 5$. By Proposition 1.1, Γ must be regular.

Furthermore, we have

Proposition 2.1 (*[14]*). *A* $(3, D, -4)$*-graph for* $D \geq 5$ *has girth at least* $2D - 1$.

In [14] it was conjectured that

Conjecture 2.1 (*[14]*). *The girth of a* $(3, D, -4)$*-graph,* $D \geq 5$*, is* $2D$.

If the girth of Γ is $2D - 1$ then there exists a vertex x in Γ such that x lies on either one or two $(2D - 1)$-cycles. Note that no vertex x can lie on more than two such cycles, otherwise $|E(N_{D-1}(x))| \geq 3$, implying $\Gamma \leq M_{3,D} - 6$, a contradiction.

Using a simple counting argument, we classify each vertex of a $(3, D, -4)$-graph according to the short cycles on which the vertex lies, as shown in Proposition 2.2.

Proposition 2.2. *Let x be a vertex of Γ. Then x lies on the short cycles specified below, and no other short cycle. We have the following cases:*

x **is contained in two** $(2D - 1)$- **cycles**. *Then*

(i) *x* **lies on exactly two** $(2D - 1)$- **cycles** *whose intersection is an l-path for some l such that* $1 \leq l \leq D - 1$. *If* $l = D - 1$ *then x is also contained in one* $2D$*-cycle; or*

x **is contained in exactly one** $(2D - 1)$- **cycle**. *Then also*

(ii) *x is a branch vertex of one* Θ_D*, or*

(iii) *x is contained in exactly two* $2D$*-cycles; or*

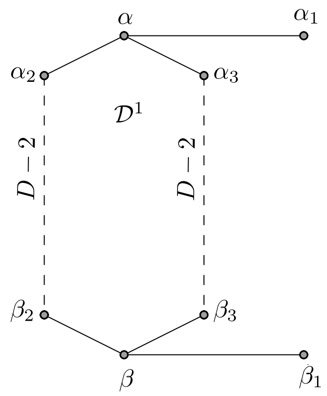

Fig. 4. Auxiliary figure for Lemma 2.1.

x **is contained in no** $(2D-1)$- **cycle**. *Then also*

(iv) *x is a branch vertex of exactly two Θ_D, or*
(v) *x is a branch vertex of one Θ_D, and is contained in two more $2D$-cycles, or*
(vi) *x is contained in exactly four $2D$-cycles.*

Each case is considered as a type. For instance, a vertex satisfying case (i) is called a vertex of Type (i).

Proof. By Proposition 2.1, we see that $N_i(x)$ is an independent set, for $i \in \{1, \dots, D-2\}$, and that $|N_{D-1}(x)| = 3 \times 2^{D-2}$.

It is clear that $|E(N_{D-1}(x))| \leq 2$, otherwise $|\Gamma| \leq M_{3,D} - 6$. We distinguish three cases according to the possible values of $|E(N_{D-1}(x))|$.

Case 1. $|E(N_{D-1}(x))| = 2$.

In this case these two edges either have a common endvertex or are independent. Therefore, (i) follows.

Case 2. $|E(N_{D-1}(x))| = 1$.

Since $|N_D(x)| = 3 \times 2^{D-1} - 4$ and $|E(N_{D-1}(x), N_D(x))| = 3 \times 2^{D-1} - 2$, we obtain (ii) or (iii).

Case 3. $|E(N_{D-1}(x))| = 0$.

Since $|N_D(x)| = 3 \times 2^{D-1} - 4$ and $|E(N_{D-1}(x), N_D(x))| = 3 \times 2^{D-1}$, it follows that x is a vertex of Type (iv), (v) or (vi). □

Observation 2.1. *If a vertex $x \in \Gamma$ belongs to exactly one $(2D-1)$-cycle C^1 then the intersection of C^1 and any $2D$-cycle is a path of length at most $D-1$.*

Next we prove a lemma that will be used repeatedly in the rest of this paper.

Lemma 2.1 (*Intersection Lemma*). *Let $\mathcal{D}^1$ be a $2D$-cycle in Γ. Let α and β be vertices on $\mathcal{D}^1$ such that $d(\alpha, \beta) = D$. Let α_1 be the neighbor of α not contained in $\mathcal{D}^1$. Let us suppose that α is not a branch vertex of a Θ_D, and that α_1 is contained in at most one $(2D-1)$-cycle, say $\mathcal{C}$, which also contains α. Then*

(i) *the intersection of $\mathcal{D}^1$ and $\mathcal{C}$ is a path of length $D-1$, or*
(ii) *there exists another $2D$-cycle, say $\mathcal{D}^2$, containing α and α_1. Furthermore, the intersection of $\mathcal{D}^1$ and $\mathcal{D}^2$ is a path of length $D-1$.*

Proof. Let $\mathcal{D}^1$ be a $2D$-cycle of Γ, and let $\alpha, \alpha_1, \alpha_2, \alpha_3, \beta, \beta_1, \beta_2$ and β_3 be as in Fig. 4.

Let $P^1 = \alpha_1 - \beta$. The length of P^1 must be D, since α is not a branch vertex of a Θ_D, α_1 is contained in at most one $(2D-1)$-cycle, and the girth of Γ is at least $2D-1$. Therefore, we have two possibilities: either P^1 goes through β_2 or β_3, or it goes through β_1. In the first case $V(P^1 \cap \mathcal{D}^1) = \{\beta_2 \text{ or } \beta_3, \beta\}$, and (i) follows. In the second case we consider the neighbor α' of α_1 such that $\alpha' \neq \alpha$ and $\alpha' \notin P^1$. A path $P^2 = \alpha' - \beta$ does not pass through β_1, otherwise α_1 would belong to a cycle of length at most $2D-1$ that does not contain α, contradicting our assumptions. Therefore, P^2 is a path of length either $D-1$ or D, which goes through β_2 or β_3, and $V(P^2 \cap \mathcal{D}^1) = \{\beta_2 \text{ or } \beta_3, \beta\}$. Consequently, if P^2 is a $(D-1)$-path then (i) follows, otherwise (ii) follows.

Note that if α_1 is contained in no $(2D-1)$-cycle then (ii) follows. □

3. On the girth of $(3, D, -4)$-graphs with $D \geq 5$

The aim of this section is to prove that the girth of Γ is exactly $2D$. This result will be obtained by ruling out the existence of vertices of Type (i), (ii) or (iii).

Theorem 3.1. *A $(3, D, -4)$-graph, $D \geq 5$, does not contain a vertex of Type* (i), (ii) *or* (iii).

We prove Theorem 3.1 by eliminating, in order, the existence of vertices of each type under consideration.

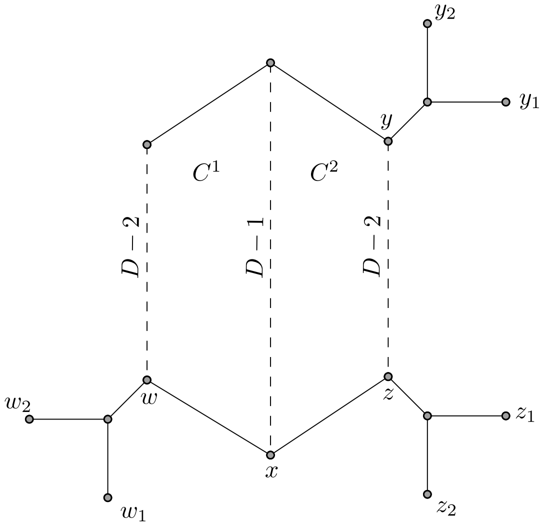

Fig. 5. Auxiliary figure for Lemma 3.1.

Non-existence of vertices of Type (i)

In the next two lemmas we give some necessary conditions for the existence of vertices of Type (i).

Lemma 3.1. *Let x be a vertex lying on two $(2D-1)$-cycles. Then the intersection of such $(2D-1)$-cycles is a path of length at most $D-2$.*

Proof. We proceed by way of contradiction. Let us consider a vertex $x \in \Gamma$ lying on two $(2D-1)$-cycles, say C^1 and C^2, and let us further suppose that the intersection of C^1 and C^2 is a path of length $D-1$. Then Γ contains the subgraph in Fig. 5.

Let $x, w, w_1, w_2, z, z_1, z_2, y, y_1, y_2$ be as in Fig. 5. A path $P = y_1 - x$ is a D-path, since $d(x, y_1) \le D$, and by Proposition 2.2(i). Besides, by Proposition 2.2(i), if P intersects with C^1 then $V(P \cap C^1) = \{x, w\}$; if instead P intersects with C^2 then $V(P \cap C^2) = \{x, z\}$. Therefore, P should pass through either w_1, w_2, z_1, or z_2.

If the path $P = y_1 - x$ passed through either z_1 or z_2, say z_1, then z would be contained in two $(2D-1)$-cycles, namely, C^2 and $C = zC^2y \ldots y_1Pz_1 \ldots z$, and in one $2D$-cycle. However, by Proposition 2.2(i), the intersection of C and C^2 should be a path of length $D-1$, and in this case, the intersection is a path of length $D-2$, namely, zC^2y, a contradiction. Therefore, $P = y_1 - x$ passes through either w_1 or w_2. Analogously, a path $Q = y_2 - x$ is a D-path, and passes through either w_1 or w_2. Thus, Γ contains a cycle of length at most $2D-2$, contradicting Proposition 2.1. □

Lemma 3.2. *Let us assume that Γ contains two non-disjoint $(2D-1)$-cycles. Then the intersection of such $(2D-1)$-cycles is a path of length exactly $D-2$.*

Proof. Let us suppose that Γ contains two non-disjoint $(2D-1)$-cycles, denoted by C^1 and C^2.

To prove this lemma we proceed by way of contradiction. Suppose that the intersection of the cycles C^1 and C^2 is a path of length l, with $l \in \{1, \ldots, D-3\}$. Recall that the case of $l = D-1$ is ruled out by Lemma 3.1.

As $C^1 \ne C^2$, there are two vertices x and x_1 such that $x \in (C^1 \cap C^2)$, $x_1 \in (C^2 - C^1)$ and $x \sim x_1$. We may also assume that x_1 has a neighbor x_3 such that $x \ne x_3$ and $x_3 \in C^2$.

Let $x_4, y, y_1, y_2, y_3, y_4, z, z_1, z_2, z_3$, and z_4 be as in Fig. 6(a).

Let us first consider a path $P^1 = x_3 - z$. Since the intersection of C^1 and C^2 is a path of length l with $1 \le l \le D-3$, we have $x_1 \notin P^1$. By assumption, P^1 cannot go through z_1. P^1 does not pass through y, and $V(P^1 \cap C^1) = \{z\}$, since x is a vertex of Type (i), and the intersection of C^1 and C^2 is a path of length l with $l \in \{1, \ldots, D-3\}$. Therefore, P^1 is a D-path that passes through either z_3 or z_4, say z_3. By following similar reasoning, a path $P^2 = x_4 - z$ goes through either z_3 or z_4. Then $V(P^1 \cap P^2) = \{z_2, z\}$, otherwise there would be a cycle of length at most $(2D-2)$. Therefore, the path P^2 uses the vertex z_4, and is a D-path. Note that $x_1x_3P^1z_3z_2z_4P^2x_4x_1$ is a $2D$-cycle, denoted by D^1.

In the same way, we can assume that the paths $Q^1 = x_3 - y$ and $Q^2 = x_4 - y$ use the vertices y_3 and y_4, respectively. Furthermore, both Q^1 and Q^2 are D-paths, and $V(Q^1 \cap Q^2) = \{y_2, y\}$. Note that $x_1x_3Q^1y_3y_2y_4Q^2x_4x_1$ is also a $2D$-cycle, denoted by D^2. Thus, x_1 and x_3 are contained in the $2D$-cycles D^1 and D^2, and in the $(2D-1)$-cycle C^2.

Let s and r be the neighbors of x_3 different from x_1 such that $s \in P^1$ and $r \notin P^1$.

Recall that the vertices x_1 and x_3 cannot be contained in any additional cycle of length at most $2D$.

We distinguish two cases: either $V(P^1 \cap Q^1) = \{x_3, s\}$ or $V(P^1 \cap Q^1) = \{x_3\}$.

Case 1. The paths P^1 and Q^1 intersect at x_3 and s.

In this case the vertices y and z lie on a further $(2D-1)$-cycle, namely, $sQ^1y_3y_2yzz_2z_3P^1s$, and consequently, the paths P^2 and Q^2 should intersect only at x_4, otherwise y and z would be contained in three $(2D-1)$-cycles; see Fig. 6(b).

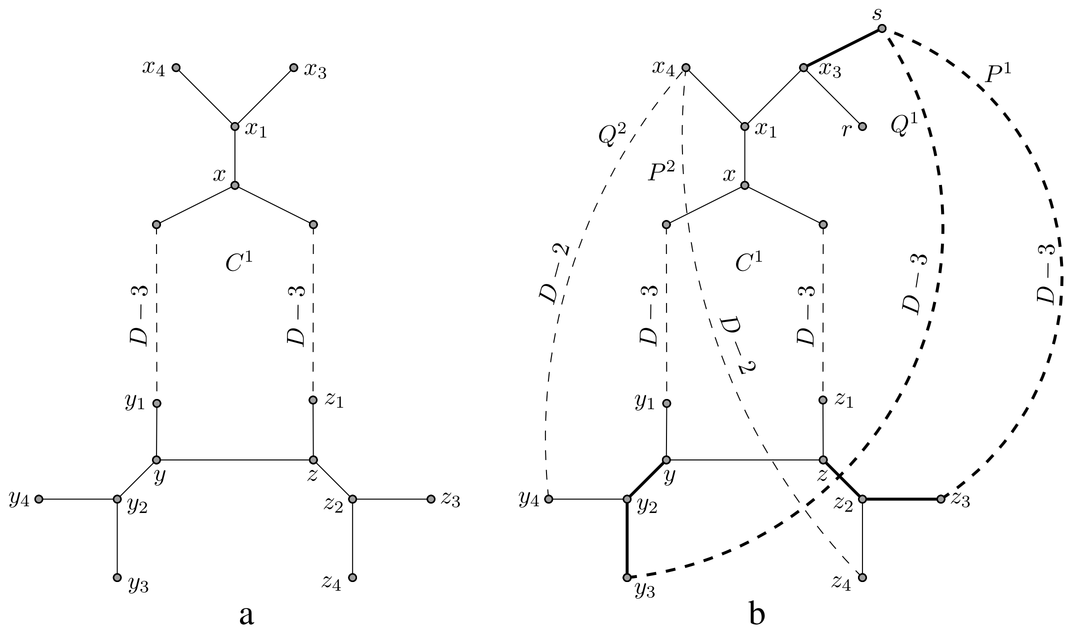

Fig. 6. (a): Auxiliary figure for Lemma 3.2. (b): Auxiliary figure for Case 1 of Lemma 3.2.

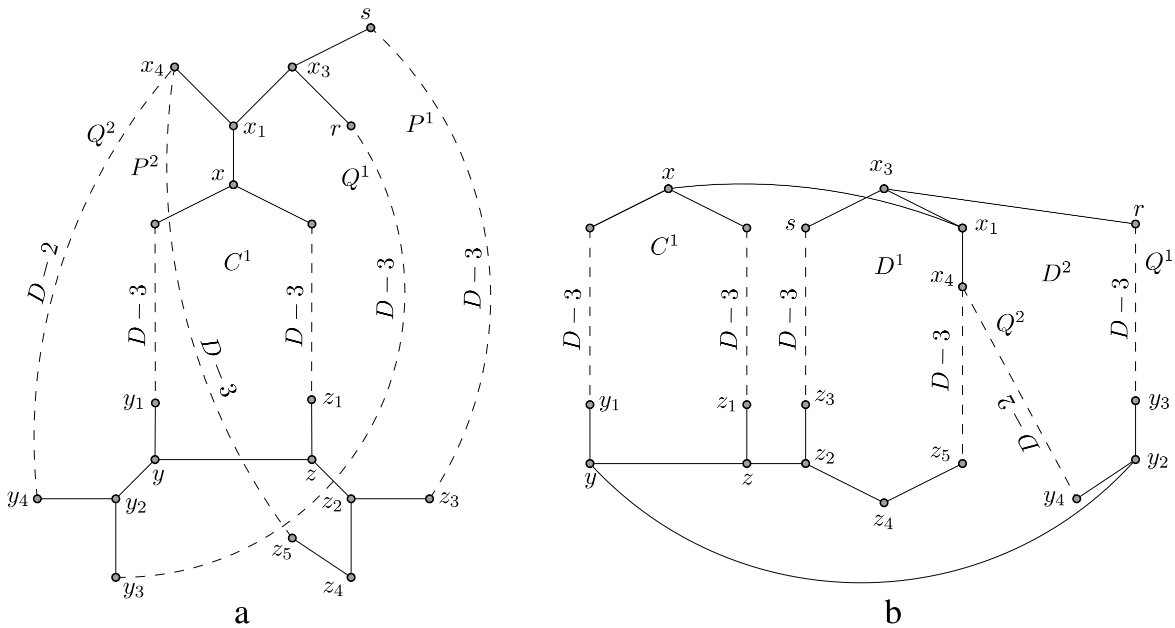

Fig. 7. Auxiliary figure for Case 2 of Lemma 3.2.

Let us consider a path $A = r - z$. Considering our assumptions, and that the vertices x_1 and x_3 cannot belong to a further short cycle, we have that the path A cannot use the vertices z_1, z_3, z_4, y_1, y_3 or y_4, and therefore, r cannot reach z in at most D steps, a contradiction.

Case 2. The paths P^1 and Q^1 intersect only at x_3.

We may assume that the paths P^2 and Q^2 intersect only at x_4; see Fig. 7(a).

Without loss of generality, as $x_3 \in C^2$, we may also assume that $r \in C^2$.

Let z_5 be as in Fig. 7(a). Note that since $D \geq 5$, Γ contains the subgraph depicted in Fig. 7(a).

To achieve a better understanding of this case, we depict Fig. 7(a) in a different way, by drawing our attention to the vertices x_3 and r, and to the $2D$-cycle D^1; see Fig. 7(b).

We see that the premises of the Intersection Lemma hold. Mapping the vertex x_3 to α, r to α_1, z_4 to β, and mapping the $2D$-cycle D^1 to $\mathcal{D}^1$, and the $(2D-1)$-cycle C^2 to $\mathcal{C}$, we obtain, by the Intersection Lemma, that one of the following cases holds. In the first case, x_3 and r are contained in a $(2D-1)$-cycle that intersects with D^1 at a path of length $D-1$. This cycle would be precisely C^2, implying $D-1=1$, a contradiction. In the second case, x_3 and r are contained in another $2D$-cycle that intersects with D^1 at a path of length $D-1$. This cycle would be precisely D^2, implying $D-1=2$, a contradiction.

Consequently, the cycles C^1 and C^2 intersect at a path of length exactly $D-2$. $\square$

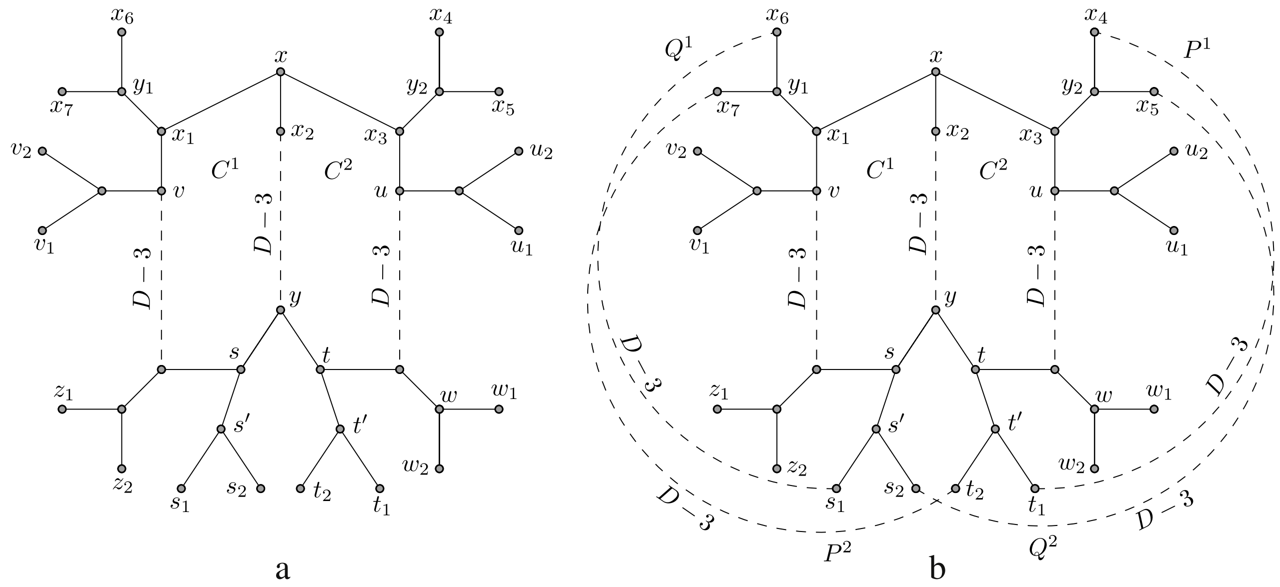

Fig. 8. Vertex of Type (i) in a $(3, D, 4)$-graph when $D \geq 5$.

Using the structural results from Lemmas 3.1 and 3.2, the next proposition rules out the existence of vertices of Type (i).

Proposition 3.1. *A $(3, D, -4)$-graph, $D \geq 5$, does not contain a vertex of Type* (i).

Proof. Let $x \in \Gamma$ be a vertex of Type (i), lying on the $(2D-1)$-cycles C^1 and C^2. In view of Lemma 3.2, the intersection of C^1 and C^2 is a path of length $D-2$. Then since $D \geq 5$, Γ contains the subgraph in Fig. 8(a).

Let $x, x_1, x_2, x_3, x_4, x_5, x_6, x_7, y, y_1, y_2, u, u_1, u_2, v, v_1, v_2, s, s', s_1, s_2, t, t', t_1, t_2, z_1, z_2, w, w_1$, and w_2 be as in Fig. 8(a).

Let us first consider a path $P^1 = t_1 - x$. Note that since all the vertices in the intersection of C^1 and C^2 are of Type (i), it follows that $t', x_2 \notin P^1$. The vertex u is not contained in P^1, otherwise there would exist a cycle of length at most $2D-2$. The vertex v is not contained in P^1 either, otherwise the vertex y would be contained in two $(2D-1)$-cycles (C^1 and C^2), and in one further cycle of length at most $2D$, a contradiction. Therefore, P^1 is a D-path, and passes through either x_4, x_5, x_6 or x_7. Analogously, a path $P^2 = t_2 - x$ is a D-path, and goes through either x_4, x_5, x_6 or x_7. Moreover, neither $x_1 \in (P^1 \cap P^2)$ nor $x_3 \in (P^1 \cap P^2)$, otherwise there would exist a cycle of length at most $2D-2$ in Γ. Without loss of generality, we may therefore assume that P^1 goes through x_4, and that P^2 goes through x_7.

In the same way, we may assume that the D-paths $Q^1 = s_1 - x$ and $Q^2 = s_2 - x$ go through x_6 and x_5, respectively. Note that $V(P^2 \cap Q^1) = \{y_1, x_1, x\}$, otherwise y would be contained in an additional cycle of length at most $2D$. Analogously, $V(P^1 \cap Q^2) = \{y_2, x_3, x\}$; see Fig. 8(b).

Note also that $D^1 = tt't_1P^1x_4y_2x_3uC^2t$ and $D^2 = ss's_1Q^1x_6y_1x_1vC^1s$ are $2D$-cycles.

Let x_8 and x_9 be the neighbors of x_6 different from y_1 such that $x_8 \in Q^1$ and $x_9 \notin Q^1$. Let r be the neighbor of v on C^1 different from x_1; see Fig. 9(a).

The paths $M^1 = w_1 - x$ and $M^2 = w_2 - x$ are D-paths. Note that $w \notin M^1$. The vertex $u \notin M^1$, otherwise there would exist a cycle of length at most $2D-3$ in Γ. The vertex $x_2 \notin M^1$, otherwise x_2 would be contained in a further cycle of length at most $2D$, contradicting Proposition 2.2(i). Since the vertex x_3 is of Type (iii) or (iv) (see Proposition 2.2(iii) and (iv)), $x_3 \notin M^1$, otherwise x_3 would be contained in an additional cycle of length at most $2D-1$ (x_3 is already contained in the $(2D-1)$-cycle C^2 and in the $2D$-cycle D^1). Furthermore, if the path M^1 went through x_7 then the vertex t would be contained in an additional cycle of length at most $2D-1$, contradicting the fact that $t \in$ Type (iii) or (iv) ($t \in C^2$ and D^1). If instead $x_8 \in M^1$ then y would be contained in a further cycle of length at most $2D-1$, contradicting Proposition 2.2(i). Therefore, M^1 passes through either v_1, v_2 or x_9. Consequently, the path $M^2 = w_2 - x$ goes also through either v_1, v_2, or x_9. If both M^1 and M^2 reached x through either v_1 or v_2 then there would exist a cycle of length at most $2D-4$ in Γ, a contradiction. We may therefore assume that M^1 goes through x_9, x_6 and y_1, and that M^2 goes through v_1; see Fig. 9(a).

Note that $D^3 = w_1M^1x_9x_6y_1x_1v \ldots v_1M^2w_2ww_1$ is a $2D$-cycle, and that after the above developments, the vertices v and x_1 cannot be contained in any further short cycle, because they are already contained in the cycles C^1, D^2 and D^3.

Let us now turn our attention to the vertex v and the cycles C^1, D^2 and D^3; see Fig. 9(b) (cycle C^1 is highlighted by a heavier line).

Let w_3 be the neighbor of w_1 different from w such that $w_3 \in M^1$. See Fig. 9(b).

We see that the premises of the Intersection Lemma hold. Mapping the vertex v to α, r to α_1, w_1 to β, and mapping the $2D$-cycle D^3 to $\mathcal{D}^1$, and the $(2D-1)$-cycle C^1 to $\mathcal{C}$, we obtain, by the Intersection Lemma, that one of the following cases holds. In the first case, v and r are contained in a $(2D-1)$-cycle that intersects with D^3 at a path of length $D-1$. This cycle would be precisely C^1, implying $D-1=1$, a contradiction. In the second case, v and r are contained in another $2D$-cycle that intersects with D^3 at a path of length $D-1$. This cycle would be precisely D^2, implying $D-1=3$, a contradiction.

As a result, when $D \geq 5$, a $(3, D, -4)$-graph does not contain a vertex of Type (i). □

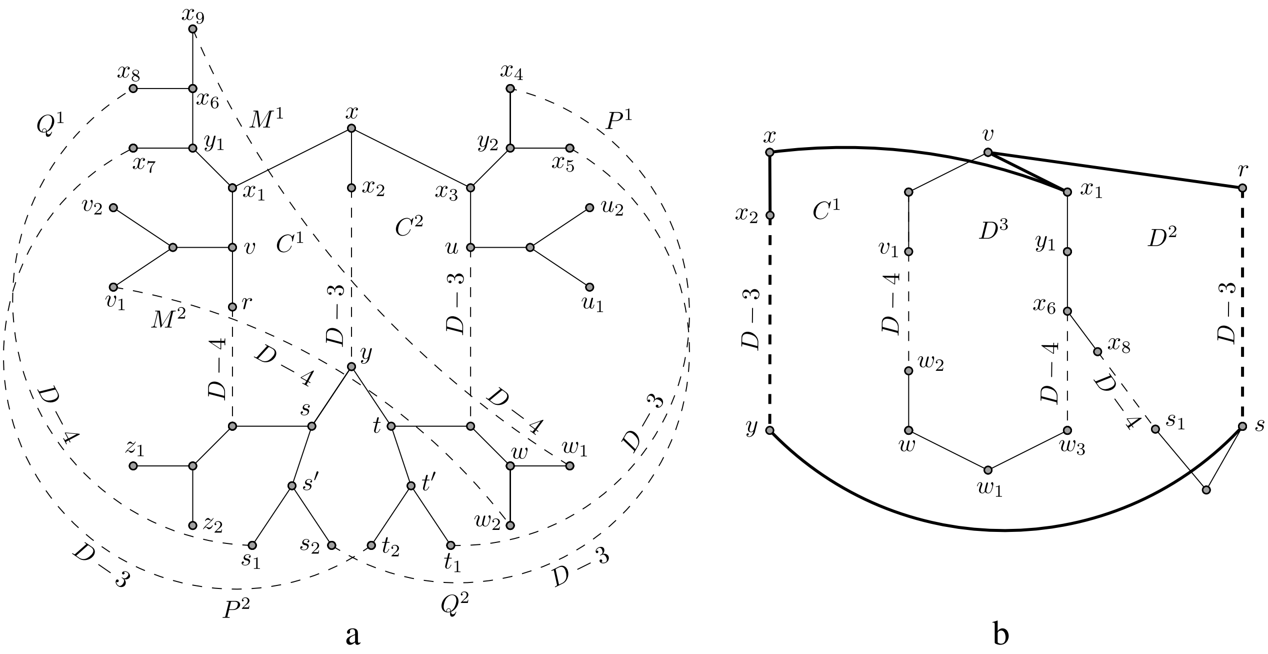

Fig. 9. Auxiliary figure for Proposition 3.1.

Non-existence of vertices of Type (ii)

Proposition 3.1 opens up a way to prove the non-existence of vertices of Type (ii), as shown below.

Proposition 3.2. *A* $(3, D, -4)$*-graph,* $D \geq 5$*, does not contain a vertex of Type* (ii).

Proof. Let x be a vertex of Type (ii). Let H be a subgraph of Γ isomorphic to Θ_D, where x and y are its branch vertices. Then H consists of three independent paths $x - y$ of length D, say P^1, P^2 and P^3. Since x is of Type (ii), x is also contained in one $(2D - 1)$-cycle, say C.

We may assume that $|V(P^1 \cap C)| > 1$, $|V(P^3 \cap C)| > 1$ and $V(P^2 \cap C) = \{x\}$. As C is a $(2D - 1)$-cycle, there is a vertex u of P^1, different from x or y, such that u and the neighbor of u not contained in P^1, say u_1, are both contained in C. Let v and w be the vertices in P^2 and P^3, respectively, at distance D from u in H. If the distance in Γ between u and either v or w was at most $D - 1$ then u would be contained in two cycles of length at most $(2D - 1)$ and in two $2D$-cycles, contradicting Proposition 2.2(iii). Therefore, the distance in Γ between u, and v or w is D. Let u_2 and u_3, v_2 and v_3, and w_2 and w_3 be the neighbors of u on P^1, the neighbors of v on P^2, and the neighbors of w on P^3, respectively; see Fig. 10(a).

A path $P = u_1 - v$ does not pass through u, otherwise some vertices of H would be contained in a cycle of length at most $2D - 3$. Suppose that P passes through v_3. If $x \sim v$ (implying $x = v_3 = w_3$) then either there would be a cycle of length at most $2D - 3$ or the distance in Γ between u and w would be at most $D - 1$, a contradiction. If instead $x \nsim v$ then x and some vertices of P^2 would be contained in a cycle of length at most $2D - 1$, but, by Proposition 2.2(ii), this cycle would be C, contradicting our assumption that $V(P^2 \cap C) = \{x\}$. Therefore, $v_3 \notin P$. If P passed through v_2 then u would be contained in a cycle of length at most $2D - 1$ that does not contain x. This is a contradiction, because, since u is a vertex of Type (iii), u can be contained in only one $(2D - 1)$-cycle, and that $(2D - 1)$-cycle is C. Therefore, P passes through the neighbor of v not contained in P_2, say v_1, and P is a D-path.

Let r be the neighbor of u_1, different from u, and not contained in P.

Reasoning as above, a path $Q = r - v$ does not contain v_2 or v_3, otherwise u would be contained in a further short cycle, contradicting Proposition 2.2(iii). As a result, Q uses the edge vv_1, and is a D-path. Thus, u_1 is contained in an additional $(2D - 1)$-cycle, namely, $C^1 = u_1 P v_1 Q r u_1$, and u_1 is therefore a vertex of Type (i) ($u \in C \cap C^1$). However, as $D \geq 5$, by Proposition 3.1, Γ does not contain such a vertex; see Fig. 10(b). □

Non-existence of vertices of Type (iii)

At this point, assuming that there are no vertices of Type (i) or (ii), our aim is to rule out the existence of vertices of Type (iii).

Lemma 3.3. *Let* C *be a* $(2D - 1)$*-cycle in* Γ*. Let* D^1 *be a* $2D$*-cycle such that* C *and* D^1 *are non-disjoint. Then the intersection of* C *and* D^1 *is a path of length at most* $D - 2$*.*

Proof. Let C be a $(2D - 1)$-cycle in Γ. Suppose, by way of contradiction, that in Γ there exists a $2D$-cycle, say D^1, such that the intersection of C and D^1 is a path of length exactly $D - 1$.

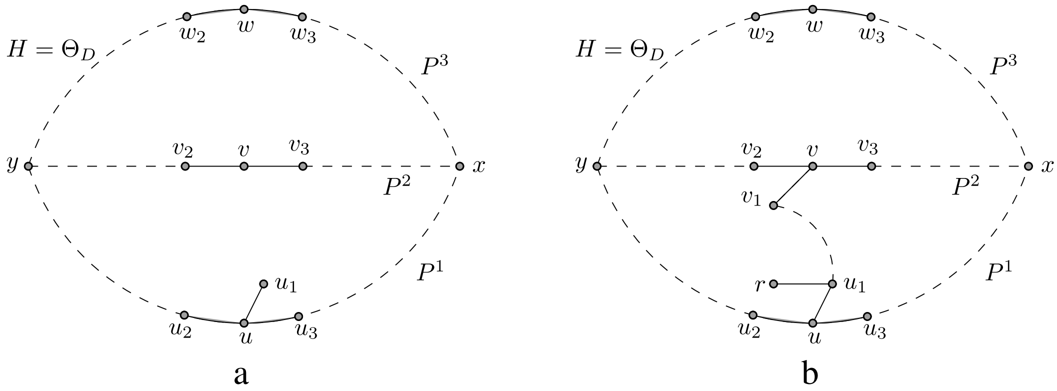

Fig. 10. Auxiliary figure for Proposition 3.2.

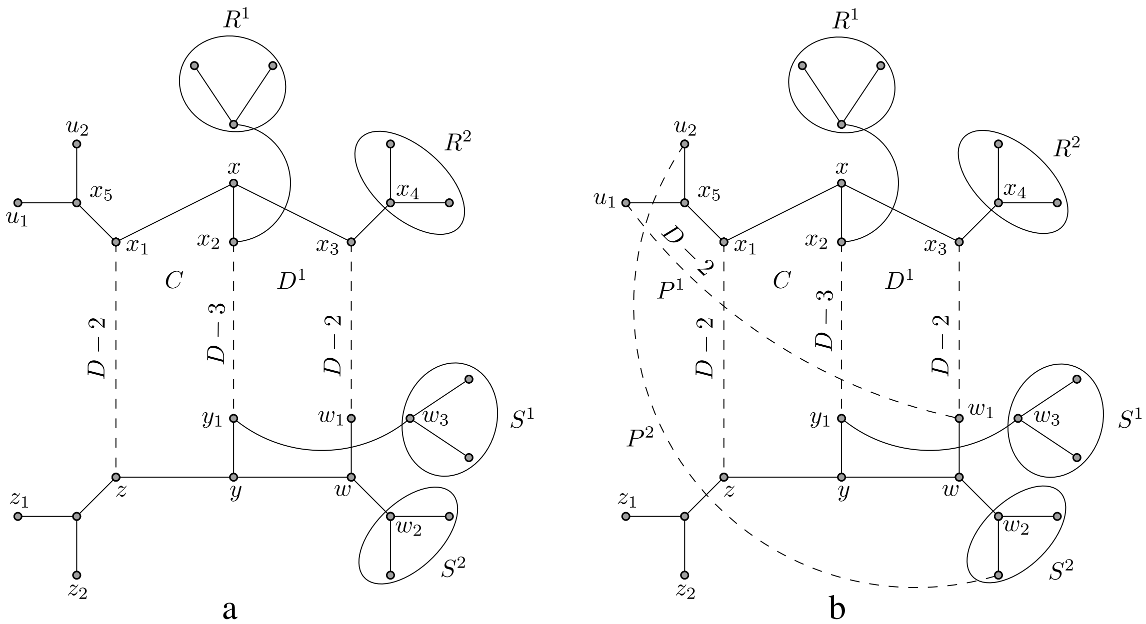

Fig. 11. Auxiliary figure for Lemma 3.3.

Since $D \geq 5$, Γ contains the subgraph depicted in Fig. 11(a). Let x be a vertex lying on C and D^1 such that x has a neighbor, say x_3, belonging to $D^1 - C$, and let y and z be the vertices at distance $D - 1$ from x in C. Let w be the vertex at distance D from x in D^1.

Let the vertices $u_1, u_2, x_1, x_2, x_4, x_5, z_1, z_2, y_1, w_1, w_2$ and w_3, and the sets S^1, S^2, R^1, and R^2 be as in Fig. 11(a).

We first consider a path $P^1 = u_1 - y$. Note that P^1 cannot pass through x_5. If P^1 passed through z then x_1 would be contained in another cycle of length at most $2D - 1$, a contradiction. Therefore, P^1 passes through either w_1, a vertex from the set S^1, or a vertex from the set S^2.

Suppose that $w_1 \in P^1$. In this case P^1 must be a D-path, otherwise x would be contained in a further cycle of length at most $2D - 1$, a contradiction. Then x is contained in an additional $2D$-cycle, namely, $D^2 = u_1x_5x_1xx_3D^1w_1P^1u_1$. Note that x is saturated.

A path $P^2 = u_2 - y$ does not contain y_1 or w_1, otherwise in Γ there would be a cycle of length at most $2D - 2$ or x would be contained in a further cycle of length at most $2D$, a contradiction. Therefore, P^2 is a path of length $D - 1$ or D which goes through S^2, and forms the cycle $D^3 = u_1P^1w_1ww_2$ (a vertex in S^2) $P^2u_2x_5u_1$. The cycle D^3 is a $(2D - 1)$-cycle or a $2D$-cycle, depending on the length of P^2. Note that, since $w_1 \in D^1, D^2, D^3$, the vertex w_1 cannot be contained in any further $(2D - 1)$-cycle. See Fig. 11(b).

In this case a path $x_4 - y$ cannot go through w (that would form a new cycle of length at most $2D - 1$ containing w_1), neither can $x_4 - y$ pass through y_1 or z (otherwise x would be contained in a further cycle of length at most $2D$). Consequently, $d(x_4, y) > D$, a contradiction.

As a result, $w_1 \notin P^1$, and P^1 reaches y through either a vertex from S^1 or a vertex from S^2. We can then assume that $P^2 = u_2 - y$ also reaches y through either a vertex from S^1 or a vertex from S^2. Note that P^1 and P^2 intersect neither in S^1 nor in S^2, otherwise there would exist a cycle of length at most $2D - 2$. We may accordingly assume that P^1 goes through a

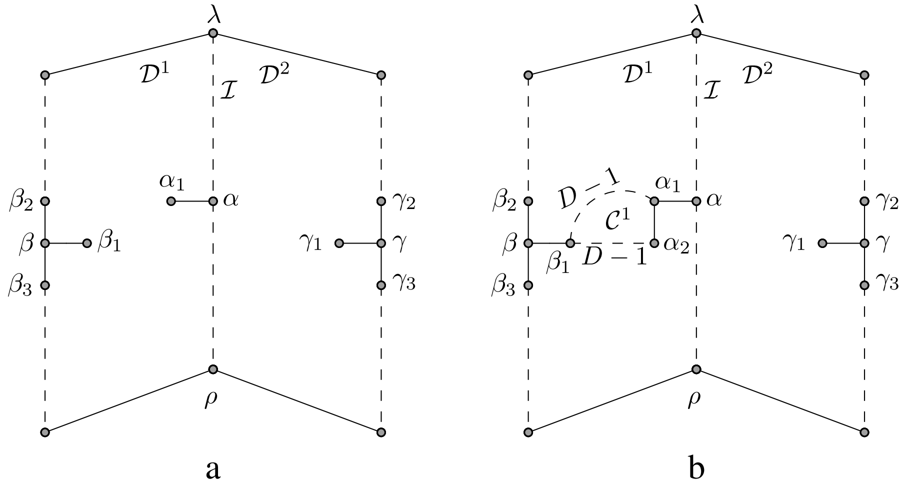

Fig. 12. Auxiliary figure for Saturation Lemma.

vertex in S^1, and that P^2 goes through a vertex in S^2. Then, P^1 must be a D-path, whereas P^2 could be either a $(D-1)$-path or a D-path.

In this way, we have obtained a new $2D$-cycle containing x and x_2, say, $D^2 = u_1P^1$ (a vertex in S^1) $w_3y_1Cx_2xx_1x_5u_1$. Note that the vertices x and x_2 are both saturated.

Finally, we consider the paths $Q^1 = z_1 - x$ and $Q^2 = z_2 - x$.

Reasoning as before, the paths Q^1 and Q^2 reach x through either a vertex from the set R^1 or a vertex from the set R^2, but both paths cannot go through the same set. Therefore, we may assume that Q^1 passes through a vertex in R^1, and that Q^2 goes through a vertex in R^2. But in this case, x_2 would be contained in a further cycle of length at most $2D$, a contradiction.

Thus, the lemma follows. □

Next we prove a lemma that will be very useful from now on.

Lemma 3.4 (*Saturation Lemma*). *Let $\mathcal{D}^1$ and $\mathcal{D}^2$ be two $2D$-cycles intersecting at a path $\mathcal{I}$ of length $D-1$. Let λ and ρ be the vertices lying on $\mathcal{I}$ at distance $D-1$ from each other. Suppose that λ is saturated and that there exists a vertex $\alpha \neq \lambda, \rho$ lying on $\mathcal{I}$ such that its neighbor α_1 not contained in $\mathcal{I}$ does not belong to any of the short cycles saturating λ. Then the following two assertions hold:*

(i) *There is at least one further short cycle $\mathcal{D}^3$ containing α, α_1 and ρ.*
(ii) *If η_1 is the neighbor not contained in $\mathcal{I}$ of a vertex $\eta \in \mathcal{I}$ such that $\eta \neq \lambda, \alpha, \rho$, then η_1 does not belong to $\mathcal{D}^3$.*

Proof. Let $\mathcal{D}^1$ and $\mathcal{D}^2$ be two $2D$-cycles intersecting at a path $\mathcal{I}$ of length $D-1$, and let λ and ρ be the vertices lying on $\mathcal{I}$ at distance $D-1$ from each other. Suppose that λ is saturated, and that there exists a vertex $\alpha \neq \lambda, \rho$ lying on $\mathcal{I}$ such that its neighbor α_1 not contained in $\mathcal{I}$ does not belong to any of the short cycles saturating λ.

Let β and γ be the vertices in $\mathcal{D}^1$ and $\mathcal{D}^2$, respectively, at distance D from α, and let the vertices $\beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2$ and γ_3 be as in Fig. 12(a).

Suppose, on the contrary, that there is no short cycle containing α, α_1 and ρ.

Consider a path $P^1 = \alpha_1 - \beta$. Note that $\alpha \notin P^1$. If P^1 went through β_2 then λ would be contained in a further short cycle, contradicting the saturation of λ. If instead P^1 passed through β_3 then ρ would belong to a short cycle that also contains α and α_1, a contradiction. Therefore, P^1 reaches β through β_1, and is D-path.

Let α_2 be the neighbor of α_1, other than α, which is not contained in P^1. Consider a path $P^2 = \alpha_2 - \beta$. Note that $\alpha_1 \notin P^2$. Then P^2 does not go through β_2, otherwise λ would be contained in a further short cycle. Neither does P^2 pass through β_3, otherwise ρ would belong to a short cycle that also contains α and α_1. Therefore, P^2 reaches β through β_1, and is D-path. Note that P^2 causes the formation of a $(2D-1)$-cycle $\mathcal{C}^1 = \alpha_1P^1\beta_1P^2\alpha_2\alpha_1$; see Fig. 12(b).

Following the same analysis as in the case of the paths P^1 and P^2, we obtain that D-paths $Q^1 = \alpha_1 - \gamma$ and $Q^2 = \alpha' - \gamma$ reach γ through γ_1, where α' is the neighbor of α_1, other than α, which is not contained in Q^1. Consequently, we obtain a new $(2D-1)$-cycle $\mathcal{C}^2 = \alpha_1Q^1\gamma_1Q^2\alpha'\alpha_1$, and thus, α_1 is contained in two $(2D-1)$-cycles $\mathcal{C}^1$ and $\mathcal{C}^2$, contradicting Proposition 3.1.

Thus, there is at least one further short cycle containing α, α_1 and ρ, and (i) follows.

Note that the second assertion follows immediately from the proof of (i). □

We are now in a position to rule out the existence of vertices of Type (iii).

Proposition 3.3. *A* $(3, D, -4)$*-graph,* $D \geq 5$*, does not contain a vertex of Type* (iii).

Proof. Let x be a vertex of Type (iii) lying on a $(2D-1)$-cycle C and two $2D$-cycles D^1 and D^2. Let x_1 be the neighbor of x such that $x_1 \in (C - D^1)$, and y_1 the vertex on D^1 at distance D from x.

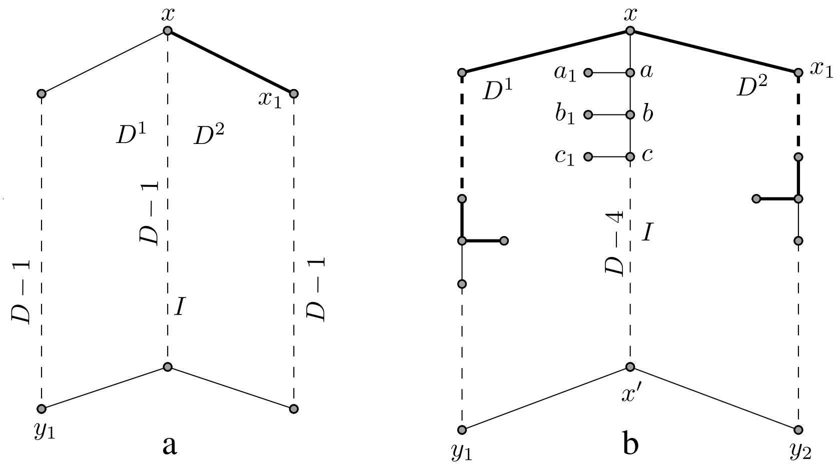

Fig. 13. Auxiliary figure for Proposition 3.3. Parts belonging to the cycle C are highlighted by a heavier line.

To prove the proposition we have prepared the following two claims.

Claim 1. *The intersection of D^1 and D^2 is a path of length $D-1$.*

Proof of Claim 1. We apply the Intersection Lemma. Mapping the vertex x to α, x_1 to α_1, y_1 to β, and mapping the $2D$-cycle D^1 to $\mathfrak{D}^1$, and the $(2D-1)$-cycle C to $\mathfrak{C}$, we see, by the Intersection Lemma (ii), that the $2D$-cycle D^2 intersects D^1 at a path I of length $D-1$. Note that the case (i) of the Intersection Lemma does not hold because of Lemma 3.3. See Fig. 13(a) (the edge $xx_1 \in C$ is highlighted by a heavier line). □

Claim 2. $C \cap I = \{x\}$.

Proof of Claim 2. We use the Intersection Lemma again. Suppose, on the contrary, that $|V(C \cap I)| > 1$. Then there are two vertices $z \neq x$ and z_1 such that $z \in (C \cap I)$, $z_1 \in (C - I)$ and $z \sim z_1$. This implies that z is a vertex of Type (iii), which belongs to C, D^1 and D^2. Therefore, z is saturated. In this case the premises of the Intersection Lemma hold again. Mapping the vertex z to α, z_1 to α_1, the vertex in D^1 at distance D from z to β, and mapping the $(2D-1)$-cycle C to $\mathfrak{C}$ and the $2D$-cycle D^1 to $\mathfrak{D}^1$, we obtain, by the Intersection Lemma (ii), that there exists an additional $2D$-cycle containing z and z_1, a contradiction. This completes the proof of the Claim. □

Let x' be the vertex on I at distance $D-1$ from x, a the neighbor of x contained in I, and a_1 the neighbor of a not contained in I. Since x is saturated, we see that the premises of the Saturation Lemma hold. Mapping the vertex x to λ, x' to ρ, a to α and a_1 to α_1, and mapping the $2D$-cycle D^1 to $\mathfrak{D}^1$ and D^2 to $\mathfrak{D}^2$, it follows that there is a further short cycle D^3 containing a, a_1 and x'.

Let $b \neq x$ be the neighbor of a contained in I, b_1 the neighbor of b not contained in I, $c \neq a$ the neighbor of b contained in I and c_1 the neighbor of c not contained in I. Since $D \geq 5$, it follows that $a \neq x, x'$, $b \neq x, x'$ and $c \neq x, x'$. See Fig. 13(b).

By Saturation Lemma (ii), we see that neither b_1 nor c_1 is contained in D^3. Therefore, we can apply the Saturation Lemma again. Mapping the vertex x to λ, x' to ρ, b to α and b_1 to α_1, and mapping the $2D$-cycle D^1 to $\mathfrak{D}^1$ and D^2 to $\mathfrak{D}^2$, it follows that there is a further short cycle D^4 containing b, b_1 and x'. Therefore, x' is saturated.

By Saturation Lemma (ii), we see that c_1 is not contained in D^4, allowing a further application of the Saturation Lemma. Mapping the vertex x to λ, x' to ρ, c to α and c_1 to α_1, and mapping the $2D$-cycle D^1 to $\mathfrak{D}^1$ and D^2 to $\mathfrak{D}^2$, it follows that there is a further short cycle D^5 containing c, c_1 and x'. But the formation of the cycle D^5 contradicts the fact that x' is saturated.

Thus, Γ does not contain a vertex of Type (iii), and the proposition follows. □

We are now in a position to prove the main result of this section.

Proof of Theorem 3.1. Combining Propositions 3.1–3.3, the theorem follows. □

As an immediate corollary of Theorem 3.1, we obtain that if a $(3, D, -4)$-graph with $D \geq 5$ exists then Conjecture 2.1 is true.

Corollary 3.1. *If a $(3, D, -4)$-graph with $D \geq 5$ exists then it must have girth $2D$.* □

4. Non-existence of $(3, D, -4)$-graphs for $D \geq 5$

From Theorem 3.1, it follows that Γ contains only vertices of Type (iv), (v) or (vi). By ruling out the existence of such vertices, we obtain the non-existence of $(3, D, -4)$-graphs for $D \geq 5$.

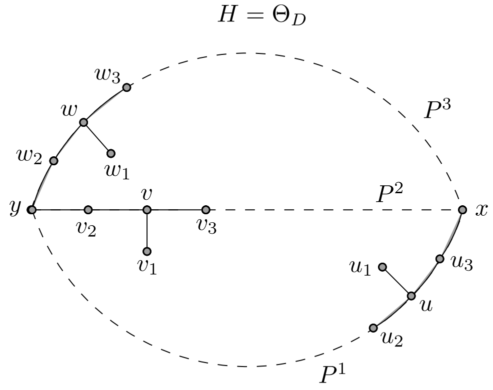

Fig. 14. Auxiliary figure for Proposition 4.1.

Proposition 4.1. *A* $(3, D, -4)$*-graph,* $D \geq 5$*, does not contain a subgraph isomorphic to* Θ_D.

Proof. Let H be a subgraph of Γ isomorphic to Θ_D, where x and y are its branch vertices.

Let u be the vertex on P^1 at distance 2 from x. Let v and w be vertices on P^2 and P^3, respectively, such that $d(u, v) = d(u, w) = D$. Let u_2 and u_3, v_2 and v_3, and w_2 and w_3 be the neighbors of u on P^1, the neighbors of v on P^2, and the neighbors of w on P^3, respectively. Let u_1, v_1 and w_1 be the neighbors of u, v and w, respectively, that do not belong to H; see Fig. 14.

First, consider a path $Q^1 = u_1 - v$. Then, Q^1 does not go through u, v_2 or v_3, otherwise there would exist a cycle of length at most $2D - 1$ in Γ. Therefore, Q^1 goes through v_1. Suppose that Q^1 is a D-path.

Let $r \neq u$ and $s \neq v$ be the neighbors of u_1 and v_1, respectively, that do not belong to Q^1.

A path $Q^2 = r - v$ does not pass through v_1, otherwise there would exist a cycle of length at most $2D - 1$ in Γ. Then Q^2 passes through either v_2 or v_3, and should be a D-path, otherwise there would exist a cycle of length at most $2D - 1$ in Γ. Analogously, a path $Q^3 = s - u$ is a D-path, and goes through either u_2 or u_3.

Note that the paths Q^2 and Q^3 form part of two $2D$-cycles, denoted by D^1 and D^2, which contain either x or y. The cycle D^1 is either $uu_3P^1xP^2v_3Q^2ru_1u$ or $uu_2P^1yP^2v_2Q^2ru_1u$, while the cycle D^2 is either $u_2P^1yP^2v_2vv_1sQ^3u_2$ or $u_3P^1xP^2v_3vv_1sQ^3u_3$.

Note that the cycles D^1 and D^2 do not contain w_1, w_2 or w_3.

Let us further suppose that a path $T^1 = u_1 - w$ is a D-path. By following the same reasoning as in the case of the paths Q^1, Q^2 and Q^3, we obtain that T^1 passes through w_1, and that there are two further $2D$-cycles, say, D_3 and D_4, containing either x or y. In this case the vertices x and y are of Type (v), and, by Proposition 2.2(v), x and y are saturated.

Since $D \geq 5$, we can find another vertex on P^1, say z, different from x, u_3, u, u_2 or y. Let p be the vertex on P^2 such that $d(z, p) = D$, and z_1 the neighbor of z that does not belong to H. Note that z_1 does not belong to D^1, D^2, D^3 or D^4.

We consider paths $R^1 = z_1 - p$ and $R^2 = q - p$, where $q \neq z$ is the neighbor of z_1 not contained in R^1. Note that the paths R^1 and R^2 must be D-paths, otherwise x and y would be contained in a further short cycle. Then we obtain a new $2D$-cycle containing z_1 and either x or y, a contradiction to Proposition 2.2(v). Therefore, T^1 is a $(D - 1)$-path.

If the path T^1 is a $(D - 1)$-path then there are two new $2D$-cycles containing w_1, different from D^1 or D^2, such that one contains x, and the other contains y. Therefore, as before, vertices x and y are both of Type (v), and are saturated. We consider again the aforementioned vertices z, z_1, p and q, and the paths $R^1 = z_1 - p$ and $q - p$. In this case z_1 does not belong to any of the cycles involving x or y. As a result, we obtain a new $2D$-cycle containing z_1 and either x or y, a contradiction.

Thus, Q^1 is a $(D - 1)$-path and so is T^1.

By analogy, if the paths Q^1 and T^1 are $(D - 1)$-paths then there are four new $2D$-cycles such that two of them contain x, and the other two contain y. Therefore, x and y cannot be contained in any additional short cycle. However, we can again use the vertices z, z_1, p and q, and the paths $R^1 = z_1 - p$ and $q - p$ to find a further $2D$-cycle containing z_1 and either x or y, contradicting Proposition 2.2(v). $\square$

Corollary 4.1. *A* $(3, D, -4)$*-graph,* $D \geq 5$*, does not contain a vertex of Type* (iv)*or* (v). $\square$

Proposition 4.2. *A* $(3, D, -4)$*-graph,* $D \geq 5$*, does not contain a vertex of Type* (vi).

Proof. Let x be a vertex of Γ. Then x is a vertex of Type (vi). Let D^1 be one of the $2D$-cycles on which x lies, and y_1 the vertex in D^1 at distance D from x. Furthermore, we denote by w_2 the neighbor of x not contained in D^1.

In this case, by the Intersection Lemma, mapping the vertex x to α, w_2 to α_1, y_1 to β, and mapping the $2D$-cycle D^1 to $\mathcal{D}^1$ (w_2 belongs to no $(2D - 1)$-cycle), we see that there exists another $2D$-cycle containing x and w_2, say D^2, such that the intersection of D^1 and D^2 is a path of length $D - 1$.

We prove this proposition by reasoning in the same way as in the proof of Proposition 4.1.

Let u and w be the vertices in $D^1 - D^2$ and in $D^2 - D^1$, respectively, at distance 2 from x. Let v be the vertex in $D^1 \cap D^2$ such that $d(u, v) = d(w, v) = D$. Let v_3 be the vertex in $D^1 \cap D^2$ at distance $D - 1$ from x. Finally, let the vertices u_1, u_2, u_3, v_1, v_2, w_1, w_3 and y_2 be as in Fig. 15.

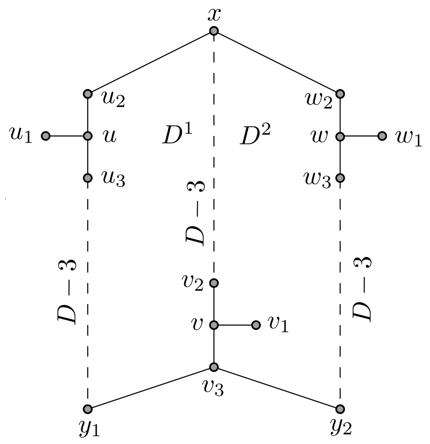

Fig. 15. Auxiliary figure for Proposition 4.2.

Consider a path $P^1 = u_1 - v$. Then P^1 does not go through u, v_2 or v_3, otherwise there would exist a cycle of length at most $2D-1$ in Γ. Therefore, P_1 goes through v_1. If P^1 was a $(D-1)$-path then both u and v would be branch vertices of a Θ_D, contradicting Proposition 4.1. As a result, P^1 is a D-path.

Let $r \neq u$ and $s \neq v$ be the neighbors of u_1 and v_1, respectively, that do not belong to P^1.

A path $P^2 = r - v$ does not pass through v_1, otherwise there would exist a cycle of length at most $2D-1$ in Γ. Then P^2 intersects D^1 at v and either v_2 or v_3, and should be a D-path, otherwise there would exist a cycle of length at most $2D-1$ in Γ. Analogously, a path $P^3 = s - u$ is a D-path, and intersects D^1 at u and either u_2 or u_3.

Note that the paths P^2 and P^3 form part of two $2D$-cycles, denoted by D^3 and D^4, which contain either x or v_3. The cycle D^3 is either $uu_3D^1y_1v_3y_2P^2ru_1u$ or $uu_2xD^1v_2P^2ru_1u$, while the cycle D^4 is either $u_2xD^1v_2vv_1sP^3u_2$ or $u_3D^1y_1v_3vv_1sP^3u_3$.

Note that the cycles D^3 and D^4 do not contain w_1, w_2 or w_3.

Consider a path $T^1 = w_1 - v$. By following the same reasoning as in the case of the paths P^1, P^3 and P^3, we obtain that T^1 passes through v_1, that T^1 is a D-path and that there are two further $2D$-cycles, say, D_5 and D_6, containing either x or v_3. In this case the vertices x and v_3 are saturated.

Since $D \geq 5$, we can find another vertex in $D^1 - D^2$, say z, different from u_2, u, u_3 or y_1. Let p be the vertex in $D^1 \cap D^2$ such that $d(z, p) = D$, and z_1 the neighbor of z that does not belong to D^1. Note that z_1 does not belong to D^3, D^4, D^5 or D^6.

Then, by considering the paths $R^1 = z_1 - p$ and $q - p$, where $q \neq z$ is the neighbor of z_1 not contained in R^1, we obtain a new $2D$-cycle containing z_1 and either x or v_3, a contradiction to Proposition 2.2(vi).

Thus in a $(3, D, -4)$-graph with $D \geq 5$ there exists no vertex of Type (vi), and the proposition follows. $\square$

Combining the results of Theorem 3.1, Corollary 4.1 and Proposition 4.2, we obtain the main result of this paper (Theorem 4.1), thus completing the catalogue of $(3, D, -4)$-graphs with $D \geq 2$.

Theorem 4.1. *For $D \geq 5$ there are no $(3, D, -4)$-graphs.* $\square$

5. Conclusions

In this paper, by proving the non-existence of $(3, D, -4)$-graphs with $D \geq 5$, we have completed the census of $(3, D, -4)$-graphs with $D \geq 2$ and $\epsilon \leq 4$, which is summarized below.

Catalogue of $(3, D, 0)$-**graphs with** $D \geq 2$. With the exceptions of the complete graph on 4 vertices and the Petersen graph, there is no cubic Moore graph.

Catalogue of $(3, D, -2)$-**graphs with** $D \geq 2$. There are only three non-isomorphic $(3, D, -2)$-graphs with $D \geq 2$; all shown in Fig. 1.

Catalogue of $(3, D, -4)$-**graphs with** $D \geq 2$. For diameter 2 there exist two regular (graphs (a) and (b) in Fig. 2) and three non-regular $(3, 2, -4)$-graphs (graphs (c), (d) and (e) in Fig. 2). When the diameter is 3, there is a unique $(3, 3, -4)$-graph; see Fig. 3. The results of this paper, combined with [11], assert that there are no $(3, D, -4)$-graphs with $D \geq 4$.

Contribution to the degree/diameter problem

Our result also improves the upper bound on $N_{3,D}$, $D \geq 5$, implying that any maximal graph of maximum degree 3 and diameter $D \geq 5$ must have order at most $M_{3,D} - 6$.

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References

[1] E. Bannai, T. Ito, On finite Moore graphs, Journal of Mathematical Sciences, The University of Tokyo 20 (1973) 191–208.
[2] J.C. Bermond, C. Delorme, G. Farhi, Large graphs with given degree and diameter II, Journal of Combinatorial Theory, Series B 36 (1) (1984) 32–48. doi:10.1016/0095-8956(84)90012-1.
[3] N.L. Biggs, Algebraic Graph Theory, 2nd ed., Cambridge University Press, Cambridge, 1993.
[4] D. Buset, Maximal cubic graphs with diameter 4, Discrete Applied Mathematics 101 (1–3) (2000) 53–61. doi:10.1016/S0166-218X(99)00204-8.
[5] R.M. Damerell, On Moore graphs, Proceedings of the Cambridge Philosophical Society 74 (1973) 227–236.
[6] R. Diestel, Graph Theory, 3rd ed., in: Graduate Texts in Mathematics, vol. 173, Springer-Verlag, New York, 2005.
[7] K.W. Doty, New designs for dense processor interconnection networks, IEEE Transactions on Computers C-33 (5) (1984) 447–450.
[8] I.A. Faradžev, Constructive enumeration of combinatorial objects, Problèmes combinatoires et théorie des graphes, Colloq. Internat. CNRS, Paris 260 (1978) 131–135.
[9] A.J. Hoffman, R.R. Singleton, On Moore graphs with diameter 2 and 3, IBM Journal of Research and Development 4 (1960) 497–504.
[10] L.K. Jørgensen, Diameters of cubic graphs, Discrete Applied Mathematics 37–38 (1992) 347–351. doi:10.1016/0166-218X(92)90144-Y.
[11] L.K. Jørgensen, Nonexistence of certain cubic graphs with small diameters, Discrete Mathematics 114 (1993) 265–273. doi:10.1016/0012-365X(93)90371-Y.
[12] B.D. McKay, G.F. Royle, Constructing the cubic graphs on up to 20 vertices, Ars Combinatoria 21A (1986) 129–140.
[13] M. Miller, J. Širáň, Moore graphs and beyond: A survey of the degree/diameter problem, The Electronic Journal of Combinatorics (2005) 1–61. Dynamic survey DS14.
[14] G. Pineda-Villavicencio, M. Miller, On graphs of maximum degree 3 and defect 4, Journal of Combinatorial Mathematics and Combinatorial Computing 65 (2008) 25–31.
[15] C. von Conta, Torus and other networks as communication networks with up to some hundred points, IEEE Transactions on Computers C 32 (7) (1983) 657–666.
[16] J. Xu, Topological Structure and Analysis of Interconnection Networks, in: Network Theory and Applications, vol. 7, Kluwer Academic Publishers, Dordrecht, 2001.