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Research Article

Global Behavior of Two Families of Nonlinear Symmetric Difference Equations

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We mainly investigate the global asymptotic stability and exponential convergence of positive solutions to two families of higher-order difference equations, one of which was recently studied in Stević's paper (2010). A new concise proof is given to a quite recent result by Stević and analogous parallel result of the other inverse equation, which extend related results of Aloqeili (2009), Berenhaut and Stević (2007), and Liao et al. (2009).

1. Introduction

The interest in investigating rational difference equations has a long history; for instance, see [1–24] and the references cited therein. More generally, it is meaningful to study not only rational recursive equations but also those with powers of arbitrary positive degrees.

For instance, at many conferences, Stević proposed to study the behavior of positive solutions of the following generic difference equation (see also [25]):

$$x_n = A + \frac{x_{n-m}^p}{x_{n-k}^q}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $A, p, q > 0$, and $k, m \in \mathbb{N}$, $k \neq m$. For some recent results in this area, see [26–32] and the references therein.

By a useful transformation method from [4], the authors of [3] confirmed that the unique positive equilibrium of the rational recursive equation

$$y_n = \frac{y_{n-k} + y_{n-m}}{1 + y_{n-k}y_{n-m}}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where $1 \leq k < m$, is globally asymptotically stable for all solutions with positive initial values. In the meantime, they also remarked that the global asymptotic stability for the unique equilibrium of the difference equation

$$y_n = \frac{1 + y_{n-k}y_{n-m}}{y_{n-k} + y_{n-m}}, \quad n \in \mathbb{N}_0, \tag{1.3}$$

can be shown through analogous calculations. Some particular cases of (1.3) had already been considered in [12, 13].

In [3] were proposed the following two conjectures.

Conjecture 1.1. *Suppose that $1 \leq k < l < m$ and that $\{y_n\}$ satisfies*

$$y_n = \frac{y_{n-k}y_{n-l}y_{n-m} + y_{n-k} + y_{n-l} + y_{n-m}}{y_{n-k}y_{n-l} + y_{n-k}y_{n-m} + y_{n-l}y_{n-m} + 1}, \quad n \in \mathbb{N}_0, \tag{1.4}$$

with positive initial values. Then, the sequence $\{y_n\}$ converges to the unique positive equilibrium point $\overline{y} = 1$.

Some special cases of (1.4) had been studied by Li [9, 10] with a semicycle analysis method, which is useful for lower-order difference equations but tedious and complicated to some extent(see the explanation in [33]). Finally, Conjecture 1.1 was also confirmed in [2] with the similar transformation method used in [3, 4]. However, it is somewhat harder to prove the following conjecture in the same way.

Conjecture 1.2. *Assume that q is odd and $1 \leq k_1 < k_2 < \cdots < k_q$, and define $S = \{1, 2, \ldots, q\}$. If $\{y_n\}$ satisfies*

$$y_n = \frac{f_1\left(y_{n-k_1}, y_{n-k_2}, \ldots, y_{n-k_q}\right)}{f_2\left(y_{n-k_1}, y_{n-k_2}, \ldots, y_{n-k_q}\right)}, \quad n \in \mathbb{N}_0, \tag{1.5}$$

with $y_{-k_q}, y_{-k_q+1}, \ldots, y_{-1} \in (0, +\infty)$, where

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_q) &= \sum_{\substack{i=1 \\ i \text{ odd}}}^{q} \sum_{\substack{\{t_1, t_2, \ldots, t_i\} \subset S \\ t_1 < t_2 < \cdots < t_i}} x_{t_1}x_{t_2}\cdots x_{t_i}, \\ f_2(x_1, x_2, \ldots, x_q) &= 1 + \sum_{\substack{i=2 \\ i \text{ even}}}^{q-1} \sum_{\substack{\{t_1, t_2, \ldots, t_i\} \subset S \\ t_1 < t_2 < \cdots < t_i}} x_{t_1}x_{t_2}\cdots x_{t_i}. \end{aligned} \tag{1.6}$$

Then the sequence $\{y_n\}$ converges to the unique positive equilibrium point $\overline{y} = 1$.

Next, we present two definitions as defined in [1].

Definition 1.3. A function of n variables is symmetric if it is invariant under any permutation of its variables. That is, a function $\varphi(x_1, \ldots, x_n)$ is called symmetric if

$$\varphi(x_1, \ldots, x_n) = \varphi\left(x_{\pi(1)}, \ldots, x_{\pi(n)}\right), \tag{1.7}$$

where $\pi(i)$ is any permutation of the numbers $\{1, 2, \ldots, n\}$.

Definition 1.4. The kth elementary symmetric function σ_k of variables $x_1, \ldots, x_n$, where $k \in \{1, 2, \ldots, n\}$ is defined by

$$\sigma_k(x_1, \ldots, x_n) = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}, \tag{1.8}$$

where the sum is taken over all C_k^n choices of the indices $i_1, \ldots, i_k$ from the set of integers $\{1, 2, \ldots, n\}$.

Obviously, the functions f_1, f_2 defined by (1.6) and (1.7) are symmetric and can be rewritten as

$$f_1(x_1, x_2, \ldots, x_q) = \sum_{\substack{i=1 \\ i \text{ odd}}}^{q} \sigma_i, \qquad f_2(x_1, x_2, \ldots, x_q) = 1 + \sum_{\substack{i=2 \\ i \text{ even}}}^{q-1} \sigma_i. \tag{1.9}$$

In this paper, we give a new proof of a quite recent result by Stević in [34] where he, among others, studied the stability of one of the following two difference equations, which are dual:

$$y_n = \frac{f_2\left(y_{n-k_1}^r, y_{n-k_2}^r, \ldots, y_{n-k_q}^r\right)}{f_1\left(y_{n-k_1}^r, y_{n-k_2}^r, \ldots, y_{n-k_q}^r\right)}, \quad n \in \mathbb{N}_0, \tag{1.10}$$

$$y_n = \frac{f_1\left(y_{n-k_1}^r, y_{n-k_2}^r, \ldots, y_{n-k_q}^r\right)}{f_2\left(y_{n-k_1}^r, y_{n-k_2}^r, \ldots, y_{n-k_q}^r\right)}, \quad n \in \mathbb{N}_0, \tag{1.11}$$

where $3 \le q \in \mathbb{N}$ is odd, $r \in (0, 1]$ and $1 \le k_1 < k_2 < \cdots < k_q$.

Apparently, Equation (1.11) is the generic form of (1.2), (1.4), and (1.5).

In [6, 18] the authors proved that the main results in some of papers [9–12] are direct consequences of a result confirmed by Kruse and Nesemann [35]. For example, in [6] was showed that the main result in [13] is also a consequence of Corollary 3 in [35]. On basis of these works, in 2008, Aloqeili [1] confirmed Conjecture 1.2 in the same way.

Later, Liao et al. [14] proved Conjecture 1.2 by using a new approach. They used a sort of "frame sequences" method(the notion suggested by Stević), which has been widely used in [5, 7, 18, 36–41]. Through careful analysis, we find that the method used in [14] can be further simplified and applied in proving Stević's result in a more concise and interesting way. Namely, we give a new proof of the following result, which generalizes related results in [1, 2, 9, 10, 14].

Theorem 1.5. *Assume that* $y_{-k_q}, y_{-k_q+1}, \dots, y_{-1} \in (0, +\infty), r \in (0, 1], 3 \le q \in \mathbb{N}$ *is odd and positive integers* $k_1, k_2, \dots, k_q$ *are satisfying* $1 \le k_1 < k_2 < \cdots < k_q$*. Then*

(1) *the unique positive equilibrium point* $\overline{y} = 1$ *of* (1.10) *is globally asymptotically stable;*

(2) *the unique positive equilibrium point* $\overline{y} = 1$ *of* (1.11) *is globally asymptotically stable.*

2. Auxiliary Results and Notation

In this section, we will introduce some useful notation and lemmas. Consider the following notation(for similar ones see [14]), which play an important role in the paper:

$$\alpha(x_1, x_2, \dots, x_q) = \frac{1}{2}\left[\prod_{i=1}^{q}(x_i^r + 1) - \prod_{i=1}^{q}(x_i^r - 1)\right],$$
$$\beta(x_1, x_2, \dots, x_q) = \frac{1}{2}\left[\prod_{i=1}^{q}(x_i^r + 1) + \prod_{i=1}^{q}(x_i^r - 1)\right]. \tag{2.1}$$

Employing α and β, define a mapping $\Phi_1 : \mathbb{R}_+^q \to \mathbb{R}$ as follows:

$$\Phi_1(x_1, x_2, \dots, x_q) = \frac{\alpha(x_1, \dots, x_q)}{\beta(x_1, \dots, x_q)} = \frac{\prod_{i=1}^{q}(x_i^r + 1) - \prod_{i=1}^{q}(x_i^r - 1)}{\prod_{i=1}^{q}(x_i^r + 1) + \prod_{i=1}^{q}(x_i^r - 1)}. \tag{2.2}$$

Then (1.10) can be rewritten as

$$y_n = \frac{\prod_{i=1}^{q}\left(y_{n-k_i}^r + 1\right) - \prod_{i=1}^{q}\left(y_{n-k_i}^r - 1\right)}{\prod_{i=1}^{q}\left(y_{n-k_i}^r + 1\right) + \prod_{i=1}^{q}\left(y_{n-k_i}^r - 1\right)}, \quad n \in \mathbb{N}_0, \tag{2.3}$$

or

$$y_n = \Phi_1\left(y_{n-k_1}, y_{n-k_2}, \dots, y_{n-k_q}\right), \quad n \in \mathbb{N}_0, \tag{2.4}$$

with $3 \le q \in \mathbb{N}$ being odd, and $r \in \mathbb{R}_+$.

By the notation defined by (2.1), define the other function $\Phi_2 : \mathbb{R}_+^q \to \mathbb{R}$ such that:

$$\Phi_2(x_1, x_2, \ldots, x_q) = \frac{\beta(x_1, \ldots, x_q)}{\alpha(x_1, \ldots, x_q)} = \frac{\prod_{i=1}^q (x_i^r + 1) + \prod_{i=1}^q (x_i^r - 1)}{\prod_{i=1}^q (x_i^r + 1) - \prod_{i=1}^q (x_i^r - 1)}. \tag{2.5}$$

Then (1.11) can be rewritten as

$$y_n = \frac{\prod_{i=1}^q \left(y_{n-k_i}^r + 1\right) + \prod_{i=1}^q \left(y_{n-k_i}^r - 1\right)}{\prod_{i=1}^q \left(y_{n-k_i}^r + 1\right) - \prod_{i=1}^q \left(y_{n-k_i}^r - 1\right)}, \quad n \in \mathbb{N}_0, \tag{2.6}$$

or

$$y_n = \Phi_2\left(y_{n-k_1}, y_{n-k_2}, \ldots, y_{n-k_q}\right), \quad n \in \mathbb{N}_0, \tag{2.7}$$

with $3 \leq q \in \mathbb{N}$ being odd, and $r \in \mathbb{R}_+$.

Lemma 2.1. *If $r \in (0,1]$, then both* (2.4) *and* (2.7) *have the unique positive equilibrium point $\overline{y} = 1$.*

Proof. Suppose that $\lambda_1 > 0$ is an equilibrium of (2.4), then

$$\lambda_1 = \Phi_1(\lambda_1, \ldots, \lambda_1) = \frac{(\lambda_1^r + 1)^q - (\lambda_1^r - 1)^q}{(\lambda_1^r + 1)^q + (\lambda_1^r - 1)^q}, \tag{2.8}$$

which implies

$$(1 - \lambda_1)(\lambda_1^r + 1)^q = (\lambda_1 + 1)(\lambda_1^r - 1)^q. \tag{2.9}$$

Obviously $\lambda_1 = 1$, due to the different signs of both sides of the last equality for the case $\lambda_1 \neq 1$. Likewise, let $\lambda_2 > 0$ be an equilibrium of (2.7); then

$$\lambda_2 = \Phi_2(\lambda_2, \ldots, \lambda_2) = \frac{(\lambda_2^r + 1)^q + (\lambda_2^r - 1)^q}{(\lambda_2^r + 1)^q - (\lambda_2^r - 1)^q}, \tag{2.10}$$

which indicates

$$(\lambda_2 - 1)(\lambda_2^r + 1)^q = (\lambda_2 + 1)(\lambda_2^r - 1)^q. \tag{2.11}$$

Assume that $\lambda_2 \neq 1$. If $\lambda_2 > 1$, then by the monotonicity of the map $h(x) = (x+1)/(x-1)$, $x > 1$ we have that

$$\frac{\lambda_2 + 1}{\lambda_2 - 1} \leq \frac{\lambda_2^r + 1}{\lambda_2^r - 1} < \left(\frac{\lambda_2^r + 1}{\lambda_2^r - 1}\right)^q, \tag{2.12}$$

which contradicts (2.11).Similarly, if $0 < \lambda_2 < 1$, then by the monotonicity of the function $l(x) = (1+x)/(1-x)$, $x \in (0,1)$, we have that

$$\frac{1+\lambda_2}{1-\lambda_2} \leq \frac{1+\lambda_2^r}{1-\lambda_2^r} < \left(\frac{1+\lambda_2^r}{1-\lambda_2^r}\right)^q, \tag{2.13}$$

which also contradicts (2.11). Thus $\lambda_2 = 1$.

The proof is complete. □

Lemma 2.2. *(1) Let Φ_1 be defined by (2.2); then Φ_1 is monotonically increasing in x_j if and only if $\prod_{i=1,i\neq j}^q (x_i - 1) < 0$, and monotonically decreasing in x_j if and only if $\prod_{i=1,i\neq j}^q (x_i - 1) > 0$, for $j = 1, 2, \ldots, q$.*

(2) Let Φ_2 be defined by (2.5), then Φ_2 is monotonically decreasing in x_j if and only if $\prod_{i=1,i\neq j}^q (x_i - 1) < 0$, and monotonically increasing in x_j if and only if $\prod_{i=1,i\neq j}^q (x_i - 1) > 0$, for $j = 1, 2, \ldots, q$.

Proof. The results follow directly from the facts below:

$$\begin{gathered}
\frac{\partial \Phi_1}{\partial x_j} = \frac{-r x_j^{r-1} \prod_{i=1,i\neq j}^q (x_i^{2r} - 1)}{\left[\beta(x_1, x_2, \ldots, x_q)\right]^2} \begin{cases} < 0, & \prod\limits_{i=1,i\neq j}^q (x_i - 1) > 0, \\ > 0, & \prod\limits_{i=1,i\neq j}^q (x_i - 1) < 0, \end{cases} \quad j \in \{1, \ldots, q\}, \\
\frac{\partial \Phi_2}{\partial x_j} = \frac{r x_j^{r-1} \prod_{i=1,i\neq j}^q (x_i^{2r} - 1)}{\left[\alpha(x_1, x_2, \ldots, x_q)\right]^2} \begin{cases} > 0, & \prod\limits_{i=1,i\neq j}^q (x_i - 1) > 0, \\ < 0, & \prod\limits_{i=1,i\neq j}^q (x_i - 1) < 0, \end{cases} \quad j \in \{1, \ldots, q\}.
\end{gathered} \tag{2.14}$$

□

Remark 2.3. The second statement (i.e., (2)) in Lemma 2.2 can also be found in Stević's paper [34] (see Lemma 1 and Corollary 1).

For $r \in \mathbb{R}^+, 3 \leq q \in \mathbb{N}$ odd, define a map $\Psi : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\Psi(x) = \frac{(1+x^r)^q - (1-x^r)^q}{(1+x^r)^q + (1-x^r)^q}, \tag{2.15}$$

which has the following simple property:

$$\Psi\left(\frac{1}{x}\right) = \frac{1}{\Psi(x)}, \quad x \in \mathbb{R}_+. \tag{2.16}$$

Lemma 2.4. *Suppose that $0 < \xi < 1$, and let $\Gamma = \Psi(\xi), \Phi \in \{\Phi_1, \Phi_2\}$. If $x_1, x_2, \ldots, x_q \in [\xi, 1/\xi]$, then*

$$\Gamma \leq \Phi(x_1, x_2, \ldots, x_q) \leq \frac{1}{\Gamma}. \tag{2.17}$$

Proof. Since $\Phi(x_1, x_2, \ldots, x_q)$ is symmetric in $x_1, x_2, \ldots, x_q$, without loss of generality, we suppose that $\xi \leq x_1 \leq x_2 \leq \cdots \leq x_q \leq 1/\xi$. If there exists $j \in \{1, \ldots, q\}$ such that $x_j = 1$, then by (2.2) and (2.5) we can easily get that $\Phi(x_1, x_2, \ldots, x_q) = 1$. Thus, assume $x_j \neq 1$ for all $j \in \{1, \ldots, q\}$.

Then we have the following $q + 1$ cases to consider:

$$\begin{aligned}
&(1) \quad \xi \leq x_1 \leq \cdots \leq x_{q-1} \leq x_q < 1 < \frac{1}{\xi};\\
&(2) \quad \xi \leq x_1 \leq \cdots \leq x_{q-1} < 1 < x_q \leq \frac{1}{\xi};\\
&(3) \quad \xi \leq x_1 \leq \cdots < 1 < x_{q-1} \leq x_q \leq \frac{1}{\xi};\\
&\quad\vdots\\
&(q) \quad \xi \leq x_1 < 1 < \cdots \leq x_{q-1} \leq x_q \leq \frac{1}{\xi};\\
&(q+1) \quad \xi < 1 < x_1 \leq \cdots \leq x_{q-1} \leq x_q \leq \frac{1}{\xi}.
\end{aligned} \tag{2.18}$$

By Lemma 2.2, for the above cases, we have that

$$\begin{aligned}
&(1) \quad 1 < \Phi_1(x_1, \ldots, x_{q-1}, x_q) \leq \frac{1}{\Gamma};\\
&(2) \quad \Gamma \leq \Phi_1(x_1, \ldots, x_{q-1}, x_q) < 1;\\
&(3) \quad 1 < \Phi_1(x_1, \ldots, x_{q-1}, x_q) \leq \frac{1}{\Gamma};\\
&\quad\vdots\\
&(q) \quad 1 < \Phi_1(x_1, \ldots, x_{q-1}, x_q) \leq \frac{1}{\Gamma};\\
&(q+1) \quad \Gamma \leq \Phi_1(x_1, \ldots, x_{q-1}, x_q) < 1.
\end{aligned} \tag{2.19}$$

Obviously, $\Gamma \leq \Phi_1(x_1, x_2, \ldots, x_q) \leq 1/\Gamma$ follows directly from the above inequalities.

The proof of the case $\Phi = \Phi_2$ is analogous and hence omitted. □

Lemma 2.5. *Suppose that $0 < r \le 1, \xi \in (0,1)$ is fixed and let $\Gamma = \Psi(\xi)$. Then we have*

$$\Gamma > \xi. \tag{2.20}$$

Proof. By the monotonicity of the function $h(x) = (1+x)/(1-x), x \in (0,1)$, we have that

$$\frac{1+\xi}{1-\xi} \le \frac{1+\xi^r}{1-\xi^r} < \left(\frac{1+\xi^r}{1-\xi^r}\right)^q, \tag{2.21}$$

which implies

$$(1-\xi)(1+\xi^r)^q > (1+\xi)(1-\xi^r)^q. \tag{2.22}$$

Therefore, $((1+\xi^r)^q - (1-\xi^r)^q)/((1+\xi^r)^q + (1-\xi^r)^q) > \xi$, that is, $\Gamma > \xi$.
The proof is complete. □

The following corollary follows directly from Lemma 2.4 and Lemma 2.5.

Corollary 2.6. *Assume that $0 < \xi < 1$. If any positive solution $(y_n)_{n=-k_q}^{+\infty}$ to (2.4) or (2.7) has the initial values*

$$y_{-k_q}, y_{-k_q+1}, \ldots, y_{-1} \in \left[\xi, \frac{1}{\xi}\right], \tag{2.23}$$

then we have $y_n \in [\xi, 1/\xi]$, for $n \in \mathbb{N}_0$.

Define two sequences $(\xi_i)_{i=0}^{+\infty}$ and $(\eta_i)_{i=0}^{+\infty}$ as follows:

$$\xi_{i+1} = \Psi(\xi_i), \quad \eta_{i+1} = \Psi(\eta_i), \quad i \in \mathbb{N}_0, \tag{2.24}$$

with initial values $\xi_0, \eta_0 > 0$.

Lemma 2.7. *For the sequences $(\xi_i)_{i=0}^{+\infty}$ and $(\eta_i)_{i=0}^{+\infty}$ defined by (2.24), if $0 < \xi_0 < 1$, and $\xi_0\eta_0 = 1$, then*

$$\lim_{i\to\infty} \xi_i = \lim_{i\to\infty} \eta_i = 1. \tag{2.25}$$

Proof. Inductively, we can simply obtain that $0 < \xi_i < 1 < \eta_i < +\infty,\ i \in \mathbb{N}_0$. Through simple calculations, by (2.16), we have that

$$\xi_i \eta_i \equiv 1, \quad \forall i \in \mathbb{N}_0. \tag{2.26}$$

Therefore by Lemma 2.4 and Lemma 2.5, we get that

$$0 < \xi_i \le \xi_{i+1} < 1 < \eta_{i+1} \le \eta_i, \quad i \in \mathbb{N}_0, \tag{2.27}$$

which implies that the sequences $(\xi_i)_{i=0}^{+\infty}$ and $(\eta_i)_{i=0}^{+\infty}$ converge to some limits (denoted by ξ^* and η^*, resp.), that is,

$$\xi^* = \lim_{i\to+\infty} \xi_i \in [\xi_0, 1], \qquad \eta^* = \lim_{i\to+\infty} \eta_i \in [1, \eta_0]. \tag{2.28}$$

By taking limits on both sides of the first identity of (2.24), we get

$$\xi^* = \frac{(1+(\xi^*)^r)^q - (1-(\xi^*)^r)^q}{(1+(\xi^*)^r)^q + (1-(\xi^*)^r)^q}, \tag{2.29}$$

which implies

$$(1-\xi^*)(1+(\xi^*)^r)^q = (1+\xi^*)(1-(\xi^*)^r)^q. \tag{2.30}$$

Suppose that $\xi^* \neq 1$; then by the monotonicity of the function $f(x) = (1+x)/(1-x)$, $x \in (0,1)$, we have that

$$\frac{1+\xi^*}{1-\xi^*} \leq \frac{1+(\xi^*)^r}{1-(\xi^*)^r} < \left(\frac{1+(\xi^*)^r}{1-(\xi^*)^r}\right)^q, \tag{2.31}$$

which contradicts (2.29). Hence, we have that $\xi^* = 1$ and then obviously it follows by (2.26) and (2.28) that $\xi^* = \eta^* = 1$.

The proof is complete. □

3. Stability

In this section, we give a new, concise and clear proof of Stević's Theorem 1.5, by the lemmas in Section 2.

Proof of Theorem 1.5. Employing Lemma 2.2, the linearized equations of (2.4) and (2.7) about the equilibrium $\overline{y} = 1$ are both

$$z_n = 0 \cdot z_{n-k_1} + 0 \cdot z_{n-k_2} + \cdots + 0 \cdot z_{n-k_q} \equiv 0, \quad n \in \mathbb{N}_0. \tag{3.1}$$

Then by the Linearized Stability Theorem, $\overline{y} = 1$ is locally stable.

Thus it suffices to confirm that $\overline{y} = 1$ is also a global attractor for all positive solutions of (2.4) and (2.7).

Let $(y_n)_{n=-k_q}^{+\infty}$ be a positive solution to (2.4) or (2.7) with initial values

$$y_{-k_q}, y_{-k_q+1}, \ldots, y_{-1} \in (0, +\infty). \tag{3.2}$$

We need to prove that $\lim_{n\to\infty} y_n = 1$.

Apparently, there exists $\xi_0 \in (0,1)$ such that

$$y_i \in [\xi_0, \eta_0], \quad i = -k_q, -k_q + 1, \dots, -1, \tag{3.3}$$

where $\eta_0 = 1/\xi_0$. Employing Corollary 2.6, we have

$$y_n \in [\xi_0, \eta_0], \quad n = -k_q, -k_q + 1, \dots. \tag{3.4}$$

Let sequences $(\xi_i)_{i=0}^{+\infty}$ and $(\eta_i)_{i=0}^{+\infty}$ be defined by (2.24). Let $\Phi \in \{\Phi_1, \Phi_2\}$; then in light of Lemma 2.4, (3.4), and (2.26), we get

$$\Psi(\xi_0) \le \Phi\left(y_{n-k_1}, y_{n-k_2}, \dots, y_{n-k_q}\right) \le \frac{1}{\Psi(\xi_0)} = \Psi(\eta_0), \quad n \in \mathbb{N}_0. \tag{3.5}$$

That is,

$$y_n \in [\xi_1, \eta_1], \quad n \in \mathbb{N}_0. \tag{3.6}$$

In view of (3.6), (2.26) and Lemma 2.4, we have that

$$\Psi(\xi_1) \le \Phi\left(y_{n-k_1}, y_{n-k_2}, \dots, y_{n-k_q}\right) \le \frac{1}{\Psi(\xi_1)} = \Psi(\eta_1), \quad n \ge k_q. \tag{3.7}$$

That is,

$$y_n \in [\xi_2, \eta_2], \quad n \ge k_q. \tag{3.8}$$

Reasoning inductively, we can get

$$y_n \in [\xi_{i+1}, \eta_{i+1}], \quad n \ge ik_q, \quad i \in \mathbb{N}_0. \tag{3.9}$$

By Lemma 2.7 and (3.9), we obtain

$$1 = \lim_{i \to \infty} \xi_{i+1} \le \lim_{n \to \infty} y_n \le \lim_{i \to \infty} \eta_{i+1} = 1, \tag{3.10}$$

which implies

$$\lim_{n \to \infty} y_n = 1. \tag{3.11}$$

The proof is complete. □

4. Exponential Convergence

In this section, we will prove that all positive solutions to (2.4) and (2.7) with $0 < r \le 1$ are exponentially convergent, by using an approach from paper [42].

Theorem 4.1. *If* $r \in (0,1]$, *then every positive solution to* (2.4) *and* (2.7) *exponentially converges to 1.*

Proof. Let $(y_n)_{n=-k_q}^{\infty}$ be a positive solution to (2.4) or (2.7); then by Theorem 1.5, there exists a sufficiently large natural number N such that for arbitrary fixed $\varepsilon > 0$ we have $|y_n - 1| < \varepsilon$ for all $n \ge N$.

Denote $T_n = |y_n - 1|, n \ge -k_q$; then $T_n < \varepsilon$ for all $n \ge N$.

(1). For (2.4).

Let $0 < \varepsilon \le 1 - \sqrt[r]{1/q}$; then by (2.4), we have

$$\begin{aligned} T_n = \left|y_n - 1\right| &= \frac{2\prod_{j=1}^{q}\left|y_{n-k_j}^r - 1\right|}{\prod_{j=1}^{q}\left(y_{n-k_j}^r + 1\right) + \prod_{j=1}^{q}\left(y_{n-k_j}^r - 1\right)} \\ &\le \frac{2\prod_{j=1}^{q}\left|y_{n-k_j}^r - 1\right|}{2\sum_{j=1}^{q} y_{n-k_j}^r} \le \frac{\prod_{j=1}^{q}\left|y_{n-k_j}^r - 1\right|}{q(1-\varepsilon)^r} \\ &\le \prod_{j=1}^{q}\left|y_{n-k_j}^r - 1\right| \le \prod_{j=1}^{q}\left|y_{n-k_j} - 1\right| \\ &= \prod_{j=1}^{q} T_{n-k_j} \le \varepsilon^{q-1} T_{n-k_q}, \quad n \ge N + k_q. \end{aligned} \tag{4.1}$$

(2). For (2.7).

Let $0 < \varepsilon < 1$ be fixed; then by (2.7), we get

$$\begin{aligned} T_n = \left|y_n - 1\right| &= \frac{2\prod_{j=1}^{q}\left|y_{n-k_j}^r - 1\right|}{\prod_{j=1}^{q}\left(y_{n-k_j}^r + 1\right) - \prod_{j=1}^{q}\left(y_{n-k_j}^r - 1\right)} \\ &\le \prod_{j=1}^{q}\left|y_{n-k_j}^r - 1\right| \le \prod_{j=1}^{q}\left|y_{n-k_j} - 1\right| \\ &= \prod_{j=1}^{q} T_{n-k_j} \le \varepsilon^{q-1} T_{n-k_q}, \quad n \ge N + r_q. \end{aligned} \tag{4.2}$$

From this inequality and Lemma 1 in [43] (see also Corollary 1 therein), the result directly follows. □

5. Other Simple Results

In this section, we will present some elementary results of (2.3) and (2.6) with $r > 1$.

Proposition 5.1. *If $r > 1$, then there is no positive solution $(y_n)_{n=-k_q}^{+\infty}$ to* (2.3) *such that* $\lim_{n\to\infty} y_n = +\infty$.

Proof. Suppose $(y_n)_{n=-k_q}^{+\infty}$ is a positive solution to (2.3) such that

$$\lim_{n\to\infty} y_n = +\infty. \tag{5.1}$$

Then for some fixed $M > 1$, there exists $N \in \mathbb{N}$ such that

$$\forall n \geq N, \quad y_n > M. \tag{5.2}$$

Employing (2.3) and (5.2), we can simply get that

$$y_{N+k_q} = \frac{\prod_{j=1}^{q}\left(y_{N+k_q-k_j}^{r}+1\right) - \prod_{j=1}^{q}\left(y_{N+k_q-k_j}^{r}-1\right)}{\prod_{j=1}^{q}\left(y_{N+k_q-k_j}^{r}+1\right) + \prod_{j=1}^{q}\left(y_{N+k_q-k_j}^{r}-1\right)} < 1 \tag{5.3}$$

which contradicts (5.2). The proof is complete. □

Proposition 5.2. *We have the following simple statements:*

(1) *if $r > 1$, then* (2.6) *has nonoscillatory positive solutions with all initial values $y_i \geq 1$, $-k_q \leq i \leq -1$, or $0 < y_i < 1, -k_q \leq i \leq -1$;*

(2) *let $H = \{y_i \mid y_i \neq 1, -k_q \leq i \leq -1\}$ and denote by $\|H\|$ the cardinality of the set H. If $\|H\| < q$, then for any positive solution $(y_n)_{n=-k_q}^{+\infty}$ to* (2.3) *or* (2.6), *we get $y_n \equiv 1$, for all $n \geq 0$.*

6. Conclusions

In the following, let $a^* = \max\{a, 1/a\}$ for any $a \in \mathbb{R}_+$ as defined in [20] and firstly we present [20, Theorem 1].

Theorem 6.1 (see [20]). *Let f, g satisfy the following two conditions:*

(H1) $[f(u_1, u_2, \ldots, u_k)]^* = f(u_1^*, u_2^*, \ldots, u_k^*)$ *and* $[g(u_1, \ldots, u_l)]^* = g(u_1^*, \ldots, u_l^*)$;

(H2) $f(u_1^*, u_2^*, \ldots, u_k^*) \leq u_1^*$.

Then $\overline{x} = 1$ is the unique positive equilibrium for equation (1) which is globally asymptotically stable.

The Equation (1) mentioned in Theorem 6.1 *is the following difference equation:*

$$x_{n+1} = \frac{f(x_{n-r_1}, \ldots, x_{n-r_k})g(x_{n-m_1}, \ldots, x_{n-m_l}) + 1}{f(x_{n-r_1}, \ldots, x_{n-r_k}) + g(x_{n-m_1}, \ldots, x_{n-m_l})}, \quad n \in \mathbb{N}, \tag{6.1}$$

where $f \in C(R_+^k, R_+)$ *and* $g \in C(R_+^l, R_+)$ *with* $k, l \in \{1, 2, \ldots\}, 0 \leq r_1 < \cdots < r_k$, *and* $0 \leq m_1 < \cdots < m_l$, *and the initial values are positive real numbers.*

Remark 6.2. Equation (1.10) is a special case of equation (1) in [20].

Proof. Let $k \geq 3$, $f_1(u) = u^r\,(u > 0)$, $r \in \mathbb{R}_+$, and define a recursive equation

$$f_j(u_1, u_2, \ldots, u_j) = \frac{f_{j-1}(u_1, u_2, \ldots, u_{j-1})u_j^r + 1}{f_{j-1}(u_1, u_2, \ldots, u_{j-1}) + u_j^r}, \tag{6.2}$$

for all $2 \leq j \leq q$. Then the following difference equation:

$$y_n = f_q\left(y_{n-r_1}, \ldots, y_{n-r_q}\right), \quad n \in \mathbb{N}_0, \tag{6.3}$$

where $1 \leq r_1 < r_2 < \cdots < r_q$ and the initial values $y_{-r_q}, y_{-r_q+1}, \ldots, y_{-1} \in (0, +\infty)$, is the very Equation (1.10) in this paper. □

Remark 6.3. Let $f(u) = u^r$, $r \in (0, 1]$, and $g(u_1, u_2, \ldots, u_k) = f_k(u_1, u_2, \ldots, u_k)$. Then through simple calculations, we have

(*H1*) $[f(u)]^* = f(u^*)$ and $[g(u_1, u_2, \ldots, u_k)]^* = g(u_1^*, u_2^*, \ldots, u_k^*)$;

(*H2*) $(H_2) f(u^*) \leq u^*$.

Thus the conditions (H1) and (H2) of [20, Theorem 1] hold. By [20, Theorem 1], we know that the unique positive equilibrium $\overline{y} = 1$ of (1.10) (also (6.3)) is globally asymptotically stable.

Remark 6.4. Although the stability of (1.10) can be also obtained as a corollary from Theorem 1 of the paper by Sun and Xi [20], the method of proof of Theorem 1.5 in this paper is distinct.

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