Applied Mathematics Letters 43 (2015) 68–71

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Reachable set bounding for nonlinear perturbed time-delay systems: The smallest bound

P.T. Nam [a,*], P.N. Pathirana [b], H. Trinh [b]

[a] Department of Mathematics, Quynhon University, Binhdinh, Viet Nam
[b] School of Engineering, Deakin University, Geelong, VIC 3217, Australia

ARTICLE INFO

Article history:
Received 4 September 2014
Received in revised form 27 November 2014
Accepted 28 November 2014
Available online 10 December 2014

Keywords:
Reachable sets
Nonlinear time-delay systems
Bounded disturbances

ABSTRACT

In this letter, we propose a new approach to obtain the smallest box which bounds all reachable sets of a class of nonlinear time-delay systems with bounded disturbances. A numerical example is studied to illustrate the obtained result.

1. Introduction

Notations: $\mathbb{R}^n(\mathbb{R}^n_+)$ is n-dimensional (nonnegative) vector space; $e_i = [0_{1\times(i-1)}\ 1\ 0_{1\times(n-i)}]^T \in \mathbb{R}^n$ is ith-unit vector in $\mathbb{R}^n$; for three vectors $x = [x_1\ x_2\ \cdots\ x_n]^T \in \mathbb{R}^n$, $y = [y_1\ y_2\ \cdots\ y_n]^T \in \mathbb{R}^n$ and $q = [q_1\ q_2\ \cdots\ q_n]^T \in \mathbb{R}^n_+$, two $n \times n$-matrices $A = [a_{ij}]$, $B = [b_{ij}]$, the following notations will be used in our development: $|x| := [|x_1|\ |x_2|\ \cdots\ |x_n|]^T$; $x \prec y(\preceq y)$ means that $x_i < y_i(\leq y_i)$, $\forall i = 1, \ldots, n$; $A \prec B(\preceq B)$ means that $a_{ij} < b_{ij}(\leq b_{ij})$, $\forall i, j = 1, \ldots, n$; A is nonnegative if $0 \preceq A$; A is *essentially nonnegative* (called a Metzler matrix) if $a_{ij} \geq 0$, $\forall i, j = 1, \ldots, n$, $i \neq j$; $\mu(A)$ stands for the spectral abscissa of matrix A; $\mathcal{B}(0, q) = \{x \in \mathbb{R}^n : |x| \preceq q\}$ is a box in $\mathbb{R}^n$.

Reachable set of dynamic systems perturbed by bounded inputs (disturbances) is the set of all the states starting from the origin by inputs with peak value [1–3]. Reachable set bounding of perturbed dynamic systems and its applications are important research areas in control theory and have attracted much attention during the past decades (see, [1–3] and the references therein). Recently, there is a growing interest in the problem of reachable set bounding for perturbed systems with time-delays [1–11] and most of the existing results are only for linear systems. In this letter, we present a new result for a class of nonlinear time-delay systems with bounded disturbances as below:

$$\begin{aligned} &\dot{x}(t) = Ax(t) + F(t, x(t), x(t-\tau_1(t)), \ldots, x(t-\tau_m(t), \omega(t)), \quad t \geq 0, \\ &x(s) = 0, \quad s \in [-h, 0] \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $\omega(t) \in \mathbb{R}^1$ is the disturbance vector satisfying

$$|\omega(t)| \leq \overline{\omega}, \tag{2}$$

* Corresponding author.
E-mail addresses: phanthanhnam@qnu.edu.vn (P.T. Nam), pubudu.pathirana@deakin.edu.au (P.N. Pathirana), hieu.trinh@deakin.edu.au (H. Trinh).

http://dx.doi.org/10.1016/j.aml.2014.11.015
0893-9659/

$\overline{\omega}$ is a given positive scalar, time-varying delays $\tau_0(t) \equiv 0$ and $0 \le \tau_k(t) \le \overline{\tau}_k \le h,\ k = 1, \dots, m$ are given continuous functions, $\overline{\tau}_k, k = 1, \dots, m$ are nonnegative scalars, $F(t, \dots) \in \mathbb{R}^n$ is a given continuous function satisfying

$$|F(t, x(t), \dots, \omega(t))| \le \sum_{k=0}^{m} A_k|x(t - \tau_k(t))| + B|\omega(t)|, \tag{3}$$

A is an essentially nonnegative matrix and $A_k, k = 0, \dots, m, B$ are nonnegative matrices. Note that there are many classes of systems such as time-varying systems, switched systems [12–14] which can be reformulated into the form of (1), (2), (3).

For linear perturbed time-delay systems whose matrices are convex combinations of constant matrices (polytopic uncertainties), the most widely used approach is based on the Lyapunov–Krasovskii (or Razumikhin) functional method [1–10]. For linear time-delay systems whose matrices are matrix functions, a novel approach which does not involve the Lyapunov–Krasovskii functional method has just been proposed in [11]. So far, there is no available approach which provides the smallest bound of reachable sets of considered time-delay systems. In this letter, inspired by the comparison method proposed in [15], we propose a new and simple approach to obtain *the smallest box* which bounds all reachable sets of the above nonlinear perturbed time-delay system. Lastly, we study the numerical example considered in [11] to illustrate the proposed approach.

2. Main result

For simplicity, we consider system (1) with one delay, i.e. $m = 1$. The obtained result can be extended to the case where system (1) has multiple delays. Let us first consider the following three linear nonnegative time-delay systems:

$$\dot{y}(t) = (A + A_0)y(t) + A_1 y(t - \tau_1(t)) + B|\omega(t)|, \tag{4}$$
$$y(s) = \varphi(s),\ s \in [-h, 0],$$
$$\dot{z}(t) = (A + A_0)z(t) + A_1 z(t - \tau_1(t)) + B\overline{\omega}, \tag{5}$$
$$z(s) = \psi(s),\ s \in [-h, 0],$$
$$\dot{u}(t) = (A + A_0)u(t) + A_1 u(t - \tau_1(t)), \tag{6}$$
$$u(s) = \phi(s),\ s \in [-h, 0].$$

Lemma 1 (*[16,17]*). *The above three linear time-delay systems are nonnegative.*

Proof. This Lemma can be seen as a natural extension of Proposition 3.1 in [16]. □

Lemma 2 (*[18]*). *If a positive scalar α exists such that one of the following conditions hold:*
(i) $\mu(\alpha I_n + (A + A_0) + A_1 e^{\alpha\overline{\tau}_1}) < 0$;
(ii) $\exists p \succ 0 : (\alpha I_n + (A + A_0) + A_1 e^{\alpha\overline{\tau}_1})^T p \preceq 0$;
then system (6) is α-exponentially stable, i.e. there is a positive vector function $\varrho(.)$ such that

$$u_\phi(t) \preceq \varrho(\phi)e^{-\alpha t}, \quad \forall t \ge 0. \tag{7}$$

Lemma 3 (*[15]*). *Let $M \in R^{n\times n}$ be a Metzler matrix. Then the following statements are equivalent*
(i) $\mu(M) < 0$.
(ii) *M is invertible and $M^{-1} \preceq 0$.*

Let us denote a solution with initial condition $y(s) = \varphi(s), s \in [-h, 0]$ of system (4) by $y(t, \varphi)$ and a solution with initial condition $z(s) = \psi(s), s \in [-h, 0]$ of system (5) by $z(t, \psi)$. The following two lemmas are useful for our development:

Lemma 4. *If $\varphi(s) \preceq \psi(s), \forall s \in [-h, 0]$ then we have $y(t, \varphi) \preceq z(t, \psi), \forall t \ge 0$.*

Proof. Denote $e(t) = z(t) - y(t), \varepsilon(t) = \overline{\omega} - |\omega(t)|$ and consider the following system

$$\dot{e}(t) = (A + A_0)e(t) + A_1 e(t - \tau_1(t)) + B\varepsilon(t), \tag{8}$$
$$e(s) = \psi(s) - \varphi(s), \quad s \in [-h, 0].$$

By Lemma 1, we have $e(t, \psi - \varphi) \succeq 0, \forall t \ge 0$. This implies that $y(t, \varphi) \preceq z(t, \psi), \forall t \ge 0$. The proof of Lemma 4 is completed. □

Lemma 5. *If $\mu(A + A_0 + A_1) < 0$ then there exist a positive $q \in \mathbb{R}^n_+$, a positive scalar α, a positive vector function $\varrho(.)$ such that*

$$q - \varrho(q)e^{-\alpha t} \preceq z(t, 0) \preceq q, \quad \forall t \ge 0, \tag{9}$$

where $z(t, 0)$ is the solution with initial condition $z(s) = 0, s \in [-h, 0]$ of system (5).

Proof. Denote

$$q := -(A + A_0 + A_1)^{-1}B\overline{\omega}. \tag{10}$$

Since $B\overline{\omega} \succeq 0$ and $-(A + A_0 + A_1)^{-1} \succeq 0$ due to Lemma 3, we have $q \succeq 0$. Taking the state transformation $v(t) = q - z(t)$, then from (5) the following system is obtained

$$\dot{v}(t) = (A + A_0)v(t) + A_1 v(t - \tau_1(t)), \qquad (11)$$
$$v(s) = q - \psi(s), \quad s \in [-h, 0],$$

and $q - z(t, 0) \equiv v(t, q), \forall t \geq 0$, where $v(t, q)$ is the solution with initial condition $v(s) = q, s \in [-h, 0]$ of system (11). Since $\mu(A + A_0 + A_1) < 0$, there is a small enough scalar $\alpha > 0$ such that $\mu(\alpha I_n + (A + A_0) + A_1 e^{\alpha\overline{\tau}_1}) < 0$. By Lemmas 1 and 2, there is a positive vector function $\varrho(.)$ such that

$$0 \preceq v(t, q) \preceq \varrho(q) e^{-\alpha t}, \quad \forall t \geq 0, \qquad (12)$$

which implies (9). The proof of Lemma 5 is completed. □

Now we are in a position to introduce the main result in the form of the following theorem.

Theorem 1. *If $\mu(A + A_0 + A_1) < 0$ then the box $\mathcal{B}(0, q) = \{x \in \mathbb{R}^n : |x| \preceq q\}$ where q defined in (10) is the smallest box which bounds reachable sets of system (1), (2), (3).*

Proof. *Step 1:* First, we prove that

$$|x(t, 0)| \preceq y(t, 0), \quad \forall t \geq 0, \qquad (13)$$

where $x(t, 0)$ is a solution with initial condition $x(s) = 0, s \in [-h, 0]$ of system (1) and $y(t, 0)$ is a solution with initial condition $y(s) = 0, s \in [-h, 0]$ of system (4). Indeed, assume on the contrary that there exists $t_0 > 0$ such that $|x(t_0, 0)| \not\preceq y(t_0, 0)$. Set $t_1 := \inf\{t \geq 0 : |x(t, 0)| \not\preceq y(t, 0)\}$, then $t_1 > 0$. By continuity, there is an index $i_0 \in \{1, \ldots, n\}$ and a positive scalar $\epsilon > 0$ such that

$$\text{(i) } |x(t, 0)| \preceq y(t, 0), \quad \forall t \leq t_1, \qquad (14)$$
$$\text{(ii) } |x_{i_0}(t_1, 0)| = y_{i_0}(t_1, 0), \qquad (15)$$
$$\text{(iii) } |x_{i_0}(t, 0)| > y_{i_0}(t, 0), \quad \forall t \in (t_1, t_1 + \epsilon). \qquad (16)$$

Taking the Dini upper-right derivative of $|x_{i_0}(t)|$ at t_1, combining with (14) and (15), we have

$$\begin{aligned} D^+|x_{i_0}(t_1, 0)| &= sgn(x_{i_0}(t_1, 0)\dot{x}_{i_0}(t_1, 0) \\ &\leq e_{i_0}^T\Big(Ae_{i_0}|x_{i_0}(t_1, 0)| + \sum_{j=1, j\neq i_0}^{n} Ae_j|x_j(t_1, 0)|\Big) + e_{i_0}^T\Big(\sum_{j=1}^{n} A_0 e_j|x_j(t_1, 0)| + A_1 e_j|x_j(t_1 - \tau_1(t_1), 0)|\Big) \\ &\leq e_{i_0}^T\Big(Ae_{i_0}|y_{i_0}(t_1, 0)| + \sum_{j=1, j\neq i_0}^{n} Ae_j|y_j(t_1, 0)|\Big) + e_{i_0}^T\Big(\sum_{j=1}^{n} A_0 e_j|y_j(t_1, 0)| + A_1 e_j|y_j(t_1 - \tau_1(t_1), 0)|\Big) \\ &= D^+|y_{i_0}(t_1, 0)|. \end{aligned} \qquad (17)$$

This conflicts with (16). Thus, inequality (13) holds.

Step 2: By Lemma 5, we can see that the box $\mathcal{B}(0, q) \bigcap R_+^n$ where q defined in (10) is the smallest box which bounds reachable sets of system (5). By Lemma 4 and Step 1, it follows that $\mathcal{B}(0, q)$ is also the smallest box which bounds reachable sets of system (1). The proof of Theorem 1 is completed. □

Remark 1. Note that the approach [11] only gives a bound of reachable sets of system (1), (2), (3). In fact, it gives a bound for all partial state vectors of system (1), (2), (3). In this paper, our approach (Theorem 1) gives the smallest bounds for each partial state vectors of system (1), (2), (3), i.e. each $q_i, i = 1, 2, \ldots, n$ is the smallest bound for each ith partial state vector $x_i(t, 0), i = 1, 2, \ldots, n$. Therefore, the box $\mathcal{B}(0, q)$ obtained by our approach is smaller than one obtained by the approach in [11].

Based on the above development, the procedure of finding the smallest box $\mathcal{B}(0, q)$ which bounds all reachable sets of a nonlinear perturbed system with multiple time-delays, is stated in the following computational algorithm.

Algorithm 1. *Step 1:* Find matrices $A, B, A_k, \ k = 0, \ldots, m$, such that the considered nonlinear system is in the form of (1), (2), (3).

Step 2: Check condition $\mu(A + \sum_{k=0}^{m} A_k) < 0$. If it holds, obtain the smallest box, $\mathcal{B}(0, q)$ with $q = -(A + \sum_{k=0}^{m} A_k)^{-1} B\overline{\omega}$. The procedure is completed.

3. A numerical example

Example 1. Consider the following linear time-varying system which was studied in [11]

$$\dot{x}(t) = A(t)x(t) + D(t)x(t - \tau(t)) + B(t)\omega(t), \quad t \geq 0 \qquad (18)$$

where $x(t) \in R^3$, $\tau(t) = 6\left|\sin\left(2\sqrt{t}\right)\right|$,

$$A(t) = \begin{pmatrix} -4 - |\sin t|e^{-t} & \cos 2t & \sin^2 t \\ e^{-t}\cos t & -6 + \sin 2t & 2\cos 3t \\ \dfrac{t\sin t}{1+t} & \dfrac{1}{\sqrt{1+|\sin t|}} & -5 - |\cos t| \end{pmatrix},$$

$$D(t) = \begin{pmatrix} \sin 3t & -e^{-2t} & 0 \\ e^{-2t}\sin t & 0 & \cos 3t \\ 0 & \cos^2 t & -e^{-t}\sin 2t \end{pmatrix}, \qquad B(t) = \begin{pmatrix} 0.1e^{\sin t} \\ 0.2\cos 2t \\ 0.1\sin 4t \end{pmatrix}$$

and $|\omega(t)| \le \overline{\omega} = 0.5$.

Solution. Let us denote

$$A = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{bmatrix}, \qquad A_0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.1e \\ 0.2 \\ 0.1 \end{bmatrix}$$

and $F(t, \ldots) = (A(t) - A)x(t) + D(t)x(t - \tau(t)) + B(t)\omega(t)$. Then system (18) is rewritten as follows

$$\dot{x}(t) = Ax(t) + F(t, x(t), x(t - \tau_1(t)), \omega(t))$$

where $F(t, \ldots)$ satisfying condition (3) with A_0, A_1, B defined as the above. By Theorem 1, the box $\mathcal{B}(0, q) = \{x \in \mathbb{R}^n : |x| \preceq q\}$ where $q = -(A + A_0 + A_1)^{-1}B\overline{\omega} = [0.3139\ \ 0.2859\ \ 0.2339]^T$, is the smallest bound of reachable sets of system (18). Note that the box derived in [11] is $\mathcal{B}(0, q_1)$ where $q_1 = [0.68\ \ 0.68\ \ 0.68]^T \gneqq q$.

4. Conclusion

This letter has presented a new and simple approach to find the smallest box which bounds all reachable sets of a class of nonlinear time delay system with bounded disturbances. This approach can be further extended to nonlinear discrete-time systems and/or to perturbed time-varying systems. A numerical example has been studied to illustrate the derived result.

Acknowledgments

This work is supported by the National Foundation for Science and Technology Development, Vietnam under grant 101.01-2014.35 and the Australian Research Council under the Discovery grant DP130101532.

References

[1] E. Fridman, U. Shaked, On reachable sets for linear systems with delay and bounded peak inputs, Automatica 39 (2003) 2005–2010.
[2] J.H. Kim, Improved ellipsoidal bound of reachable sets for time-delayed linear systems with disturbances, Automatica 44 (2008) 2940–2943.
[3] Z. Zuo, D.W.C. Ho, Y. Wang, Reachable set bounding for delayed systems with polytopic uncertainties: The maximal Lyapunov–Krasovskii functional approach, Automatica 46 (2010) 949–952.
[4] P.T. Nam, P.N. Pathirana, Further result on reachable set bounding for linear uncertain polytopic systems with interval time-varying delays, Automatica 47 (2011) 1838–1841.
[5] O.M. Kwon, S.M. Lee, J.H. Park, On the reachable set bounding of uncertain dynamic systems with time-varying delays and disturbances, Inf. Sci. 181 (2011) 3735–3748.
[6] C. Shen, S. Zhong, The ellipsoidal bound of reachable sets for linear neutral systems with disturbances, J. Franklin Inst. 348 (2011) 2570–2585.
[7] Z. Zuo, Y. Chen, Y. Wang, D.W.C. Ho, M.Z.Q. Chen, H. Li, A note on reachable set bounding for delayed systems with polytopic uncertainties, J. Franklin Inst. 350 (2013) 1827–1835.
[8] N.D. That, P.T. Nam, Q.P. Ha, Reachable set bounding for linear discrete-time systems with delays and bounded disturbances, J. Optim. Theory Appl. 157 (2013) 96–107.
[9] B. Zhang, J. Lam, S. Xu, Reachable set estimation and controller design for distributed delay systems with bounded disturbances, J. Franklin Inst. 351 (2014) 3068–3088.
[10] J. Lam, B. Zhang, Y. Chen, S. Xu, Reachable set estimation for discrete-time linear systems with time delays, Int. J. Robust Nonlinear Control (2014) http://dx.doi.org/10.1002/rnc.3086.
[11] L.V. Hien, H. Trinh, A new approach to state bounding for linear time-varying systems with delay and bounded disturbances, Automatica 50 (2014) 1735–1738.
[12] J. Qi, Y. Sun, Global exponential stability of certain switched systems with time-varying delays, Appl. Math. Lett. 26 (2013) 760–765.
[13] V.N. Phat, Y. Khongtham, K. Ratchagit, LMI approach to exponential stability of linear systems with interval time-varying delays, Linear Algebra Appl. 436 (2012) 243–251.
[14] G. Rajchakit, Exponential stability of switched linear systems with interval time-varying delays, in: IEEE Int. Conf. Ro. Bio., 2012, pp. 1502–1506.
[15] P.H.A. Ngoc, On exponential stability of nonlinear differential systems with time-varying delay, Appl. Math. Lett. 25 (2012) 1208–1213.
[16] W.M. Haddad, V. Chellaboina, T. Rajpurohit, Dissipativity theory for nonnegative and compartmental dynamical systems with time delay, IEEE Trans. Automat. Control 49 (2004) 747–751.
[17] X. Liu, W. Yu, L. Wang, Stability analysis for continuous-time positive systems with time-varying delays, IEEE Trans. Automat. Control 55 (2010) 1024–1028.
[18] S. Zhu, M. Meng, C. Zhang, Exponential stability for positive systems with bounded time-varying delays and static output feedback stabilization, J. Franklin Inst. 350 (2013) 617–636.