GAYE WILLIAMS

CHAPTER NINE

Student-Created Tasks Inform Conceptual Task Design

INTRODUCTION

Over fifteen years as a teacher of secondary mathematics, I progressively developed a teaching approach that involved students beginning a topic in groups, and working on 'complex tasks' after a five to ten-minute introduction. By complex tasks, I mean, tasks that can be approached in many different ways, that value mathematical exploration more than moving in a linear direction towards a single solution, and that require students to explain and generalise. I developed this approach to make mathematics accessible to a greater number of students, and to decrease their anxiety about it. I was very surprised when I found able students reported that they understood mathematics much better, and enjoyed this process of learning.

As a teacher, I found I could design tasks that achieved these outcomes and that I could articulate some of the features these tasks possessed (Williams, 1996) but, I did not know why they 'worked'. Later, as a teacher-researcher, I studied student responses to my senior secondary calculus tasks and found that students were not so focussed on my task, as on questions they asked themselves as they worked with these tasks (Williams, 2000a). They were creating and exploring their own tasks!

In this chapter, I explore the types of questions students ask themselves to achieve such deep understanding. I use data from my research within the Learner's Perspective Study (LPS) to explore these ideas. I joined the Learner's Perspective Study (LPS) because I considered that classrooms of teachers who displayed 'good teaching practice' should be a rich source of data about student-created tasks. I was fascinated to find that when students did create their own tasks within the LPS (in Australia and the USA), and develop new understandings as a result, their teachers had not explicitly intended such activity, and were frequently not aware it had happened.

In my study within the LPS (Williams, 2005), the thinking of eighty-six students was studied in detail. These students came from six different classrooms in Australia (4) and the USA (2). Only eight student-created tasks were identified in total and these tasks were created by five of the eighty-six students, and seven of these tasks were created and solved individually. These five students varied in their mathematical performances in class. Kerri was in a class for students identified as

Y. Shimizu et al. (Eds.), Mathematical Tasks in Classrooms around the World, pp. 165–184.

gifted, Leon, Pepe, and Eden were identified (by their teachers) as above average performance, and Dean stated that he struggled to pass mathematics which fitted with his teacher's description.

These five students created their own 'conceptual tasks' (Williams, 2005) where the term 'conceptual tasks' has previously been used to describe tasks designed by teachers and/or researchers to support student development of new mathematical understandings (see Lampert, 2001).

Student-created tasks from Kerri (USA) and Leon (Australia) were selected for the focus of this chapter. More detail about these cases can be found in Williams (2006; 2007a) respectively. Other students in my LPS study (Williams, 2005) who created their own tasks undertook the same types of thinking as they explored their self-created tasks. Dean differed to the other students in that his exploration was interrupted because he did not possess sufficient background knowledge to complete his 'constructing' process. Constructing is an 'observable cognitive element' of the process of abstracting (Dreyfus, Hershkowitz, & Schwarz, 2001b) where abstracting is the process of "vertically reorganising previously constructed mathematical knowledge into a new structure" (p. 377), and "vertical" refers to the forming a new mathematical structure as opposed to strengthening connections between a mathematical structure and a context ("horizontal", Treffers & Goffree, 1985). Thus, Dean was able to commence the developing of a new mathematical structure and progress a considerable way towards this, but a gap in his background knowledge meant he was unable to complete the process.

As these student-created tasks led to deep understanding, and the ways in which these students worked with these tasks included a progressive structuring of questions that elicited more and more complex processes of thinking, there is the potential to learn from these students. The following inquiry is the focus of this chapter: "How can conceptual tasks formulated by students, and their activity associated with them, inform the design of conceptual tasks more generally?"

PREVIOUS RESEARCH ON CONCEPTUAL TASKS

Over the past thirty years, research focused around tasks that provide opportunities for students to develop conceptual knowledge has increased (e.g., Cobb, Wood, Yackel, & McNeal, 1992; Krutetskii, 1976; Lampert, 2001; Tabach, Hershkowitz, & Schwarz, 2005; Williams, 2002b).

Conceptual tasks designed by Krutetskii (1976) included the following characteristics:

- They could not be solved using known procedures; and
- They could be solved in more than one way.

His findings about student thinking in responses to these tasks included:

- The 'mental activities' students employed in processing information during problem solving could be categorised
- Some students solved these tasks without the assistance of an 'expert other' (they 'spontaneously' created their own ZPD, Vygotsky, 1933/1966)

- These students posed questions that structured their future exploratory activity within the task
- Some students needed to be asked more explicit questions to focus them on relevant task features (they required an expert other, Vygotsky, 1978)
- Highly-capable students employed some mental activities not employed by others
- Highly-capable students curtailed solution processes and remembered in generalities
- Other students retained specific values and procedures not general principles.

In summary, Krutetskii found that students presented with unfamiliar problems in interview situations and asked to think 'out loud' about them, employed 'mental activities' that led to the development of new knowledge that was sometimes conceptual and sometimes procedural in nature. Students who developed conceptual knowledge structured their exploration with spontaneous questions. 'Spontaneity' (e.g., Thornton, 1999; Steffe & Thompson, 2000) involves student activity, that is not directly caused by an expert other but that can result from situations set up by expert others. Spontaneous questions that progressively structure exploration have been identified during individual written responses to unfamiliar tasks (Cifarelli, 1999), individual student responses to technology-supported games (Kieran & Guzmàn, 2003), and group work (Williams, 2000a, 2002a).

LINKS BETWEEN STUDENT THINKING AND CONCEPTUAL TASKS

We know some task features can provide opportunities for students to engage in thinking that leads to new conceptual knowledge (e.g., Hershkowitz, Schwarz, & Dreyfus, 2001; Lampert, 2001; Williams, 2002b; Wood, Williams, & McNeal, 2006). For example, the student studied by Hershkowitz, Schwarz and Dreyfus (2001) used multiple representations in a technological environment (including graphs and tables) to develop an understanding of how the curve of the graph represented the rate of change. It is time to focus in more detail on the questions students ask themselves during these exploratory processes, and how new understandings develop as a result.

To study student thinking during the development of insight, I integrated the observable cognitive elements of the process of 'abstracting' (Hershkowitz, Schwarz, & Dreyfus, 2001) with Krutetskii's (1976) 'mental activities' (cognitive activities). Hershkowitz, Schwarz and Dreyfus (2001) found the genesis of an abstraction passes through (a) a need for a new structure; (b) the construction of a new abstract entity; and (c) the consolidation of the abstract entity in using it in further activities with increasing ease (Hershkowitz, Schwarz, & Dreyfus, 2001). Dreyfus, Hershkowitz and Schwarz (2001b) found three observable cognitive elements within the process of abstracting: 'recognising' (seeing what mathematics could be used to assist the exploration), 'building-with' (using this mathematics in unfamiliar ways to progress the exploration), and constructing (the integrating of mathematical ideas during the developing of insight). These processes are 'nested'

within each other. For example, when constructing new knowledge, 'recognising' the mathematics needed and 'building-with' this mathematics are both nested within the constructing process.

The cognitive activities associated with information processing during the problem solving activity of high ability students were student initiated and student controlled (Krutetskii, 1976) and fitted as sub-categories of the observable cognitive elements identified by Hershkowitz, Schwarz and Dreyfus (2001). From least complex to most complex, these cognitive activities have been described as 'analysis', 'synthetic-analysis', 'evaluative-analysis', 'synthesis', and 'evaluation'. The hierarchical nature of these thought processes are implicit in Krutetskii's descriptions of them and supported by his empirical data (Williams, 2000b, p. 18).

Krutetskii (1976) described analysis as a process of examining a problem element by element, commenting that "to generalise mathematical relations one must first dismember them" (p. 228). Analysis can involve recognising or both recognising and building-with (Williams, 2005). Simultaneous analysis of several diagrams, graphs, representations, procedures, or areas of mathematics for the purpose of making connections between them (synthetic-analysis), and making judgements as a result (evaluative-analysis) are subcategories of building-with. Krutetskii described synthesising as identifying "generality hidden behind various particular details" or "'grasp[ing]' what was main, basic, and general in the externally different and distinctive [and finding] elements of the familiar in the new" (p. 240). In other words, synthesising involves integrating what is known to form something new. He described evaluating as considering the mathematics generated in terms of its consistency with what is already known and also how these new ideas can be used for other purposes. Both synthesising and evaluating occur as part of the constructing process and recognising is nested within this constructing in various ways including recognising a new purpose for the mathematical structure just constructed. These thought processes are illustrated through the cases described in this chapter.

This chapter examines the activity of Kerri and Leon for the purpose of identifying the questions they formulated to structure their explorations, the progressive complexifying of thinking that resulted, the new mathematical structures developed, and how the types of thought processes supported this development. By examining these processes, we should learn more about how to design tasks to promote such activity.

RESEARCH DESIGN

Context

Of the eighty-six students whose thinking was studied in detail (Williams, 2005), only five students spontaneously created their own tasks. The two students whose self-created tasks were selected as illustrative cases for this chapter are Kerri and Leon who each created their own tasks on more than one occasion. These five students who created their own tasks (Williams, 2005) were the only students

identified developing new conceptual understandings rather than just learning new mathematical procedures. They came from schools that differed in perceived educational status, mathematical performances, cultural mix of the students, socio-economic status of school community members, and also in the teaching approaches used in their classes.

The other eighty-one students were not identified undertaking activity more complex than analysis. In other words, they did not simultaneously consider two or more pathways, representations, diagrams, mathematical topics (or some combination of these) for the purpose of making connections between them. Instead, they used mathematics that their teachers identified as relevant (recognised externally rather than spontaneously) to undertake procedures that the teacher had taught them to use (building-with but not spontaneous building-with).

The features common to the lessons in which students created tasks are:

- Mathematical topic commenced with exploration involving hands on activity
- Time to think without interruption
- Class members had the behavioural autonomy to think alone or with others.

Further descriptions of the activities that were common to these students, and personal factors that contributed to these activities can be found in Williams (2007a, 2006). It should be noted that the personal characteristic 'optimism' was found crucial to spontaneous thinking but is not a focus of this chapter.

The cases selected contained rich data to illuminate Kerri's and Leon's questions, thought processes, and insights, because both students were particularly reflective and articulate. These types of thought processes and structuring questioning were evident for the other three students (Eden, Dean, Pepe) but Dean did not progress beyond evaluative-analysis because he did not possess the cognitive artefacts needed to proceed further. Even so, Dean displayed thought processes more complex than analysis which was the most complex thinking identified in the activity of the other eighty-one students; most of whom were higher performers than Dean on their class tests.

Data Collection

The Learner's Perspective Study (LPS) (Clarke, 2006) research design was ideal for identifying students who created their own tasks, the questions they asked themselves, the thinking they undertook as they pursued their explorations, and the insights they developed. There were features of the LPS interview probes in the Australian interviews and some of the interviews in the USA that fitted with Ericsson and Simon's (1980) findings about how to generate high quality verbal data to study cognitive activity. These interviews: (a) stimulated student reconstruction of their lesson activity using salient stimuli (mixed image lesson video of the student [centre screen] and the teacher [in the corner]); (b) allowed the student to focus the content of the interview; (c) provided a sketch pad so that students who preferred to do so, could communicate using images and symbols to assist them; and (d) encouraged students to focus on lesson activity rather than on

general questions that were not related to their specific activity. This guarded against researchers asking specific questions that included constructs the subject had not previously reported and "generat[ing] answers without consulting memory traces" (Ericsson & Simon, 1980, p. 217).

In their video stimulated interviews, the students identified when new learning occurred during the lesson and reconstructed their thinking during that time. The interview was my primary data source; video analysis was informed by these students' reconstructing their classroom activity and what they had attended to in the classroom. Teacher interviews, interviews with other students, and photocopies of student work added detail where insufficient detail was available from the student interview and lesson video. The conceptual understandings of other students in the class (who had not created their own tasks but rather had focused on the task as set by the teacher) were studied using lesson video, student interviews, teacher interviews, and student worksheets. This was used to demonstrate the learning advantages of the student-created tasks.

Identifying Spontaneous Activity

Spontaneous thinking was considered to occur when the social elements of the process of abstracting (Dreyfus, Hershkowitz, & Schwarz, 2001a) were internal (Williams, 2004). In other words, when the student:

- *Controlled* the recognising process
- *Controlled* the mathematical directions they took
- *Explained* and *elaborated* mathematical ideas for themselves
- Made their own decisions about whether they *agreed* with or *queried* mathematics they had generated.

Thus, where social elements of the process of abstracting arose from an internal rather than external source, student thinking was taken to be spontaneous. The sixth social element of the process of abstraction (attention) was crucial to the spontaneous questions students formulated to structure their future explorations. Students attended to complexities that became evident during their work with the task and spontaneous questions arose from these foci (Williams, 2000a).

ILLUSTRATIVE CASES: STUDENT-CREATED TASKS

This section describes Kerri's self-created task about linear functions, and Leon's self-created task about areas of triangles, and the subsequent activity of each student through:

- A narrative of student activity as they formulated and solved their task
- The types of questions they asked to structure their exploratory activity
- How these questions complexified their thinking and how this thinking contributed to developing new mathematical structures.

Narrative of Kerri's Linear Functions Task

Before the research period, Kerri's class was taught to find equations of linear graphs when two points were given by:

- Plotting two points
- Ruling a line through them
- Drawing a right-angled triangle with the line segment between points as hypotenuse
- Measuring the lengths of the other two sides of this triangle ('rise' and 'run')
- Finding the gradient of the line (ratio of rise to run)
- Finding the y-intercept of the graph by inspection
- Substituting the gradient and y-intercept into $y = mx + b$ to find the equation of the line.

When the students were tested on this procedure just prior to the research period, Kerri forgot her graph paper so she found another way to proceed. The following night while doing her homework, her understandings crystallised. The next day in class, the teacher taught procedures that were associated with Kerri's new understandings: finding the linear equation when given two points 'without using graph paper'.

In her interview after Lesson1, US School 3, Kerri reconstructed what she had done in the test and the insights she had developed during her subsequent homework. In her test, Kerri had:

- Made a sketch of the line between the two points
- 'Seen' a slope triangle "cuz you can picture a line in a little right triangle on it"
- Subtracted y values and subtracted x values of co-ordinates of the points to find lengths
- Calculated the gradient as the ratio of these lengths
- Substituted the gradient and a set of co-ordinates into the equation to find the y-intercept
- Found the equation by substituting the gradient and y-intercept into $y = mx + b$.

The method Kerri developed in her test involved steps she had previously been taught and new mathematical ideas she developed. Work she had undertaken previously included: a) calculating the gradient as a ratio, and b) substituting the gradient and the y-intercept into the general equation. New mathematical ideas she developed included:

- Recognised a right-angled triangle could be drawn on sketch of a line between the points
- Used her understanding of the Cartesian Coordinate System to find lengths a new way
- Combined previous knowledge of substituting constants *or* values of variables to find an unknown by substituting a constant *and* values of variables.

Kerri was pleased with what she achieved:

Kerri If you find the slope .... [using the] difference of the points and .... then we can substitute, oh perfect. So I just wrote the equation.

Key to symbols used in transcripts in this chapter

.... Omitted text that does not alter the meaning of the quote

[text] Comments and annotations, often descriptions of non-verbal action

By creating new ways of working mathematically, Kerri developed a procedure that enabled her to answer the test question. At this stage, she had not fully realised the implications: that the horizontal and vertical lengths could be found without the need for a diagram. When doing homework after the test, she gained insight. This homework involved plotting graphs and measuring lengths to find equations to lines when two points were given, and in addition, measuring the length of the line segment between the points given. Whilst doing her homework, Kerri measured the rise and the run on the graph paper for each question as required by the teacher, and simultaneously calculated the lengths using her own method. This thinking is an example of synthetic-analysis because there was simultaneously focusing on two methods; she measured and calculated at the same time, and thought about both methods as she did so. As a result, she made a judgement (evaluative-analysis) that each method always gave the same answer:

Kerri I was doing my graph, and then I like realised like- really solidly, … I got the same answer, … [by measuring as] if you do the subtraction.

Kerri's understanding crystallised (synthesis): she realised that she did not need a diagram because her operations on the numerical values of the coordinates of two points always gave the measure of the lengths of horizontal and vertical lines separating them. She 'saw' the equivalence of attributes (line length) in the numerical and graphical methods and integrated the representations as a result.

Once, she realised that the Cartesian Axes System could be used as a tool to find vertical and horizontal lengths, she used this new insight along with Pythagoras' Theorem to extend the usefulness of this tool: the Cartesian Axes System could be used as a tool to find lengths of any segments where the coordinates of the endpoints were known. Kerri completed her homework:

Kerri And then also … we had to find the distance between the two plots, and it was supposed to graph them too-… I was using Pythagoras' Theorem.

What Kerri knew 'really solidly' about finding lengths using coordinates was evident in class the next day when she queried whether the teacher was 'Finding the Equation of a Line without a Graph' as stated in the heading on the board. The teacher had demonstrated her procedure by making a sketch and using it to find the lengths by subtracting x values and y. Kerri queried the teacher's procedure: "You still graphed it". Kerri explained that it could be done without the sketch because the gradient could be found by operating on elements of the ordered pairs representing the points on the line:

Kerri It would just be like the difference in y is two, and the difference in x is one. So that's [what you need to calculate] your slope.

Towards the end of this lesson, the other members of Kerri's group plotted graphs and measured line segments and queried the validity of Kerri's approach to finding the distance between the two points on the line using Pythagoras. In her interview, Kerri described difference between what she was doing and what other students were doing thus demonstrating her deep understanding:

Kerri [The questions] said graph and find the distance- and most people would graph the line, and then do the little thing [right-angled slope triangle]. But I would find what- see that'd be two and then one [subtracting *y* values, and then *x* values in co-ordinates], so you do um, *a* squared plus *b* squared equals *c* squared. … if you make it a right triangle- it's the hypotenuse- not just the distance

Kerri's comments showed her generalised understanding: she could find rise and the run by operating on the x and y values in the co-ordinates, and saw the equivalence of the algebraic expressions for the hypotenuse (from Pythagoras' Theorem), and the length of the line segment between the two points. In other words, she had subsumed the line segment length into the algebraic expression for the hypotenuse thus extending how the Cartesian Axes System could be used as a tool for finding lengths by operating with x and y values of co-ordinates. The other students in this class had not developed such understandings. They still needed to plot and measure.

Types of Questions Kerri Asked

During her spontaneous explorations prior to the lesson described above, Kerri progressively asked questions to structure her future activity. I use 'future' as used by Cifarelli (1999) to capture the need for these questions to structure the way forward within an exploration that had already commenced.

In the test, when she found she did not have graph paper and so could not measure to find lengths, she asked herself "What can I do instead?" Once she had sketched the two points and the line between them, she analysed her sketch to find mathematics relevant to the situation: "What maths can I use to help?" She recognised she could use her knowledge of the Cartesian Axes System to find the vertical and horizontal lengths and proceeded to do so (recognising, and building-with).

Table 1. Types of questions Kerri posed and how they contributed to her developing understanding

Questions in Order Posed	*What Was Involved in Answering the Question?*	*Complexity of Thinking Involved in Answering this Question*
"What can I do instead?"	Drew a sketch and marked given information on it. Recognised lengths could not be measured.	Analysis: recognised the same diagram could be drawn as in the graphing method.
"What maths can I use to help?"	Recognising the difference between what was known in this instance and what was able to be found previously. This time the lengths were not known.	Analysis: Considered what was known in the previous situation that was not known here and looking for mathematics that could help find what was unknown this time.
"Can it help me find those lengths that I cannot find the other way?"	Recognised the properties of the Coordinate Axes System could be useful and used a numerical representation in conjunctions with the Cartesian Axes System to find lengths.	Building-with (synthetic-analysis) by simultaneously considering the sketch, the Cartesian Axes System, and a numerical representation of these, a relevant procedure was developed.
"Do both methods give the same answer?" "Does it always work?"	Simultaneous considering of operating with numbers in the Cartesian Axes System and measuring side lengths in graphical representation.	Evaluative-analysis: Synthetic-analysis for the purpose of making decisions about reasonableness, and comparability of methods.
"Is there an easier way?" "Can the process be curtailed?"	Saw the equivalence of attributes (side lengths and operations on x and y values in coordinates of points). Realised she no longer needed to diagram to find lengths.	Synthesis: Subsuming the side lengths into an algebraic formula using x values and y values in coordinates.
"Can this be used for anything else?"	As the length of the line segment between the two points is the hypotenuse of a right-angled triangle where the other two side lengths can be found, Pythagoras' Theorem can be used in conjunction with this new insight to find the length of the line segment.	Evaluation: Using the new insight for another purpose. Subsuming the hypotenuse in Pythagoras' Theorem into the slope triangle by 'seeing' the equivalence of the hypotenuse and the line segment between the two points and extending the numerical operations used to find line segments within the Cartesian Axes System (synthetic-analysis and synthesis nested within evaluation)

This involved simultaneous analysis of the sketch, and its placement in the axes system, (synthetic-analysis as part of building-with) to answer the question "Can it help me find those lengths I cannot find the other way?" Once she had recognised the Cartesian Axes System was relevant, she formulated appropriate numerical operations to find the lengths she could not measure. Crucial to Kerri developing insight was her activity during her homework that night. She compared answers she generated by the teacher's method with answers she generated by her own method and decided they were always the same (evaluative-analysis): "Do both methods give the same answer?" "Does it always work?" Her thinking crystallised at this stage (synthesis). She realised she had something that always worked that could be expressed generally through an algebraic representation. Kerri made judgments based on her simultaneous analysis of four representations (diagrammatic, Cartesian Axes System, numeric, and algebraic) (evaluative-analysis) by asking questions of the nature of: "Is there an easier way?" "Can the process be curtailed?" She subsumed the other representations into the algebraic representation, and 'realised really solidly' that she did not need the diagram (synthesis) because she recognised the equivalence of the line lengths, and the algebraic representations derived from her knowledge of the Cartesian Axes System.

Kerri's thinking was curtailed (Krutetskii, 1976) when she operated on the values in the coordinates without needing the diagram and was able to express this generally (synthesis as part of constructing). Kerri continued to think further once she had developed insight: "Can this be used for anything else?" She rapidly developed an additional insight: she recognised the relevance of Pythagoras' Theorem and subsumed line segment into the algebraic representation of the hypotenuse of the right-angled slope triangle because she 'saw' their equivalence.

Kerri Complexifies Her Thinking through Structuring Questions

Unlike the questions that Cifarelli (1999) identified that were specific to the problem at hand, most of the questions Kerri asked had broader applicability. They included:

a) What can I do instead?
b) What mathematics could help?
c) What does this tell me?
d) Will it always work?
e) Is there a simpler way?
f) Can I use what I have found for anything else?

Questions a) and b) elicit analysis of the context, Question c) elicits evaluative-analysis with synthetic-analysis nested within it, whilst Question e) elicits thinking about the connecting of ideas that could result in synthesis, and Question f) elicits evaluation. Task features that stimulated this constructing process included the absence of a resource used previously (graph paper), and the possibility to use the mathematical context differently by recognising other relevant mathematics (the

Cartesian Axes System). The hands on activity prior to the test supported Kerri's creative activity because she had become familiar with the mathematics involved with the context, and the representation she had worked in earlier provided opportunities to consider alternative pathways.

Narrative of Leon's Area of Triangles Task

Prior to Australian School 1, Lesson 12 in which Leon created the task under study, the class had found the areas of their hands by tracing them, then drawing and counting squares. In Lesson 12, the teacher:

- Placed three large coloured triangles on the board (see Figure 1)
- Allocated triangles to pairs of students
- Asked pairs to find the area of their triangles without using a rule.

The class did not know the rule for finding areas of triangles and all students except Leon focused on counting squares. This was probably because they had just completed such an activity. Leon and Pepe worked on Triangle 1 and although Pepe counted squares for the pair to produce their solution, Leon searched for a faster way to proceed. Once Pepe commenced work on the task Leon did not write or draw anything but instead focused idiosyncratically on the three triangles on the board asking himself: "which triangle is easiest [to find the area of]". This question elicited synthetic-analysis through simultaneously considering the triangles in Figure 1, and evaluative-analysis in making the judgement that Triangle 2 was the easiest. He then developed Method A:

- Juxtapose two right-angled triangles to form a rectangle
- Find its area
- Halve this to find the area of the triangle.

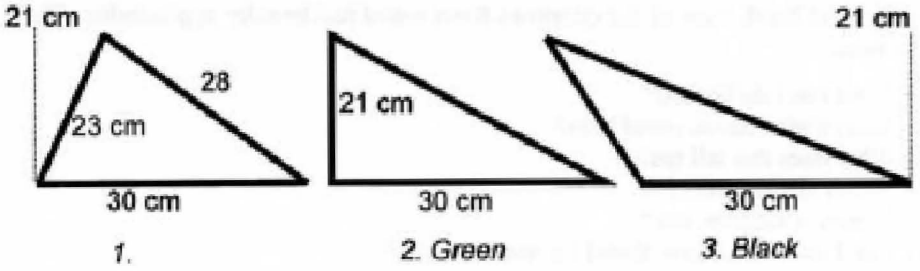

Figure 1. Triangles placed on the board in Leon's Class

Leon reconstructed his thinking about how to find the area of a right-angled triangle in his post-lesson interview: "Figure out what a rectangle is that has ... [that] length and width and ... then you can just halve it". He did not rely solely on

visual images to justify the shapes formed: he considered the figures' properties as well.

To try to find the area of the acute-angled triangle (Method B), Leon drew upon one of the strategies he used in Method A (juxtaposing triangles). He:

- Juxtaposed two acute-angled triangles
- Recognised the shape was a parallelogram by its properties
- Sectioned the parallelogram into four right-angled triangles to find its area (Figure 2, top left-hand diagram)
- Used Method A four times to find the areas of the four rectangles made by juxtaposing right-angled triangles (see Figure 2)
- Total areas and halve result.

Figure 2 shows the parallelogram sectioned into four right-angled triangles as indicated by Leon's statement "figure out what it would be if it was four" and the rectangles formed by juxtaposition of another congruent right-angled triangle beside each (multiple use of Method A as part of Method B) to form four rectangles. He had not written or drawn anything whilst undertaking this activity. He was trying to find the area through multiple calculations of areas of rectangles, totalling them, "and then halve it".

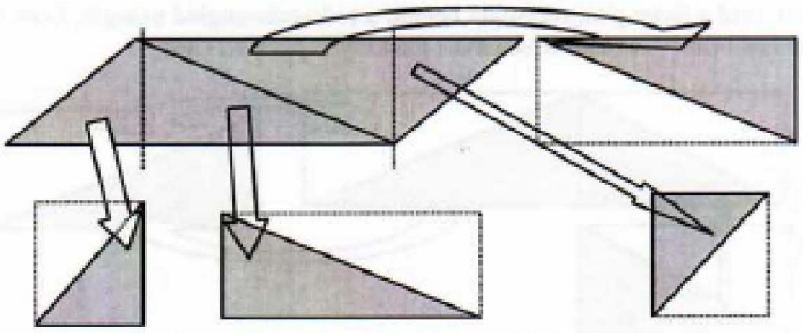

Figure 2. Interpretation of Leon's Method B

Leon expressed the fragility of his thinking about Method B in his interview: "I understood it – I didn't understand it then I understood it then I didn't understand it". The process had become difficult to retain in his head.

Before he had completed his calculations, he 'looked-in' on what was happening around him, and wondered: "Ooh! Maybe my method isn't the best". The term 'looking-in' (Williams, 2004) was used by Leon to describe his focusing on dynamic visual displays generated by others and extracting mathematical ideas implicit within them. It is an idiosyncratic student activity that can occur when a student does not possess appropriate 'cognitive artefacts' to progress their exploration. Leon described looking-in as a common part of his classroom activity:

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Leon        When you look around the classroom and see how everyone
            else is doing it and you are doing it a completely
            different way- … and you think ooh! [soft] maybe my method
            isn't the best and … you think about everyone's ... and
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then you think about your own and they all sort of piece together and you just sort of go oh! and it pops into your head.

Leon described how he used dynamic visual displays visible around him in class in conjunction with the ideas he was developing himself to 'see' something new:

Leon People were drawing the actual rectangles around it [triangles] - I don't know whether they knew they [triangles] were coming from rectangles … the way they were drawing it made it look like they did … made me think about it.

As other students drew grids across their triangles in preparation for counting squares, they formed rectangles during this process. Leon knew they probably had not realised what he saw when he looked-in. He was the only student identified using methods other than counting squares for the acute-angled triangle. As Leon 'looked-in' his goal changed from laboriously *finishing* the work to *understanding* the work. By looking-in, he extracted a big idea that he had not previously been aware of: "triangles come in rectangles" and wondered how he could use this.

Leon was excited by what he had found. He might no longer need to section a parallelogram and laboriously apply Method A four times, total areas, and halve. There could be a more 'elegant' way. Half way through Lesson 13, when the teacher held a large pink rectangle behind a red acute-angled triangle, Leon softly exclaimed "Oh!" He reconstructed his thinking in his interview:

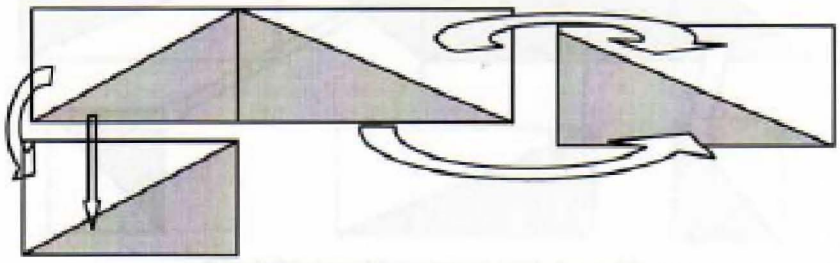

Figure 3. Leon applied Method A twice within his elegant Method

Leon I sort of- sort of thought a little bit about why it was happening [triangle area half rectangle area] … if you take one part out like a triangle that's set at an angle if you take both parts out and put them together it equals the rectangle. … I was sort of looking at them and then I just realised, like I (pause) sort of just in my head I pulled it apart and put them together so that they equalled the same (see Figure 3).

Figure 3 represents the mental images Leon generated. He mentally moved the two right-angled-triangle sections of the acute angled triangle (shaded parts) out of the larger rectangle. The triangles congruent to each of these shaded sections were then taken out of the initial diagram and juxtaposed with the appropriate shaded triangle to make two smaller rectangles: "if you take one part out like a triangle that's set at an angle if you take both parts out and put them together it equals the rectangle". He had simultaneously drawn upon what he knew from Method A and shifted

pieces in his head to justify that the area of the acute angled triangle was always half the area of the enclosing rectangle.

Like Kerri, Leon realised almost immediately that his new insight could be used for something else. He clapped his hand against his cheek as he realised different shaped triangles had the same area if they were enclosed by rectangles of the same size. The following excerpt of whole class discussion captures Leon's justification:

Leon	That's [Triangle 1] half of the rectangle as well [as Triangle 2]
Leon	That would be three hundred and thirty centimetres squared
Teacher	Three hundred and thirty. Why?
Leon	It would be exactly the same as the first one [Triangle 2]
Teacher	Why?
Leon	Because the green one is half of the rectangle too [see Figure 3]

Leon explained that both Triangle 1 and Triangle 2 have areas that are half of equivalent rectangles so they must be the same size. The exercise set by the teacher towards the end of Lesson 13 involved finding areas of triangles in different orientations. The teacher found that (other than Leon) the class struggled to find areas of these triangles: "I *assumed* ... they knew ... base and height ... and how to recognise it ... [it's] *very obvious* ... they don't understand". The other students, including those with higher mathematical performances on class tests, knew the rule but not why it worked (Skemp, 1976) so were not able to 'see' the perpendicular height that Leon could recognise easily.

Wertheimer (1959) identified similar problems with students not recognising perpendicular heights in parallelograms. Leon knew more than the rule, he knew why it worked. Leon could do examples with triangles in any orientation because he could 'see' the perpendicular height of the triangle (even though he did not use this terminology). Leon had subsumed the attributes of the rectangle into equivalent attributes of triangles so he could operate with attributes of triangles instead, and not need to draw the enclosing rectangle. The questions Leon asked himself to structure the future parts of his exploration, and the more complex thinking that resulted are captured in Table 2.

Types of Questions Leon Asked

Unlike Kerri, whose synthetic-analysis involved considering the same attributes in graphical, diagrammatic, numerical, and algebraic representations, Leon's early synthetic-analysis focused on more than one diagrammatic representation arising from his question: "Which triangle is easiest?" Leon's quest for elegance led to his questioning whether there was an easier way (synthetic-analysis nested within evaluative-analysis) once his calculations in Method B became too messy to easily undertake in his head.

Table 2. Types of questions Leon posed and how they contributed to his developing understanding

Questions in Order Posed	*What Was Involved in Answering Them?*	*Complexity of Thinking Involved in Answering this Question*
"Which triangle is easiest?"	Analysed the three triangles on the board to find which was the easiest area to find.	Analysis: recognising that two right-angled triangles put together make a rectangle (knowing properties of rectangles).
"What maths could help?"	Recognised rule for finding areas of rectangles could be used and the answer halved to find the area of the right-angled triangle.	Analysis: Considering the use of previously known processes one after the other to find an answer (building-with but not yet synthetic-analysis because procedures occur one after another).
"Can I use similar strategies here?"	Recognised parallelogram was formed when two acute-angled triangles were juxtaposed but had not found the areas of parallelograms previously.	Building-with (synthetic-analysis) by simultaneously considering the acute angled triangles and the properties that showed a parallelogram was formed.
"What maths can I use to help?"	Recognised the previously developed process to find the area of a right-angled triangle could be used. Made four right-angled triangles within the parallelogram (drop perpendiculars from two vertices to opposite sides). Process became too messy to easily complete in his head.	Synthetic-analysis: Considered what had been previously developed that might be useful to this new situation. Analysis nested within: Considered the use of previously known process one after the other to find an answer
"Is there an easier way?"	Simultaneously considered ideas developed and diagrams produced by others to form grids for counting squares. Insight: Triangles always come in rectangles.	Synthesis: Knew right triangles were useful to him and that there was always a rectangle that could be made to enclose any triangle.
Can 'triangles always come in rectangles' help me to find the area of any rectangle?	Recognised two right-angled triangles are embedded in the diagram so knew how to find the area of each. In addition, the triangles needed for the juxtaposing process were also within this rectangle. Thus, the area of this rectangle can be halved to find the area of the triangle.	Synthesis: Insight that the rectangle is not needed to find the area of the acute angled triangle because there are attributes of this triangle that are equivalent to the length and width of the rectangle. Subsumed attributes (length, width) of one representation into another (base, perpendicular height).
"Can this be used to save work in finding the areas of the other triangles?"	Simultaneous considering another triangle with the same width and height and his new insight. As the two triangles had the same enclosing rectangle, they should have the same area.	Evaluation: Using the previously developed insight for another purpose

Although Leon did not use algebra to express what he had found, his generalised understanding was indicated by his insight that triangles enclosed by the same rectangle having the same area. Task features that stimulated Leon's creating of a conceptual task were: the display of three very different triangles with the same areas, the opportunity to look-in, knowing the answer but not the reason (from empirical explorations of others), and the teacher's juxtaposition of an acute angled triangle with its enclosing rectangle in sharply contrasting colours. As with Kerri, the task set by the teacher (that was not intended to elicit creative thinking) lead to the idiosyncratic formulating of a student-created conceptual task.

Leon Complexifies His Thinking through Structuring Questions

Table 2 shows the complexifying of thinking that was stimulated by the questions Leon asked himself. Again, as for Kerri, Leon subsumed some representations within others. In Leon's case he subsumed the attributes of the rectangle into the triangle and this occurred because the synthetic-analysis and evaluative-analysis he undertook brought him closer and closer to a realisation of this equivalence as he found the enclosing rectangle was useful for different types of triangles.

CONCLUSIONS

This chapter highlights student-formulated questions that elicit complex thinking that supports the development of new mathematical structures. It suggests pedagogical advantages to integrating these types of questions into tasks to scaffold student entry to idiosyncratic exploration. In these cases, students exploring self-created tasks developed deeper understandings than those who undertook the task as set by the teacher. The finding (Williams, 2005) that a student who struggled in mathematics could think creatively to develop a greater understanding than other students with higher performances on class tests (Williams, 2005) negates assumptions that only highly able students should be presented with rich explorations.

The process of evaluative-analysis appeared crucial to the process of subsuming representations to form new mathematical structures. Questions eliciting this type of thinking were focused around finding more elegant ways to proceed (Leon), and checking the reasonableness of mathematics generated (Kerri). This seemed to highlight the equivalence of attributes that was needed for the subsuming process (synthesis). The questions these students asked themselves could be built in to tasks to scaffold students who are not yet able to ask such questions for themselves. This should increase the likelihood that students undertake synthetic-analysis and evaluative analysis and this should support synthesis. Increasing looking-in opportunities could be achieved by structuring reporting sessions *during* a task, rather than only after the task (e.g., Williams, 2007b). Introducing these aspects into task design should increase opportunities for students to discovering mathematical complexities that were not evident to them at the start of the task and

this could provide the impetus for student-created tasks. The following section uses these findings to suggest a possible task and the rationale for different features.

SPECULATING: A CONCEPTUAL TASK ABOUT AREAS OF TRIANGLES

A conceptual task could be built around the triangles in Figure 1 because they look so different yet their areas are the same so there is potential for surprise that could create the impetus to explore. Structuring a task that requires students to predict the areas, thus commit to a position, could create this element of surprise. For students of differing abilities to have a chance to access the task, it should be set prior to finding areas of:

- Irregular shapes by counting squares
- Triangles using the rule $A = bh$.

The background knowledge needed includes a conceptual rather than procedural understanding of area as the amount of space within the boundaries of a two-dimensional figure. The task could be undertaken with or without knowledge of how to find areas of rectangles because it is possible for students to find these areas without formal knowledge. Thus, the type of task suggested could be appropriate in late elementary school or early secondary school. The type of wording in the questions below is intentionally predominantly common language and tentative, rather than demanding and technical. This should increase the likelihood of informal exploration. Questions like the following could be embedded in the task to stimulate idiosyncratic thinking:

- Predict the areas of the three triangles giving reasons for your predications
- Find a way to find the areas using any method that you can explain
- Find a way to quickly check that your answers are reasonable (explain how you did this)
- Can you see any patterns? Describe them
- Can you add another triangle that fits this pattern? Explain and test
- Can you work out why this pattern works?
- Work out how to tell a friend what you have found as simply as possible.

This task includes the ideas behind the structuring questions that Leon and Kerri asked. The first dot point should elicit: "What can I use?" The second dot point: "What mathematics might help?" The third dot point stimulates evaluative-analysis through comparing of two methods to make a judgment. The fourth dot point elicits synthetic-analysis through the search for patterns by considering various aspects simultaneously. The fifth dot point involves making a judgment (evaluative-analysis) and the sixth dot point can elicit synthesis because knowing why involves developing a new mathematical structure. The seventh dot-point is intended to assist students to express what they know by asking them to focus on communicating their ideas to a friend. Some students are likely to create their own tasks earlier than others because the mathematics needed earlier in the task will be unfamiliar to some. Student-created tasks, elicited by this classroom task, might

focus around topics including: constructing triangles, counting squares, approximating when counting squares, areas of composite shapes, juxtaposing shapes to find ways to find areas, recognising the significance of enclosing rectangles, areas of triangles, and areas of parallelograms.

Even though the study upon which this chapter was based identified only eight student-created tasks, the diversity of the educational settings, pedagogical approaches, and student abilities adds strength to the usefulness of these student-formulated questions for task design in mainstream classes.

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