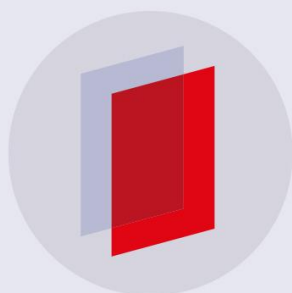# Stress update algorithm based on finite difference method and its application to homogenous anisotropic hardening (HAH) model

To cite this article: H S Choi *et al* 2018 *J. Phys.: Conf. Ser.* **1063** 012011

View the article online for updates and enhancements.

## Related content

# Stress update algorithm based on finite difference method and its application to homogenous anisotropic hardening (HAH) model

**H S Choi[1], J W Yoon[1,2,*] and F Barlat[3]**

[1] Department of Mechanical Engineering, Korea Advanced Institute of Science and Technology (KAIST), 291 Daehak-ro, Yuseong-gu, Daejeon 34141, Republic of Korea
[2] School of Engineering, Deakin University, Waurn Ponds, VIC 3220, Australia
[3] Graduate Institute of Ferrous Technology (GIFT), Pohang University of Science and Technology (POSTECH), 77 Chengam-ro, Nam-gu, Pohang 37673, Republic of Korea

[*]Corresponding autor
E-mail: j.yoon@kaist.ac.kr,  j.yoon@deakin.edu.au

**Abstract**. This paper deals with stress update algorithm based on finite difference method. The proposed algorithm is based on Euler backward method with multi-step return mapping approach. Central difference method was utilized for the approximation of the first and second derivatives of yield function. With the proposed algorithm, it is possible to perform elastic-plastic finite element simulation without any analytical derivatives of yield function. General yield functions including Hill's (1948) and Yld2000-2d, and also the HAH distortion hardening model were implemented to an implicit finite element code (ABAQUS/STANDARD). For the verification purpose, various finite element simulations were performed. With the anisotropic functions, single element loading-unloading and cup-drawing simulation were carried out. The results obtained from the proposed algorithm were compared with the results from analytical derivatives and reference data. The availability of the proposed algorithm for distortional plasticity such as HAH model was evaluated by single element loading-reloading simulations: tension-compression and tension-orthogonal tension. The effectiveness of the proposed algorithm compared to the classical Euler backward method was identified from the simulation results.

## 1. Introduction
Stress update algorithm is an indispensable step to integrate the advanced constitutive models with rate or incremental form of equation. Among many algorithms, Euler backward method based on multi-step return mapping method [1] called hereafter the Euler backward method (analytical derivatives) has gained popularity because of its accuracy and unconditional stability. For example, by using the algorithm, Yoon et al. [2, 3] successfully implemented Yld2000-2d [4] and Yld2004-18p [5] anisotropic functions under isotropic hardening assumption and they accurately predicted severe anisotropy of aluminium alloy. Recently, Lee et al. [6] derived the Euler backward method for the homogenous anisotropic hardening (HAH) model [7]. Throughout the stress update procedure using the Euler backward method, the calculation and implementation of first and second derivatives are the most time consuming and tedious components. In this work, finite difference method was applied to the Euler

backward method to skip the calculation and implementation of any derivatives of yield function. Single element loading-unloading and cup-drawing simulations were performed with Hill's (1948) [8] and Yld2000-2d [4] anisotropic yield functions based on the proposed numerical algorithm and the Euler backward method (analytical derivatives). The accuracy and efficiency of the proposed algorithm were evaluated by comparing the simulation results with the two numerical and analytical methods. Finally, the strength of the proposed algorithm for the HAH model which has quite complicated forms of analytical derivatives was estimated with a single element loading-reloading simulation.

## 2. Stress update algorithm based on finite difference method (FDM)

In the Euler backward method (analytical derivatives), increment of the plastic multiplier is obtained at each sub step ($k$) and iteration ($i$) during stress update procedure as follows:

$$\delta(\Delta\lambda_{(k)}^{(i)}) = \frac{R_1(\Delta\lambda_{(k)}^{(i)}) - \mathbf{m}_{(k)}^{(i)}\mathbf{E}_{(k)}^{(i)-1}R_2(\Delta\lambda_{(k)}^{(i)}) + R_3(\Delta\lambda_{(k)}^{(i)})H_{(k)}^{(i)}}{\mathbf{m}_{(k)}^{(i)}\mathbf{E}_{(k)}^{(i)-1}\mathbf{m}_{(k)}^{(i)} + H_{(k)}^{(i)}} \tag{1}$$

where

$$\mathbf{m}_{(k)}^{(i)} = \frac{\partial f}{\partial\boldsymbol{\sigma}}(\boldsymbol{\sigma}_{(k)}^{(i)}) \tag{2}$$

$$R_1(\Delta\lambda_{(k)}^{(i)}) = f(\boldsymbol{\sigma}_{(k)}^{(i)}) - \rho(\Delta\overline{\varepsilon}^{\mathrm{p}} + \Delta\lambda_{(k)}^{(i)}) - F_{(k)} = 0 \tag{3}$$

$$R_2(\Delta\lambda_{(k)}^{(i)}) = \mathbf{C}^{\mathrm{e}-1}(\boldsymbol{\sigma}_{(k)}^{(i)} - \boldsymbol{\sigma}^{T}) + \Delta\lambda_{(k)}^{(i)}\mathbf{m}_{(k)}^{(i)} = 0 \tag{4}$$

$$R_3(\Delta\lambda_{(k)}^{(i)}) = H^{-1}(\rho_{(k)}^{(i)} - \rho_{(k)}^{(i)}) - \Delta\lambda_{(k)}^{(i)} = 0 \tag{5}$$

and $H$ is the instantaneous slope of hardening curve.

Based on the updated $\delta(\Delta\lambda_{(k)}^{(i)})$ at each iteration ($i$) and sub step ($k$), other variables are updated as:

$$\Delta\lambda_{(k)}^{(i+1)} = \Delta\lambda_{(k)}^{(i)} + \delta(\Delta\lambda_{(k)}^{(i)})$$

$$\Delta\overline{\varepsilon}^{p(i+1)}_{(k)} = \Delta\overline{\varepsilon}^{p(i)}_{(k)} + \delta(\Delta\lambda_{(k)}^{(i)})$$

$$\boldsymbol{\sigma}_{(k)}^{(i+1)} = \boldsymbol{\sigma}_{(k)}^{(i)} + \Delta\boldsymbol{\sigma}_{(k)}^{(i)} \tag{6}$$

$$\rho_{(k)}^{(i+1)} = \rho_{(k)}^{(i)} + \Delta\rho_{(k)}^{(i)}$$

$$X_{(k)}^{(i+1)} = X_{(k)}^{(i)} + \delta(\Delta X_{(k)}^{(i)})$$

It is note that additional state variables (*X*) for the HAH model consisting of equivalent plastic strain could be updated. The stress update procedure will be end when the potential residual defined as Eq. (3) is satisfied within the numerical tolerance.

Consistent tangent modulus is also defined to ensure the quadratic convergence rate of Newton`s iteration as follows:

$$d\boldsymbol{\sigma}_{n+1} = \overline{\mathbf{C}}^{\mathrm{ep}}d\boldsymbol{\varepsilon}_{n+1} = \left(\overline{\mathbf{C}}^{-1} - \frac{\overline{\mathbf{C}}^{-1}\mathbf{m}_{n+1}\otimes\overline{\mathbf{C}}^{-1}\mathbf{m}_{n+1}}{\mathbf{m}_{n+1}\overline{\mathbf{C}}^{-1}\mathbf{m}_{n+1} + H}\right)d\boldsymbol{\varepsilon}_{n+1} \tag{7}$$

where

$$\overline{\mathbf{C}}^{-1} = \left[\mathbf{C}^{\mathrm{e}} + \Delta\lambda_{n+1}\frac{\partial\mathbf{m}_{n+1}}{\partial\boldsymbol{\sigma}_{n+1}}\right]^{-1} \tag{8}$$

As shown in Eqs. (1) and (8), both first and second derivatives of yield function are required during the stress integration procedure. Central difference method was utilized for the approximation of the derivatives. With the method, both derivatives are approximated under the plane stress assumption as follows:

$$\frac{\delta f}{\delta\sigma_{11}} \approx \frac{f(\sigma_{11} + \Delta\sigma_{11}, \sigma_{22}, \sigma_{12}) - f(\sigma_{11} - \Delta\sigma_{11}, \sigma_{22}, \sigma_{12})}{2\Delta\sigma_{11}} \tag{9}$$

$$\frac{\delta^2 f}{\delta^2 \sigma_{11}} \approx \frac{f(\sigma_{11} + \Delta\sigma_{11}, \sigma_{22}, \sigma_{12}) - 2f(\sigma_{11}, \sigma_{22}, \sigma_{12}) + f(\sigma_{11} - \Delta\sigma_{11}, \sigma_{22}, \sigma_{12})}{(\Delta\sigma_{11})^2} \tag{10}$$

As indicated in Eqs. (9) and (10), three sets of stress components are required for the approximation and the way to define three sets during stress integration procedure strongly influences the convergence of FE simulation. The method to define three sets is explained with a simple example as below.

Let's assume that the total number of sub steps is only one ($N$=1). Then, the stress ($\sigma_1 = \sigma_{n+1}$) is updated by satisfying the consistency condition. For the definition during stress integration procedure, the stress lower limit should be defined based on the step size of stress as follows:

$$\Delta\sigma_{\text{limit}} = (\sigma_{n+1}^{\text{trial}} - \sigma_n) / 2 * N \tag{11}$$

and

$$\sigma^{\text{limit}.k} = \sigma_{n+1}^{\text{trial}} - 2 * k \cdot \Delta\sigma_{\text{limit}} = \sigma_n \tag{12}$$

where $k$ indicates the corresponding sub step. Since the total number of sub steps is one in this demonstration, only one stress limit is defined, which is the same as the last converged stress ($\sigma_n$).

For the first iteration at the step ($i$=1, $k$=1), the step size ($\Delta\sigma_{1.\text{initial}}$) is newly defined for three sets as follows:

$$\Delta\sigma_{1.\text{initial}} = (\sigma_{n+1}^{\text{trial}} - \sigma^{\text{limit}.1}) / (2 * N) \tag{13}$$

Then, three sets for the first iteration are obtained as:

$$\sigma + \Delta\sigma = \sigma_{n+1}^{\text{trial}}$$
$$\sigma = \sigma_{n+1}^{\text{trial}} - \Delta\sigma_{1.\text{initial}} \tag{14}$$
$$\sigma - \Delta\sigma = \sigma_{n+1}^{\text{trial}} - 2 \cdot \Delta\sigma_{1.\text{initial}}$$

By the three sets, both derivatives are approximated for the first iteration. If the updated stress ($\sigma^{1.\text{new}}$) at the first iteration does not satisfy the convergence condition, the sets for the next iteration ($i$=2) are defined based on the step size for the next iteration as follows:

$$\Delta\sigma_{1.\text{next}} = (\sigma^{1.\text{new}} - \sigma^{\text{limit}.1}) / (2 * N) \tag{15}$$

$$\sigma + \Delta\sigma = \sigma^{1.\text{new}}$$
$$\sigma = \sigma^{1.\text{new}} - \Delta\sigma_{1.\text{next}} \tag{16}$$
$$\sigma - \Delta\sigma = \sigma^{1.\text{new}} - 2 \cdot \Delta\sigma_{1.\text{next}}$$

Again, above three sets are utilized for the approximation of the derivatives. This procedure will continue until updated stress satisfy the convergence condition. If the updated stress ($\sigma_1 = \sigma_{n+1}$) satisfies the convergence condition, three sets for both derivatives employed for the calculation of consistent tangent modulus are defined by following relations.

$$\Delta\sigma_{\text{final}} = (\sigma_{n+1} - \sigma^{\text{limit}.1}) / (2 * N) \tag{17}$$

$$\sigma + \Delta\sigma = \sigma_{n+1}$$
$$\sigma = \sigma_{n+1} - \Delta\sigma_{\text{final}} \tag{18}$$
$$\sigma - \Delta\sigma = \sigma_{n+1} - 2 \cdot \Delta\sigma_{\text{final}}$$

Since the definition of the three sets is based on the step size of stress components, both denominator and numerator of the derivatives can be zero resulting in divergence of FE simulation. Therefore, with the numerical tolerance of step size of stress components, the derivatives within the tolerance step size should be zero as following example:

$$\text{If } \Delta\sigma_{12} < Tol \approx 10^{-8}, \text{ then } \frac{\partial f}{\partial \sigma_{12}} = 0 \text{ and } \frac{\partial^2 f}{\partial^2 \sigma_{12}}, \frac{\partial}{\partial \sigma_{11}}\left(\frac{\partial f}{\partial \sigma_{12}}\right), \frac{\partial}{\partial \sigma_{22}}\left(\frac{\partial f}{\partial \sigma_{12}}\right) = 0 \qquad (19)$$

## 3. Verifications and numerical results

### 3.1. Single element loading-unloading simulation

The r-values of AA6022-T4E32 were predicted by a single element loading-unloading simulation. Hill's (1948) and Yld2000-2d anisotropic functions under Voce isotropic hardening assumption were utilized for the simulation. The predicted r-values with the anisotropic functions are presented in figures 1 and 2, respectively. In the figures, the solid lines are the theoretical results and the black, red, and green symbols are the simulation results corresponding to time increment 0.1, 0.01, and 0.001 where the total time step is equal to 1 for each step. As shown in the figures the predicted r-value based on both Euler backward method (analytical derivatives) and the proposed numerical algorithm show very good agreement regardless of time increment and yield function.
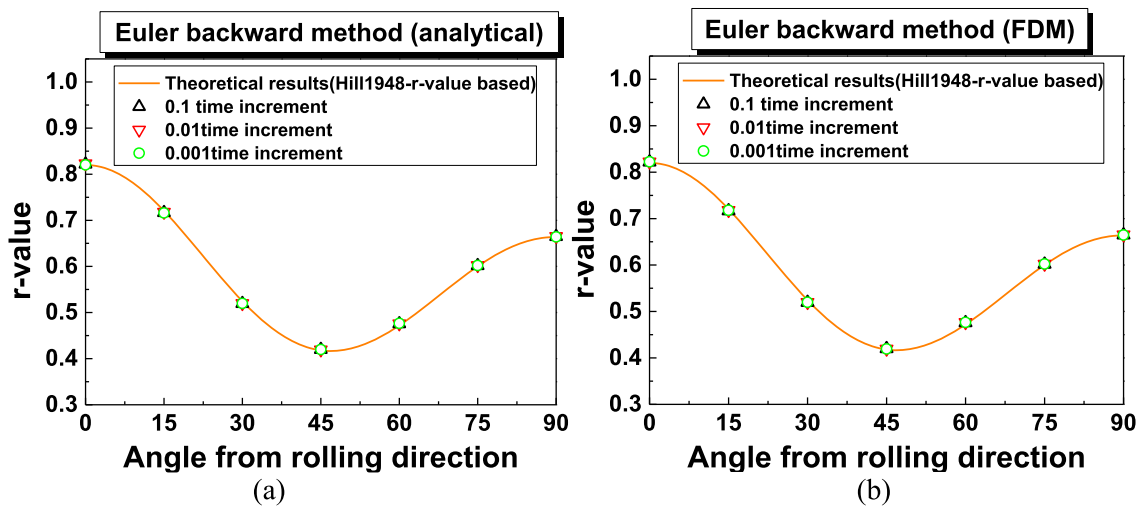


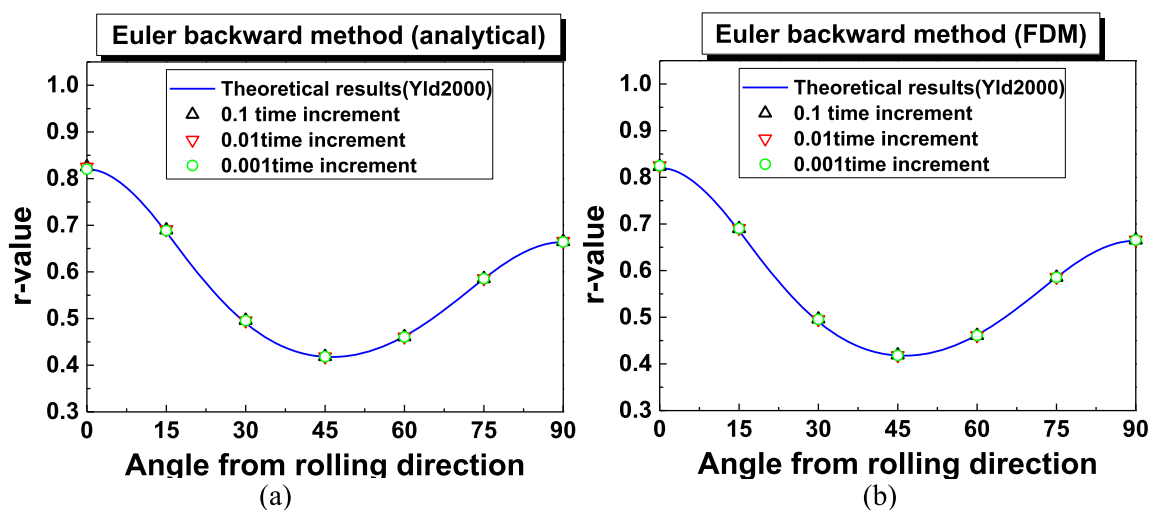**Figure 1.** The predicted r-value based on Hill's 1948 function.



**Figure 2.** The predicted r-value based on Yld2000-2d function.

### 3.2. Cup-drawing simulation

Cup-drawing simulation was performed with two algorithms to predict the earing profile for AA2090-T3. The same geometry, tool dimension, and material properties were utilized as reported by Yoon et al. [3]. Figure 3 shows the predicted results based on Hill's (1948) and Yld2000-2d anisotropic functions, respectively. In the graph, black and red color indicate the simulation results based on each anisotropic yield functions. In addition, solid lines and symbols are the simulation results obtained from the Euler backward method (analytical derivatives) and the proposed numerical algorithm. As shown in the figure, the proposed algorithm shows the same level of accuracy as the Euler backward method (analytical derivatives). As for time efficiency, the proposed algorithm takes a longer simulation time as 1.2 times as the Euler backward method (analytical derivatives) for both constitutive models.
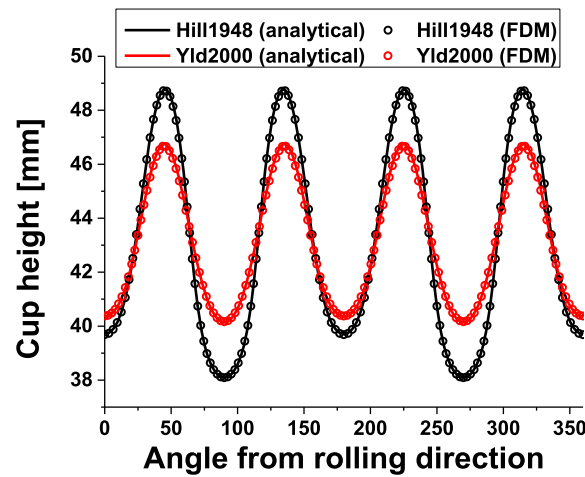


**Figure 3.** Comparison of predicted earing profile.

### 3.3. Single element loading-reloading simulation

Single element loading-reloading simulations were performed with two loading conditions: tension-compression and tension-orthogonal tension. The original HAH model and 'Generic material' proposed by Barlat et al. [7] were utilized for the simulations. Figures 4(a) and 4(b) show the predicted true stress-effective plastic strain curves for each loading condition. The predicted curves well describe Bauschinger effect and material behavior after loading path change for both loading conditions. In addition, the distortion of yield surface during the simulations were well described as shown in figure 5.
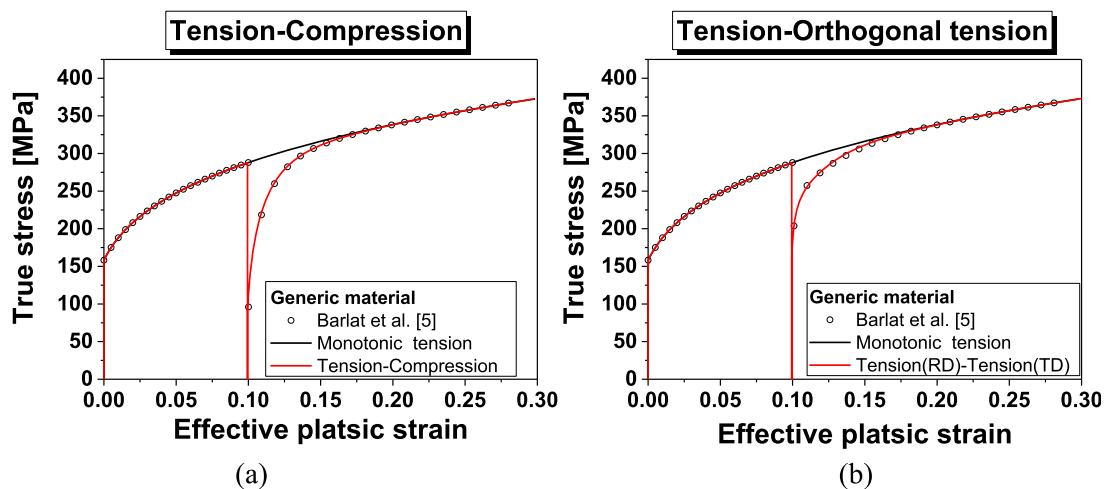


(a)                                                            (b)

**Figure 4.** The predicted true stress-effective plastic strain curves from the original HAH model when the proposed numerical integration is used.
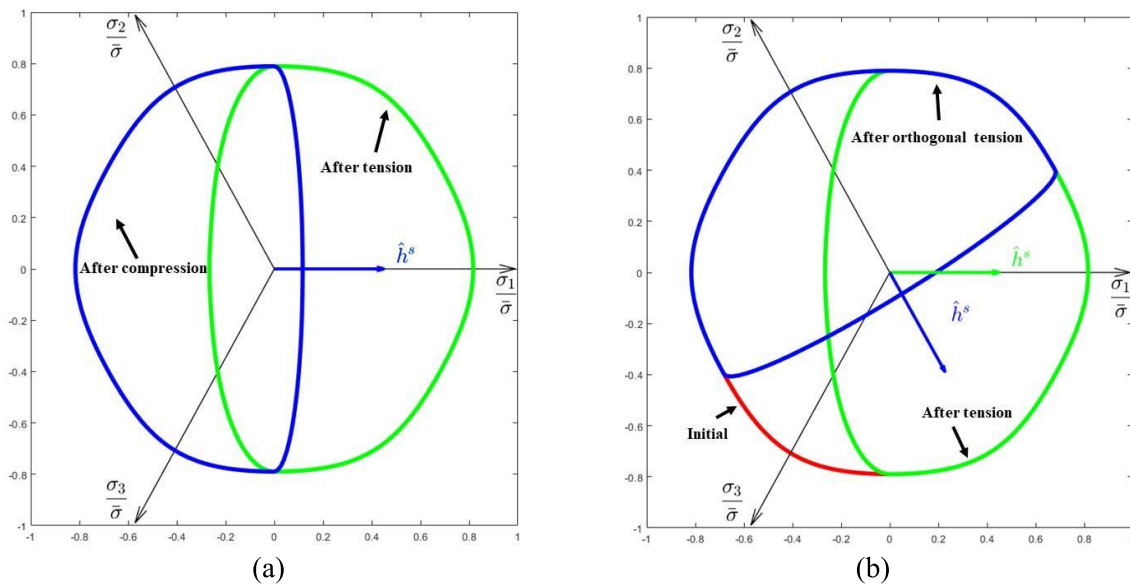
**Figure 5.** The predicted distortion of yield surface by the original HAH model from the original HAH model when the proposed numerical integration is used

## 4.  Conclusions

Stress integration algorithm based on finite difference method was proposed. The accuracy and time efficiency was identified by comparing the simulation results obtained from the proposed algorithm with those obtained from the Euler backward method (analytical derivatives). The applicability of the proposed algorithm for distortional plasticity was successfully evaluated by applying the algorithm to the original HAH model. Considering accuracy and convenience for implementation, the proposed algorithm could be an efficient way to implement advanced constitutive models with moderate delay of simulation time.

## 5.  References

[1]     Yoon J, Yang D and Chung K 1999 *Comput. Methods. Appl. Mech. Eng.* **174** 23-56
[2]     Yoon J, Barlat F, Dick RE and Chung K 2004 *Int. J. Plast.* **20** 495-522
[3]     Yoon J, Barlat F, Dick R and Karabin M 2006 *Int. J. Plast.* **22** 174-93
[4]     Barlat F, Aretz H, Yoon J, Karabin ME, Brem JC and Dick RE 2006 *Int. J. Plast.* **21** 1009-39
[5]     Barlat F, Brem JC, Yoon J, Chung K, Dick RE, Lege DJ, Pourboghrat F, Choi SH and Chu E 2003 *Int. J. Plast.* **19** 1297-319
[6]     Lee J, Lee MG, Barlat F and Kim JH 2012 *Comput. Methods. Appl. Mech. Eng.* **247** 73-92
[7]     Barlat F, Gracio JJ, Lee MG, Rauch EF and Vincze G 2011 *Int. J. Plast.* **27** 1309-27
[8]     Hill R 1948 *Ser. A. Math. Phys. Sci.* **193** 281-297